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A BIFURCATION – TYPE THEOREM FOR SINGULAR NONLINEAR ELLIPTIC EQUATIONS*

NIKOLAOS S. PAPAGEORGIOU[†] AND GEORGE SMYRLIS[†]

Abstract. We consider a parametric nonlinear Dirichlet problem driven by the p-Laplacian and exhibiting the combined effects of singular and superlinear terms. Using variational methods combined with truncation and comparison techniques, we prove a bifurcation - type theorem. More precisely, we show that there exists a critical parameter value $\lambda^* > 0$ s.t. for all $\lambda \in (0,\ \lambda^*)$ (λ being the parameter) the problem has at least two positive smooth solutions, for $\lambda = \lambda^*$ the problem has at least one positive smooth solution and for $\lambda > \lambda^*$ the positive solutions disappear.

Key words. Singular term, superlinear term, weak and strong comparison principles, bifurcation type theorem, positive solution.

AMS subject classifications. 35J20, 35J25, 35J67.

1. Introduction. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Dirichlet equation with singular terms

$$\left\{ \begin{array}{c} -\Delta_p u(z) = \xi(z)u(z)^{-\eta} + \lambda f(z,u(z)) \quad \text{in } \Omega, \\ u\,|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0, \quad \eta > 0. \end{array} \right\} \qquad (P)_\lambda$$

Here Δ_p denotes the p -Laplacian differential operator defined by

$$\Delta_p u(z) = \mathrm{div}(\ ||Du(z)||^{p-2}Du(z)) \quad \text{for all} \quad u \in W_0^{1,p}(\Omega), \quad 1 < p < \infty.$$

Also $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ a.e. in Ω, $\xi \neq 0$, $\lambda > 0$ is the parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory perturbation (i.e., for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous). Let $F(z,x) = \int_0^x f(z,s)ds$ (the primitive of $f(z,\cdot)$). We assume that for a.a. $z \in \Omega$, $F(z,\cdot)$ is p -superlinear near $+\infty$. This is the case, if we assume that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$ -superlinear near $+\infty$. However, we do not employ the usual in such cases Ambrosetti-Rabinowitz condition (AR-condition for short). Instead, we use a weaker "superlinearity" condition which permits the consideration of perturbations $f(z,\cdot)$ with "slower" growth near $+\infty$. Therefore, in problem $(P)_\lambda$ we have the combined (competing) effects of singular and superlinear terms. We prove a "bifurcation-type" theorem describing the dependence of the positive solutions of $(P)_\lambda$ on the parameter $\lambda > 0$.

More precisely, we show that there exists a critical parameter value $\lambda^* > 0$ such that for all $\lambda \in (0,\ \lambda^*)$, problem $(P)_\lambda$ has at least two nontrivial solutions, for $\lambda = \lambda^*$ there exists at least one solution and the solutions of $(P)_\lambda$ disappear when $\lambda > \lambda^*$.

Our result is analogous to the ones concerning parametric elliptic equations involving the combined effects of concave and convex nonlinearities proved by Garcia Azorero-Manfredi-Peral Alonso [6], Guo-Zhang [10], Hu-Papageorgiou [12] and

*Received August 29, 2012; accepted for publication July 3, 2014.
[†]National Technical University of Athens, Department of Mathematics, Zografou Campus, Athens 157 80, Greece ({npapg; gsmyrlis}@math.ntua.gr).

Papageorgiou-Smyrlis [18]. For singular equations, such problems were studied by Coclite-Palmieri [4], Crandall-Rabinowitz-Tartar [5], Hirano-Saccon-Shioji [11], Lair-Shaker [14], Lazer-MacKenna [15], Sun-Wu-Long [21] for $p = 2$ (semilinear equations) and Giacomoni-Schindler-Takac [8], Perera-Zhang [19] for $p \neq 2$ (nonlinear equations). However, in the aforementioned works, the authors either they do not prove the precise dependence on the parameter $\lambda > 0$ (i.e., they do not prove a bifurcation-type result) or they have a perturbation of very special form (i.e., $f(z,x) = f(x) = x^{r-1}$ with

$$p < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if} \quad p < N \\ +\infty & \text{if} \quad p \geq N \end{cases})$$

Our approach is variational based on the critical point theory and uses also truncation and comparison techniques. In the next section, for the convenience of the reader, we recall the main mathematical tools that we will be used in this paper.

2. Mathematical background – Auxiliary results. Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Suppose that $\varphi \in C^1(X)$. We say that φ satisfies the Cerami condition (the C- condition for short), if the following holds:

"Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||x_n||)\varphi'(x_n) \to 0 \ \text{ in } \ X^* \ \text{ as } \ n \to \infty,$$

admits a strongly convergent subsequence".

This compactness-type condition, is in general weaker than the more common Palais-Smale condition (PS- condition for short). However, the C- condition suffices to prove a deformation theorem and from it derive the minimax theory for certain critical values of $\varphi \in C^1(X)$ (see, for example, Gasinski-Papageorgiou [7]). In particular, we can state the following theorem, known in the literature as the "mountain pass theorem".

THEOREM 1. *If $\varphi \in C^1(X)$ satisfies the C- condition, $x_0, \ x_1 \in X, \ ||x_0 - x_1|| > \rho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf [\ \varphi(x) : ||x - x_0|| = \rho\ ] = \eta_\rho$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t)) \ \ \textit{where} \ \ \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0 \ , \ \gamma(1) = x_1 \ \},$$

<u>then</u> $c \geq \eta_\rho$ and c is a critical value of φ.

Consider the nonlinear map

$$A : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^* = W^{-1,\,p'}(\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$

defined by

$$\langle A(u),\ y\rangle = \int_\Omega ||Du||^{p-2}(Du, Dy)_{\mathbb{R}^N} dz \quad \text{for all } \ u, y \in W_0^{1,p}(\Omega). \tag{1}$$

PROPOSITION 2. *The map* $A : W_0^{1,\,p}(\Omega) \to W^{-1,\,p'}(\Omega)$ *defined by (1) is continuous, strictly monotone (hence maximal monotone too), bounded (i.e., maps bounded sets to bounded ones) and of type* $(S)_+$, *i.e., if* $u_n \xrightarrow{w} u$ *in* $W_0^{1,p}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle A(u_n),\, u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* $W_0^{1,p}(\Omega)$.

REMARK. The above result remains true, if we view A as a map from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)^*$.

In the analysis of problem $(P)_\lambda$, in addition to the Sobolev spaces $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$, we will also use the ordered Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\,|_{\partial\Omega} = 0\}.$$

The positive cone of $C_0^1(\overline{\Omega})$ is given by

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \ge 0, \text{ for all } z \in \Omega\,\}.$$

This cone has a nonempty interior given by

$$\mathrm{int}C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega \right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

In what follows, by $\widehat{\lambda}_1$ we denote the first eigenvalue of the negative Dirichlet p -Laplacian. We know that $\widehat{\lambda}_1$ admits the following variational characterization

$$\widehat{\lambda}_1 = \inf\left[\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ \ u \ne 0\right]. \tag{2}$$

The infimum in (2) is realized on the eigenspace of $\widehat{\lambda}_1$. We know that $\widehat{\lambda}_1 > 0$ and it is simple (i.e., the corresponding eigenspace is one dimensional) and isolated. Let $\widehat{u}_1 \in W_0^{1,p}(\Omega)$ be the L^p -normalized eigenfunction corresponding to $\widehat{\lambda}_1 > 0$ (i.e., $||\widehat{u}_1||_p = 1$). It is clear from (2) that $\widehat{u}_1$ has a constant sign and we take $\widehat{u}_1 \ge 0$. Nonlinear regularity theory (see, for example, Gasinski-Papageorgiou [7] (pp. 737-738)) and the nonlinear maximum principle of Vazquez [23], imply that $\widehat{u}_1 \in \mathrm{int}C_+$.

Suppose $\underline{u}$, $\overline{u} \in \mathrm{int}C_+$ with $\underline{u} \le \overline{u}$ and consider

$$\beta(z,x) = \begin{cases} \xi(z)\underline{u}(z)^{-\eta} & \text{if} \quad x < \underline{u}(z) \\ \xi(z)x^{-\eta} & \text{if} \quad \underline{u}(z) \le x \le \overline{u}(z) \\ \xi(z)\overline{u}(z)^{-\eta} & \text{if} \quad \overline{u}(z) < x \end{cases} \qquad \text{with } 0 < \eta < 1.$$

Evidently this is a Carathéodory function. We set $B(z,x) = \displaystyle\int_0^x \beta(z,s)ds$ and consider the functional $\sigma : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) = \int_\Omega \beta(z,u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next proposition is useful in establishing the continuous differentiability of the energy functionals that we consider in the sequel.

PROPOSITION 3. *$\sigma \in C^1(W_0^{1,p}(\Omega))$ and $\sigma'(u) = \beta(\cdot, u(\cdot))$ for all $u \in W_0^{1,p}(\Omega)$.*

Proof. Let $u, y \in W_0^{1,p}(\Omega)$ and $\vartheta \in \mathbb{R} \setminus \{0\}$. From the integral form of the mean value theorem we have

$$\frac{1}{\vartheta}[\,\sigma(u + \vartheta y) - \sigma(u)\,] = \int_\Omega \left(\int_0^1 \beta(z, u + s\vartheta y) ds \right) y dz. \tag{3}$$

We know that

$$\int_0^1 \beta(z, u + s\vartheta y) ds \to \beta(z, u) \quad \text{for a.a. } z \in \Omega \text{ as } \vartheta \to 0. \tag{4}$$

Also, we have

$$\begin{aligned}
&\int_0^1 \beta(z, u + s\vartheta y) ds \\
&= \xi(z)\underline{u}(z)^{-\eta} \mathcal{X}_{\{u+s\vartheta y\,<\,\underline{u}\,\}}(z) + \mathcal{X}_{\{\underline{u}\,\le\,u+s\vartheta y\,\le\,\overline{u}\,\}}(z) \int_0^1 |u + s\vartheta y|^{-\eta} ds + \\
&\quad + \xi(z)\overline{u}(z)^{-\eta} \mathcal{X}_{\{\overline{u}\,<\,u+s\vartheta y\,\}}(z) \le \\
&\le 2\xi(z)\underline{u}(z)^{-\eta} + c_\eta \left(\max_{0 \le s \le 1} |u + s\vartheta y|^{-\eta} \right) \mathcal{X}_{\{\underline{u}\,\le\,u+s\vartheta y\,\le\,\overline{u}\,\}}(z) \\
&\quad \text{for some } c_\eta > 0 \ \text{(since } \underline{u} \le \overline{u} \text{ and from Takac [22] (Appendix))} \\
&\le 2\xi(z)\underline{u}(z)^{-\eta} + c_\eta\, \underline{u}(z)^{-\eta} \\
&\le \widehat{c}_\eta\, \underline{u}(z)^{-\eta} \quad \text{for a.a. } z \in \Omega \text{ and some } \widehat{c}_\eta > 0.
\end{aligned} \tag{5}$$

