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001

A REPRESENTATION THEOREM AND ITS APPLICATIONS TO CHARACTERIZE BOUNDED AND COMPACT COMPOSITION OPERATORS ON BESOV SPACES ON DOMAINS IN $\mathbb{C}^N$*

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Dedicated to Professor Stephen Yau on the occasion of his 60th birthday

Abstract. We prove a representation theorem for Besov space on smooth bounded strictly pseudoconvex domain. As applications we characterize the boundedness and the compactness of composition operators on Besov spaces.

Key words. Representation theorem, Composition operator, Besov space, Strictly pseudoconvex domain.

AMS subject classifications. Primary 32A10; Secondary 47B33.

1. Introduction. Given a holomorphic self-mapping ϕ on a domain D in $\mathbb{C}^n$, the composition operator C_ϕ acting on function u on D is defined as $C_\phi(u)(z) = u(\phi(z))$. Theory of composition operators on holomorphic function spaces over a domain D in $\mathbb{C}^n$ has been developed by many authors in the last four decades (for examples, see the book of Shapiro [33], the book of Cowen and MacCuler [8], the book of Zhu [39], survey papers by Li [18], Russo [31] and references therein). Major researches have been concentrated on the function theoretic characterization of bounded, compact and Schatten-von-Neumann class composition operators as well as the spectral distribution of composition operators. When $n = 1$, the composition operator is always bounded on Hardy space $H^p(D)$ (see Li [18] for a proof and references therein). However, it is no longer true when $n > 1$ and counter examples were discovered by Shapiro, and Cima, Stanton and Wogen [6], MacCluer [29], Cima and Wogen [7].

A complete function theoretic characterization of a compact composition operator on Hardy space over the unit disk was given by Shapiro [32]. For results of compact composition operators and Schatten-von-Neumann class composition operators on Bergman spaces or Hardy spaces in several complex variables can be found (for examples) from the references: [8], [9], [17], [18, 19, 20], [27], [25], [30], [31], [34] and [39].

Compact composition operators on Bloch space over classical bounded symmetric domains were characterized by Zhou and Shi [37] and compact composition operators on BMOA space over the unit disc were characterized by Bourdon, Cima and Matheson [3], Smith [35] and Wulan [36]. Characterization of compact composition operators on $BMOA(B_n)$ was given by Li and Long [21] and many others.

In this paper we characterize bounded and compact composition operators on holomorphic Besov space $B^p(D)$ over a smoothly bounded strictly pseudoconvex domain D in $\mathbb{C}^n$. A representation theorem for $B^p(D)$, which we develop along the way, plays a key role in our proof and also is of independent interests itself. Such a

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representation for Hardy and weighted Bergman spaces on the bounded symmetric domains in $\mathbb{C}^n$ was given by Coifman and Rochberg in [4], and for A_p-Hardy and Bergman space as well as for BMOA and Bloch spaces over the unit ball in $\mathbb{C}^n$ by Luecking in [26].

To state our theorems we need to introduce function spaces and some notation. Let $K(z,w)$ be the Bergman kernel function for a domain D and $K(z) = K(z,z)$. Let dv be the normalized Lebesgue measure on D and $d\lambda(z) = K(z)dv$ which is a biholomorphic invariant measure over D. For each $z \in D$, we let

$$k_z(w) = K(z)^{-1}K_z(w), \quad K_z(w) = K(w,z). \tag{1.1}$$

From the estimates for Bergman kernel function given by Fefferman in [10] we have

$$C^{-1}\delta^{-n-1}(z) \le K(z) \le C\delta^{-n-1}(z), \tag{1.2}$$

for some fixed constant C depending only on D where $\delta(z) = \mathrm{dist}(z, \partial D)$. For a holomorphic function f on D, let

$$|\nabla^{n+1} f(z)| = \sum_{|\alpha| \le n+1} \Big|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z)\Big|.$$

For $1 \le p \le \infty$, we say that $f \in B^p(D)$ if

$$\|f\|_{B^p(D)} = \Big(\int_D |\nabla^{n+1} f(z)|^p \delta(z)^{p(n+1)} d\lambda(z)\Big)^{1/p} < \infty. \tag{1.3}$$

One can show that $k_z \in B^p(D)$ and $\|k_z\|_{B^p(D)}$ is comparable to 1 for all $z \in D$ and $1 \le p \le \infty$, and it can be found in [23, 24] and references therein that if D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ (or more general), one has the following inclusions:

$$B^p(D) \subset B^s(D) \subset VMOA(D) \subset BMOA(D) \subset H^q(D)$$

for $1 \le p \le s < \infty$ and $0 < q < \infty$, and

$$VMOA(D) \subset \mathcal{B}_0(D) \subset \mathcal{B}(D) \subset A^q(D)$$

for $1 \le p < s < \infty, 0 < q < \infty$. It is well known that $B^\infty(D) = \mathcal{B}(D)$, the holomorphic Bloch space over D while $\mathcal{B}_0(D)$ is the little Bloch space.

The ideas of Coifman and Rochberg in [4] and Luecking in [26] show that one can have a represention theorems for Hardy and Bergman space and VMO and little Block space. Using their approach we prove a representation theorem for Besov space over bounded strictly pseudoconvex domains in $\mathbb{C}^n$. To describe the representation theorem for functions in $B^p(D)$, let $\{z_{j,k}\}_{j=1,k=1}^{\infty,n_j}$ be a sequence satisfying ϵ-separation and δ_ϵ-density conditions(see Definition 2.2 for definition) and let $\mathcal{B}^p(D)$ be spaces of all holomorphic functions $f(z)$ having the representations:

$$f(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} c_{j,k} k_{z_{j,k}}(z), \quad \{c_{j,k}\}_{j=1,k=1}^{\infty,\ n_j} \in \ell^p \tag{1.4}$$

with the norm

$$\|f\|_{\mathcal{B}^p(D)} = \inf\Big\{\|\{c_{j,k}\}\|_{\ell^p} : f(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} c_{j,k}k_{z_{j,k}}(z)\Big\}. \tag{1.5}$$

We first prove the following representation theorem for functions in Besov spaces.

Theorem 1.1. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $1 \le p < \infty$. Then, there is a sequence of points $\{z_{j,k}\}_{j=1,k=1}^{\infty,\ n_j}$ satisfying ϵ-separation and δ_ϵ-density conditions so that*

$$(B^p(D), \|\cdot\|_{B^p}) = (\mathcal{B}^p(D), \|\cdot\|_{\mathcal{B}^p}) \tag{1.6}$$

with norms equivalent.

We apply this representation theorem to composition operators on Besov spaces. First, we introduce some notation. For $1 \le p \le \infty$ and for each $z \in D$, let

$$M_{\phi,p}(z) = \|C_\phi(k_z)\|_{B^p(D)}. \tag{1.7}$$

For any $\epsilon > 0$, let

$$D_\epsilon = \{z \in D : \operatorname{dist}(z, \partial D) > \epsilon\} \quad \text{and} \quad D_\epsilon^c = D \setminus D_\epsilon. \tag{1.8}$$

As applications of Theorem 1.1, we prove the following.

