

SOME GENERALIZATIONS OF THE SECOND MAIN THEOREM INTERSECTING HYPERSURFACES*

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Dedicated to Professor Stephen Yau on the occasion of his 60th birthday

Abstract. In [18], the author established a defect relation for holomorphic mappings from \mathbb{C} into the projective variety $X \subset \mathbb{P}^N(\mathbb{C})$ intersecting divisors of X coming from hypersurfaces in \mathbb{P}^N . The purpose of this paper is to do some further generalizations. It contains two major parts: (I) We extend the defect relation to effective divisors on X with $Pic(X) = \mathbb{Z}$. (II) We extend the defect relation to meromorphic mappings from parabolic manifolds into $X \subset \mathbb{P}^N$.

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1. Introduction. Let $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map, and $H_j, 1 \leq j \leq q$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. In 1933, H. Cartan (see [4]) proved the defect relation (or a Second Main Theorem)

$$\sum_{j=1}^q \delta_f(H_j) \leq n + 1.$$
 Recently, the author (see [18]) extended H. Cartan's result to

algebraically nondegenerate holomorphic curves $f : \mathbb{C} \rightarrow X \subset \mathbb{P}^N(\mathbb{C})$ and obtained

$$\sum_{j=1}^q \delta_f(D_j) \leq \dim X + 1,$$

where X is a smooth projective variety and D_1, \dots, D_q are hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ located in general position in X .

To get a more precise statement, we introduce some standard notations in Nevanlinna theory: Let $f : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ be a holomorphic map. Let $\mathbf{f} = (f_0, \dots, f_N)$ be a reduced representative of f , where f_0, \dots, f_N are entire functions on \mathbb{C} and have no common zeros. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| d\theta$$

where

$$\|\mathbf{f}(z)\| = \max\{|f_0(z)|, \dots, |f_N(z)|\}.$$

The above definition is independent, up to an additive constant, of the choice of the reduced representation of f . Let D be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ of degree d and let Q be the homogeneous polynomial of degree d defining D . The *proximity function* is defined as

$$m_f(r, D) = \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\|^d \|Q\| d\theta}{|Q(\mathbf{f})(re^{i\theta})|} \frac{d\theta}{2\pi},$$

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where $\|Q\|$ is the maximum of the absolute values of the coefficients of Q . To define the counting function, let $n_f(r, D)$ be the number of zeros of $Q \circ f$ in the disk $|z| < r$, counting multiplicity. The *counting function* is then defined by

$$N_f(r, D) = \int_1^r \frac{n_f(t, D)}{t} dt.$$

The Poisson-Jensen formula implies:

THE FIRST MAIN THEOREM. *Let $f : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ be a holomorphic map, and let D be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ of degree d . If $f(\mathbb{C}) \not\subset D$, then for every real number r with $0 < r < \infty$,*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r .

Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n \geq 1$. Let D_1, \dots, D_q be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$, where $q > n$. The hypersurfaces D_1, \dots, D_q are said to be *in general position in X* if for every subset $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$,

$$X \cap \text{supp}D_{i_0} \cap \dots \cap \text{supp}D_{i_n} = \emptyset,$$

where $\text{supp}(D)$ means the support of the divisor D . A map $f : \mathbb{C} \rightarrow X$ is said to be *algebraically non-degenerate* if the image of f is not contained in any proper subvarieties of X . The following result, which extends the result of H. Cartan, was proved in [18].

THEOREM A [18]. *Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety. Let D_1, \dots, D_q be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree d_j , located in general position in X . Let $f : \mathbb{C} \rightarrow X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^q d_j^{-1} m_f(r, D_j) \leq (\dim X + 1 + \epsilon) T_f(r) \parallel_E$$

where \parallel_E means that the inequality holds for all r except a set $E \subset (0, +\infty)$ with finite Lebesgue measure.

Define the *defect of f with respect to a hypersurface D* of degree d by

$$\delta_f(D) = \liminf_{r \rightarrow +\infty} \frac{m_f(r, D)}{dT_f(r)} = 1 - \limsup_{r \rightarrow +\infty} \frac{N_f(r, D)}{dT_f(r)}.$$

Then Theorem A gives

COROLLARY A (Defect Relation). *Under the same assumptions in Theorem A, we have*

$$\sum_{j=1}^q \delta_f(D_j) \leq \dim X + 1.$$

The purpose of this paper is to generalize the above results. This paper contains two major parts: (I) We extend Theorem A to effective divisors on X with $\text{Pic}(X) = \mathbb{Z}$, where $\text{Pic}(X)$ is the Picard group of X , by verifying that *in the case $\text{Pic}(X) = \mathbb{Z}$ every effective divisor on X is cut out by a hypersurface in the ambient projective space, as assumed in Theorem A*; (II) We extend Theorem A to meromorphic mappings from parabolic manifolds into $X \subset \mathbb{P}^N$.

2. Divisors on projective variety X with $Pic(X) = \mathbb{Z}$. Theorem A (so the defect relation in Corollary A) only applies to those divisors on X which are cut out by hypersurfaces in an ambient projective space. For a general non-singular projective variety, it is far from true that (multiples of) effective divisors are cut out by hypersurfaces in the ambient projective space. In this section we prove that, for a nonsingular projective variety X with $Pic(X) = \mathbb{Z}$, up to taking multiples, effective divisors are cut out by hypersurfaces. Therefore Theorem A in above can be applied.

LEMMA 2.1. *Let $X \subseteq \mathbb{P}^N$ be a nonsingular projective variety. Assume that $Pic(X) = \mathbb{Z}$. Let D be an effective divisor on X . Then there exists a positive integer k such that kD is the restriction to X of some hypersurface in \mathbb{P}^N .*

Proof. Consider the line bundle $\mathcal{O}_X(D)$, which represents an element in $Pic(X)$. Due to $Pic(X) = \mathbb{Z}$, there exist positive integers m, κ such that $\mathcal{O}_X(mD) = \mathcal{O}_X(\kappa)$ as elements of $Pic(X)$. We are free to multiply this equality with an arbitrary positive integer \tilde{m} , obtaining $\mathcal{O}_X(\tilde{m}mD) = \mathcal{O}_X(\tilde{m}\kappa)$. Consider the short exact sequence

$$0 \rightarrow \mathcal{I}_X(\tilde{m}\kappa) \rightarrow \mathcal{O}_{\mathbb{P}^N}(\tilde{m}\kappa) \rightarrow \mathcal{O}_X(\tilde{m}\kappa) \rightarrow 0.$$

The corresponding long exact sequence starts with

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{P}^N, \mathcal{I}_X(\tilde{m}\kappa)) &\rightarrow H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\tilde{m}\kappa)) \rightarrow H^0(X, \mathcal{O}_X(\tilde{m}\kappa)) \\ &\rightarrow H^1(\mathbb{P}^N, \mathcal{I}_X(\tilde{m}\kappa)). \end{aligned}$$

Due to Serre’s Vanishing Theorem, $H^1(\mathbb{P}^N, \mathcal{I}_X(\tilde{m}\kappa)) = 0$ when \tilde{m} is sufficiently large. Thus, all sections in $H^0(X, \mathcal{O}_X(\tilde{m}\kappa))$ extend to sections in $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\tilde{m}\kappa))$ when \tilde{m} is sufficiently large. In particular, sections $s \in H^0(X, \mathcal{O}_X(\tilde{m}mD)) = H^0(X, \mathcal{O}_X(\tilde{m}\kappa))$ with

$$(s) = \tilde{m}mD$$

extend. However, the sections of $\mathcal{O}_{\mathbb{P}^N}(\tilde{m}\kappa)$ correspond to homogeneous polynomials of degree $\tilde{m}\kappa$ in $\mathbb{C}[X_0, \dots, X_N]$. Thus, $\tilde{m}mD$ is indeed defined by the restriction to X of a homogeneous polynomial in $\mathbb{C}[X_0, \dots, X_N]$ when \tilde{m} is sufficiently large. This proves the lemma. \square

Let $X \subseteq \mathbb{P}^N$ be a nonsingular projective variety and let $L \rightarrow X$ be an ample line bundle over X . Denote by $\|\cdot\|$ a Hermitian fiber metric on L and ω be its Chern form. Let $f : \mathbb{C} \rightarrow X$ be a holomorphic mapping. We set

$$T_f(r, L) = \int_1^r \frac{dt}{t} \int_{\{z \mid |z| < t\}} f^* \omega,$$

and call it the *characteristic function of f with respect to L* . It is easy to check that it is independent, up to a bounded term, of the choice of the Hermitian fiber metric on L . Let $D = (\sigma) \in |L|$ with $\|\sigma\| < 1$ on X , where $|L|$ is the complete system of L . Assume that $f(\mathbb{C})$ is not contained in $\text{Supp}D$. We define the *approximity function of f with respect to D* by

$$m_f(r, D) = \int_0^{2\pi} \log \left(\frac{1}{\|\sigma(f(re^{i\theta}))\|} \right) \frac{d\theta}{2\pi}.$$

To define the *counting function* for $D = (\sigma) \in |L|$, let $n_f(r, D)$ be the number of zeros of $\sigma \circ f$ in the disk $|z| < r$, counting multiplicity. The *counting function* is then defined by

$$N_f(r, D) = \int_1^r \frac{n_f(t, D)}{t} dt.$$

The Poincare-Lelong formula implies that (*The First Main Theorem*)

$$T_f(r, L) = m_f(r, D) + N_f(r, D) + O(1)$$

for all $D \in |L|$ with $f(\mathbb{C}) \not\subset \text{Supp}(D)$, where $O(1)$ stands for a bounded term as $r \rightarrow +\infty$.