Let $d(z) = \text{dist}(z, \partial\Omega)$. Since $\underline{u} \in \text{int}C_+$ we can find $c_1 > 0$ s.t. $\underline{u}(z) \ge c_1 d(z)$ for all $z \in \overline{\Omega}$. We have

$$\begin{aligned}
\widehat{c}_\eta\, \underline{u}(z)^{-\eta}\, y(z) &= \widehat{c}_\eta\, \underline{u}(z)^{1-\eta}\, \frac{y(z)}{\underline{u}(z)} \quad (\text{since } \underline{u} \in \text{int}C_+\,) \\
&\le \frac{\widehat{c}_\eta}{c_1} \frac{y(z)}{d(z)}, \\
&\Rightarrow \widehat{c}_\eta\, u(\cdot)^{-\eta}\, y(\cdot) \in L^p(\Omega)
\end{aligned}$$

(by Hardy's inequality, see Brezis [3] (p.313)).

So, from (4) and (5), we see that we can apply the Lebesgue dominated convergence theorem and have

$$\begin{aligned}
&\langle \sigma'(u),\ y \rangle = \int_\Omega \beta(z, u(z)) y(z) dz \quad \text{for all } u,\ y \in W_0^{1,p}(\Omega), \\
&\Rightarrow \sigma \in C^1(W_0^{1,p}(\Omega)) \text{ and } \sigma'(u) = \beta(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).
\end{aligned}$$

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The next auxiliary result, is a "singular" version of the strong comparison principle of Arcoya-Ruiz [2] (Proposition 2.6) and also extends Theorem 2.3 of Giacomoni-Schindler-Takac [8]. First we introduce the following notation. Let $\vartheta_1,\ \vartheta_2 \in L^\infty(\Omega)$. We write $\vartheta_1 \prec \vartheta_2$ if, for any $K \subseteq \Omega$ compact, we can find $\varepsilon > 0$ s.t.

$$\vartheta_1(z) + \varepsilon \le \vartheta_2(z) \quad \text{for a.a. } z \in K.$$

Evidently, if $\vartheta_1,\ \vartheta_2 \in C(\Omega)$ and $\vartheta_1(z) < \vartheta_2(z)$ for all $z \in \Omega$, then $\vartheta_1 \prec \vartheta_2$.

PROPOSITION 4. *If* $\sigma \geq 0,\ \vartheta_1,\ \vartheta_2 \in L^\infty(\Omega),\ \vartheta_1 \prec \vartheta_2$ *and* $u \in C_+,\ u(z) > 0$ *for all* $z \in \Omega,\ v \in intC_+$ *satisfy the following equalities*

$$-\Delta_p u(z) - \xi(z)u(z)^{-\eta} + \sigma\, u(z)^{p-1} = \vartheta_1(z) \quad in\ \ \Omega$$

$$-\Delta_p v(z) - \xi(z)v(z)^{-\eta} + \sigma\, v(z)^{p-1} = \vartheta_2(z) \quad in\ \ \Omega \quad with\ \ 0 < \eta < 1,$$

then $v - u \in intC_+$.

Proof. From the weak comparison principle (see Pucci-Serrin [20](p.61)), we have $u \leq v$.

We set

$$D_0 = \{\ z \in \Omega : u(z) = v(z)\ \} \quad \text{and} \quad D = \{\ z \in \Omega : Du(z) = Dv(z) = 0\ \}.$$

CLAIM. $D_0 \subseteq D$.

Let $z \in D_0$. Then $w = v - u \in C_+ \setminus \{0\}$ attains its minimum at $z \in \Omega$ and so

$$Dw(z) = 0,$$
$$\Rightarrow Du(z) = Dv(z).$$

Arguing by contradiction, suppose that the Claim is not true and we have $Du(z) \neq 0$. Consider an open ball B centered at $z \in \Omega$ s.t. $B \subseteq \Omega$ and

$$||Du(z')|| > 0, \quad ||Dv(z')|| > 0, \quad (Du(z'),\ Dv(z'))_{\mathbb{R}^N} > 0 \quad \text{for all}\ \ z' \in B.$$

As in Guedda-Veron [9] (see the proof of Proposition 2.1), we have

$$\begin{aligned} -\text{div}(A(z)Dw(z)) &= -\sigma\ (v(z)^{p-1} - u(z)^{p-1}) + \xi(z)\ (v(z)^{-\eta} - u(z)^{-\eta}) + \\ &\quad + \vartheta_2(z) - \vartheta_1(z) \quad \text{in}\ \ B, \end{aligned} \tag{6}$$

where $A(z) = (a_{ij})_{i,\, j=1}^N$ and the entries of this matrix satisfy

$$a_{ij}(z) = \int_0^1 \frac{\partial k_i}{\partial y_j}((1-t)Du(z) + tDv(z))dt,$$

where $k(y) = (k_i(y))_{i=1}^N = ||y||^{p-2}y$ for all $y \in \mathbb{R}^N$ (hence $k \in C^1(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N)$).

By making the ball B even smaller if necessary, we may assume that in (6) the differential operator is strictly elliptic and the right hand side is positive. Then from the strong maximum principle (see, for example, Pucci-Serrin [20] (p.35)), we have

$$u(z') < v(z') \quad \text{for all}\ \ z' \in B.$$

But $z \in D_0 \cap B$, a contradiction. This proves the Claim.

Since by hypothesis $v \in \text{int}C_+$, we see that D is compact. Then D_0 being a closed subspace of D (see the Claim), it is itself compact. So, we can find $\Omega_1 \subseteq \Omega$ open s.t.

$$D_0 \subseteq \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega. \tag{7}$$

Choose $\varepsilon \in (0,\ 1)$ small s.t.

$$u(z)+\varepsilon \le v(z) \quad \text{for all } z \in \partial\Omega_1 \quad \text{(see (7))} \tag{8}$$

$$\vartheta_1(z)+\varepsilon \le \vartheta_2(z) \quad \text{for a.a. } z \in \overline{\Omega}_1 \quad (\text{recall that } \vartheta_1 \prec \vartheta_2\ ). \tag{9}$$

Let $\delta \in (0,\ \varepsilon)$ be s.t.

$$\sigma \mid |s|^{p-2}s - |s'|^{p-2}s' \mid \le \varepsilon/2 \quad \text{and} \quad ||\xi||_\infty \left| \frac{1}{s^\eta} - \frac{1}{(s')^\eta} \right| \le \varepsilon/2 \tag{10}$$

for all $s, s' \in \left[\min\limits_{\overline{\Omega}_1} u,\ ||v||_\infty \right]$ with $|s-s'| \le \delta$ (by hypothesis $\min\limits_{\overline{\Omega}_1} u > 0$ (see (7)) and so $s \to \dfrac{1}{s^\eta}$ is uniformly continuous on $\left[\min\limits_{\overline{\Omega}_1} u,\ ||v||_\infty \right]$).

Then we have

$$\begin{aligned}
&-\Delta_p(u+\delta) \ - \ \xi(u+\delta)^{-\eta} + \sigma\ (u+\delta)^{p-1} \\
&= -\Delta_p u \ - \ \xi(u+\delta)^{-\eta} + \sigma\ (u+\delta)^{p-1} \\
&= \xi u^{-\eta} - \sigma\ u^{p-1} - \xi(u+\delta)^{-\eta} + \sigma\ (u+\delta)^{p-1} + \vartheta_1 \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \vartheta_1 \quad \text{(see (10))} \\
&\le \vartheta_2 \quad \text{(see (9))} \\
&= -\Delta_p v \ - \ \xi v^{-\eta} + \sigma\ v^{p-1} \quad \text{in } \Omega_1\ , \\
&\Rightarrow u+\delta \le v \quad \text{in } \Omega_1 \ \text{(by the weak maximum principle (see (8) and [20] (p.61)),}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow D_0 = \emptyset \quad \text{(see (7))}, \\
&\Rightarrow u(z) < v(z) \quad \text{for all } z \in \Omega.
\end{aligned} \tag{11}$$

Now, let $z_0 \in \partial\Omega$. Since by hypothesis $\partial\Omega$ is a C^2 -manifold, we can find $\rho > 0$ s.t.

$$B_{2\rho}(\widehat{z}\ ) \subseteq \Omega \quad \text{and} \quad z_0 \in \partial B_{2\rho}(\widehat{z}\ ) \cap \partial\Omega \quad (\widehat{z} \in \Omega).$$

By virtue of Lemma 2 of Lewis [16], we can find $\widehat{w} \in C^1(B_{2\rho}(\widehat{z}\ ))$ s.t.

$$\begin{aligned}
&-\mathrm{div}(A(z)D\widehat{w}(z)) = 0 \quad \text{in} \quad B_{2\rho}(\widehat{z}\ ) \setminus \overline{B}_\rho(\widehat{z}\ ), \\
&\widehat{w}\ |_{\partial B_\rho(\widehat{z}\ )} = 1, \qquad \widehat{w}\ |_{\partial B_{2\rho}(\widehat{z}\ )} = 0,
\end{aligned} \tag{12}$$

$0 < \widehat{w} < 1$ and $||D\widehat{w}(z)|| \ge \widehat{c} > 0$ for all $z \in B_{2\rho}(\widehat{z}\ ) \setminus \overline{B}_\rho(\widehat{z}\ )$.