Theorem 1.2. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let ϕ be a holomorphic self map on D. Then,*

(1) C_ϕ is bounded on $B^1(D)$ if and only if $M_{\phi,1}(z) \in L^\infty(D)$;
(2) C_ϕ is compact on $B^1(D)$ if and only if $M_{\phi,1}(z) \in C_0(D)$;
(3) If C_ϕ is bounded on $B^1(D)$, then C_ϕ is bounded on $B^p(D)$ for $1 < p < \infty$;
(4) If C_ϕ is compact on $B^1(D)$ then C_ϕ is compact on $B^p(D)$ for $1 < p < \infty$;
(5) If $M_{\phi,p} \in L^{p'}(D, d\lambda)$ for $1 < p < \infty$, then C_ϕ is bounded on $B^p(D)$;
(6) If $\lim_{\epsilon \to 0^+} \|M_{\phi,p}\|_{L^{p'}(D_\epsilon^c, d\lambda)} = 0$ for $1 < p < \infty$, then C_ϕ is compact on $B^p(D)$.

The paper is organized as follows. In Section 2, we provide some basic results on strictly convex domains in $\mathbb{C}^n$, and as a consequence we prove Corollary 2.6 which gives one directional inclusion for Theorem 1.1. In Section 3, we prove duality theorems for Besov space $B^p(D)$ with an appropriate paring, and complete our proof of Theorem 1.1. Theorem 1.2 is proved in Section 5.

Constants. In the rest of the paper we use the same letter C to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants C will be often specified in the parenthesis or as a subscript. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \le CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminary. For the rest of the paper we let D be a fixed domain in $\mathbb{C}^n$ which is smoothly bounded and strictly pseudoconvex. Let $\delta(z) := \operatorname{dist}(z, \partial D)$ be the distance function from z to ∂D. For each $z \in D$ near ∂D, let $\nu(z) = (\frac{\partial\delta(z)}{\partial\bar{z}_1}, \cdots, \frac{\partial\delta(z)}{\partial\bar{z}_n})$, the complex normal vector at z of $\partial D_{\delta(z)}$. The orthogonal complement of $\nu(z)$ in $\mathbb{C}^n$ is

$$\mathbb{C}^{n-1}_{\nu(z)} = \{w \in \mathbb{C}^n : \langle w, \nu(z)\rangle = 0\}.$$

Let $\{w_{1,z}, \cdots, w_{n-1,z}\}$ be an orthonormal basis for $\mathbb{C}^{n-1}_{\nu(z)}$. Then a non-isotropic polydisc (or ball) in D centered at $z \in D$ and radius $\epsilon > 0$ is defined as follows:

$$E_\epsilon(z) = \left\{ \sum_{j=1}^{n-1} \lambda_j w_{j,z} + \lambda_n \nu(z) : |\lambda_j|^2 < \epsilon^2\, \delta(z), |\lambda_n| \le \epsilon\, \delta(z), 1 \le j < n \right\}. \tag{2.1}$$

Since D is strictly pseudoconvex with C^2 boundary, there are two positive numbers ϵ_D and C_D so that if $0 < \epsilon \le \epsilon_D$, then $E_{2\epsilon}(z) \subset D$ and

(i) if $0 < r_1 < r_2$, then $E_{r_1}(z) \subset E_{r_2}(z)$,

(ii) if $E_{r_1}(z) \cap E_{r_2}(w) \ne \emptyset$ with $r_1 \le r_2$, then $E_{r_1}(z) \subset E_{C_D r_2}(w)$,

(iii) $\delta(w) \approx \delta(z)$ for $w \in E_\epsilon(z)$.

Applying the idea of the Whitney Covering Lemma in [1] and decomposition of D in [4] for bounded symmetric domain in $\mathbb{C}^n$ and the unit ball of $\mathbb{C}^n$ in [26], one has the following lemma.

LEMMA 2.1. *There is $c > 0$ depending only on D so that for any $\epsilon > 0$ with $\delta_\epsilon =: c\epsilon < \epsilon_D/2$, there is a sequence of points $\{z_{j,k}\}_{k=1,j=1}^{n_j,\ \infty}$ of D satisfying*

$$\delta(z_{j,k}) = \delta(z_{j,m}), \quad \frac{\epsilon}{c2^j} \le \delta(z_{j,k}) \le \frac{c}{\epsilon\, 2^j}, \quad 1 \le k, m \le n_j,\ j = 1, 2, 3, \cdots, \tag{2.2}$$

and

$$\Big\{E_\epsilon(z_{j,k})\Big\}_{k=1,j=1}^{n_j,\infty} \text{ are disjoint and } D = \cup_{j=1}^{\infty} \cup_{k=1}^{n_j} E_{c\epsilon}(z_{j,k}). \tag{2.3}$$

Note that by (2.2) and (2.3) we have

$$n_j \approx \delta(z_{j,k})^{-n}, \quad 1 \le k \le n_j, \quad j = 1, 2, 3, \cdots. \tag{2.4}$$

DEFINITION 2.2. *A sequence $\{z_{j,k}\}_{k=1,j=1}^{n_j,\ \infty}$ of points in D is said to be satisfying ϵ-separation and δ_ϵ- density conditions if it satisfies* (2.2) *and* (2.3).

For any positive defining function $r(z)$ for D with $r \in C^2(\overline{D})$, denote by $K_r(z, w)$, the Bergman kernel function for the weighted Bergman space $A^2(D, r(z)^{n+1}dv)$. Let

$$V_r(f)(z) = r(z)^{n+1} \int_D f(w) K_r(z, w) dv(w), \quad z \in D$$

and let P be the Bergman projection defined as

$$P(f)(z) = \int_D f(w) K(z, w) dv(w), \quad z \in D.$$

PROPOSITION 2.3. *For $1 \le p < \infty$, we have*

(1) $V_r P = V_r$ on L^p;

(2) $V_r(K_\xi)(z) = r(z)^{n+1} K_r(z, \xi)$.

Proof. For any $f \in C_0^\infty(\overline{D})$, one has

$$\begin{aligned} V_r(P(f))(z) &= r(z)^{n+1} \int_D K_r(z,w)P(f)(w)dv(w) \\ &= r(z)^{n+1} \int_D K_r(z,w) \int_D f(\xi)K(w,\xi)dv(\xi)dv(w) \\ &= r(z)^{n+1} \int_D f(\xi) \int_D K_r(z,w)K(w,\xi)dv(w)dv(\xi) \\ &= r(z)^{n+1} \int_D f(\xi)K_r(z,\xi)dv(\xi) \\ &= V_r(f)(z). \end{aligned}$$

Thus, (1) follows from this since $C_0^\infty(\overline{D})$ is dense in L^p.