Let f and D be as above (with the assumption that D is ample), we define Nevvvanlinna’s deficiency $\delta_f(D)$ by

$$\delta_f(D) = \liminf_{r \rightarrow +\infty} \frac{m_f(r, D)}{T_f(r, \mathcal{O}_X(D))} = 1 - \limsup_{r \rightarrow +\infty} \frac{N_f(r, D)}{T_f(r, \mathcal{O}_X(D))},$$

where $\mathcal{O}_X(D)$ is the line bundle over X associated to the divisor D . Note that the Nevvvanlinna’s deficiency $\delta_f(D)$ only depends on f and D (independent of the choice of the metrics on $\mathcal{O}(D)$). We also note that for any integer positive integer k , $n_f(r, kD) = kn_f(r, D)$ and $T_f(r, \mathcal{O}_X(kD)) = kT_f(r, \mathcal{O}_X(D))$, thus, for any integer positive integer k ,

$$\delta_f(kD) = \delta_f(D).$$

This notion extends to any effective divisor D since we can always write $D = D_1 - D_2$ with D_1, D_2 being very ample.

The first result, among the two main results in this paper, is the following Theorem.

THEOREM 1 [Defect Relation]. *Let X be a non-singular projective variety of dimension n with $\text{Pic}(X) = \mathbb{Z}$. Let D_1, \dots, D_q be effective divisors on X located in general position (i.e. for every subset $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$, $\text{supp}D_{i_0} \cap \dots \cap \text{supp}D_{i_n} = \emptyset$). Let $f : \mathbb{C} \rightarrow X$ be an algebraically nondegenerate holomorphic map. Then*

$$\sum_{j=1}^q \delta_f(D_j) \leq \dim X + 1.$$

In particular, if $q \geq \dim X + 2$, then every holomorphic map $f : \mathbb{C} \rightarrow X \setminus D$, where $D = \sum_{j=1}^q D_j$, must be algebraically degenerate, i.e., the image of f must be contained in a proper subvariety of X .

Proof. By Lemma 2.1, there exist positive integers, say $k_j, 1 \leq j \leq q$, such that $k_j D_j$ are defined by the restriction to X of homogeneous polynomials Q_j in $\mathbb{C}[X_0, \dots, X_N]$. From Corollary A, we get

$$\sum_{j=1}^q \delta_f(k_j D_j) \leq \dim X + 1.$$

But, as we noted in above,

$$\delta_f(k_j D_j) = \delta_f(D_j),$$

hence

$$\sum_{j=1}^q \delta_f(D_j) \leq \dim X + 1.$$

This proves Theorem 1. \square

Next, we exhibit a large class of nonsingular projective varieties that satisfy the condition $\text{Pic}(X) = \mathbb{Z}$, namely those of sufficiently *small codimension*. We refer the reader to [10] or [13, section 3.2] for more on the algebraic geometry behind this question, which is essentially the problem of extending the Lefschetz Hyperplane Theorem to nonsingular varieties that are not complete intersections. In particular, the theorems of [2], [12], and, in the case of a general ground field of characteristic zero, [14] apply. For our purposes, we simply state the following proposition, which is an immediate consequence of these theorems.

PROPOSITION 2.1. *Let $X \subset \mathbb{P}^N$ be a nonsingular projective variety of dimension n . If $2n - N \geq 2$, then restriction yields an isomorphism*

$$\text{Pic}(\mathbb{P}^N) \xrightarrow{\cong} \text{Pic}(X).$$

Since $\text{Pic}(\mathbb{P}^N) = \mathbb{Z}$, the proposition gives the following immediate consequence of Corollary.

COROLLARY 2.1. *Let $X \subseteq \mathbb{P}^N$ be a nonsingular projective variety. Assume that $2 \dim X - N \geq 2$ holds. Let $D = \sum_{j=1}^q D_j$ be an effective divisor on X such that $D_j, 1 \leq j \leq q$, are in general position. Let $f : \mathbb{C} \rightarrow X$ be an algebraically nondegenerate holomorphic map. Then*

$$\sum_{j=1}^q \delta_f(D_j) \leq \dim X + 1.$$

In particular, if $q \geq \dim X + 2$, then every holomorphic map $f : \mathbb{C} \rightarrow X \setminus D$ must be algebraically degenerate, i.e., the image of f must be contained in a proper subvariety of X .

We make a remark here about the nature of the condition $2 \dim X - N \geq 2$. It is well-known that, by general linear projections, any nonsingular projective variety of dimension n can be embedded in \mathbb{P}^N with $N = 2n + 1$. Thus, for an arbitrary nonsingular projective variety, one can always find an embedding with $2n - N = -1$. In particular, Corollary 2.1 applies to many interesting special projective varieties such as hypersurfaces, appropriate complete intersections, and certain Grassmannians, such as $G(2, 5)$ embedded into \mathbb{P}^9 under the Plücker embedding. Note that the latter is not a complete intersection due to Bézout's Theorem, because its degree is (the prime number) 5, while it is not contained in any hyperplane.

3. The second main theorem on parabolic manifolds. In this section, we extend Theorem A to meromorphic mappings from parabolic manifolds to algebraic variety $X \subset \mathbb{P}^N$.

3.1. The result of Stoll-Wong. H. Cartan’s defect relation mentioned in the introduction was also proved by L. Ahlfors in 1941 (see [1]) with a different method. Ahlfors method was indeed more geometrical, which is a magnificent demonstration of the interplay between topology, geometry and analysis, on holomorphic curves. Ahlfors’ theory was extended to parabolic Riemann surfaces by H. Weyl and J. Weyl in the famous monograph [25] published in 1943. Modern treatment of this theory can also be found in Wu [28]. Extension of the theory to higher dimension was first achieved by Stoll in a series of papers [21], [22] in late forties and early fifties. In the sixties and early seventies, value distribution theory attracted the attention of many mathematicians, notably the works of Carlson-Griffiths [3], Griffiths-King [9] extending the theory to algebraic manifolds in the equi-dimensional case. However in this article we shall restrict ourselves to the ideas of Stoll in extending the defect relation to a larger case of manifolds other than \mathbf{C}^m .

In extending the defect relation, the class of manifolds should be large enough so as to include all affine algebraic manifolds and, on the other hand, manifolds in this class should not admit any non-constant bounded holomorphic functions (otherwise we cannot expect to have a nice simple defect relation). Such manifolds can be produced by the following procedure (cf. Stoll [21]). For simplicity we consider only the case of Kahler manifolds. Let M be a complex manifold of dimension m with Kahler metric g and ω the associated Kahler form. Choose a sequence of relatively compact domains with smooth boundaries $\{G_j\}$, $G_j \subset\subset G_{j+1}$ and $\cup G_j = M$. Solve the Dirichlet problem for the Laplace-Beltrami operator Δ on $G_j - G_0$,

$$(1.1) \quad \Delta u_j = 0 \text{ on } G_j - G_0$$

$$(1.2) \quad u_j = \begin{cases} 0 & \text{on } M - G_j \\ 1 & \text{on } \overline{G_0} \end{cases} .$$

The harmonic condition (1.1) is equivalent to

$$(1.3) \quad dd^c u_j \wedge \omega^{m-1} = 0$$

where $d^c = \sqrt{-1}(\bar{\partial} - \partial)$. Define constants

$$C_j = - \int_{\partial G_j} d^c u_j \wedge \omega^{m-1}.$$

These constants are strictly positive (by Hopf’s lemma) and satisfies $C_j \geq C_k$ if $k \geq j$ (by maximum principle). Thus the limit

$$C_\infty(M) = \lim_{j \rightarrow \infty} C_j \geq 0$$

exists and is called the *capacity* of M at infinity.

PROPOSITION 3.1.1. *A connected Kahler manifold M with zero capacity at infinity does not admit any non-constant bounded holomorphic function.*

Proof. Let ω be the fundamental Kahler form and suppose there is a bounded non-constant holomorphic function f , say $|f| \leq 1$. Then by (1.2) we have,

$$0 < \lambda = \int_{G_0} dd^c \log(1 + |f|^2) \wedge \omega^{m-1} = \int_{G_0} u_j dd^c \log(1 + |f|^2) \wedge \omega^{m-1}.$$

From (1.1), (1.2) and Stoke’s theorem, the last term above is equal to

$$\int_{\partial G_0 - \partial G_j} \log(1 + |f|^2) d^c u_j \wedge \omega^{m-1} \leq - \int_{\partial G_j} \log(1 + |f|^2) d^c u_j \wedge \omega^{m-1} \leq C_j \log 2.$$

Thus we have $0 < \lambda \leq C_j \log 2$, for any j . This is a contradiction since $C_j \rightarrow 0$ as $j \rightarrow \infty$. \square

The concept of zero capacity above depends on the metric, thus it is desirable to have more intrinsic notion of “parabolicity”. The following definition defining parabolicity via the homogeneous complex Monge-Ampere equation is due to Stoll (cf. [23]).

DEFINITION 3.1.1. *A continuous non-negative function τ on a complex manifold M^m is parabolic if*

- (i) τ is C^∞ on $M_* = M - \{\tau = 0\}$;
- (ii) τ is plurisubharmonic on M and is strictly plurisubharmonic at some point of M_* ;
- (iii) the function $u = \log \tau$ satisfies the complex homogeneous Monge-Ampere equation on M_* , i.e.

$$\det(u_{\alpha\bar{\beta}}) \equiv 0 \quad \text{on } M_*$$

where $u_{\alpha\bar{\beta}} = \partial^2 u / \partial z \partial \bar{z}$.

A complex manifold M is *parabolic* if there exists a continuous exhaustion $\tau : M \rightarrow [0, \infty)$ which is parabolic on the complement of some compact set K . Examples of parabolic manifolds include (1) Open parabolic Riemann surfaces (as in the classical sense); (2) Euclidean space \mathbb{C}^m (with $\tau(z) = |z|^2$); (3) An affine algebraic manifold as it can be realized a finite branched cover $\pi : M \rightarrow \mathbb{C}^m$ (with $\tau(z) = |\pi(z)|^2$).