From (11) we see that $w(z) > 0$ for all $z \in \Omega$. Hence

$$m_\rho = \inf\left[\ w(z) : z \in \partial B_\rho(\widehat{z}\ )\ \right] > 0.$$

Let $\widetilde{w} = m_\rho \widehat{w}$. Then from (12) we have

$$\begin{aligned}
&-\mathrm{div}(A(z)D\widetilde{w}(z)) = 0 \quad \text{in} \quad B_{2\rho}(\widehat{z}\ ) \setminus \overline{B}_\rho(\widehat{z}\ ) \\
&\widetilde{w}\ |_{\partial B_\rho(\widehat{z}\ )} = m_\rho, \qquad \widetilde{w}\ |_{\partial B_{2\rho}(\widehat{z}\ )} = 0.
\end{aligned}$$

The weak comparison principle implies that $\widetilde{w}(z) \le w(z)$ for all $z \in B_{2\rho}(\widehat{z}\,) \setminus \overline{B}_\rho(\widehat{z}\,)$. Also, we have $\widetilde{w}(z_0) = w(z_0) = 0$. Therefore

$$\begin{aligned}&\frac{\partial w}{\partial n}(z_0) \;\le\; \frac{\partial \widetilde{w}}{\partial n}(z_0) = m_\rho\,\frac{\partial \widehat{w}}{\partial n}(z_0) < 0 \quad \text{(see (12))},\\ \Rightarrow\; & w = v - u \in \mathrm{int}C_+ \,.\end{aligned}$$

∎

In the sequel we will use the following inequalities valid for all $y, h \in \mathbb{R}^N$:

$$\begin{aligned}&(\,||y||^{p-2}y - ||h||^{p-2}h,\; y-h\,)_{\mathbb{R}^N} \ge \\ &\ge \gamma \begin{cases} \dfrac{||y-h||^2}{(1+||y||+||h||)^{2-p}} & \text{if} \quad 1 < p \le 2 \\ ||y-h||^p & \text{if} \quad 2 \le p \end{cases} \qquad \text{with } \gamma > 0. \end{aligned} \tag{13}$$

Finally, throughout this work, by $||\cdot||$ we denote the norm of $W_0^{1,p}(\Omega)$ defined by

$$||u|| = ||Du||_p \quad \text{for all } u \in W_0^{1,p}(\Omega) \quad \text{(by Poincare's inequality)}.$$

The notation $||\cdot||$ will also be used to denote the $\mathbb{R}^N$-norm. However, no confusion is possible, since it will always be clear from the context which norm is used.

For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$.

We know that $u^\pm \in W_0^{1,p}(\Omega)$, $u = u^+ - u^-$ and $|u| = u^+ + u^-$.

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Given $h : \Omega \times \mathbb{R} \to \mathbb{R}$ a measurable function (for example a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

3. Bifurcation-type theorem. The hypotheses on the data of problem $(P)_\lambda$ are the following:

$\mathbf{H_1}$: $\xi \in L^\infty(\Omega)$, $\operatorname{ess\,inf}_\Omega \xi > 0$ and $0 < \eta < 1$.

$\mathbf{H_2}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function s.t. for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$ and

(i) $f(z,x) \le \alpha(z) + c\,x^{r-1}$ for a.a. $z \in \Omega$, all $x \ge 0$, with $\alpha \in L^\infty(\Omega)_+$, $c > 0$ and $p < r < p^*$;

(ii) if $F(z,x) = \displaystyle\int_0^x f(z,s)ds$, then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\tau \in \left(\,(r-p)\max\left\{1, \dfrac{N}{p}\right\},\; p^*\right)$ and $\beta_0 > 0$ s.t.

$$\beta_0 \;\le\; \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^\tau} \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\gamma_\rho > 0$ s.t. for a.a. $z \in \Omega$, $x \to f(z,x) + \gamma_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$;

(v) $0 \le \liminf_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} \le \limsup_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} \le \vartheta^*$ uniformly for a.a. $z \in \Omega$, where ϑ^* is a nonnegative real number.

Remark. Since we are interested in positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may (and will) assume that

$$f(z,x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \le 0.$$

Hypothesis H_2(ii) implies that for a.a. $z \in \Omega$, the primitive $F(z,\cdot)$ is p- superlinear near $+\infty$. Note that this asymptotic condition is true, if we assume that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$ -superlinear near $+\infty$, namely if

$$\lim_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

However, as we already indicated in the Introduction, we do not use the usual in such cases AR-condition. We recall that the AR-condition (unilateral version) says that there exist $q > p$ and $M > 0$ s.t.

$$0 < qF(z,x) \le f(z,x)x \ \text{ for a.a. } z \in \Omega, \ \text{ all } x \ge M, \ \operatorname*{ess\,inf}_{\Omega} F(\cdot, M) > 0. \tag{14}$$

Integrating the first inequality in (14) and using the second inequality, we obtain the weaker growth condition

$$c_0 x^q \le F(z,x) \ \text{ for a.a. } z \in \Omega, \ \text{ all } x \ge M, \quad \text{with } c_0 > 0. \tag{15}$$

This leads to the much weaker condition which is hypothesis H_2(ii).

Moreover, if the AR-condition holds (see (14)), then we may assume that $q > (r-p)\max\left\{1, \frac{N}{p}\right\}$ and we have

$$\frac{f(z,x)x - pF(z,x)}{x^q} = \frac{f(z,x)x - qF(z,x)}{x^q} + \frac{(q-p)F(z,x)}{x^q} \ge (q-p)c_0$$

(see (14), (15)).

Therefore hypothesis H_2(iii) is satisfied.

The function

$$f(x) = x^{p-1}\left(\ln(1+x) + \frac{1}{p}\frac{x}{1+x}\right) \quad \text{for } x \ge 0,$$

satisfies hypotheses H_2, but not the AR-condition. Of course, the function $f(x) = x^{q-1}$ with $p < q < p^*$, which we encounter in the literature (see [4], [6], [8], [10]) satisfies hypotheses H_2 and the AR-condition.

We start by considering the following auxiliary purely singular Dirichlet problem (i.e., no perturbation term is present):

$$-\Delta_p u(z) = \xi(z)u(z)^{-\eta} \quad \text{in } \Omega, \quad u\,|_{\partial\Omega} = 0, \ u > 0. \tag{16}$$

PROPOSITION 5. *If hypotheses H_1 hold, then problem (16) has a unique solution $\underline{u} \in intC_+$.*

Proof. We consider the following Dirichlet problem with nonhomogeneous boundary condition

$$(17)_k \qquad -\Delta_p u(z) = \xi(z) \quad \text{in } \Omega, \quad u\,|_{\partial\Omega} = \frac{1}{k}\,, \quad u > 0, \quad k \geq 1.$$

Evidently, for every integer $k \geq 1$, problem $(17)_k$ has a unique solution $u_k \in W^{1,p}(\Omega)$ (with $u_k - \frac{1}{k} \in W_0^{1,p}(\Omega)$). Nonlinear regularity theory (see Ladyzhenskaya-Uraltseva [13] (p.286)) and Lieberman [17] (Theorem 1)), implies that $u_k \in C^1(\overline{\Omega})$, while from the weak comparison principle (see [20] (p.61)), we have $u_k(z) \geq \frac{1}{k}$ for all $z \in \overline{\Omega}$. Let u_* be the unique solution of problem $(17)_\infty$ (i.e., of the problem with homogeneous Dirichlet boundary condition). For u_*, via the strong maximum principle of Vazquez [23], we have $u_* \in \text{int}C_+$. Moreover, the weak comparison principle implies that $u_* \leq u_k$ for all $k \geq 1$. So, we have

$$u_* \leq u_k \quad \text{for all } k \geq 1, \quad \{u_k\}_{k\geq 1} \text{ is decreasing}$$

and

$$\frac{1}{k} \leq u_k(z) \quad \text{for all } z \in \overline{\Omega}, \quad \text{all } k \geq 1.$$

We choose $t \in (0,1]$ small s.t. $tu_1(z) \in (0,1]$ for all $z \in \overline{\Omega}$. Then $tu_k(z) \in (0,1]$ for all $z \in \overline{\Omega}$, all $k \geq 1$. We set $\widehat{u}_k = tu_k \in C^1(\overline{\Omega}) \setminus \{0\}$, $k \geq 1$. Then $\underline{u}_* = tu_* \leq \widehat{u}_k$ for all $k \geq 1$ and

$$(18)_k \qquad \begin{cases} -\Delta_p \widehat{u}_k(z) = t^{p-1}(-\Delta_p u_k(z)) = t^{p-1}\xi(z) \leq \xi(z) \leq \\ \qquad \leq \xi(z)\widehat{u}_k(z)^{-\eta} \quad \text{a.e. in } \Omega, \\ \widehat{u}_k\,|_{\partial\Omega} = \frac{t}{k}\,, \quad k \geq 1. \end{cases}$$

We consider the following nonhomogeneous Dirichlet problem

$$(19)_k \qquad -\Delta_p u(z) = \xi(z)u(z)^{-\eta} \quad \text{in } \Omega, \quad u\,|_{\partial\Omega} = \frac{t}{k}\,, \quad u > 0, \quad k \geq 1.$$

Note that, if $y = u_1 \in C^1(\overline{\Omega})$, then from the weak comparison principle, we have $1 \leq y(z)$ for all $z \in \overline{\Omega}$ and so

$$(20) \qquad -\Delta_p y(z) = \xi(z) \geq \xi(z)y(z)^{-\eta} \quad \text{a.e. in } \Omega, \quad y\,|_{\partial\Omega} \geq \frac{t}{k} \quad \text{for all } k \geq 1.$$

We introduce the function

$$(21) \qquad g(z,x) = \begin{cases} \xi(z)\widehat{u}_k(z)^{-\eta} & \text{if} \quad x < \widehat{u}_k(z) \\ \xi(z)x^{-\eta} & \text{if} \quad \widehat{u}_k(z) \leq x \leq y(z) \\ \xi(z)y(z)^{-\eta} & \text{if} \quad y(z) < x. \end{cases}$$

Clearly this is a Carathéodory function. Moreover, since

$$\frac{t}{k} \leq \widehat{u}_k(z) \leq y(z) \quad \text{for all } z \in \overline{\Omega}, \text{ all } k \geq 1$$

we have

$$0 \ < \ g(z,x) \ \leq \ \widehat{\alpha}(z) \quad \text{for a.a} \quad z \in \Omega, \ \text{ all } \ x \in \mathbb{R}, \ \text{ with } \ \widehat{\alpha} \in L^{\infty}(\Omega)_{+} \ .$$