For (2) note that for $\xi \in D$

$$\begin{aligned} V_r(K_\xi)(z) &= r(z)^{n+1} \int_D K_r(z,w)K_\xi(w)dv(w) \\ &= r(z)^{n+1} K_r(z,\xi). \end{aligned}$$

□

The following is proved by Li and Luo in [23]:

THEOREM 2.4. *For $1 \le p < \infty$, we have*
(1) $P(L^p(D,d\lambda)) = B^p(D)$;
(2) $V_r : B^p(D) \to L^p(D,d\lambda)$ is bounded;
(3) $PV_r = I$ on $B^p(D)$.

With Theorem 2.4 at hand, we prove the following proposition from which we can easily deduce one implication of Theorem 1.1.

PROPOSITION 2.5. *Let $\{z_{j,k}\}_{k=1,j=1}^{n_j,\ \infty}$ be a sequence on points of D satisfying ϵ-separation and δ_ϵ-density condition. Then, the following holds;*

(1) For $1 \le p < \infty$ and $\{c_{j,k}\}_{k=1,j=1}^{n_j,\ \infty} \in \ell^p$, we have

$$f(z) := \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} c_{j,k} k_{z_{j,k}}(z) \in B^p(D)$$

and

$$\|f\|_{B^p(D)} \lesssim \|\{c_{j,k}\}_{k=1,j=1}^{n_j,\ \infty}\|_{\ell^p}.$$

(2) If f is given as in (1) with $\{c_{j,k}\}_{k=1,j=1}^{n_j,\ \infty} \in c_0$, then $f \in \mathcal{B}_0(D)$.

Proof. Let

$$F(z) = \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} c_{j,k} K(z_{j,k})^{-1} |E_\epsilon(z_{j,k})|^{-1} \chi_{E_\epsilon(z_{j,k})}(z).$$

Since $\{E_\epsilon(z_{j,k})\}_{k=1,j=1}^{n_j,\ \infty}$ are disjoint and $\delta(w) \approx \delta(z)$ for $w \in E_\epsilon(z)$, we have

$$\begin{aligned} \int_D |F(z)|^p d\lambda(z) &= \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} |c_{j,k}|^p \Big(K(z_{j,k})^{-1} |E_\epsilon(z_{j,k})|^{-1} \Big)^p \int_{E_\epsilon(z_{jk})} d\lambda(z) \\ &\lesssim \epsilon^{2n(1-p)} \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} |c_{j,k}|^p \end{aligned}$$

and

$$
\begin{aligned}
P(F)(z) &= \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} c_{j,k}K(z_{j,k})^{-1}|E_\epsilon(z_{j,k})|^{-1}\int_{E_\epsilon(z_{j,k})} K(z,w)dv(w) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} c_{j,k}k_{z_{j,k}}(z) \\
&= f(z).
\end{aligned}
$$

Therefore, by (1) of Theorem 2.4, we get

$$
\|f\|_{B^p(D)} = \|P(F)\|_{B^p(D)} \lesssim \|F\|_{L^p(D,d\lambda)} \lesssim \left(\epsilon^{2n(1-p)}\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}|c_{j,k}|^p\right)^{1/p}
$$

which proves (1).

To prove (2), let

$$
F_m(z) = \sum_{j=1}^{m}\sum_{k=1}^{n_j} c_{j,k}K(z_{j,k})^{-1}|E_\epsilon(z_{j,k})|^{-1}\chi_{E_\epsilon(z_{j,k})}(z).
$$

Since $\{c_{j,k}\}_{k=1,j=1}^{n_j,\ \infty} \in c_0$, we get

$$
\lim_{m\to\infty}\|F_m(z) - F(z)\|_{L^\infty(D)} = 0.
$$

Also, note that from the estimates of the derivatives of the Bergman kernel we get

$$
\begin{aligned}
\|f - P(F_m)\|_{\mathcal{B}(D)} &\lesssim \sup_{z\in D}\ \delta(z)\int_D |\nabla_z K(z,w)||F(w) - F_m(w)|dv(w) \\
&\lesssim \|F - F_m\|_{L^\infty(D)}.
\end{aligned}
$$

Therefore, we have

$$
\lim_{m\to\infty}\|f - P(F_m)\|_{\mathcal{B}(D)} \lesssim \lim_{m\to\infty}\|F - F_m\|_{L^\infty(D)} = 0.
$$

Hence, we get $f = P(F) \in \mathcal{B}_0(D)$ since $P(F_m) \in C^\infty(\overline{D})$. ☐

As a corollary we get a proof for one directional inclusion of Theorem 1.1

COROLLARY 2.6. *For $1 \le p < \infty$, we have*

$$
\mathcal{B}^p(D) \subset B^p(D)
$$

and

$$
\|f\|_{B^p(D)} \lesssim \|f\|_{\mathcal{B}^p(D)}.
$$

3. Duality and Decomposition for $B^p(D)$**.** In this section, through certain relations between B^p and ℓ^p we prove the duality theorem for B^p and complete the proof of Theorem 1.1. We follow the scheme of Coifman and Rochberg in [4] using Theorem 2.4.

Consider a linear operator $T : B^p(D) \to \ell^p$ defined as follows: For $f \in B^p(D)$, let

$$T(f) = \{V_r(f)(z_{j,k})\}_{k=1,j=1}^{n_j,\,\infty}, \quad r(z)^{n+1} = K(z)^{-1}. \tag{3.1}$$

This linear operator T plays a key role in the proof of Theorem 1.1 and the following holds.

Theorem 3.1. *The following holds:*
(1) $T : B^p(D) \to \ell^p$ *is bounded and* $\|T(f)\|_{\ell^p} \approx \|f\|_{B^p(D)}$ *if* $1 < p < \infty$*;*
(2) $T : \mathcal{B}_0(D) \to c_0$ *is bounded and injective.*

Proof. Let $1 < p < \infty$ and

$$T_{K_r}(f)(z) = \int_D K_r(z,w)f(w)dv(w).$$

Then, from the plurisubmarmonicity of $T_{K_r}(f)$ and the fact that $r(z) \approx r(z_{j,k})$ on $E_\epsilon(z_{j,k})$, we have

$$|V_r(f)(z_{j,k})| \le |E_\epsilon(z_{j,k})|^{-1} \int_{E_\epsilon(z_{j,k})} |V_r(f)(z)|dv(z).$$

Since $\{E_\epsilon(z_{j,k})\}_{k=1,j=1}^{n_j,\,\infty}$ are disjoint subsets in D, we thus have

$$\begin{aligned}
\|T(f)\|_{\ell^p}^p &= \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |V_r(f)(z_{j,k})|^p \\
&\le \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \frac{1}{|E_\epsilon(z_{j,k})|} \int_{E_\epsilon(z_{j,k})} |V_r(f)(z)|^p dv(z) \\
&\lesssim \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \int_{E_\epsilon(z_{j,k})} |V_r(f)(z)|^p d\lambda(z) \\
&\le \int_D |V_r(f)(z)|^p d\lambda(z) \\
&\lesssim \|f\|_{B^p(D)}^p.
\end{aligned}$$

Here, the last inequality follows from (2) of Theorem 2.4. This proves that $T : B^p(D) \to \ell^p$ is bounded when $1 < p < \infty$.