On a parabolic manifold (M, τ) take r_0 so that the set $M_{r_0} = \{\tau < r_0\}$ contains the exceptional compact set K . For any $r > r_0$, the function

$$u_r = \begin{cases} 0 & \text{on } M - M_r \\ \frac{\log r - \log \tau}{\log r - \log r_0} & \text{on } M_r - M_{r_0} \\ 1 & \text{on } M_{r_0} \end{cases}$$

solves the Dirichlet problem (1.2) and (1.3) with $\omega = dd^c \log \tau$. Capacity can be defined as before. It is easy to verify that, since τ is unbounded, the capacity at infinity $C_\infty(M)$ vanishes. We thus have, from Proposition 3.1.1,

COROLLARY 3.1.1. *Let M be a parabolic manifold then $C_\infty(M) = 0$ so it does not admit any non-constant bounded holomorphic function.*

The following defect relation on parabolic manifolds is due to Stoll [23] and Wong [26]).

THEOREM 3.1.1 (Stoll-Wong). *Let (M, τ) be a parabolic manifold and let $f : M \rightarrow \mathbb{P}^n$ be a linearly non-degenerate meromorphic map. Let H_1, \dots, H_q be hyperplanes in general position in \mathbb{P}^n , then*

$$\sum_{j=1}^q \delta_f(H_j) \leq n + 1 + \frac{n(n+1)}{2} R_f + 3n(n+1) Y_f.$$

The term R_f , which vanishes in the case of \mathbf{C}^m , reflects the topology of the manifold M . The appearance of the term Y_f (called the majorant defect) is due mainly to technicality. Due to the global nature of the theorem, it is necessary to define the derivatives (first and higher orders) of f by a global (meromorphic) vector field Z in such a way that $f \wedge Zf \wedge \cdots \wedge Z^n f \neq 0$. The term Y_f occurs in the estimate of the growth of Z , for which one does not have good control in general. A natural way of constructing Z with "good" growth condition is to take the (complex) gradient vector field Z of τ . However, Z is in general not holomorphic. If Z is holomorphic, then the term Y_f vanishes, this is the case for parabolic Riemann surfaces and for affine algebraic manifolds.

COROLLARY 3.1.2. *If M is algebraic, theorem 3.1.1 reduces to*

$$(i) \sum_{j=1}^q \delta_f(H_j) \leq n + 1 \text{ if } f \text{ is transcendental}$$

$$(ii) \sum_{j=1}^q \delta_f(H_j) \leq n + 1 + \frac{n(n+1) \deg(M)}{2 \deg(f)}, \text{ if } f \text{ is rational.}$$

3.2. Meromorphic mappings on M . Throughout the rest of the paper, we shall assume that M is parabolic with the parabolic exhaustion τ . Assume that $\dim M = m$. Take $r_0 > 1$ so that the set $M_{r_0} = \{\tau < r_0\}$ contains the exceptional compact set K . For $r_0 \leq r \in \mathbb{R}$ and $A \subseteq M$ define

$$A[r] = \{x \in A \mid r_0 < \tau(x) \leq r^2\}, \quad A(r) = \{x \in A \mid r_0 < \tau(x) < r^2\},$$

$$A\langle r \rangle = \{x \in A \mid \tau(x) = r^2\}, \quad A_* = \{x \in A \mid \tau(x) > r_0\},$$

$$v = dd^c \tau, \quad \omega = dd^c \log \tau, \quad \sigma = d^c \log \tau \wedge \omega^{m-1}.$$

Define

$$\hat{\mathbb{R}}_\tau = \{r \in \mathbb{R}^+ \mid d\tau(x) \neq 0 \text{ for all } x \in M\langle r \rangle\}.$$

Then $\mathbb{R}^+ \setminus \hat{\mathbb{R}}_\tau$ has measure zero. If $r \in \hat{\mathbb{R}}_\tau$, the boundary $\partial M(r) = M\langle r \rangle$ is a compact, real, $(2m - 1)$ -dimensional submanifold of class C^∞ of M , oriented to the exterior of $M\langle r \rangle$. By Stoll ([23], p. 133), for all $r \in \hat{\mathbb{R}}_\tau$,

$$(1.4) \quad \varsigma := \int_{M\langle r \rangle} \sigma$$

is a positive constant, independent of r . From now on, we always assume that $r > r_0$ and $r \in \hat{\mathbb{R}}_\tau$.

Let $A \neq \emptyset$ be a nonempty open subset of M such that $S = M - A$ is analytic. Let $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ be holomorphic on A . The closure Γ of the graph $\{(x, f(x)) \mid x \in A\}$ in $M \times \mathbb{P}^n(\mathbb{C})$ is called the *closed graph* of f . The map f is said to be *meromorphic* on M if (i) Γ is analytic in $M \times \mathbb{P}^n(\mathbb{C})$ and (ii) $\Gamma(f) \cap (K \times \mathbb{P}^n(\mathbb{C}))$ is compact for each compact subset $K \subseteq M$, i.e. the projection $\rho : \Gamma(f) \rightarrow M$ is proper. If f is meromorphic, then the set of *indeterminacy* $I_f = \{x \in M \mid \#\rho^{-1}(x) > 1\}$ is analytic with $\dim I_f \leq m - 2$

and is contained in S . f extends holomorphically to $f : M - I_f \rightarrow \mathbb{P}^n(\mathbb{C})$ such that we can assume, a posteriori, that $S = I_f$. If $m = 1$, I_f is necessarily empty and $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ is holomorphic. Given M, A, S and a holomorphic map $f : A \rightarrow \mathbb{P}^n(\mathbb{C})$ as above. A holomorphic map $\mathbf{f} (\neq 0) : U \rightarrow \mathbb{C}^{n+1}$ on an open and connected subset U of M is said to be a *representation* of f if $f(x) = \mathbb{P}(\mathbf{f}(x))$ for all $x \in A \cap U$ with $\mathbf{f}(x) \neq 0$. A representation \mathbf{f} is said to be *reduced* if $\dim \mathbf{f}^{-1}(0) \leq m - 2$. The map f is meromorphic if and only if for every point $p \in M$, there is a representation $\mathbf{f} : U \rightarrow \mathbb{C}^{n+1}$ of f with $p \in U$. If so, a representation \mathbf{f} is reduced if and only if $U \cap I_f = \mathbf{f}^{-1}(0)$. There is also a reduced representation at every point $p \in M$.

3.3. The associated maps. To define the associated maps of f , we need to assume that there exists a holomorphic form B of bidegree $(m - 1, 0)$ on M . Let \mathbf{f} be a holomorphic vector-valued function on an open subset U of M . If $z = (z_1, \dots, z_m)$ is a chart with $U_z \cap U \neq \emptyset$, then the B -derivative $\mathbf{f}'_{B,z} = \mathbf{f}'$ on $U \cap U_z$ for z is defined by $d\mathbf{f} \wedge B = \mathbf{f}' dz_1 \wedge \dots \wedge dz_m$. The operation can be iterated so that the k -th B -derivative $\mathbf{f}^{(k)}$ is defined: $\mathbf{f}^{(k)} = (\mathbf{f}^{(k-1)})'$. Put $\mathbf{f}^{(0)} = \mathbf{f}$. Abbreviate

$$\mathbf{f}_k = \mathbf{f} \wedge \mathbf{f}' \wedge \dots \wedge \mathbf{f}^{(k)} : U \rightarrow \wedge^{k+1} \mathbb{C}^{n+1}.$$

Let $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map. If $\mathbf{f}_k \neq 0$ for one choice of a reduced representation $\mathbf{f} : U \rightarrow \mathbb{C}^{n+1}$ on a chart U_z , then $\mathbf{f}_k \neq 0$ for all possible choices and f is said to be *general of order k for B* . In this case, the k -th associated map $f_k : M \rightarrow \mathbb{P}(\wedge^{k+1} \mathbb{C}^{n+1})$ is well-defined as a meromorphic map by $f_k|U = \mathbb{P}(\mathbf{f}_k)$ for all possible choices of \mathbf{f} and chart z . We say that f is *general for B* if f is general of order k for B for all $k, 1 \leq k \leq n$. The basic existence theorem for a holomorphic $(m - 1)$ -form B on M is due to W. Stoll. He (see [23]) proved the following statement: *Let M be a connected Stein manifold and let $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic map. Then there exists a holomorphic $(m - 1)$ -form B on M such that f is general for B . If $\dim M = 1$, we may take $B \equiv 1$. If M is affine algebraic with the exhaustion τ defined as above and $\dim M \geq 2$, then the form B can be chosen so that*

$$mi_{m-1} B \wedge \bar{B} \leq (1 + \tau)^{n-1} (dd^c \tau)^{m-1}$$

where $i_{m-1} = \left(\frac{\sqrt{-1}}{2\pi} \right) (m - 1)! (-1)^{(m-1)(m-2)/2}$.

3.4. The majorant function. For a general parabolic Stein manifold (M, τ) , even though the existence of B exists by Stoll's result above, in general, it may not have a polynomial type estimate as for affine algebraic manifolds. To overcome this difficulty, Stoll [23] postulates the existence of a majorant function such that

$$(1.5) \quad mi_{m-1} B \wedge \bar{B} \leq Y(r) v^{m-1}$$

on $M[r]$. Such function $Y(r)$ is called a *majorant for B* . The majorant function $Y(r)$ introduces an extra term in the Second Main Theorem.

3.5. The Plücker formula. Let $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is linearly non-degenerate, hence f is general for B . Let f_k be the k th associated map of f . Let Ω_k be the Fubini-Study form on $\mathbb{P}^n(\wedge^{k+1} \mathbb{C}^{n+1})$. Then we have (see [23], or [27] Lemma 1.10)

$$(1.6) \quad mi_{m-1} f_k^*(\Omega_k) B \wedge \bar{B} = \frac{\|\mathbf{f}_{k-1}\| \|\mathbf{f}_{k+1}\|}{\|\mathbf{f}_{k-1}\|} m! dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_m \wedge d\bar{z}_m.$$

Define the k th characteristic function

$$T_{f_k}(r) = \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f_k^*(\Omega_k) \wedge \nu^{m-1}.$$

It is known that $T_{f_n}(r) \equiv 0$. Denote $T_{f_{-1}}(r) \equiv 0$.