If we consider the nonhomogeneous Dirichlet problem

$$-\Delta_p u(z) = g(z,u(z)) \quad \text{in} \ \ \Omega, \quad u\,|_{\partial\Omega} = \frac{t}{k} \ , \ \ u > 0, \ \ k \geq 1,$$

then this problem has a unique solution $v_k \in C^1(\overline{\Omega})$ (nonlinear regularity theory, see [13], [17]). We have

$$A(v_k) = N_g(v_k) \quad \text{for all} \ \ k \geq 1. \tag{22}$$

On(22) we act with $(\widehat{u}_k - v_k)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
&\langle A(v_k), \ (\widehat{u}_k - v_k)^+ \rangle \\
&= \int_\Omega g(z,v_k)(\widehat{u}_k - v_k)^+ dz \\
&= \int_\Omega \xi \ \widehat{u}_k^{-\eta}(\widehat{u}_k - v_k)^+ dz \quad \text{(see (21))} \\
&\geq \langle A(\widehat{u}_k), \ (\widehat{u}_k - v_k)^+ \rangle \quad \text{(see } (18)_k\text{)}, \\
&\Rightarrow \langle A(\widehat{u}_k) - A(v_k), \ (\widehat{u}_k - v_k)^+ \rangle \leq 0, \\
&\Rightarrow \int_{\{\widehat{u}_k \ > v_k\}} (\ ||D\widehat{u}_k||^{p-2} D\widehat{u}_k \ - \ ||Dv_k||^{p-2} Dv_k \ , \ D\widehat{u}_k \ - \ Dv_k)_{\mathbb{R}^N} dz \leq 0, \\
&\Rightarrow |\{\widehat{u}_k \ > v_k\}|_N = 0 \quad \text{(see (13)) and so} \quad \widehat{u}_k \leq v_k \quad \text{for all} \ \ k \geq 1.
\end{aligned}$$

Similarly, acting on (22) with $(v_k - y)^+ \in W_0^{1,p}(\Omega)$ and using (20), we show that $v_k \leq y$, for all $k \geq 1$. So, we have proved that

$$v_k \in [\ \widehat{u}_k, \ y \ ] = \{u \in W^{1,p}(\Omega) : \widehat{u}_k(z) \ \leq \ u(z) \ \leq \ y(z) \ \text{ a.e. in } \ \Omega \ \}.$$

Then from (22) and (21) it follows that $v_k \in C^1(\overline{\Omega}) \setminus \{0\}$ is a solution of problem $(19)_k$, $k \geq 1$.

Following the above reasoning, we can find $v_{k+1} \in W^{1,p}(\Omega)$ a solution of $(19)_{k+1}$ s.t.

$$v_{k+1} \in [\ \widehat{u}_{k+1}, \ y \ ] \cap C^1(\overline{\Omega}).$$

This way, we obtain $\{v_k\}_{k \geq 1}$ a decreasing sequence of solutions of the problems $\{(19)_k\}_{k \geq 1}$ and all these solutions belong in $[\ \underline{u}_* \ , \ y \ ] \cap C^1(\overline{\Omega})$.

Let $\underline{u}(z) = \lim\limits_{k \to \infty} v_k(z)$ for all $z \in \overline{\Omega}$. Then $\underline{u} \in [\ \underline{u}_* \ , \ y \ ]$. We have

$$-\Delta_p v_k(z) = \xi(z) v_k(z)^{-\eta} \quad \text{in} \ \ \Omega \quad \text{for all} \ \ k \geq 1.$$

We act with the test function $v_k - \dfrac{t}{k} \in C_0^1(\overline{\Omega})$ for all $k \geq 1$. Using the nonlinear

Green's identity (see, for example, Gasinski-Papageorgiou [7] (p.211)), we obtain

$$\begin{aligned}||Dv_k||_p^p &= \int_\Omega \xi\, v_k^{-\eta}\left(v_k - \frac{t}{k}\right) dz \\ &\le \int_\Omega \xi\, v_k^{1-\eta} dz \\ &\le \int_\Omega \xi\, y^{1-\eta} dz \quad (\text{since } v_k \le y) \\ &\le ||\xi||_\infty ||y||_\infty^{1-\eta}\, |\Omega|_N\,, \\ &\Rightarrow \{v_k\}_{k\ge 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.}\end{aligned}$$

So, we have

$$v_k \xrightarrow{w} \underline{u} \text{ in } W^{1,\,p}(\Omega), \quad v_k \to \underline{u} \text{ in } L^p(\Omega) \quad \text{and} \quad \underline{u} \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) \tag{23}$$

(recall that $v_k\,|_{\partial\Omega}=\dfrac{t}{k},\ \ v_k \in [\,\underline{u}_*\,,\ y\,],\ \ k\ge 1.$)

Recall that

$$A(v_k) = \xi v_k^{-\eta} \quad \text{in } W_0^{1,p}(\Omega) \quad \text{for all } k \ge 1 \ \ (\text{see } (19)_k).$$

We act with $v_k - \underline{u} - \dfrac{t}{k} \in W_0^{1,p}(\Omega)$. Then

$$\left\langle A(v_k),\ v_k - \underline{u} - \frac{t}{k}\right\rangle = \int_\Omega \lambda\xi v_k^{-\eta}\left(v_k - \underline{u} - \frac{t}{k}\right) dz. \tag{24}$$

Note that

$$\begin{aligned}0 &\le \lambda\xi v_k^{-\eta}\left(v_k - \underline{u} - \frac{t}{k}\right) \le \lambda\xi v_k^{-\eta}\left(v_k - \frac{t}{k}\right) \le \lambda\xi v_k^{1-\eta} \le \\ &\le \lambda||\xi||_\infty ||y||_\infty^{1-\eta}.\end{aligned} \tag{25}$$

So, given $\varepsilon > 0$, we can find $\Omega' \subseteq \Omega$ s.t.

$$\int_{\Omega\setminus\Omega'} \lambda\xi v_k^{-\eta}\left(v_k - \underline{u} - \frac{t}{k}\right) dz \le \varepsilon \quad \text{for all } k \ge 1. \tag{26}$$

Also we have

$$\lambda\xi v_k^{-\eta}\left(v_k - \underline{u} - \frac{t}{k}\right) \to 0 \quad \text{a.e. in } \Omega'.$$

Because of (25), we can apply the dominated convergence theorem and obtain

$$\int_{\Omega'} \lambda\xi v_k^{-\eta}\left(v_k - \underline{u} - \frac{t}{k}\right) dz \to 0. \tag{27}$$

So, returning to (24), passing to the limit as $k \to \infty$ and using (26), (27), we have

$$\limsup_{k\to\infty}\left\langle A(v_k),\ v_k - \underline{u} - \frac{t}{k}\right\rangle \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that

$$\limsup_{k\to\infty} \left\langle A(v_k),\ v_k - \underline{u} - \frac{t}{k} \right\rangle \ \le\ 0, \tag{28}$$
$$\Rightarrow \quad v_k \to \underline{u} \quad \text{in } \ W^{1,\,p}(\Omega)$$

(see Proposition 2, its Remark and (23)).

Let $h \in C_c^{\infty}(\Omega)$. Then we have

$$\int_{\Omega} ||Dv_k||^{p-2}(Dv_k, Dh)_{\mathbb{R}^N}\,dz \to \int_{\Omega} ||D\underline{u}||^{p-2}(D\underline{u},\ Dh)_{\mathbb{R}^N}\,dz \quad \text{(see (28))} \tag{29}$$

$$\int_{\Omega} \xi v_k^{-\eta} h dz \to \int_{\Omega} \xi \underline{u}^{-\eta} h dz. \tag{30}$$

We know (see $(19)_k$) that

(31)
$$\int_{\Omega} ||Dv_k||^{p-2}(Dv_k, Dh)_{\mathbb{R}^N}\,dz = \int_{\Omega} \xi v_k^{-\eta} h dz \quad \text{for all } \ h \in C_c^{\infty}(\Omega), \ \text{ all } \ k \ge 1,$$
$$\Rightarrow \quad \int_{\Omega} ||D\underline{u}||^{p-2}(D\underline{u},\ Dh)_{\mathbb{R}^N}\,dz = \int_{\Omega} \xi \underline{u}^{-\eta} h dz \quad \text{for all } \ h \in C_c^{\infty}(\Omega)$$

(see (29), (30)).

Let $h \in W_0^{1,p}(\Omega)$. Since $h = h^+ - h^-$ and $h^+, h^- \in W_0^{1,p}(\Omega)$, we may assume that $h \ge 0$. We can find $\{\vartheta_n\}_{n\ge1} \subseteq C_c^{\infty}(\Omega)$ with $\vartheta_n \ge 0$ s.t. $\vartheta_n \to h$ in $W_0^{1,p}(\Omega)$. By virtue of Fatou's lemma, we have

$$\int_{\Omega} \xi \underline{u}^{-\eta} h dz \ \le\ \liminf_{n\to\infty} \int_{\Omega} \xi \underline{u}^{-\eta} \vartheta_n dz. \tag{32}$$

From (31) we have

$$\int_{\Omega} ||D\underline{u}||^{p-2}(D\underline{u},\ D\vartheta_n)_{\mathbb{R}^N}\,dz = \int_{\Omega} \xi \underline{u}^{-\eta}\ \vartheta_n dz \quad \text{for all } \ n \ge 1,$$
$$\Rightarrow \int_{\Omega} \xi \underline{u}^{-\eta}\ \vartheta_n dz \ \le\ ||\underline{u}||^{p-1}||\vartheta_n|| \quad \text{for all } \ n \ge 1.$$

So, finally we infer that

$$\left| \int_{\Omega} \xi \underline{u}^{-\eta} h dz \right| \ \le\ ||\underline{u}||^{p-1}||h|| \tag{33}$$

(see (32) and recall $\vartheta_n \to h$ in $W_0^{1,p}(\Omega)$).