Next, by choosing $\epsilon > 0$ small enough we show that

$$\|V_r(f)\|_{L^p(D,d\lambda)} \lesssim \|T(f)\|_{\ell^p}. \tag{3.2}$$

This completes proof of (1) since we have

$$\|f\|_{B^p(D)} \lesssim \|V_r(f)\|_{L^p(D,d\lambda)} \tag{3.3}$$

by (1) and (3) of Theorem 2.4. First, we choose $\epsilon > 0$ small enough so that

$$c(\epsilon) := \sup\Big\{|E_{\delta_\epsilon}(z_{j,k})|K(z) : z \in E_{\delta_\epsilon}(z_{j,k}) : 1 \le k \le n_j, j = 1,2,3,\cdots\Big\} \lesssim 1.$$

Choose $\xi_{j,k} \in E_{\delta_\epsilon}(z_{j,k})$ so that

$$|V_r(f)(\xi_{j,k})| = \max\{|V_r(f)(z)| : z \in E_{\delta_\epsilon}(z_{j,k})\}.$$

Since $D \subset \cup_{j=1}^{\infty} \cup_{k=1}^{n_j} E_{\delta_\epsilon}(z_{j,k})$, we see that

$$\begin{aligned}
\|V_r(f)\|^p_{L^p(D,d\lambda)} &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \int_{E_{\delta_\epsilon}(z_{j,k})} |V_r(f)|^p d\lambda(z) \\
&\lesssim c(\delta) \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |V_r(f)(\xi_{j,k})|^p \\
&\lesssim \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |V_r(f)(z_{j,k})|^p + \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |V_r(f)(\xi_{j,k}) - V_r(f)(z_{j,k})|^p.
\end{aligned}$$

Thus, it suffices to show that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |V_r(f)(\xi_{j,k}) - V_r(f)(z_{j,k})|^p \lesssim \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |V_r(f)(z_{j,k})|^p. \tag{3.4}$$

From (1) of Theorem 2.4 we can choose $f_0 \in L^p(D, d\lambda)$ so that

$$\|f_0\|_{L^p(D,d\lambda)} \approx \|f\|_{B^p(D)} \quad \text{and} \quad f = P(f_0). \tag{3.5}$$

Let

$$A(z_{j,k}, \xi_{j,k}) = \int_D |K_r(z_{j,k}, w) - K_r(\xi_{j,k}, w)| dv(w).$$

If f is holomorphic, then $f(a) - f(b) = \langle \nabla f(a + t(b-a)), \overline{a-b} \rangle$ for for some $t \in [0, 1]$. Thus, from the estimates of the (weighted) Bergman kernel we have

$$|K(z_{j,k}) - K(\xi_{j,k})| \lesssim \epsilon\, K(z_{j,k}) \qquad \text{and} \qquad A(z_{j,k}, \xi_{j,k}) \lesssim \epsilon\, K(z_{j,k}). \tag{3.6}$$

Since $V_r P = V_r$ by (1) of Proposition 2.3, we then have

$$\begin{aligned}
&|V_r(f)(\xi_{j,k}) - V_r(f)(z_{j,k})| \\
&= |V_r(f_0)(\xi_{j,k}) - V_r(f_0)(z_{j,k})| \\
&= \left|K(\xi_{j,k})^{-1} T_{K_r}(f_0)(\xi_{j,k}) - K(z_{j,k})^{-1} T_{K_r}(f_0)(z_{j,k})\right| \\
&\leq |(K(\xi_{j,k})^{-1} - K(z_{j,k})^{-1}) T_{K_r}(f_0)(z_{j,k})| \\
&\qquad + K(\xi_{j,k})^{-1} |T_{K_r}(f_0)(\xi_{j,k}) - T_{K_r}(f_0)(z_{j,k})| \\
&\lesssim \epsilon K(z_{j,k})^{-1} |T_{K_r}(f_0)(z_{j,k})| \\
&\qquad + K(\xi_{j,k})^{-1} \int_D |K_r(\xi_{j,k}, w) - K_r(z_{j,k}, w)||f_0(w)| dv(w)| \\
&\lesssim \epsilon |V_r(f)(z_{j,k})| + K(\xi_{j,k})^{-1} \int_D |K_r(\xi_{j,k}, w) - K_r(z_{j,k}, w)||f_0(w)| dv(w)|.
\end{aligned}$$

Also, note that by Hölder's inequality we get

$$\begin{aligned}&\Big(|K(z_{j,k})^{-1}\int_D |K_r(\xi_{j,k},w)-K_r(z_{j,k},w)||f_0(w)|dv(w)|\Big)^p\\&\le K(z_{j,k})^{-p'}A(z_{j,k},\xi_{j,k})^{p'-1}\int_D |K_r(\xi_{j,k},w)-K_r(z_{j,k},w))||f_0(w)|^p dv(w)\\&\lesssim \epsilon\ K(z_{j,k})^{-1}\int_D |K_r(\xi_{j,k},w)-K_r(z_{j,k},w))||f_0(w)|^p dv(w)\\&\lesssim \epsilon\ K(z_{j,k})^{-1}\int_D |K_r(z_{j,k},w))||f_0(w)|^p dv(w)\\&\lesssim \epsilon\ \int_D\int_{E_{\delta_\epsilon}(z_{j,k})} |K_r(z_{j,k},w))||f_0(w)|^p dv(z)dv(w).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}|V_r(f)(\xi_{j,k})-V_r(f)(z_{j,k})|^p\\&\lesssim \epsilon\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}|V_r(f)(z_{j,k})|^p+\epsilon\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}K(z_{j,k})^{-1}\int_D |K_r(z_{j,k},w))||f_0(w)|^p dv(w)\\&\lesssim \epsilon\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}|V_r(f)(z_{j,k})|^p+\epsilon\int_D\int_D |K_r(z,w))||f_0(w)|^p dv(w)dv(z)\\&\lesssim \epsilon\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}|V_r(f)(z_{j,k})|^p+\epsilon\int_D |f_0(w)|^p d\lambda(z)d\lambda(w).\end{aligned}$$

By (3.3) and (3.5), we then get (3.4) and hence completes proof of (1).