Let ν be a divisor on M with $S = \text{supp } \nu$. The *counting function* of ν is defined to be

$$(1.7) \quad N_\nu(r) = \int_{r_0}^r n_\nu(t) \frac{dt}{t}$$

where

$$n_\nu(t) = t^{2-2m} \int_{S[t]} \nu \nu^{m-1} = \int_{S_*[t]} \nu \omega^{m-1} + n_\nu(0), \quad \text{if } m > 1,$$

$$n_\nu(t) = \sum_{z \in S[t]} \nu(z), \quad \text{if } m = 1.$$

Let d_k be the zero divisor of f_k (when $k = n$, we obtain the Wronskian divisor d_n). The divisor $l_k = d_{k-1} - 2d_k + d_{k+1} \geq 0$ is called the k th *stationary index*, here we assume that $d_{-1} = 0$. Let I_k be the indeterminacy of f_k . On $M - I_k$ define,

$$(1.8) \quad h_k^2 := \frac{mi_{m-1} f_k^*(\Omega_k) \wedge B \wedge \bar{B}}{\nu^m}.$$

and define

$$(1.9) \quad S_k(r) = \int_{M\langle r \rangle} \log h_k^2 \sigma.$$

THEOREM 3.5.1 [Plücker Formula] ([23], Theorem 7.6).

$$N_{l_k}(r) + T_{f_{k-1}}(r) - 2T_{f_k}(r) + T_{f_{k+1}}(r) = S_k(r) - S_k(r_0) + Ric_\tau(r).$$

The function $Ric_\tau(r)$ is called the Ricci function (c.f. [23], or see (3.2) in [27]). It depends only on the geometry (topology) of the manifold M . The other terms of Plucker Formula are analytic invariants. The Ricci function is the integrated Euler characteristic. For affine algebraic manifolds $\pi : M \rightarrow \mathbb{C}^m$, we have $\lim_{r \rightarrow +\infty} \frac{Ric_\tau(r)}{\log r} = d_\pi = \text{degree of the branching divisor of } \pi$.

3.6. Projective distance. Denote by \mathbb{C}^{*n+1} the dual space of \mathbb{C}^{n+1} . For $0 \leq k \leq n$, let $[\cdot] : \left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right) \times \mathbb{C}^{*n+1} \rightarrow \bigwedge^k \mathbb{C}^{n+1}$ be the interior product defined in the usual way. Let $x \in \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right)$ with representative $\xi \in \bigwedge^{k+1} \mathbb{C}^{n+1} - \{0\}$ and let $a \in \mathbb{P}(\mathbb{C}^{*n+1})$ with representative $\alpha \in \mathbb{C}^{*n+1} - \{0\}$, the **projective distance** between x and a is defined by

$$0 \leq \|x; a\| = \frac{\|\xi \lrcorner \alpha\|}{\|\xi\| \|\alpha\|} \leq 1,$$

where the norm on $\bigwedge^k \mathbb{C}^{n+1}$ is induced by the standard norm on \mathbb{C}^{n+1} . Note that the above definition is independent of choice of the representatives α and ξ . Note that a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ can also be regarded as a point in $\mathbb{P}^n(\mathbb{C}^*)$. Hence, for every meromorphic map $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$, $\|f_k(z); H\|$ is defined for $z \in M$. This gives a distance function (from $f_k(z)$ to H) on M . For a hyperplane H in $\mathbb{P}^n(\mathbb{C})$, define the *proximity function*

$$m_{f_k}(r, H) = \int_{M\langle r \rangle} \log \frac{1}{\|f_k; H\|} \sigma.$$

For a hyperplane H in $\mathbb{P}^n(\mathbb{C})$, define an H -divisor $\nu = \mu_{f_k}^H$ as in Stoll [23] let $N_{f_k}(r, H) = N_\nu(r)$ as defined in (1.7).

THEOREM 3.6.1 [First Main Theorem] ([23], (8.21), P.153). *Let $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is general for B . Then, for every hyperplane $H \in \mathbb{P}^n(\mathbb{C})$ and for every $0 \leq k \leq n$, we have*

$$T_{f_k}(r) \geq N_{f_k}(r, H) + m_{f_k}(r, H) - m_{f_k}(r_0, H).$$

Similarly, let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d . Assume that $f(M) \not\subset D$. Let Q be the homogeneous polynomial (form) of degree d defining D . The proximity function $m_f(r, D)$ is defined as,

$$m_f(r, D) = \int_{M\langle r \rangle} \log \frac{\|\mathbf{f}(z)\|^d \|Q\|}{|Q(\mathbf{f})(z)|} \sigma.$$

Let $N_f(r, D) = N_\nu(r)$, with $\nu = \mu_{Q(\mathbf{f})}$ as defined in (1.7).

THEOREM 3.6.2 [First Main Theorem]. *Let $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d . If $f(M) \not\subset D$, then*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r .

3.7. The Ahlfors' Estimate. We first recall the following theorem.

THEOREM 3.7.1 [Ahlfors Estimate] ([23], Theorem 10.3). *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Then for any $0 < \lambda < 1$, we have*

$$\int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\lambda}} h_k^2 v^m \leq \frac{8Y(r)}{\lambda^2} (\lambda T_{f_k}(r) + \varsigma),$$

where ς is defined in (1.4).

To re-formulate Theorem 3.7.1 to meet our purpose, we recall the following Calculus lemma.

LEMMA 3.7.1 (Calculus Lemma). *Let $T(r) \geq 0$ ($r \geq r_0 \geq 0$) be a monotone increasing function. For an arbitrary $\epsilon > 0$*

$$\frac{d}{dr} T(r) \leq T(r) \cdot \log^{1+\epsilon} T(r) \parallel_{E(\epsilon)}$$

where $\|_{E(\epsilon)}$ means that the inequality holds for all r except a set E_ϵ (depends on ϵ) with finite Lebesgue measure.

Proof. Since $T(r)$ is monotone increasing, the derivative $T'(r)$ exists almost everywhere. We may assume that $T(r) \not\equiv 0$. Set

$$E(\epsilon) = \{r \geq r_0 \mid T'(r) \geq T(r) \cdot \log^{1+\epsilon} T(r)\}.$$

Then

$$\text{meas}(E(\epsilon)) = \int_{E(\epsilon)} dr \leq \int_{r_1}^\infty \frac{T'(r)}{T(r) \cdot \log^{1+\epsilon} T(r)} dr < +\infty$$

which proves the lemma. \square

The typical use of the calculus lemma is as follows:

LEMMA 3.7.2. *Let h be a nonnegative measurable function on M such that $h\nu^m$ is locally integrable. Let T be a function defined by*

$$T(r) = \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} h\nu^m.$$

Then, for an arbitrary $\epsilon > 0$,

$$\int_{M\langle r \rangle} h\sigma \leq T(r) \log^{1+\epsilon} T(r) (\log(r^{2m-1}T(r) \log^{1+\epsilon} T(r)))^{1+\epsilon} \|_{E(\epsilon)}.$$

Proof. Notice that

$$\int_{M[r]} h\nu^m = m \int_{M[r]} h\tau^{m-1} d\tau \wedge \sigma = 2m \int_0^r \left(\int_{M\langle r \rangle} h\sigma \right) t^{2m-1} dt.$$

Hence

$$2m \int_{M\langle r \rangle} h\sigma = r^{-(2m-1)} \frac{d}{dr} \left(r^{2m-1} \frac{dT}{dr} \right).$$

Thus may apply the Calculus lemma twice, first to the function $r^{2m-1}T'(r)$ and then to the function $T'(r)$. This proves the result. \square

LEMMA 3.7.3. *Let $S_k(r)$ be defined in (1.9). Then*

$$S_k(r) \leq \varsigma (\log^+ T_{f_k}(r) + \log^+ Y(r) + 2 \log^+ \log r) \|_E$$

where $E \subset [r_0, +\infty)$ is a set with finite measure, and ς is given in (1.4).

Proof. We wish to applying Lemma 3.7.2 to h_k^2 which is defined in (1.8). To do so, we first notice

$$\begin{aligned} \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} h_k^2 \nu^m &= m i_{m-1} \int_{M[t]} f_k^* \Omega \wedge B \wedge \bar{B} \\ &\leq Y(r) \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f_k^* \Omega \wedge \nu^{m-1} = Y(r) T_{f_k}(r). \end{aligned}$$

Hence, Lemma 3.7.2 with $\epsilon = 1$

$$\int_{M\langle r \rangle} h_k^2 \sigma \leq Y(r) T_{f_k}(r) [\log(Y(r) T_{f_k}(r))]^2 \{ \log[r^{2m-1} Y(r) T_{f_k}(r) \log^2(Y(r) T_{f_k}(r))] \}^2 \parallel_E$$

where $E \subset [r_0, +\infty)$ is a set with finite measure. Hence

$$\begin{aligned} S_k(r) &= \frac{\varsigma}{2} \int_{M\langle r \rangle} \log h_k^2 \frac{\sigma}{\varsigma} \leq \frac{\varsigma}{2} \log \int_{M\langle r \rangle} h_k^2 \frac{\sigma}{\varsigma} \\ &\leq \frac{\varsigma}{2} [\log^+ T_{f_k}(r) + \log^+ Y(r) + 4 \log^+ \log T_{f_k}(r) + 4 \log^+ \log^+ Y(r) + 2 \log^+ \log r \\ &\quad + 2 \log^+ \log^+ \log^+ Y(r) + 2 \log^+ \log^+ \log^+ T_{f_k}(r) + O(1)] \parallel_E. \end{aligned}$$

By enlarging E if necessary, we can assume that $4 \log^+ \log^+ \log T_{f_k}(r) \leq \log^+ T_{f_k}(r) \parallel_E$, $8 \log^+ \log Y(r) \leq \log T_{f_k}(r) \parallel_E$, $4 \log^+ \log^+ \log Y(r) \leq \log^+ Y(r) \parallel_E$, $8 \log^+ \log Y(r) \leq \log^+ Y(r) \parallel_E$ and $O(1) \leq \log^+ \log r \parallel_E$. Thus we get

$$S_k(r) \leq \varsigma [\log^+ Y(r) + \log^+ T_{f_k}(r) + 2 \log^+ \log r] \parallel_E.$$

This proves the Lemma. \square

THEOREM 3.7.2. For $0 \leq k \leq n - 1$,

$$T_{f_k}(r) \leq 2(k + 1)T_f(r) + k(k + 1)Ric_\tau(r) + k(k + 1)\varsigma[\log^+ Y(r) + 2 \log^+ \log r] \parallel_E.$$

where $E \subset (r_0, \infty)$ is a set with finite measure.