For any $w \in W_0^{1,p}(\Omega)$ we know that we can find $\{w_n\}_{n\ge1} \subseteq C_c^{\infty}(\Omega)$ s.t. $w_n \to w$ in $W_0^{1,p}(\Omega)$. So, if in (33) we choose $h = w - w_n$, then we have

$$\int_{\Omega} \xi \underline{u}^{-\eta} w_n dz \to \int_{\Omega} \xi \underline{u}^{-\eta} w dz. \tag{34}$$

Also, we have that

$$\int_\Omega ||D\underline{u}||^{p-2}(D\underline{u},\ Dw_n)_{\mathbb{R}^N}dz \to \int_\Omega ||D\underline{u}||^{p-2}(D\underline{u},\ Dw)_{\mathbb{R}^N}dz. \tag{35}$$

Recall that

$$\int_\Omega ||D\underline{u}||^{p-2}(D\underline{u},\ Dw_n)_{\mathbb{R}^N}dz = \int_\Omega \xi\underline{u}^{-\eta}w_n dz \quad \text{for all } n \geq 1,$$

$$\Rightarrow \int_\Omega ||D\underline{u}||^{p-2}(D\underline{u},\ Dw)_{\mathbb{R}^N}dz = \int_\Omega \xi\underline{u}^{-\eta}w dz \quad \text{for all } w \in W_0^{1,p}(\Omega)$$

(see (34) and (35)),

$\Rightarrow \underline{u} \in [\,\underline{u}_*,\ y\,]$ is a nontrivial positive solution of (16).

Since $\underline{u}_* \in \text{int}C_+$, we can find $c_2 > 0$ s.t.

$$c_2 d(z) \ \leq\ \underline{u}_*(z) \quad \text{for all } z \in \overline{\Omega} \quad (\text{recall } d(z) = d(z,\ \partial\Omega))$$

$$\begin{aligned} \Rightarrow\ \ \xi(z)\underline{u}(z)^{-\eta} &\leq\ \xi(z)\underline{u}_*(z)^{-\eta} \quad (\text{since } \underline{u}_* \leq \underline{u}) \\ &\leq \xi(z)(c_2 d(z))^{-\eta} \\ &\leq c_3 d(z)^{-\eta} \quad \text{for a.a. } z \in \Omega, \end{aligned} \tag{36}$$

with $c_3 = ||\xi||_\infty c_2^{-\eta} > 0$.

Also we have $\xi(\cdot)\underline{u}(\cdot)^{-\eta} \in L^\infty_{loc}(\Omega)$ (recall that $\underline{u} \in [\,\underline{u}_*,\ y\,]$).

Finally since $\underline{u} \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, we can find $c_4 > 0$ s.t. $\underline{u}(z) \leq c_4 d(z)$ for all $z \in \overline{\Omega}$. Therefore, from the regularity result of Giacomoni-Schindler-Takac [8], we infer that $\underline{u} \in C_+ \setminus \{0\}$. We have

$$-\Delta_p\underline{u}(z) = \xi(z)\underline{u}(z)^{-\eta} \ \geq\ 0 \quad \text{a.e. in } \Omega,$$

$$\Rightarrow \underline{u} \in \text{int}C_+ \quad (\text{see Vazquez [23]}).$$

Next we show the uniqueness of the solution $\underline{u}$. Suppose that $\underline{v} \in W_0^{1,p}(\Omega)$ is another nontrivial positive solution of (16). Then, we have

$$\begin{aligned} 0 &\leq \int_{\{\underline{u}\,>\,\underline{v}\}} (\ ||D\underline{u}||^{p-2}D\underline{u} - ||D\underline{v}||^{p-2}D\underline{v},\ D\underline{u} - D\underline{v}\ )_{\mathbb{R}^N}dz = \\ &= \int_{\{\underline{u}\,>\,\underline{v}\}} \left(\frac{\xi}{\underline{u}^\eta} - \frac{\xi}{\underline{v}^\eta}\right) dz \ <\ 0, \end{aligned}$$

$$\Rightarrow |\{\underline{u}\ >\ \underline{v}\}|_N = 0 \quad \text{and so} \quad \underline{u}\ \leq\ \underline{v}.$$

By interchanging the roles of $\underline{u}$ and $\underline{v}$ in the above argument we also show that $\underline{v}\ \leq\ \underline{u}$. Therefore, we conclude that $\underline{v} = \underline{u}$ and so $\underline{u} \in \text{int}C_+$ is the unique nontrivial positive solution of (16). ∎

Now let $\mathcal{P} = \{\lambda > 0 :$ problem $(P)_\lambda$ has a nontrivial positive solution$\}$. Also, for every $\lambda \in \mathcal{P}$, by $S(\lambda)$ we denote the corresponding solution set of $(P)_\lambda$.

Proposition 6. *If hypotheses H_1 and H_2 hold, then $\mathcal{P} \neq \emptyset$ and for every $\lambda \in \mathcal{P},\ S(\lambda) \subseteq intC_+$.*

Proof. Let $\underline{u} \in \mathrm{int}C_+$ be the unique nontrivial positive solution of (16) produced in Proposition 5. We have

$$-\Delta_p \underline{u}(z) = \xi(z)\underline{u}(z)^{-\eta} \le \xi(z)\underline{u}(z)^{-\eta} + \lambda f(z, \underline{u}(z)) \quad \text{a.e. in } \Omega \tag{37}$$

(see H_2).

We consider the following auxiliary Dirichlet problem

$$-\Delta_p u(z) = \xi(z)\underline{u}(z)^{-\eta} + 1 \quad \text{in } \Omega, \quad u\,|_{\partial\Omega} = 0, \quad u > 0. \tag{38}$$

Setting $h(z) = \xi(z)\underline{u}(z)^{-\eta} + 1$, we see that $h \in L^\infty_{loc}(\Omega)$ and

$$0 \le h(z) \le c_5 d(z)^{-\eta} \quad \text{for a.a. } z \in \Omega, \quad \text{some } c_5 > 0 \quad (\text{see (36)}).$$

So, (38) has a unique solution $\overline{u} \in \mathrm{int}C_+$ (see [8]). Moreover, the weak comparison principle (see [20] (p.61)) implies that $\underline{u} \le \overline{u}$. By virtue of hypothesis H_2(i), we can find $\lambda > 0$ small s.t.

$$\lambda f(z, \overline{u}(z)) \le 1 \quad \text{for a.a. } z \in \Omega. \tag{39}$$

Then we have

$$\begin{aligned} -\Delta_p \overline{u}(z) &= \xi(z)\underline{u}(z)^{-\eta} + 1 \\ &\ge \xi(z)\overline{u}(z)^{-\eta} + 1 \quad (\text{since } \underline{u} \le \overline{u}) \\ &\ge \xi(z)\overline{u}(z)^{-\eta} + \lambda f(z, \overline{u}(z)) \quad \text{a.e. in } \Omega \quad (\text{see (39)}). \end{aligned} \tag{40}$$

We introduce the following truncation of the reaction in problem $(P)_\lambda$:

$$g_\lambda(z,x) = \begin{cases} \xi(z)\underline{u}(z)^{-\eta} + \lambda f(z, \underline{u}(z)) & \text{if } x < \underline{u}(z) \\ \xi(z)x^{-\eta} + \lambda f(z,x) & \text{if } \underline{u}(z) \le x \le \overline{u}(z) \\ \xi(z)\overline{u}(z)^{-\eta} + \lambda f(z, \overline{u}(z)) & \text{if } \overline{u}(z) < x. \end{cases} \tag{41}$$

Evidently $g_\lambda(z,x)$ is a Carathéodory function. Let $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the functional $\psi_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p - \int_\Omega G_\lambda(z, u(z))dz \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

By virtue of Proposition 3, we have $\psi_\lambda \in C^1(W^{1,p}_0(\Omega))$. Also, it is clear from (41) that ψ_λ is coercive. Moreover, using the Sobolev embedding theorem, we can easily check that ψ_λ is sequentially weakly lower semicontinuous. Therefore by the Weierstrass theorem, we can find $u_0 \in W^{1,p}_0(\Omega)$ s.t.

$$\begin{aligned} &\psi_\lambda(u_0) = \inf[\, \psi_\lambda(u) : u \in W^{1,p}_0(\Omega)\,], \\ \Rightarrow\ &\psi'_\lambda(u_0) = 0, \\ \Rightarrow\ &A(u_0) = N_{g_\lambda}(u_0). \end{aligned} \tag{42}$$

On (42) we act with $(\underline{u} - u_0)^+ \in W^{1,p}_0(\Omega)$. Then

$$
\begin{aligned}
&\langle A(u_0),\ (\underline{u}-u_0)^+\rangle \\
&= \int_\Omega g_\lambda(z,u_0)(\underline{u}-u_0)^+dz \\
&= \int_\Omega [\ \xi\underline{u}^{-\eta}+\lambda f(z,\underline{u})\ ](\underline{u}-u_0)^+dz \quad \text{(see (41))} \\
&\geq \langle A(\underline{u}),\ (\underline{u}-u_0)^+\rangle \quad \text{(see (37))}, \\
&\Rightarrow \int_{\{\underline{u}\,>\,u_0\}} (\ ||D\underline{u}||^{p-2}D\underline{u}-||Du_0||^{p-2}Du_0\ ,\ D\underline{u}-Du_0\ )_{\mathbb{R}^N}dz \leq 0, \\
&\Rightarrow |\{\underline{u}\ >\ u_0\}|_N = 0 \quad \text{(see (13)), hence} \quad \underline{u}\ \leq\ u_0\ .
\end{aligned}
$$

Similarly, acting on (42) with $(u_0-\overline{u})^+ \in W_0^{1,p}(\Omega)$ and using this time (40), we show that $u_0 \leq \overline{u}$. Therefore, we have

$$u_0 \in [\ \underline{u},\ \ \overline{u}\ ] = \{u \in W_0^{1,p}(\Omega) : \underline{u}(z) \leq u(z) \leq \overline{u}(z) \quad \text{a.e. in } \Omega\}.$$

This means that (42) becomes

$$
\begin{aligned}
&A(u_0) = \xi u_0^{-\eta} + N_f(u_0) \quad \text{(see (41))} \\
&\Rightarrow u_0 \text{ is a nontrivial positive solution of } (P)_\lambda \text{ for } \lambda > 0 \text{ small}, \\
&\Rightarrow \mathcal{P} \neq \emptyset.
\end{aligned}
$$

Moreover, as before we have $u_0 \in \mathrm{int}C_+$ (see [8], [23]) and so for all $\lambda \in \mathcal{P}$, $S(\lambda) \subseteq \mathrm{int}C_+$. □

Now let $\lambda^* = \sup \mathcal{P}$.