To prove (2), first note that $\|V_r(f)\|_{L^\infty}\lesssim\|f\|_{L^\infty}$ which implies $T: B^\infty(D)\to\ell^\infty$ is bounded. Thus, for any $f\in\mathcal{B}_0(D)$ we have $T(f)\in\ell^\infty$. It is known from [14] that there is an $f_0\in C_0(D)$ so that $P(f_0)=f$ and $\|f_0\|_{C(\overline{D})}\approx\|f\|_{\mathcal{B}_0(D)}$. Thus for any $\epsilon>0$, there is $\delta>0$ so that

$$|f_0(z)|\le\epsilon,\ \text{when } z\in D\setminus D_\delta=\{w\in D:\delta(w)\le\delta\}. \tag{3.15}$$

By (1) of Proposition 2.3, we have

$$V_r(f)(z)=V_r(P(f_0))(z)=V_r(f_0)(z)=r(z)^{n+1}T_{K_r}(f_0)(z). \tag{3.16}$$

Therefore,

$$\begin{aligned}&|V_r(f)(z_{j,k})|\\&=r(z_{j,k})^{n+1}\Big|\int_D f_0(w)K_r(z_{j,k},w)dv(w)\Big|\\&\le r(z_{j,k})^{n+1}\Big[\Big|\int_{D_\delta} f_0(w)K_r(z_{j,k},w)dv(w)\Big|+\Big|\int_{D\setminus D_\delta} f_0(w)K_r(z_{j,k},w)dv(w)\Big|\Big]\\&\lesssim r(z_{j,k})^{n+1}\Big[\|f_0\|_{C(\overline{D})}\delta^{-2(n+1)}+\epsilon r(z_{j,k})^{-n-1}\Big].\end{aligned}$$

Therefore, we see that

$$\lim_{j\to\infty}|V_r(f)(z_{j,k})|=0.$$

This implies that $T(f) \in c_0$. With a similar argument, one has $T : \mathcal{B}_0(D) \to c_0$ is injective and proof of (2) is complete. □

For $1 \le p < \infty$ let p' be the conjugate of p defined by the relation $1/p + 1/p' = 1$.

THEOREM 3.2. *With the following pairing for functions in* $B^p(D)$,

$$(f, g)_0 = \int_D V_r(f)(z)\overline{V_r(g)(z)}d\lambda(z), \tag{3.7}$$

we have the following duality relations:

(1) $B^p(D)^* = B^{p'}(D)$ *for* $1 \le p < \infty$;

(2) $\mathcal{B}_0(D)^* = B^1(D)$.

Proof. We first prove (1). It is obvious that $B^{p'}(D) \subset (B^p(D))^*$ with respect to the pairing (3.7) since $V_r : B^p(D) \to L^p(D, d\lambda)$ is bounded for $1 \le p < \infty$ by (2) of Theorem 2.4 which is also true for $p = \infty$ from the estimate of the weighted Bergman kernel. On the other hand, if $\mathcal{L}$ is a bounded linear functional on $B^p(D)$ with respect to (3.7), which is interpreted as that $\mathcal{L}$ is a bounded linear functional on the subspace $\{V_r(f) : f \in B^p(D)\}$ of $L^p(D, d\lambda)$. By Hahn Banach theorem, $\mathcal{L}$ can be extended as a bounded linear functional on $L^p(D, d\lambda)$ with the same norm. Since $L^p(D)^* = L^{p'}(D)$, there is a $g \in L^{p'}(D, d\lambda)$ so that

$$\mathcal{L}(f) = \int_D V_r(f)(z)\overline{g(z)}d\lambda(z), \quad \|\mathcal{L}\| = \|g\|_{L^p(D,d\lambda)}$$

for all $f \in B^p(D)$. Since $r(z)^{n+1}d\lambda = dv(z)$ and $PV_r = I$, we have

$$\begin{aligned}
\mathcal{L}(f) &= \int_D T_{K_r}(f)(z)\overline{g}(z)dv(z) \\
&= \int_D T_{K_r}(f)(z)\overline{P(g)(z)}dv(z) \\
&= \int_D f(w)\overline{T_{K_r}(P(g))}(w)dv(w) \\
&= \int_D PV_r(f)(w)\overline{T_{K_r}(P(g))}(w)dv(w) \\
&= \int_D V_r(f)(w)\overline{T_{K_r}(P(g))}(w)dv(w) \\
&= \int_D V_r(f)(w)\overline{V_r(P(g))(w)}d\lambda(w).
\end{aligned}$$

Since $P : L^p(D, d\lambda) \to B^p(D)$ is bounded for $1 \le p \le \infty$, we have $P(g) \in B^{p'}(D)$ and $\mathcal{L}$ can be identified as $P(g)$. The proof of (1) is complete.

Next we prove (2). By (1) with $p = 1$, we have $B^1(D)^* = \mathcal{B}(D)$ with respect to the pairing (3.7). Thus, one can easily see that $B^1(D) \subset \mathcal{B}_0(D)^*$. On the other hands, for any linear functional $\mathcal{L}$ on $\mathcal{B}_0(D)$ with respect to the pairing (3.7), which means that $\mathcal{L}$ is viewed as a bounded linear functional on the linear subspace $\{V_r(f) : f \in \mathcal{B}_0(D)\}$ of $C_0(D)$. By Hahn-Banach theorem, $\mathcal{L}$ can be extended as a linear functional on $C_0(D)$ with the same norm. By Riesz representation theorem, there is a finite complex Borel measure $d\mu$ on D so that

$$\mathcal{L}(g) = \int_D V_r(g)(z)d\mu(z) = \int_D V_r(g)(w)\overline{P(r^{n+1}\overline{d\mu})(w)}d\lambda,$$

where

$$f(z) := P(r^{n+1}\overline{d\mu})(z) = \int_D K(z,w)r(w)^{n+1}d\overline{\mu}(w).$$

Then, following the arguments of the proof of (1) we get

$$\begin{aligned}\mathcal{L}(g) = \int_D V_r(g)(z)\overline{f(z)}d\lambda(z) &= \int_D V_r(g)(z)\overline{V_r(f)(z)}d\lambda(z)\\ &= \int_D V_r(g)(z)\overline{V_rP[V_r(f)](z)}d\lambda(z).\end{aligned}$$

Notice that

$$V_r(f)(z) = r(z)^{n+1}\int_D K_r(z,w)f(w)dv(w) = r(z)^{n+1}\int_D K_r(z,w)r(w)^{n+1}d\overline{\mu}(w)$$

and thus

$$\begin{aligned}\int_D |V_r(f)(z)|d\lambda(z) &= \int_D \Big|\int_D K_r(z,w)r(w)^{n+1}d\overline{\mu}(w)\Big|dv(z)\\ &\le \int_D\int_D \Big|K_r(z,w)\Big|r(w)^{n+1}\ dv(z)\ |d\overline{\mu}|(w)\\ &\lesssim \int_D r(w)^{-n-1}r(w)^{n+1}|d\overline{\mu}|(w)\\ &\lesssim \|d\mu\|.\end{aligned}$$

This implies that $V_r(f) \in L^1(D, d\lambda)$, so $PV_r(f) \in B^1(D)$ with $\|PV_r(f)\|_{B^1(D)} \lesssim \|d\mu\|$. Therefore, $\mathcal{B}_0(D)^* = B^1(D)$, the proof of (2) is complete, and so is Theorem 3.2. ∎

We now are ready to complete the proof for Theorem 1.1.