Proof. Summing Theorem 3.5.1 (the Plucker formula) from $p = 0$ to $p = k$, we get

$$T_{f_k}(r) - (k + 1)T_0(r) \leq \frac{k(k + 1)}{2} Ric_\tau(r) + \sum_{0 \leq p \leq k-1} (k - p)S_p(r),$$

where $T_0(r) = T_f(r)$. This, together with Lemma 3.7.3, yields

$$\begin{aligned} T_{f_k}(r) &\leq (k + 1)T_f(r) + \frac{k(k + 1)}{2} Ric_\tau(r) \\ &\quad + \frac{k(k + 1)}{2} \varsigma [\log^+ Y(r) + \log^+ T(r) + 2 \log^+ \log r] \parallel_E \end{aligned}$$

where $T(r) = \max_{0 \leq p \leq k} T_{f_p}(r)$. Notice that above inequality holds for all k with $k = 0, 1, \dots, n - 1$, hence

$$\begin{aligned} T(r) &= \max_{0 \leq p \leq k} T_{f_p}(r) \leq (k + 1)T_f(r) + \frac{k(k + 1)}{2} Ric_\tau(r) \\ &\quad + \frac{k(k + 1)}{2} \varsigma [\log^+ Y(r) + \log^+ T(r) + 2 \log^+ \log r] \parallel_E. \end{aligned}$$

Using $\frac{k(k+1)}{2} \varsigma \log^+ T(r) \leq \frac{1}{2} T(r) \parallel_E$ for some $E \subset (r_0, \infty)$ with finite measure since f is linearly non-degenerate, we derive

$$T(r) \leq 2(k + 1)T_f(r) + k(k + 1)Ric_\tau(r) + k(k + 1)\varsigma[\log^+ Y(r) + 2 \log^+ \log r] \parallel_E.$$

Noticing that $T_{f_k}(r) \leq T(r)$, the above implies that the inequality in Theorem 3.7.2 holds. \square

THEOREM 3.7.3 (Ahlfors Estimate II). *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Let $\Lambda(r) = \min_k \{1/(1 + T_{f_k}(r))\}$. Then, for every $0 \leq k \leq n - 1$,*

$$\begin{aligned} & \log^+ \int_{M\langle r \rangle} \left(\frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k^2 \right) \sigma \\ & \leq 2 \log^+ T_f(r) + 2 \log^+ Ric_\tau(r) + \log^+ Y(r) + 10 \log^+ \log T_f(r) + 9 \log^+ \log^+ Ric_\tau(r) \\ & \quad 10 \log^+ \log^+ Y(r) + 4 \log^+ \log r \parallel_E. \end{aligned}$$

where $E \subset [r_0, +\infty)$ is a set with finite measure.

Proof. Define functions

$$K_k(r) = \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda^*}} h_k^2 v^m$$

where $\Lambda^* = \Lambda \circ \tau^{1/2}$. By Lemma 3.7.2 with $\epsilon = 1$, we have

$$(1.10) \quad \int_{M\langle r \rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k^2 \sigma \leq K_k(r) \log^2 K_k(r) [\log(r^{2m-1} K_k(r) \log^2 K_k(r))]^2 \parallel_E.$$

On the other hand, noticing that Λ is a decreasing function, we have $\|f_k; H\|^{\Lambda^*} \leq \|f_k; H\|^{\Lambda(r)}$. Hence by Ahlfors' estimate with $\lambda = \Lambda(r)$, we have

$$\begin{aligned} (1.11) \quad K_k(r) &= \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda^*}} h_k^2 v^m \\ &\leq \frac{8Y(r)}{\Lambda^2(r)} (\Lambda(r) T_{f_k}(r) + \varsigma) \\ &\leq 8Y(r) (1 + T_{f_k}(r))^2 \left(\frac{T_{f_k}(r)}{1 + T_{f_k}(r)} + \varsigma \right) \\ &\leq 9(1 + \varsigma) T_k^2(r) Y(r). \end{aligned}$$

Combining (1.10) and (1.11),

$$\begin{aligned} & \log^+ \int_{M\langle r \rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k^2 \sigma \\ & \leq 2 \log^+ T_{f_k}(r) + \log^+ Y(r) + 8 \log^+ \log^+ T_{f_k}(r) + 4 \log^+ \log^+ Y(r) \\ & \quad + 2 \log^+ \log^+ \log Y(r) + 2 \log^+ \log^+ \log^+ T_{f_k}(r) + 2 \log^+ \log r + O(1) \parallel_E. \end{aligned}$$

By enlarge E if necessary, we can assume that $2 \log^+ \log^+ \log^+ T_{f_k}(r) \leq \log^+ \log T_{f_k}(r) \parallel_E$, $2 \log^+ \log^+ \log^+ Y(r) \leq \log^+ \log^+ Y(r) \parallel_E$ and $O(1) \leq \log^+ \log r \parallel_E$. Thus we get

$$\begin{aligned} & \log^+ \int_{M\langle r \rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k^2 \sigma \\ & \leq 2 \log^+ T_{f_k}(r) + \log^+ Y(r) + 9 \log^+ \log^+ T_{f_k}(r) + 5 \log^+ \log^+ Y(r) + 3 \log^+ \log r \parallel_E. \end{aligned}$$

Now use Theorem 3.7.2, the above inequality becomes

$$\begin{aligned} & \log^+ \int_{M\langle r \rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k^2 \sigma \\ & \leq 2 \log^+ T_f(r) + 2 \log^+ Ric_\tau(r) + \log^+ Y(r) + 9 \log^+ \log^+ T_f(r) + 7 \log^+ \log Y(r) \\ & \quad + 9 \log^+ \log^+ Ric_\tau(r) + 9 \log^+ \log^+ \log^+ Y(r) + 9 \log^+ \log^+ \log r \\ & \quad + 2 \log^+ \log r + O(1) \|_E. \end{aligned}$$

Again, by enlarge E if necessary, we can assume that $9 \log^+ \log^+ \log Y(r) \leq \log^+ \log^+ Y(r) \|_E$, $9 \log^+ \log^+ \log r \leq \log^+ \log r \|_E$, and $O(1) \leq \log^+ \log r \|_E$. Hence

$$\begin{aligned} & \log^+ \int_{M\langle r \rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k^2 \sigma \\ & \leq 2 \log^+ T_f(r) + 2 \log^+ Ric_\tau(r) + \log^+ Y(r) + 10 \log^+ \log T_f(r) \\ & \quad + 9 \log^+ \log^+ Ric_\tau(r) + 10 \log^+ \log^+ Y(r) + 4 \log^+ \log r \|_E. \end{aligned}$$

This concludes the proof. \square

3.8. A slight generalization of Wong-Stoll’s theorem. In this subsection, we extend the Second Main Theorem of Wong-Stoll to the case where the given hyperplanes H_1, \dots, H_q in $\mathbb{P}^n(\mathbb{C})$ are not necessarily in general position.

To establish the value distribution theory on parabolic manifolds M , similar to [23], we make the following assumptions on M (c.f. subsection 3.1-3.4): (i) M is a connected complex manifold of dimension m ; (ii) there exists a parabolic exhaustion function τ on M ; (iii) For every integer n and every linearly non-degenerate map $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$, there is a holomorphic differential form B of degree $(m - 1, 0)$ on M such that f is general for B and

$$mi_{m-1}B \wedge \bar{B} \leq Y(r)v^{m-1}$$

on $M[r]$ for some real positive valued function $Y(r)$ on M , which is independent of f (Y is called a *majorant for B*) where, for any positive integer m ,

$$i_m = \left(\frac{\sqrt{-1}}{2\pi} \right)^m (-1)^{\frac{m(m-1)}{2}} m!.$$

A complex manifold M with the assumptions (i)-(iii) is called an *admissible parabolic manifold*. Throughout this paper, we shall work on admissible parabolic manifolds.

THEOREM 3.8.1. *Let M be an admissible parabolic manifold of complex dimension m . Let $f : M \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is linearly non-degenerate. Let $\epsilon > 0$ and let H_1, \dots, H_q be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then*

$$\begin{aligned} & \int_{M\langle r \rangle} \max_K \sum_{j \in K} \log \frac{1}{\|f(z); H_j\|} \sigma \leq (n + 1)T_f(r) + \frac{n(n + 1)}{2} Ric_\tau(r, s_0) \\ & - N_{d_n}(r, s_0) + \varsigma \frac{n(n + 1)}{2} [\log^+ T_f(r) + \log^+ Ric_\tau(r) + \frac{1}{2} \log^+ Y(r) \\ & + 5 \log^+ \log T_f(r) + 5 \log^+ \log^+ Ric_\tau(r) + 5 \log^+ \log^+ Y(r) + 3 \log^+ \log^+ r] \|_E \end{aligned}$$

where $E \subset [r_0, +\infty)$ is a set with finite measure, and the max is taken over all subsets K of $\{1, \dots, q\}$ such that the linear forms $H_j, j \in K$, are linearly independent.