Proposition 7. *If hypotheses H_1 and H_2 hold, then $\lambda^* < +\infty$.*

Proof. Hypotheses H_1 and H_2 (i), (ii), (iii) and (v) imply that we can find $\lambda_0 > 0$ s.t.

$$\xi(z)x^{-\eta} + \lambda_0 f(z,x) \geq \widehat{\lambda}_1 x^{p-1} \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \geq 0. \tag{43}$$

Let $\lambda > \lambda_0$ and suppose that $\lambda \in \mathcal{P}$. Then we can find $u_\lambda \in \mathrm{int}C_+$ solution of $(P)_\lambda$. Let $\beta > 0$ be s.t. $\beta\widehat{u}_1 \leq u_\lambda$. Choose $\beta > 0$ to be the biggest such positive real. We have

$$
\begin{aligned}
-\Delta_p u_\lambda(z) &= \xi(z)u_\lambda(z)^{-\eta} + \lambda f(z,u_\lambda(z)) \\
&> \xi(z)u_\lambda(z)^{-\eta} + \lambda_0 f(z,u_\lambda(z)) \quad \text{(recall that } \lambda > \lambda_0 \text{ and } f(z,x) > 0 \\
&\quad\ \text{for a.a. } z \in \Omega, \text{ all } x > 0) \\
&\geq \widehat{\lambda}_1 u_\lambda(z)^{p-1} \quad \text{(see (43))} \\
&\geq \widehat{\lambda}_1(\beta\widehat{u}_1(z))^{p-1} \\
&= -\Delta_p(\beta\widehat{u}_1(z)) \quad \text{a.e. in } \Omega, \\
&\Rightarrow u_\lambda - \beta\widehat{u}_1 \in \mathrm{int}C_+ \quad \text{(see Guedda-Veron [9])}.
\end{aligned}
$$

This contradicts the maximality of $\beta > 0$. Therefore $\lambda \notin \mathcal{P}$ and so $\lambda^* \leq \lambda_0 < \infty$. □

Proposition 8. *If hypotheses H_1 and H_2 hold and $\lambda \in (0,\ \lambda^*)$, then problem $(P)_\lambda$ has at least two nontrivial positive solutions*

$$u_0,\ \widehat{u} \in intC_+, \quad u_0 \leq \widehat{u}, \quad u_0 \neq \widehat{u}.$$

Proof. Let $\underline{u} \in \mathrm{int}C_+$ be the unique nontrivial positive solution of (16) produced in Proposition 5. We have

$$-\Delta_p \underline{u}(z) = \xi(z)\underline{u}(z)^{-\eta} \ \leq\ \xi(z)\underline{u}(z)^{-\eta} + \lambda f(z, \underline{u}(z)) \quad \text{a.e. in } \Omega. \tag{44}$$

Let $\mu \in (\lambda,\ \lambda^*) \cap \mathcal{P}$ and let $u_\mu \in S(\mu) \subseteq \mathrm{int}C_+$. Note that

$$-\Delta_p u_\mu(z) = \xi(z)u_\mu(z)^{-\eta} + \mu f(z, u_\mu(z)) \ \geq\ \xi(z)u_\mu(z)^{-\eta} \quad \text{a.e. in } \Omega. \tag{45}$$

Hence by truncating the singular term $x \to \xi(z)x^{-\eta}$ at $u_\mu(z)$ and using the direct method, (45) and the uniqueness of the solution $\underline{u} \in \mathrm{int}C_+$ of (16) (see Proposition 5), we obtain that $u_\mu \geq \underline{u}$. We have

$$\begin{aligned} -\Delta_p u_\mu(z) &= \xi(z)u_\mu(z)^{-\eta} + \mu f(z, u_\mu(z)) \\ &\geq \xi(z)u_\mu(z)^{-\eta} + \lambda f(z, u_\mu(z)) \quad \text{a.e. in } \Omega \ \ (\text{recall } \ \lambda < \mu). \end{aligned} \tag{46}$$

As before (see the proof of Proposition 6), truncating the reaction of problem $(P)_\lambda$ at $\{\underline{u}(z),\ u_\mu(z)\}$ and using the direct method we obtain

$$u_0 \in S(\lambda) \subseteq \mathrm{int}C_+$$

and

$$u_0 \in [\ \underline{u},\ u_\mu\ ] = \{u \in W_0^{1,p}(\Omega) : \underline{u}(z) \leq u(z) \leq u_\mu(z) \ \text{ a.e. in } \ \Omega\}.$$

We use the solution $u_0 \in \mathrm{int}C_+$ to truncate the reaction of problem $(P)_\lambda$ as follows:

$$\widehat{g}_\lambda(z,x) = \begin{cases} \xi(z)u_0(z)^{-\eta} + \lambda f(z, u_0(z)) & \text{if} \quad x \leq u_0(z) \\ \xi(z)x^{-\eta} + \lambda f(z,x) & \text{if} \quad u_0(z) < x. \end{cases} \tag{47}$$

This is a Carathéodory function. Let $\widehat{G}_\lambda(z,x) = \displaystyle\int_0^x \widehat{g}_\lambda(z,s)ds$ and consider the functional $\widehat{\psi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p}||Du||_p^p \ - \ \int_\Omega \widehat{G}_\lambda(z, u(z))dz \quad \text{for all } \ u \in W_0^{1,p}(\Omega).$$

We have $\widehat{\psi}_\lambda \in C^1(W_0^{1,p}(\Omega))$ (see Proposition 3).

Claim 1. $\widehat{\psi}_\lambda$ satisfies the C- condition.
Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence s.t.

$$|\widehat{\psi}_\lambda(u_n)| \ \leq \ M_1 \quad \text{for some } \ M_1 > 0, \ \text{ all } \ n \geq 1 \tag{48}$$

and

$$(\,1 + ||u_n||\,)\widehat{\psi}_\lambda^{\,'}(u_n) \to 0 \quad \text{in} \quad W^{-1,\,p'}(\Omega) \quad \text{as } \ n \to \infty. \tag{49}$$

From (49) we have

$$\left|\langle A(u_n),\ h\rangle\ -\ \int_\Omega \widehat{g}_\lambda(z,u_n)h dz\right|\ \leq\ \frac{\varepsilon_n||h||}{1+||u_n||}\ . \tag{50}$$

In (50) we choose $h=-u_n^-\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&||Du_n^-||_p^p\ \leq\ \varepsilon_n \quad \text{for all}\ \ n\geq 1\ \ \text{(see (47))},\\ &\Rightarrow\quad u_n^-\to 0 \quad \text{in}\ \ W_0^{1,p}(\Omega) \quad \text{as}\ \ n\to\infty.\end{aligned} \tag{51}$$

Next, in (50) we choose $h=u_n^+\in W_0^{1,p}(\Omega)$. Then

$$-||Du_n^+||_p^p\ +\ \int_{\{u_n>u_0\}}\xi(u_n^+)^\eta dz\ +\ \int_{\{u_n>u_0\}}\lambda f(z,u_n^+)u_n^+dz\ \leq\ M_2 \tag{52}$$

for some $M_2>0$, all $n\geq 1$ (see (47)).

On the other hand from (48) and (51) we have

$$||Du_n^+||_p^p\ -\ \frac{p}{p-\eta}\int_{\{u_n>u_0\}}\xi(u_n^+)^{1-\eta}dz\ -\ \int_{\{u_n>u_0\}}\lambda pF(z,u_n^+)dz\ \leq\ M_3 \tag{53}$$

for some $M_3>0$, all $n\geq 1$.

Adding (52) and (53), we obtain

$$\begin{aligned}&\lambda\int_{\{u_n>u_0\}}[\,f(z,u_n^+)u_n^+\ -\ pF(z,u_n^+)\,]dz\ \leq\\ &\leq\ M_4+\left(\frac{p}{1-\eta}-1\right)\int_{\{u_n>u_0\}}\xi(u_n^+)^{1-\eta}dz \quad \text{for some}\ \ M_4>0,\ \ \text{all}\ \ n\geq 1,\\ &\Rightarrow\ \ \lambda\int_\Omega[\,f(z,u_n^+)u_n^+\ -\ pF(z,u_n^+)\,]dz\ \leq\\ &\leq\ M_5+\left(\frac{p}{1-\eta}-1\right)\int_\Omega\xi(u_n^+)^{1-\eta}dz \quad \text{for some}\ M_5>0,\ \ \text{all}\ \ n\geq 1.\end{aligned} \tag{54}$$

By virtue of hypothesis H_2 (i), (iii), we can find $\beta_1\in(0,\ \beta_0)$ and $c_6>0$ s.t.

$$\beta_1x^\tau-c_6\ \leq\ f(z,x)x-pF(z,x) \quad \text{for a.a.}\ \ z\in\Omega,\ \ \text{all}\ \ x\geq 0. \tag{55}$$

Also, for $\vartheta\in(1,\ \tau)$ we have

$$\begin{aligned}\int_\Omega\xi(u_n^+)^{1-\eta}dz\ &\leq\ \int_{\{0\leq u_n\leq 1\}}\xi dz\ +\ \int_{\{u_n>1\}}\xi(u_n^+)^\vartheta dz\\ &\leq\ c_7(\,1+||u_n^+||_\tau^\vartheta\,) \quad \text{for some}\ \ c_7>0,\ \ \text{all}\ \ n\geq 1\end{aligned} \tag{56}$$

(recall $\vartheta<\tau$).

Returning to (54) and using (55) and (56), we have

$$\begin{aligned}&\lambda\beta_1||u_n^+||_\tau^\tau\ \leq\ c_8(\,1+||u_n^+||_\tau^\vartheta\,) \quad \text{for some}\ \ c_8>0,\ \ \text{all}\ \ n\geq 1,\\ &\Rightarrow\quad \{u_n^+\}_{n\geq 1}\ \subseteq\ L^\tau(\Omega) \quad \text{is bounded} \quad \text{(recall}\ \ \vartheta<\tau).\end{aligned} \tag{57}$$

First suppose that $N \neq p$. It is clear from hypothesis H_2(iii) that we may assume that $\tau \leq r < p^*$. So, we can find $t \in [0,1)$ s.t.

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}\,.$$

The interpolation inequality (see, for example, Gasinski-Papageorgiou [7] (p.905)) implies that

$$||u_n^+||_r \leq ||u_n^+||_\tau^{1-t}\, ||u_n^+||_{p*}^t\,,$$

$$\Rightarrow\ ||u_n^+||_r^r \leq M_6||u_n^+||^{tr} \quad \text{for some } M_6 > 0, \text{ all } n \geq 1 \text{ (see (57))}. \tag{58}$$

From (50) with $h = u_n^+ \in W_0^{1,p}(\Omega)$, we have

$$\left| \, ||Du_n^+||_p^p - \int_\Omega \widehat{g}_\lambda(z, u_n^+)u_n^+ \, dz \right| \leq \varepsilon_n \quad \text{for all } n \geq 1,$$

$$\Rightarrow\ ||Du_n^+||_p^p \leq \int_\Omega \lambda f(z, u_n^+)u_n^+ \, dz + c_9 \quad \text{for some } c_9 > 0, \tag{59}$$

all $n \geq 1$ (see (47)).