Proof of Theorem 1.1. By Corollary 2.6, it suffices to show that for $1 \le p < \infty$,

$$B^p(D) \subset \mathcal{B}^p(D)$$

and

$$\|f\|_{\mathcal{B}^p(D)} \lesssim \|f\|_{B^p(D)}.$$

First, we prove the case $1 < p < \infty$. Let $T : B^{p'}(D) \to \ell^{p'}$ be a linear operator defined by (3.1). By (1) of Theorem 3.1, we see that $T : B^{p'}(D) \to \ell^{p'}$ is isomorphic to a subspace of $\ell^{p'}$. Therefore, $T^* : \ell^p \to B^p(D)$ is a bounded and onto linear operator.

We will evaluate $T^*(\alpha)(z)$ for $\alpha = \{\lambda_{j,k}\}_{k=1,j=1}^{n_j,\,\infty} \in \ell^{p'}$. For any $g \in B^{p'}(D)$, we have

$$\begin{aligned}(T^*(\alpha), g)_0 &= (\alpha, T(g))\\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k}\overline{V_r(g)(z_{j,k})}\\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} r(z_{j,k})^{n+1}\overline{T_{K_r}(g)(z_{j,k})}\\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} r(z_{j,k})^{n+1}\int_D \overline{T_{K_r}(g)(w)}K_r(w, z_{j,k})r(w)^{n+1}dv(w)\\ &= \int_D \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} r(z_{j,k})^{n+1} r(w)^{n+1} K_r(w, z_{j,k})\overline{T_{K_r}(g)(w)}dv(w)\\ &= \int_D \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} r(z_{j,k})^{n+1} r(w)^{n+1} K_r(w, z_{j,k})\overline{V_r(g)(w)}d\lambda(w).\end{aligned}$$

Let

$$f(w) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} r(z_{j,k})^{n+1} K(w, z_{j,k}),$$

then

$$\begin{aligned}V_r(f)(z) &= r(z)^{n+1}\int_D f(w)K_r(z,w)dv(w)\\ &= r(z)^{n+1}\sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} r(z_{j,k})^{n+1} K_r(z, z_{j,k}).\end{aligned}$$

Therefore, for any $g \in B^{p'}(D)$ we have

$$\begin{aligned}(T^*(\alpha), g)_0 &= \int_D \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} r(z_{j,k})^{n+1} r(w)^{n+1} K_r(w, z_{j,k})\overline{V_r(g)(w)}d\lambda(w)\\ &= \int_D V_r(f)(w)\overline{V_r(g)(w)}d\lambda(w)\\ &= (f, g)_0.\end{aligned}$$

This implies that

$$T^*(\{\lambda_{j,k}\}_{k=1,j=1}^{n_j,\infty})(z) = f(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} k_{z_{j,k}}(z).$$

This implies that $f \in \mathcal{B}^p(D)$. Since $T^* : \ell^p \to B^p(D)$ is onto, we have $B^p(D) \subset \mathcal{B}^p(D)$. Since $T : B^{p'}(D) \to \ell^{p'}$ is isomorphic to a subspace of $\ell^{p'}$, we see that

$$\|f\|_{\mathcal{B}^p(D)} \lesssim \|f\|_{B^p(D)}.$$

This completes proof for $1 < p < \infty$.

Finally, suppose $p = 1$. Since $T : \mathcal{B}_0(D) \to c_0$ is bounded and injective. Since $\mathcal{B}_0(D)^* = B^1(D)$ by (2) of Theorem 3.2 and $(c_0)^* = \ell^1$, we see that $T^* : \ell^1 \to B^1(D)$ is bounded and onto. Then, following the same argument for the case $1 < p < \infty$, we can see that for any $f \in B^1(D)$, there is $\{c_{j,k}\}_{k=1,j=1}^{n_j,\,\infty} \in \ell^1$ so that

$$f(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} c_{j,k} k_{z_{j,k}}(z) \quad \text{with} \quad \|f\|_{B^1(D)} \approx \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |c_{j,k}|.$$

From this we can deduce

$$\|f\|_{\mathcal{B}^1(D)} \lesssim \|f\|_{B^1(D)}$$

which completes proof for $p = 1$, and proof of Theorem 1.1 is complete. □

4. Composition operators on $B^p(D)$. In this section, we prove Theorem 1.2. For $1 \le p < \infty$, let

$$M_p(\phi) := \sup\Big\{\|C_\phi(k_{z_{j,k}})\|_{B^p(D)} : 1 \le k \le n_j,\ 1 \le j < \infty\Big\}.$$

Throughout this section, we assume $\phi : D \to D$ is a holomorphic.

We start with the case $p = 1$.

THEOREM 4.1. *For any holomorphic mapping $\phi : D \to D$, we have*
(1) C_ϕ is bounded on $B^1(D)$ if and only if

$$M_1(\phi) < \infty;$$

(2) C_ϕ is compact on $B^1(D)$ if and only if

$$\lim_{j\to\infty} \sup\Big\{\|C_\phi(k_{z_{j,k}})\|_{B^1(D)} : 1 \le k \le n_j\Big\} = 0.$$

Proof. Since $\|k_{z_{j,k}}\|_{B^1(D)} \approx 1$, the necessity condition for (1) or (2) is obvious. Thus, it is enough to prove the sufficiency condition.

For (1), suppose $M_1(\phi) < \infty$ and let

$$f(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} k_{z_{j,k}}(z) \quad \text{with } \|f\|_{\mathcal{B}^1(D)} = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |\lambda_{j,k}| < \infty.$$

Then, we have

$$\|C_\phi(f)\|_{B^1(D)} \le \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |\lambda_{j,k}| \|C_\phi(k_{z_{j,k}})\|_{B^1(D)} \le M_1(\phi)\|f\|_{\mathcal{B}^1(D)}.$$

To prove (2), suppose

$$\lim_{j\to\infty} \sup\Big\{\|C_\phi(k_{z_{j,k}})\|_{B^1(D)} : 1 \le k \le n_j\Big\} = 0.$$

To show the compactness of C_ϕ on $B^1(D)$, it suffices to show that $\lim_{s\to\infty} \|C_\phi(f_s)\|_{B^1(D)} = 0$ for all sequence $\{f_s\}_{s=1}^\infty$ in $B^1(D)$ with norm 1 which converges to 0 uniformly on any compact subset of D. Let $\{f_s\}_{s=1}^\infty$ be a sequence in

$B^1(D)$ with norm 1 which converges to 0 uniformly on any compact subset of D. Then, by Theorem 3.2

$$f_s(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda^s_{j,k} k_{z_{j,k}}(z), \quad \text{with } \|\{\lambda^s_{j.k}\}\|_{\ell^1} \approx 1.$$

Let $\epsilon > 0$, then by assumption there is j_0 such that

$$\|C_\phi(k_{z_j,k})\|_{B^1} < \epsilon, \text{ for all } 1 \le k \le n_j, j \ge j_0.$$