Proof. Denote by $K \subset \{1, \dots, q\}$ such that linear forms $\{H_k, k \in K\}$, are linearly independent. Without loss of generality, we may assume $q \geq n + 1$ and that $\#K = n + 1$. Let T be the set of all the injective maps $\mu : \{0, 1, \dots, n\} \rightarrow \{1, \dots, q\}$ such that $H_{\mu(0)}, \dots, H_{\mu(n)}$ are linearly independent. Denote by $\Gamma = \max_{1 \leq j \leq q} \{ \sum_{k=0}^{n-1} m_{f_k}(s_0, H_j) \}$ and $\Lambda(r) = \min_k \{1/(1 + T_{f_k}(r))\}$. For any $\mu \in T, z \notin I_f$, the Product to Sum Estimate (see [27] Lemma 1.12), with $\lambda = \Lambda(r)$, reads

$$\prod_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \leq c_k \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \right)^{n-k},$$

where $c_k > 0$ is a constant. Since $\|f_n; H_{\mu(j)}\|$ is a constant for any $0 \leq j \leq n$, we have

$$\begin{aligned} \prod_{j=0}^n \frac{1}{\|f(z); H_{\mu(j)}\|^2} &= \prod_{k=0}^{n-1} \prod_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \cdot \prod_{k=0}^{n-1} \prod_{j=0}^n \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \\ &\leq c \prod_{k=0}^{n-1} \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \right)^{n-k} \cdot \prod_{k=0}^{n-1} \prod_{j=0}^n \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}}, \end{aligned}$$

where $c > 1$ is a constant. Therefore, we have

$$\begin{aligned} (1.12) \quad &\int_{M\langle r \rangle} \max_K \sum_{j \in K} \log \frac{1}{\|f(z); H_j\|^2} \sigma \\ &= \int_{M\langle r \rangle} \max_{\mu \in T} \log \left(\prod_{j=0}^n \frac{1}{\|f(z); H_{\mu(j)}\|^2} \right) \sigma \\ &\leq \sum_{k=0}^{n-1} \int_{M\langle r \rangle} \max_{\mu \in T} \log \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \right)^{n-k} \sigma \\ &\quad + \sum_{k=0}^{n-1} \sum_{j=0}^n \int_{M\langle r \rangle} \max_{\mu \in T} \log \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \sigma + O(1) \\ &= \sum_{k=0}^{n-1} (n-k) \int_{M\langle r \rangle} \log \max_{\mu \in T} \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k^2 \right) \sigma \\ &\quad - 2 \sum_{k=0}^{n-1} (n-k) S_k(r) + \sum_{k=0}^{n-1} \sum_{j=0}^n \int_{M\langle r \rangle} \max_{\mu \in T} \log \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \sigma + O(1) \end{aligned}$$

where, in above, h_k is defined by (1.8), $S_k(r)$ is defined by (1.9). We now estimate

each term appearing the above inequality. First,

$$\begin{aligned} & \int_{M\langle r \rangle} \log \max_{\mu \in T} \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k^2 \right) \sigma \\ &= \kappa \int_{M\langle r \rangle} \log \max_{\mu \in T} \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k^2 \right) \frac{\sigma}{\kappa} \\ &\leq \kappa \log \int_{M\langle r \rangle} \max_{\mu \in T} \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k^2 \right) \frac{\sigma}{\kappa} \\ &\leq \kappa \max_{1 \leq j \leq q} \log^+ \int_{M\langle r \rangle} \left(\frac{\|f_{k+1}(z); H_j\|^2}{\|f_k(z); H_j\|^{2-2\Lambda(r)}} h_k^2 \right) \sigma + C'. \end{aligned}$$

By Theorem 3.7.3,

$$\begin{aligned} & \max_{1 \leq j \leq q} \log^+ \int_{M\langle r \rangle} \left(\frac{\|f_{k+1}(z); H_j\|^2}{\|f_k(z); H_j\|^{2-2\Lambda(r)}} h_k^2 \right) \sigma \\ &\leq 2 \log^+ T_f(r) + 2 \log^+ Ric_\tau(r) + \log^+ Y(r) + 10 \log^+ \log T_f(r) + 9 \log^+ \log^+ Ric_\tau(r) \\ &\quad + 10 \log^+ \log^+ Y(r) + 4 \log^+ \log r. \quad \|_E. \end{aligned}$$

Hence,

$$\begin{aligned} (1.13) \quad & \sum_{k=0}^{n-1} \int_{M\langle r \rangle} \log \max_{\mu \in T} \left(\sum_{j=0}^n \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k^2 \right)^{n-k} \sigma \\ &\leq n(n+1)\zeta [\log^+ T_f(r) + \log^+ Ric_\tau(r) + \frac{1}{2} \log^+ Y(r) + 5 \log^+ \log T_f(r) \\ &\quad + 5 \log^+ \log^+ Ric_\tau(r) + 5 \log^+ \log^+ Y(r) + 2 \log^+ \log r] \|_E. \end{aligned}$$

Next, using Theorem 3.5.1 (the Plücker formula), we have

$$N_{l_k}(r) + T_{f_{k-1}}(r) - 2T_{f_k}(r) + T_{f_{k+1}}(r) = S_k(r) - S_k(r_0) + Ric_\tau(r).$$

Noticing that $T_{f_n}(r) = 0$,

$$(1.14) \quad \sum_{k=0}^{n-1} (n-k)S_k(r) = N_{d_n}(r) - (n+1)T_f(r) - \frac{n(n+1)}{2} Ric_\tau(r) + O(1).$$

Finally, by the First Main Theorem,

$$\begin{aligned}
 (1.15) \quad & \sum_{k=0}^{n-1} \sum_{j=0}^n \int_{M\langle r \rangle} \max_{\mu \in T} \log \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \sigma \\
 & \leq \sum_{\mu \in T} \sum_{k=0}^{n-1} \sum_{j=0}^n \int_{M\langle r \rangle} 2\Lambda(r) \log \frac{1}{\|f_k(z); H_{\mu(j)}\|} \sigma + O(1) \\
 & = \sum_{\mu \in T} \sum_{k=0}^{n-1} \sum_{j=0}^n 2\Lambda(r) m_{f_k}(r, H_{\mu(j)}) + O(1) \\
 & \leq \sum_{k=0}^{n-1} \sum_{j=0}^n 2q! \Lambda(r) (T_{f_k}(r) + m_{f_k}(s_0, H_{\mu(j)})) + O(1) \\
 & \leq O(1).
 \end{aligned}$$

Combining (1.12), (1.13), (1.14), and (1.15), we have

$$\begin{aligned}
 & \int_{M\langle r \rangle} \max_K \sum_{j \in K} \log \frac{1}{\|f(z); H_j\|} \sigma \leq (n+1)T_f(r) + \frac{n(n+1)}{2} Ric_\tau(r, s_0) - N_{d_n}(r, s_0) \\
 & + \varsigma \frac{n(n+1)}{2} [\log^+ T_f(r) + \log^+ Ric_\tau(r) + \frac{1}{2} \log^+ Y(r) + 5 \log^+ \log T_f(r) \\
 & + 5 \log^+ \log^+ Ric_\tau(r) + 5 \log^+ \log^+ Y(r) + 3 \log^+ \log^+ r] \ll_E.
 \end{aligned}$$

□

3.9. The main result. We are now ready to state the second result of our two main theorems in the paper which extends Theorem A to meromorphic mappings on parabolic manifolds.

THEOREM 2. *Let M be an admissible parabolic manifold of dimension m . Let $f : M \rightarrow X \subset \mathbb{P}^N(\mathbb{C})$ be an algebraically non-degenerate meromorphic map, where X is a smooth algebraic variety. Let D_1, \dots, D_q be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree d_j in general position in X . Then for every $\epsilon > 0$, we have*

$$\begin{aligned}
 & \sum_{j=1}^q d_j^{-1} m_f(r, D_j) \leq (\dim X + 1 + \epsilon) T_f(r) + c_{1,\epsilon} Ric_\tau(r) \\
 & + c_{2,\epsilon} [\log^+ T_f(r) + \log^+ Ric_\tau(r) + \frac{1}{2} \log^+ Y(r) + 5 \log^+ \log^+ T_f(r) \\
 & + 5 \log^+ \log^+ Ric_\tau(r) + 5 \log^+ \log^+ Y(r) + 3 \log^+ \log^+ r] \ll_E
 \end{aligned}$$

where $E \subset [r_0, +\infty)$ is a set with finite measure, $c_{1,\epsilon}$ and $c_{2,\epsilon}$ are two positive constants depending on ϵ which are explicitly given in the proof below.

We note that $Ric_\tau(r)$ is called the Ricci function which depends only on the geometry (topology) of the manifold M . An important class of admissible parabolic manifolds consists of affine algebraic manifolds. In this case, there exists a finite branched covering $\pi : M \rightarrow \mathbb{C}^m$. If we take $\tau = \|\pi\|^2$, Stoll [23] showed that there exists a holomorphic differential form B of degree $(m-1, 0)$ on M (c.f. [23] or [27]) such that

$$mi_{m-1} B \wedge \bar{B} \leq (1 + |\tau|)^{m-1} (dd^c \tau)^{m-1}.$$

Hence, we can take $Y(r) = (1 + r^2)^{m-1}$. Further, we can show (c.f. [23] or [27]) that

$$d_\pi = \lim_{r \rightarrow +\infty} \frac{Ric_\tau(r, s_0)}{\log r}$$

exists and equals the degree of the branching divisor of π . The Theorem 2 thus implies the following corollary in the case that M is affine algebraic manifold.