Hypothesis H_2(i) implies that

$$f(z,x)x \leq \widehat{a}(z) + \widehat{c}\, x^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0, \tag{60}$$

with $\widehat{a} \in L^\infty(\Omega)_+$, $\widehat{c} > 0$.

Using (60) in (59), we obtain

$$||u_n^+||^p \leq c_{10}(\,1 + ||u_n^+||_r^r\,) \quad \text{for some } c_{10} > 0, \text{ all } n \geq 1,$$

$$\leq c_{11}(\,1 + ||u_n^+||^{tr}\,) \quad \text{for some } c_{11} > 0, \text{ all } n \geq 1 \quad \text{(see (58))}. \tag{61}$$

The condition on τ (see hypothesis H_2(iii)) implies that $tr < p$. Hence from (61) it follows that

$$\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded,}$$

$$\Rightarrow \{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded} \quad \text{(see (51))}.$$

Therefore, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^r(\Omega). \tag{62}$$

From (51) and (62) we see that $u \geq 0$ and

$$u_n^+ \xrightarrow{w} u \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{and} \quad u_n^+ \to u \quad \text{in} \quad L^r(\Omega). \tag{63}$$

In (50) we choose $h = u_n^+ - u \in W_0^{1,p}(\Omega)$. We have

$$\langle A(u_n^+),\, u_n^+ - u\rangle - \int_{\{u_n > u_0\}} \xi(u_n^+)^{-\eta}(u_n^+ - u)dz -$$

$$- \lambda \int_{\{u_n > u_0\}} f(z, u_n^+)(u_n^+ - u)dz \leq \varepsilon_n' \tag{64}$$

with $\varepsilon_n' \to 0$ (see (63)).

Note that

$$\lambda \int_{\{u_n > u_0\}} f(z, u_n^+)(u_n^+ - u)dz \to 0 \quad \text{as } n \to \infty \quad (\text{see } H_2(\text{i}) \text{ and } (63)). \tag{65}$$

Also, we have

$$\int_{\{u_n > u_0\}} \xi (u_n^+)^{-\eta}(u_n^+ - u)dz \ \le\ \int_{\{u_n > u_0\}} \xi (u_n^+)^{1-\eta} dz \quad (\text{recall } u \ge 0),$$

$$\Rightarrow \ \limsup_{n\to\infty} \int_{\{u_n > u_0\}} \xi (u_n^+)^{-\eta}(u_n^+ - u)dz \ \le\ 0. \tag{66}$$

So, if in (64) we pass to the limit as $n \to \infty$ and use (65) and (66), then

$$\begin{aligned} &\limsup_{n\to\infty} \langle A(u_n^+),\ u_n^+ - u\rangle \ \le\ 0, \\ &\Rightarrow u_n^+ \to u \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{see Proposition 2}), \\ &\Rightarrow u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{see } (51)). \end{aligned}$$

If $N = p$, then in this case $p^* = +\infty$ while by the Sobolev embedding theorem (see [7]), we have that $W_0^{1,p}(\Omega)$ is embedded compactly in $L^s(\Omega)$ for all $s \in [1, +\infty)$. So, in the above argument, we replace p^* by $q > r$ big s.t.

$$tr = \frac{q(r-\tau)}{q-\tau} \ <\ p.$$

Then again we establish that $u_n \to u$ in $W_0^{1,p}(\Omega)$. This proves Claim 1.

Let $K_{\widehat{\psi}_\lambda}$ denote the set of critical points of $\widehat{\psi}_\lambda$, i.e., $K_{\widehat{\psi}_\lambda} = \{u \in W_0^{1,p}(\Omega) : \widehat{\psi}_\lambda'(u) = 0\}$.

Using (47), we can easily check that

$$K_{\widehat{\psi}_\lambda} \subseteq [u_0) = \{u \in W_0^{1,p}(\Omega) : u_0(z) \ \le\ u(z) \ \text{ a.e. in } \Omega\}. \tag{67}$$

Also, we may assume that

$$K_{\widehat{\psi}_\lambda} \cap [u_0,\ u_\mu] = \{u_0\} \quad (\text{recall } \lambda < \mu < \lambda^*) \tag{68}$$

or otherwise we already have a second nontrivial positive solution of $(P)_\lambda$ (see (47) and (67)) with $u_0 \le \widehat{u}$ and so we are done.

Claim 2. u_0 is a local minimizer of $\widehat{\psi}_\lambda$.

We consider the following truncation of $\widehat{g}_\lambda(z,\cdot)$ (see (47))

$$\widetilde{g}_\lambda(z,x) = \begin{cases} \widehat{g}_\lambda(z,x) & \text{if } \ x \le u_\mu(z) \\ \widehat{g}_\lambda(z, u_\mu(z)) & \text{if } \ u_\mu(z) < x. \end{cases} \tag{69}$$

This is a Carathéodory function. We set $\widetilde{G}_\lambda(z,x) = \displaystyle\int_0^x \widetilde{g}_\lambda(z,s)ds$ and consider the functional $\widetilde{\psi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\psi}_\lambda(u) = \frac{1}{p}||Du||_p^p \ -\ \int_\Omega \widetilde{G}_\lambda(z, u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3 implies that $\widetilde{\psi}_\lambda \in C^1(W_0^{1,p}(\Omega))$. Also, from (69) it is clear that $\widetilde{\psi}_\lambda$ is coercive and it is easy to see that it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$ s.t.

$$\begin{aligned} & \widetilde{\psi}_\lambda(\widehat{u}_0) = \inf[\, \widetilde{\psi}_\lambda(u) : u \in W_0^{1,p}(\Omega)\,], \\ \Rightarrow\ & \widetilde{\psi}_\lambda'(\widehat{u}_0) = 0, \\ \Rightarrow\ & A(u_0) = N_{\widetilde{g}_\lambda}(\widehat{u}_0). \end{aligned} \tag{70}$$

On (70) we act with $(u_0 - \widehat{u}_0)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(\widehat{u}_0),\ (u_0 - \widehat{u}_0)^+ \rangle \\ &= \int_\Omega \widetilde{g}_\lambda(z, \widehat{u}_0)(u_0 - \widehat{u}_0)^+ dz \\ &= \int_\Omega [\, \xi u_0^{-\eta} + \lambda f(z, u_0)\,](u_0 - \widehat{u}_0)^+ dz \quad \text{(see (47) and (69))} \\ &= \langle A(u_0),\ (u_0 - \widehat{u}_0)^+ \rangle, \\ &\Rightarrow \int_{\{u_0 > \widehat{u}_0\}} (\, ||Du_0||^{p-2} Du_0 \ -\ ||D\widehat{u}_0||^{p-2} D\widehat{u}_0\ ,\ Du_0 - D\widehat{u}_0\,)_{\mathbb{R}^N} dz = 0, \\ &\Rightarrow |\{u_0 > \widehat{u}_0\}|_N = 0 \quad \text{(see (13)), hence} \quad u_0 \le \widehat{u}_0\,. \end{aligned}$$

Similarly, acting on (70) with $(\widehat{u}_0 - u_\mu)^+ \in W_0^{1,p}(\Omega)$ and using this time (46), we show that $\widehat{u}_0 \le u_\mu$. Therefore, we have proved that

$$\widehat{u}_0 \in [\, u_0,\ u_\mu\,] = \{u \in W_0^{1,p}(\Omega) : u_0(z) \le u(z) \le u_\mu(z) \quad \text{a.e. in } \Omega\}.$$

Hence (70) becomes

$$\begin{aligned} & A(\widehat{u}_0) = \xi \widehat{u}_0^{-\eta} \ +\ \lambda N_f(\widehat{u}_0) \quad \text{with } u_0 \le \widehat{u}_0\,, \\ \Rightarrow\ & \widehat{u}_0 \in K_{\widehat{\psi}_\lambda} \cap [\, u_0,\ u_\mu\,], \\ \Rightarrow\ & \widehat{u}_0 = u_0 \quad \text{(see (68))}. \end{aligned}$$

Let $\rho = ||u_\mu||_\infty$ and $\gamma_\rho > 0$ be as postulated by hypothesis H_2(iv). Let $\widehat{\gamma}_\rho > \gamma_\rho$. We have

$$\begin{aligned} & -\Delta_p u_0(z) - \xi(z) u_0(z)^{-\eta} + \lambda \widehat{\gamma}_\rho u_0(z)^{p-1} \\ &= \lambda f(z, u_0(z)) + \lambda \widehat{\gamma}_\rho u_0(z)^{p-1} \\ &\le\ \lambda f(z, u_\mu(z)) + \lambda \widehat{\gamma}_\rho u_\mu(z)^{p-1} \quad \text{(see hypothesis } H_2\text{(iv) and recall } u_0 \le u_\mu) \\ &<\ \mu f(z, u_\mu(z)) + \lambda \widehat{\gamma}_\rho u_\mu(z)^{p-1} \quad \text{(recall } u_\mu \in \mathrm{int} C_+) \\ &= -\Delta_p u_\mu(z) - \xi(z) u_\mu(z)^{-\eta} + \lambda \widehat{\gamma}_\rho u_\mu(z)^{p-1} \quad \text{a.e. in } \Omega. \end{aligned} \tag{71}$$

We set $\vartheta_1(z) = \lambda f(z, u_0(z)) + \lambda \widehat{\gamma}_\rho u_0(z)^{p-1}$ and $\vartheta_2(z) = \mu f(z, u_\mu(z)) + \lambda \widehat{\gamma}_\rho u_\mu(z)^{p-1}$. Then $\vartheta_1, \vartheta_2 \in L^\infty(\Omega)$ and $\vartheta_1 \prec \vartheta_2$ (see hypothesis H_2(iv) and recall $\widehat{\gamma}_\rho > \gamma_\rho$). So, from (71) and Proposition 4 it follows that $u_\mu - u_0 \in \mathrm{int} C_+$. Since $\widetilde{\psi}_\lambda\, |_{[0,\ u_\mu]} = \widehat{\psi}_\lambda\, |_{[0,\ u_\mu]}$ (see (47) and (69)), it follows that $u_0 \in \mathrm{int} C_+$ is a local $C_0^1(\overline{\Omega})$ - minimizer of $\widehat{\psi}_\lambda$. Hence u_0 is also a local $W_0^{1,p}(\Omega)$ - minimizer of $\widehat{\psi}_\lambda$ (see [8]). This proves Claim 2.