Since $\{f_s\}_{s=1}^{\infty}$ converges to 0 uniformly on any compact subset of D, there is $s_0 > 1$ so that if $s \ge s_0$ then

$$\sum_{j=1}^{j_0}\sum_{k=1}^{n_j} |\lambda^s_{j,k}| < \epsilon.$$

Therefore, if $s \ge s_0$ then

$$\begin{aligned}\|C_\phi(f_s)\|_{B^1(D)} &\le \sum_{j=1}^{j_0}\sum_{k=1}^{n_j} |\lambda^s_{j,k}| \|C_\phi(k_{z_{j,k}})\|_{B^1(D)} + \sum_{j=j_0+1}^{\infty}\sum_{k=1}^{n_j} |\lambda^s_{j,k}| \|C_\phi(k_{z_{j,k}})\|_{B^1(D)} \\ &\lesssim \|C_\phi\|\epsilon.\end{aligned}$$

Therefore, $\lim_{s\to\infty} \|C_\phi(f_s)\|_{B^1(D)} = 0$ which complete proof of (2). □

Next, we consider the case when $1 < p < \infty$. Let M^ϕ be the $\infty \times \infty$ matrix defined by

$$M^\phi = \left[\frac{\int_D K(\phi(z), z_{j,k}) K_r(z_{\ell,m}, z) dv(z)}{K(z_{j,k}) K(z_{\ell,m})}\right]. \tag{4.1}$$

Then, we have the following characterization for the boundedness and the compactness which is not part of Theorem 1.2.

THEOREM 4.2. *For any holomorphic mapping $\phi : D \to D$ and $1 < p < \infty$, we have*

(1) C_ϕ is bounded on $B^p(D)$ if and only if M^ϕ is bounded on ℓ^p;
(2) C_ϕ is compact on $B^p(D)$ if and only if M^ϕ is compact on ℓ^p.

Proof. Let

$$f(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} c_{j,k} k_{z_{j,k}}(z) \in B^p(D) \quad \text{with } \|f\|^p_{B^p(D)} \approx \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |c_{j,k}|^p$$

and

$$g(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} k_{z_{j,k}}(z) \in B^{p'}(D) \quad \text{with } \|g\|^{p'}_{B^{p'}(D)} \approx \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |c_{j,k}|^{p'}.$$

Then

$$
\begin{aligned}
(C_\phi(f), g)_0 &= \int_D V_r(C_\phi(f))(z)\overline{V_r(g))(z)}d\lambda(z) \\
&= \int_D C_\phi(f)(z)\overline{V_r(g(z)}d\lambda(z) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} c_{j,k} \sum_{\ell=1}^{\infty}\sum_{m=1}^{n_\ell} \overline{\lambda}_{\ell,m} \frac{\int_D C_\phi(K_{z_{j,k}})(z)K_r(z, z_{\ell,m})dv(z)}{K(z_{j,k})K(z_{\ell,m})} \\
&= (M^\phi\{c_{j,k}\}, \{\lambda_{\ell,m}\}).
\end{aligned}
$$

Using the duality theorems: $B^p(D)^* = B^{p'}(D)$ under the pairing $(\cdot,\cdot)_0$ and the fact $(\ell^p)^* = \ell^{p'}$, we see that C_ϕ is bounded on $B^p(D)$ if and only if M^ϕ is bounded on ℓ^p. Similarly, we can deduce that C_ϕ is compact on $B^p(D)$ if and only if M^ϕ is compact on ℓ^p. □

Questions about if one has a nicer characterization on boundedness and compactness for C_ϕ on $B^p(D)$ as the case $p = 1$ and $p = \infty$ are not completely answered here. We provide some partial results, and leave the problem for further study.

THEOREM 4.3. *For $1 < p < \infty$, we have*
(1) If C_ϕ is bounded on $B^p(D)$, then $\sup_{z\in D} \|C_\phi(k_z)\|_{B^p(D)} < \infty$;
(2) If C_ϕ is compact on $B^p(D)$, then $\lim_{z\to\partial D} \|C_\phi(k_z)\|_{B^p(D)} = 0$;
(3) If C_ϕ is bounded on $B^1(D)$, then C_ϕ is bounded on $B^p(D)$;
(4) If C_ϕ is compact on $B^1(D)$, then C_ϕ is compact on $B^p(D)$.

Proof. (1) and (2) easily follow from the fact that $\|k_z\|_{B^p(D)} \approx 1$ for all $z \in D$ and $k_z(w) \to 0$ uniformly on any compact subset of D.

To prove (3), recall the interpolation theorem for $B^p(D)$,

$$B^p(D) = [B^1(D), B^\infty(D) = \mathcal{B}(D)]_\theta, \quad \theta = \frac{1}{p}.$$

Thus

$$\|C_\phi\|_{B^p\to B^p} \le \|C_\phi\|^{\theta}_{B^1(D)\to B^1(D)} \|C_\phi\|^{1-\theta}_{\mathcal{B}(D)\to\mathcal{B}(D)}.$$

Therefore, the boundedness of C_ϕ on $B^1(D)$ implies that C_ϕ is bounded on $B^p(D)$ for all $1 \le p \le \infty$ since C_ϕ is always bounded on $\mathcal{B}(D)$.

Next, to prove (4) suppose C_ϕ is compact on $B^1(D)$. Let $\{g_m\}_{m=1}^\infty$ be a sequence in $B^p(D)$ with $\|g_m\|_{B^p(D)} = 1$ and assume $\{g_m\}$ converges uniformly to 0 on any compact subset in D. It suffices to show that

$$\lim_{m\to\infty} \|C_\phi(g_m)\|_{B^p(D)} = 0.$$

First, we claim the following: For any $\epsilon > 0$, there is $M_\epsilon \ge 1$ such that

$$\|C_\phi(f)\|_{B^1(D)} < \epsilon \|f\|_{B^1(D)} \tag{4.2}$$

for all

$$f(z) = \sum_{j=M_\epsilon}^{\infty}\sum_{k=1}^{n_j} c_{j,k} k_{z_{j,k}}(z).$$

Suppose the claim is not true, then there is a constant $\epsilon_0 > 0$, a sequence $\{M_m\}$ such that $\lim_{m\to\infty} M_m = \infty$ and

$$f_m(z) = \sum_{j=M_m}^{\infty} \sum_{k=1}^{n_j} c_{j,k} k_{z_{j,k}}(z) \quad \text{with } \|f_m\|_{B^1(D)} = 1,$$

but $\|C_\phi(f_m)\|_{B^1(D)} \geq \epsilon_0$ for all m. Note that $f_m(z)$ converges uniformly to 0 on any compact subset of D since $\sum_{j=M_m}^{\infty} \sum_{k=1}^{n_j} |c_{j,k}| \lesssim 1$ by Theorem 1.1 and $\lim_{m\to\infty} M_m = \infty$. From the compactness of C_ϕ on $B^1(D)$, we see that $\|C_\phi(f_m)\|_{B^1(D)} \to 0$ as $m \to \infty$. This contradicts with $\|C_\phi(f_m)\|_{B^1(D)} \geq \epsilon_0 > 0$. So the claim is proved.