COROLLARY. *Let M be an affine algebraic manifold of complex dimension m . Let $\pi : M \rightarrow \mathbb{C}^m$ be a finite branched covering. Let $f : M \rightarrow X \subset \mathbb{P}^N(\mathbb{C})$ be an algebraically non-degenerate meromorphic map. Let D_1, \dots, D_q be a finite collection of hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ in general position in X . Assume that f is transcendental. Then*

$$\sum_{j=1}^q \delta_f(D_j) \leq \dim X + 1.$$

Proof of Theorem 2. For given hypersurfaces D_1, \dots, D_q defined by Q_1, \dots, Q_q , assuming they have the same degree (WLOG), say d . The proof depends on the **Key inequality** (see below) which allows us to the transfer from the (study of) hypersurfaces to the hyperplanes. To state and prove the Key inequality, we consider the map $\chi : X \rightarrow \mathbb{P}^{q-1}(\mathbb{C})$ defined by $\chi(w) = [Q_1(w) : \dots : Q_q(w)]$. It is a finite morphism since D_1, \dots, D_q are in general position. Let $Z = \chi(X)$. We have that Z is a complex subvariety of \mathbb{P}^{q-1} , $\dim Z = \dim X = n$ and

$$\Delta := \deg Z \leq d^n \deg X.$$

We shall study the map $\chi \circ f : \mathbb{C} \rightarrow Z \subset \mathbb{P}^{q-1}(\mathbb{C})$.

We first introduce some notation: Let $Z \subset \mathbb{P}^l$ be a projective variety of dimension n (with $l = q - 1$ in the above case). Denote by $\mathbb{C}[x_0, \dots, x_l]_m$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_l]$ of degree m (including 0). Let I_Z be the prime ideal in $\mathbb{C}[x_0, \dots, x_l]$ defining Z . Put $(I_Z)_m = \mathbb{C}[x_0, \dots, x_l]_m \cap I_Z$. The Hilbert polynomial $H_Z(m)$ of Z is defined by

$$H_Z(m) := \dim(\mathbb{C}[x_0, \dots, x_l]_m / I_Z(m)).$$

The Hilbert Weight of Z with respect to the weight $\mathbf{c} \in \mathbb{R}^{l+1}$ is defined by

$$S_Z(m, \mathbf{c}) = \max \left(\sum_{i=1}^{H_Z(m)} \mathbf{a}_i \cdot \mathbf{c} \right),$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{H_Z(m)}}$ whose residue classes modulo I_Z form a basis of $\mathbb{C}[x_0, \dots, x_l]_m / (I_Z)_m$, where for $\mathbf{x} = (x_0, \dots, x_l)$, $\mathbf{a} = (a_0, \dots, a_l) \in \mathbb{Z}_+^{l+1}$, $\mathbf{x}^{\mathbf{a}} = x_0^{a_0} \dots x_l^{a_l}$. We need the following lemma to estimate (get a lower bound) of Hilbert Weight $S_Z(m, \mathbf{c})$.

LEMMA 3.9.1 (see [18]). *Let $Z \subset \mathbb{P}^l$ be an algebraic variety of dimension n and degree Δ . Let $m > \Delta$ be an integer and let $\mathbf{c} = (c_0, \dots, c_l) \in \mathbb{R}^{l+1}$. Let $\{i_0, \dots, i_n\}$ be a subset of $\{0, \dots, l\}$ such that*

$$\{x = [x_0 : \dots : x_l] \in \mathbb{P}^l \mid x_{i_0} = \dots = x_{i_n} = 0\} \cap Z = \emptyset.$$

Then

$$\frac{1}{mH_Z(m)}S_Z(m, \mathbf{c}) \geq \frac{1}{(n+1)}(c_{i_0} + \dots + c_{i_n}) - \frac{(2n+1)\Delta}{m} \left(\max_{0 \leq i \leq l} c_i \right).$$

Now with the above $Z = \chi(X) \subset \mathbb{P}^{q-1}$, consider the vector space $V_m = \mathbb{C}[x_1, \dots, x_q]_m / (I_Z)_m$, and fix a basis $\phi_0, \dots, \phi_{n_m}$ where $n_m = H_Z(m) - 1$. Consider the map

$$F = [\phi_0(\chi \circ f) : \dots : \phi_{n_m}(\chi \circ f)] : M \rightarrow \mathbb{P}^{n_m}.$$

Let $\{U_\lambda, \lambda \in \Lambda\}$ be an open covering of M , and let $\mathbf{f}_\lambda : U_\lambda \rightarrow \mathbb{C}^{N+1}$ be a reduced representation of f on U_λ . Write, on U_λ , $\mathbf{F}_\lambda = (F_{0,\lambda}, \dots, F_{n_m,\lambda})$ where $F_{i,\lambda} = \phi_i(Q_1(\mathbf{f}), \dots, Q_q(\mathbf{f}))$ for $i = 0, 1, \dots, n_m$. Note that \mathbf{F}_λ is a reduced representation of F on U_λ . Define,

$$c_j(z) = \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_j\|}{|Q_j(\mathbf{f}_\lambda)(z)|}, 1 \leq j \leq q,$$

which, we note that, is independent of the choice of f_λ (hence is independent of λ), so c_j are globally defined functions on M . Let $\mathbf{c}(z) = (c_1(z), \dots, c_q(z))$. With this given weight $\mathbf{c}(z)$, take monomials (which depends on z) $\mathbf{x}^{\mathbf{a}^1}, \dots, \mathbf{x}^{\mathbf{a}^{H_Z(m)}}$ whose residue classes modulo I_Z form a basis of $V_m = \mathbb{C}[x_1, \dots, x_q]_m / (I_Z)_m$ and which reaches the max. in the Hilbert weight $S_Z(m, \mathbf{c}(z))$. For each $1 \leq j \leq H_Z(m)$, write $\mathbf{x}^{\mathbf{a}^j} = L_{j,z}(\phi_0, \dots, \phi_{n_m})$ then, for $L_{j,z}$ are linear forms which are linearly independent for each fixed z . Note that the linear forms depends on z , but there are only finitely many of them in total. We then have the following **key inequality** which transfers the study of hypersurfaces for f into the study of linear forms for the map F (so we can apply Cartan’s result for F).

KEY INEQUALITY. For each $z \in U_\lambda$ and linear forms $L_{j,z}, 1 \leq j \leq H_Z(m) = n_m + 1$, chosen above, we have

$$\begin{aligned} \sum_{j=1}^q \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_j\|}{|Q_j(\mathbf{f}_\lambda)(z)|} &\leq \frac{n+1}{mH_Z(m)} \sum_{j=1}^{H_Z(m)} \log \frac{\|\mathbf{F}_\lambda(z)\|}{|L_{j,z}(\mathbf{F}_\lambda)(z)|} \\ &+ \frac{(2n+1)\Delta}{m} \left(\max_{0 \leq i \leq q} \log \frac{\|\mathbf{f}_\lambda(z)\| \|Q_i\|}{|Q_i(\mathbf{f}_\lambda)(z)|} \right) + O(1), \end{aligned}$$

where $O(1)$ denotes a bounded term which is independent of z .

We note that terms on both left and right hand sides are independent of the choice of \mathbf{f}_λ (hence is independent of λ). Thus the inequality indeed holds on M .

Proof. For every $\mathbf{b} = [b_0 : \dots : b_N] \in \mathbb{P}^N(\mathbb{C})$, consider the function

$$\|\mathbf{b}, D_j\| = \frac{|Q_j(\mathbf{b})|}{\|\mathbf{b}\|^d \|Q_j\|},$$

where $\|\mathbf{b}\| = \max_{0 \leq j \leq N} |b_j|$ and $\|Q_j\|$ is the maximum of the absolute values of the coefficients of Q_j . By the “in general position” condition, at each point $\mathbf{b} \in X$, there $\|\mathbf{b}, D_j\|$ can be zero for no more than n indices $j \in \{1, \dots, q\}$. For the remaining indices j , we have $\|\mathbf{b}, D_j\| > 0$ and by the continuity of these functions and the

compactness of V , there exists $C > 0$ such that $\|\mathbf{b}, D_j\| > C$ for all $\mathbf{b} \in V$ and all D_j , except for at most n of them. Hence, for $z \in U_\lambda$, there is i_0, \dots, i_n (which may depend on z) such that

$$\sum_{j=1}^q \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_j\|}{|Q_j(\mathbf{f}_\lambda)(z)|} \leq \sum_{t=0}^n \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_{i_t}\|}{|Q_{i_t}(\mathbf{f}_\lambda)(z)|}.$$

With $c_j(z) = \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_j\|}{|Q_j(\mathbf{f}_\lambda)(z)|}$ and $\mathbf{c}(z) = (c_1(z), \dots, c_q(z))$, and by applying Lemma 3.9.1, we get

$$\frac{1}{mH_Z(m)} S_Z(m, \mathbf{c}(z)) \geq \frac{1}{(n+1)} (c_{i_0}(z) + \dots + c_{i_n}(z)) - \frac{(2n+1)\Delta}{m} \left(\max_{1 \leq i \leq q} c_i(z) \right),$$

which gives

$$\sum_{t=0}^n \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_{i_t}\|}{|Q_{i_t}(\mathbf{f}_\lambda)(z)|} \leq \frac{(n+1)S_Z(m, \mathbf{c}(z))}{mH_Z(m)} + \frac{(n+1)(2n+1)\Delta}{m} \left(\max_{0 \leq i \leq q} \log \frac{\|\mathbf{f}_\lambda(z)\| \|Q_i\|}{|Q_i(\mathbf{f}_\lambda)(z)|} \right).$$

Therefore,

$$\sum_{j=1}^q \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_j\|}{|Q_j(\mathbf{f}_\lambda)(z)|} \leq \frac{(n+1)S_Z(m, \mathbf{c}(z))}{mH_Z(m)} + \frac{(n+1)(2n+1)\Delta}{m} \left(\max_{0 \leq i \leq q} \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_i\|}{|Q_i(\mathbf{f}_\lambda)(z)|} \right).$$