By virtue of Claim 2, as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0,\ 1)$ small s.t.

$$\widehat{\psi}_\lambda(u_0) \ < \ \inf[\ \widehat{\psi}_\lambda(u) : ||u - u_0|| = \rho\ ] = \widehat{\eta}_\rho. \tag{72}$$

Claim 3. $\widehat{\psi}_\lambda(t\widehat{u}_1) \to -\infty$ as $t \to +\infty$.
Since $\widehat{u}_1,\ u_0 \in \mathrm{int}C_+$, we can find $t > 0$ s.t. $u_0 \le t\widehat{u}_1$. Then

$$\widehat{\psi}_\lambda(t\widehat{u}_1) \ \le \ \frac{t^p}{p}\widehat{\lambda}_1 \ - \ \frac{t^{1-\eta}}{1-\eta}\int_\Omega \xi\widehat{u}_1^{1-\eta}dz \ - \ \lambda\int_\Omega F(z, t\widehat{u}_1)dz \ + \ c_{12} \tag{73}$$

for some $c_{12} > 0$ (see (2) and (47)).
Hypotheses H_2(i), (ii) imply that given any $\zeta > 0$, we can find $c_\zeta > 0$ s.t.

$$F(z,x) \ \ge \ \frac{\zeta}{p}x^p \ - \ c_\zeta \quad \text{for a.a. } z \in \Omega, \ \text{ all } x \ge 0. \tag{74}$$

Using (74) in (73), we obtain

$$\widehat{\psi}_\lambda(t\widehat{u}_1) \ \le \ \frac{t^p}{p}\widehat{\lambda}_1 \ - \ \frac{t^p}{p}\lambda\zeta \ + \ \lambda c_\zeta|\Omega|_N \ + \ c_{12} \quad (\text{recall } ||\widehat{u}_1||_p = 1).$$

Choosing $\zeta > \dfrac{\widehat{\lambda}_1}{\lambda}$, from this last inequality we see that

$$\widehat{\psi}_\lambda(t\widehat{u}_1) \to -\infty \quad \text{as} \quad t \to +\infty.$$

This proves Claim 3.

Claims 1,3 and (72) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ s.t.

$$\widehat{\eta}_\rho \ \le \ \widehat{\psi}_\lambda(\widehat{u}) \tag{75}$$

$$\widehat{\psi}_\lambda'(\widehat{u}) = 0. \tag{76}$$

From (72) and (75) it follows that $\widehat{u} \ne u_0$, while from (76) and (67) we have $u_0 \le \widehat{u}$. Therefore, $\widehat{u}$ is the second nontrivial positive solution of problem $(P)_\lambda$ (see (47)) distinct from u_0 and $\widehat{u} \in \mathrm{int}C_+$ (see Proposition 6). □

Proposition 9. *If hypotheses H_1 and H_2 hold, then $\lambda^* \in \mathcal{P}$.*

Proof. Let $\{\lambda_n\}_{n\ge1} \subseteq \mathcal{P}$ s.t. $\lambda_n < \lambda^*$ and $\lambda_n \uparrow \lambda^*$. As in the proof of Proposition 8, we consider the truncation of the reaction of problem $(P)_{\lambda_n}$ at $\{\underline{u}(z),\ u_{\lambda_{n+1}}(z)\}$ with $u_{\lambda_{n+1}} \in S(u_{\lambda_{n+1}})$ (see Proposition 8). We denote this truncation by $\overline{g}^n_{\lambda_n}(z,x)$ and set $\overline{G}^n_{\lambda_n}(z,x) = \int_\Omega \overline{g}^n_{\lambda_n}(z,s)ds$ and consider the C^1 - functional $\overline{\psi}^n_{\lambda_n} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\overline{\psi}^n_{\lambda_n}(u) = \frac{1}{p}||Du||_p^p \ - \ \int_\Omega \overline{G}^n_{\lambda_n}(z, u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

(see Proposition 3). Using the direct method, we obtain solutions $u_n \in [\ \underline{u},\ u_{\lambda_{n+1}}\ ]$ minimizers of $\overline{\psi}^n_{\lambda_n}$ (see the first part of the proof of Proposition 8). Then we have

$$\begin{aligned} \overline{\psi}^n_{\lambda_n}(u_n) &\le \overline{\psi}^n_{\lambda_n}(\underline{u}) \\ &= \frac{1}{p}||D\underline{u}||_p^p - \int_\Omega \xi \underline{u}^{1-\eta} dz - \lambda \int_\Omega f(z,\underline{u})\underline{u} dz. \end{aligned} \tag{77}$$

From Proposition 5, we have

$$||D\underline{u}||_p^p = \int_\Omega \xi \underline{u}^{1-\eta} dz.$$

Therefore

$$\begin{aligned} &\overline{\psi}^n_{\lambda_n}(u_n) < 0 \quad (\text{for all } n \ge 1 \ (\text{see } (77)), \\ &\Rightarrow \ ||Du_n||_p^p - \frac{p}{1-\eta}\int_\Omega \xi u_n^{1-\eta} dz - \lambda_n p \int_\Omega F(z,u_n) dz \le M_7 \end{aligned} \tag{78}$$

for some $M_7 > 0$, for all $n \ge 1$.

Also, we have

$$\underline{u} \le u_n \quad \text{and} \quad A(u_n) = \xi u_n^{-\eta} + \lambda_n N_f(u_n) \quad \text{for all } n \ge 1, \tag{79}$$

$$\Rightarrow \quad -||Du_n||_p^p + \int_\Omega \xi u_n^{1-\eta} dz + \lambda_n \int_\Omega f(z,u_n)u_n dz = 0 \quad \text{for all } n \ge 1. \tag{80}$$

Adding (78) and (80), we obtain

$$\begin{aligned} &\lambda_n \int_\Omega [\ f(z,u_n)u_n - pF(z,u_n)\ ] dz \le \left(\frac{p}{1-\eta} - 1\right)\int_\Omega \xi u_n^{1-\eta} dz + c_{13} \\ &\text{for some } c_{13} > 0, \ \text{all } n \ge 1, \\ &\Rightarrow \int_\Omega [\ f(z,u_n)u_n - pF(z,u_n)\ ] dz \le \frac{1}{\lambda_1}\left(\frac{p}{1-\eta} - 1\right)\int_\Omega \xi u_n^{1-\eta} dz + \\ &+ \frac{1}{\lambda_1} c_{13} \quad \text{for all } n \ge 1. \end{aligned} \tag{81}$$

Using (81) and reasoning as in the proof of Proposition 8 (see Claim 1 and in particular the part of the proof after (54)), we show that (at least for a subsequence), we have

$$u_n \to u^* \quad \text{in} \quad W_0^{1,p}(\Omega), \quad \underline{u} \le u^* \quad (\text{see } (79)). \tag{82}$$

For every $h \in W_0^{1,p}(\Omega)$, we have

$$\langle A(u_n),\ h\rangle = \int_\Omega \xi u_n^{-\eta} h dz + \lambda_n \int_\Omega f(z,u_n) h dz \quad \text{for all } n \ge 1 \tag{83}$$

(see (79)).

Also, since $\underline{u} \in \text{int}C_+$ (see Proposition 5), we have

$$c_{14} d(z) \le \underline{u}(z) \le u_n(z) \quad \text{for all } z \in \overline{\Omega}, \ \text{all } n \ge 1 \text{ and some } \ c_{14} > 0.$$

Therefore

$$\xi(z)\frac{h(z)}{u(z)^{\eta}} \leq \frac{||\xi||_{\infty}}{c_{14}}\frac{h(z)}{d(z)^{\eta}} \leq c_{15}\frac{h(z)}{d(z)} \quad \text{a.e. in } \Omega \text{ for some } c_{15}>0.$$

Hardy's inequality implies that $\frac{h}{d} \in L^p(\Omega)$ (recall $h \in W_0^{1,p}(\Omega)$). Since

$$\xi(z)u_n(z)^{-\eta}h(z) \to \xi(z)u^*(z)^{-\eta}h(z) \quad \text{a.e. in } \Omega,$$

invoking the Lebesgue dominated convergence theorem, we have

$$\int_{\Omega}\xi u_n^{-\eta}hdz \to \int_{\Omega}\xi(u^*)^{-\eta}hdz. \tag{84}$$

Also, we have

$$\lambda_n\int_{\Omega}f(z,u_n)hdz \to \lambda^*\int_{\Omega}f(z,u^*)hdz \text{ and } \langle A(u_n),h\rangle \to \langle A(u^*),h\rangle \tag{85}$$

(see (82)).

So, if in (83) we pass to the limit as $n\to\infty$ and use (84) and (85), then

$$\begin{aligned}&\langle A(u^*),\ h\rangle = \int_{\Omega}\xi(u^*)^{-\eta}hdz + \int_{\Omega}f(z,u^*)hdz \quad \text{for all } h\in W_0^{1,p}(\Omega),\\ &\underline{u} \leq u^*,\\ \Rightarrow\ & u^*\in S(\lambda^*) \subseteq \operatorname{int}C_+.\end{aligned}$$

Therefore $\lambda^*\in\mathcal{P}$. ☐

So, summarizing the situation for problem $(P)_\lambda$, we can state the following bifurcation-type theorem.

THEOREM 10. *If hypotheses H_1 and H_2 hold, then we can find $\lambda^*>0$ s.t.*

(i) for every $\lambda\in(0,\ \lambda^*)$ problem $(P)_\lambda$ has at least two nontrivial positive solutions $u_0,\ \widehat{u}\in intC_+,\ u_0\leq\widehat{u},\ u_0\neq\widehat{u}$;

(ii) for $\lambda=\lambda^*$ problem $(P)_\lambda$ has one nontrivial positive solution $u^*\in intC_+$;

(ii) for $\lambda>\lambda^*$ problem $(P)_\lambda$ has no nontrivial positive solution.

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