But by our selection of the linear forms $L_{j,z}, L_{j,z}(\phi_0, \dots, \phi_{n_m}) = \mathbf{x}^{\mathbf{a}_j}$ and

$$S_Z(m, \mathbf{c}(z)) = \sum_{i=1}^{H_Z(m)} \mathbf{a}_i \cdot \mathbf{c}(z).$$

Notice that from $L_{i,z}(\phi_0, \dots, \phi_{n_m}) = \mathbf{x}^{\mathbf{a}_i}$, we get $L_{i,z}(\mathbf{F}_\lambda(z)) = (Q_1(\mathbf{f}_\lambda)(z))^{a_{i1}} \dots Q_q(\mathbf{f}_\lambda)(z))^{a_{iq}}$ where $\mathbf{a}_i = (a_{i1}, \dots, a_{iq})$. So

$$\begin{aligned} -\log |L_{i,z}(\mathbf{F}_\lambda(z))| &= -(a_{i1} \log |Q_1(\mathbf{f}_\lambda)(z)| + \dots + a_{iq} \log |Q_q(\mathbf{f}_\lambda)(z)|) \\ &= a_{i1} c_1(z) + \dots + a_{iq} c_q(z) - m \log \|\mathbf{f}_\lambda(z)\|^d \\ &= \mathbf{a}_i \cdot \mathbf{c}(z) - m \log \|\mathbf{f}_\lambda(z)\|^d. \end{aligned}$$

Thus

$$-\sum_{i=1}^{H_Z(m)} \log |L_{i,z}(\mathbf{F}_\lambda(z))| = S_Z(m, \mathbf{c}(z)) - mH_Z(m) \log \|\mathbf{f}_\lambda(z)\|^d.$$

Hence

$$\begin{aligned} \sum_{j=1}^q \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_j\|}{|Q_j(\mathbf{f}_\lambda)(z)|} &\leq \frac{(n+1)}{mH_Z(m)} \sum_{i=1}^{H_Z(m)} \log \frac{\|\mathbf{F}_\lambda(z)\|}{|L_{i,z}(\mathbf{F}_\lambda)(z)|} + \frac{n+1}{m} \log \frac{\|\mathbf{f}_\lambda(z)\|^{dm}}{\|\mathbf{F}_\lambda(z)\|} \\ &+ \frac{(n+1)(2n+1)\Delta}{m} \left(\max_{0 \leq i \leq q} \log \frac{\|\mathbf{f}_\lambda(z)\|^d \|Q_i\|}{|Q_i(\mathbf{f}_\lambda)(z)|} \right). \end{aligned}$$

By the definition of \mathbf{F} , we easily see that $C_1 \|\mathbf{f}_\lambda\|^{dm} \leq \|\mathbf{F}_\lambda\| \leq C_2 \|\mathbf{f}_\lambda\|^{dm}$ for some positive C_1 and C_2 independent of λ . This proves our key inequality. \square

We now continue our proof. By integrating the key inequality and applying Theorem 3.8.1 to F , we get

$$\begin{aligned}
 (1.16) \quad \sum_{j=1}^q m_f(r, D_j) &\leq \frac{n+1}{mH_Y(m)} \left\{ (n_m+1)T_F(r) + \frac{n_m(n_m+1)}{2} Ric_\tau(r) \right. \\
 &+ \varsigma \frac{n_m(n_m+1)}{2} [\log^+ T_F(r) + \log^+ Ric_\tau(r) + \frac{1}{2} \log^+ Y(r) + 5 \log^+ \log T_F(r) \\
 &+ 5 \log^+ \log^+ Ric_\tau(r) + 5 \log^+ \log^+ Y(r) + 3 \log^+ \log^+ r] \Big\} \|_E \\
 &+ \frac{(2n+1)(n+1)\Delta}{m} \int_0^{2\pi} \left(\max_{1 \leq j \leq q} \log \frac{\|\mathbf{f}_\lambda^d(re^{i\theta})\| \|Q_j\|}{|Q_j(\mathbf{f}_\lambda(re^{i\theta}))|} \right) \frac{d\theta}{2\pi} + O(1).
 \end{aligned}$$

For each $j \in \{1, \dots, q\}$, by the First Main Theorem,

$$(1.17) \quad \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\| \|Q_j\|}{|Q_j(re^{i\theta})|} \frac{d\theta}{2\pi} \leq dT_f(r) + O(1).$$

It remains to compare $T_f(r)$ and $T_F(r)$. Let $\{U_\lambda, \lambda \in \Lambda\}$ be an open covering of M , and let $\mathbf{f}_\lambda : U_\lambda \rightarrow \mathbb{C}^{N+1}$ be a reduced representation of f on U_λ , then there is a holomorphic function $g_{\lambda\mu} : U_\lambda \cap U_\mu \rightarrow \mathbb{C}_*$ such that

$$\mathbf{f}_\lambda = g_{\lambda\mu} \mathbf{f}_\mu \quad \text{on } U_\lambda \cap U_\mu.$$

It is easy to check that $\{g_{\lambda\mu}\}$ is a basic cocycle (cf. [23]). Therefore there exists a holomorphic line bundle L_f on M with a holomorphic frame atlas $\{U_\lambda, s_\lambda\}_{\lambda \in \Lambda}$ such that

$$s_\lambda = g_{\lambda\mu} v_\mu \quad \text{on } U_\lambda \cap U_\mu.$$

The line bundle L_f is called the *hyperplane section bundle of f* . Also define $\tilde{\mathbf{f}}_\lambda \in \Gamma(U_\lambda, M \times \mathbb{C}^{N+1})$ by $\tilde{\mathbf{f}}_\lambda(z) = (z, \mathbf{f}_\lambda(z))$ for $z \in U_\lambda$. Notice that $\tilde{\mathbf{f}}_\lambda \otimes s_\lambda = g_{\lambda\mu} \tilde{\mathbf{f}}_\mu \otimes s_\lambda = \tilde{\mathbf{f}}_\mu \otimes g_{\lambda\mu} s_\lambda = \tilde{\mathbf{f}}_\mu \otimes s_\mu$ on $U_\lambda \cap U_\mu$. Therefore there exists a holomorphic section Γ_f of $(M \times \mathbb{C}^{N+1}) \otimes L_f$ such that $\Gamma_f|_{U_\lambda} = \tilde{\mathbf{f}}_\lambda \otimes s_\lambda$. Γ_f is called the *standard reduced representation section of f* . Let ℓ be the standard hermitian metric along the fibers of the trivial bundle $M \times \mathbb{C}^{N+1}$ and κ be a hermitian metric along the fibers of L_f . Then

$$dd^c \log \|\Gamma_f\|_{\ell \otimes \kappa}^2 = dd^c \log \|\mathbf{f}_\lambda\|^2 + dd^c \log \|s_\lambda\|_\kappa^2 = f^* \Omega_{FS} - c_1(L_f, \kappa),$$

where Ω_{FS} is the Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$. Hence, by Green's formula(cf. [23]), we have

$$\begin{aligned}
 T_f(r) &= \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f^* \Omega_{FS} \wedge v^{m-1} = \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f^* c_1(L_f, \kappa) \wedge v^{m-1} \\
 &+ \int_{M\langle r \rangle} \log \|\Gamma_f\|_{\ell \otimes \kappa} - \int_{M\langle r_0 \rangle} \log \|\Gamma_f\|_{\ell \otimes \kappa} \sigma.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 T_F(r) &= \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f^* c_1(L_F, \kappa) \wedge v^{m-1} \\
 &+ \int_{M\langle r \rangle} \log \|\Gamma_F\|_{\ell \otimes \kappa} - \int_{M\langle r_0 \rangle} \log \|\Gamma_F\|_{\ell \otimes \kappa} \sigma.
 \end{aligned}$$

By comparing the transition functions of L_f and L_F , it is clear that $L_F = L_f^{\otimes dm}$ and also

$$\frac{\|\Gamma_F\|_{\ell^{\otimes \kappa}}}{\|\Gamma_L\|_{\ell^{\otimes \kappa}}^{dm}} = \frac{\|\mathbf{F}_\lambda\|}{\|\mathbf{f}_\lambda\|^{dm}} \leq C.$$

We thus conclude that

$$(1.18) \quad T_F(r) \leq dmT_f(r) + O(1).$$

By combing (1.16), (1.17) and (1.18), we get, noticing that $H_Z(m) = n_m + 1$,

$$\begin{aligned} \sum_{j=1}^q m_f(r, D_j) &\leq (n+1)dT_f(r) + \frac{(n+1)n_m}{2m} Ric_\tau(r) \\ &+ \frac{\zeta n_m}{2m} [\log^+ T_f(r) + \log^+ Ric_\tau(r) + \frac{1}{2} \log^+ Y(r) + 5 \log^+ \log^+ T_f(r) \\ &+ 5 \log^+ \log^+ Ric_\tau(r) + 5 \log^+ \log^+ Y(r) + 3 \log^+ \log^+ r] \\ &+ \frac{(2n+1)(n+1)d\Delta}{m} T_f(r) \|_E. \end{aligned}$$

For any given $\epsilon > 0$, take m_0 such that $\frac{(2n+1)(n+1)\Delta}{m_0} < \epsilon$, then

$$\begin{aligned} \sum_{j=1}^q m_f(r, D_j) &\leq (n+1+\epsilon)dT_f(r) + \frac{(n+1)n_{m_0}}{2m_0} Ric_\tau(r) \\ &+ \frac{\zeta n_{m_0}}{2m_0} [\log^+ T_f(r) + \log^+ Ric_\tau(r) + \frac{1}{2} \log^+ Y(r) + 5 \log^+ \log^+ T_f(r) \\ &+ 5 \log^+ \log^+ Ric_\tau(r) + 5 \log^+ \log^+ Y(r) + 3 \log^+ \log^+ r] \|_E. \end{aligned}$$

This proves Theorem 2 with

$$c_{1,\epsilon} = \frac{(n+1)n_{m_0}}{2m_0}, \quad c_{2,\epsilon} = \frac{\zeta n_{m_0}}{2m_0}.$$

□

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