Let $\epsilon > 0$ and choose M_ϵ as in the claim. By Theorem 1.1, we can write

$$g_m(z) = \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} c_{j,k;m} k_{z_{j,k}}(z) \quad \text{with } \|g_m\|^p_{B^p(D)} \approx \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} |c_{j,k;m}|^p.$$

Since $\{g_m\}$ converges uniformly to 0 on any compact subset in D, for any $\epsilon > 0$ there is m_0 such that

$$\sum_{j=1}^{M_\epsilon} \sum_{k=1}^{n_j} |c_{j,k;m}| < \epsilon \tag{4.3}$$

for all $m \geq m_0$.

Consider the following subspaces of $B^p(D)$

$$B^p(D, \epsilon) = \Big\{ f = \sum_{j=M_\epsilon}^{\infty} \sum_{k=1}^{n_j} c_{j,k} k_{z_{j,k}} : \{c_{j,k}\} \in \ell^p \Big\}.$$

Then, C_ϕ is bounded on $B^1(D, \epsilon)$ with $\|C_\phi\|_{B^1(D,\epsilon)} \leq \epsilon$. Applying the interpolation theorem for space $B^p(D, \epsilon)$, we see that

$$\| \sum_{j=M_\epsilon}^{\infty} \sum_{k=1}^{n_j} c_{j,k;m} \|_{B^p(D)} \leq C^{1/p'} \epsilon^{1/p}. \tag{4.4}$$

From (4.3) and (4.4) we can deduce that $\lim_{m\to\infty} \|C_\phi(g_m)\|_{B^p(D)} \to 0$ as $m \to \infty$. So, C_ϕ is compact on $B^p(D)$ and (4) is proved, and so is Theorem 4.3. □

We now are ready to prove Theorem 1.2.

Proof of Theorem 1.2. For (1) and (2), note that

$$M_1(\phi) \leq \|M_{\phi,1}(\cdot)\|_{L^\infty(D)}$$

where $M_{\phi,1}(z)$ is defined as in (1.7). Note that the boundedness of C_ϕ on $B^1(D)$ implies $\|M_{\phi,1}(\cdot)\|_{L^\infty(D)} \lesssim \|C_\phi\| < \infty$. Thus, (1) follows from (1) of Theorem 4.1. Similarly, we can deduce (2) from (2) of Theorem 4.1.

Note that (3) and (4) follows from those of Theorem 4.3. To prove (5), let $f \in B^p(D)$. Then, by Theorem 1.1 we get

$$f(z) = \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \lambda_{j,k} k_{z_{j,k}}(z) \quad \text{with } \|f\|_{B^p(D)} \approx \Big(\sum_{j=1}^{\infty} \sum_{k=1}^{n_j} |\lambda_{j,k}|^p \Big)^{1/p}.$$

Thus, we have

$$\begin{aligned}\|C_\phi(f)\|_{B^p(D)} &\le \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |\lambda_{j,k}| \|C_\phi(k_{z_{j,k}})\|_{B^p(D)} \\ &\le \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |\lambda_{j,k}|^p\Big)^{1/p} \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \|C_\phi(k_{z_{j,k}})\|_{B^p(D)}^{p'}\Big)^{1/p'} \\ &\lesssim \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |\lambda_{j,k}|^p\Big)^{1/p} \Big(\int_D \|C_\phi(k_z)\|_{B^p(D)}^{p'} d\lambda(z)\Big)^{1/p'} \\ &\lesssim \Big(\int_D \|C_\phi(k_z)\|_{B^p(D)}^{p'} d\lambda(z)\Big)^{1/p'} \|f\|_{B^p(D)}.\end{aligned}$$

This implies (5) of Theorem 1.2.

To prove (6), it suffices to show that $\|C_\phi(f_s)\|_{B^p(D)} \to 0$ as $s \to \infty$ for every bounded sequence $\{f_s\}$ in $B^p(D)$ which converges to 0 uniformly on any compact subset of D. Let $\{f_s\}$ be a such sequence, then by Theorem 1.1

$$f_s(z) = \sum_{j=1}^{\infty}\sum_{k=1}^{n_j} \lambda_{j,k} k_{z_{j,k;s}}(z) \quad \text{with } \|f\|_{B^p(D)} \approx \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{n_j} |\lambda_{j,k;s}|^p\Big)^{1/p} \approx 1.$$

By the assumption of (6), for any $\eta > 0$ there is $\delta > 0$ so that

$$\left(\int_{D_\delta^c} \|C_\phi(k_z)\|_{B^p(D)}^{p'}\ dv(z)\right)^{1/p'} < \eta \tag{4.5}$$

where D_δ^c is the set defined in (1.8). Therefore, there is N so that $z_{j,k} \in D_\delta^c$ for all $k = 1, \cdots, n_j$ if $j \ge N$. Moreover, from the assumption that $\{f_s\}$ converges uniformly on compact subsets of D, there is $s_0 \ge 1$ so that if $s \ge s_0$ then

$$\Big(\sum_{j=1}^{N}\sum_{k=1}^{n_j} |\lambda_{j,k;s}|^p\Big)^{1/p} < \eta. \tag{4.6}$$

From (4.5) and (4.6), we have

$$\begin{aligned}\|C_\phi(f_s)\|_{B^p(D)} &\le \sum_{j=1}^{\infty}\sum_{k=1}^{n_j}|\lambda_{j,k;s}|\|C_\phi(k_{z_{j,k}})\|_{B^p(D)}\\ &\le \Big(\sum_{j=1}^{N}\sum_{k=1}^{n_j}|\lambda_{j,k;s}|^p\Big)^{1/p}\Big(\sum_{j=1}^{N}\sum_{k=1}^{n_j}\|C_\phi(k_{z_{j,k}})\|_{B^p(D)}^{p'}\Big)^{1/p'}\\ &\quad+\Big(\sum_{j=N}^{\infty}\sum_{k=1}^{n_j}|\lambda_{j,k;s}|^p\Big)^{1/p}\Big(\sum_{j=N}^{\infty}\sum_{k=1}^{n_j}\|C_\phi(k_{z_{j,k}})\|_{B^p(D)}^{p'}\Big)^{1/p'}\\ &\lesssim \Big(\sum_{j=1}^{N}\sum_{k=1}^{n_j}|\lambda_{j,k;s}|^p\Big)^{1/p}\Big(\int_D\|C_\phi(k_z)\|_{B^p(D)}^{p'}d\lambda(z)\Big)^{1/p'}\\ &\quad+\Big(\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}|\lambda_{j,k;s}|^p\Big)^{1/p}\Big(\int_{D_\delta^c}\|C_\phi(k_z)\|_{B^p(D)}^{p'}d\lambda(z)\Big)^{1/p'}\\ &\le \Big(\|M_{\phi,p}(\cdot)\|_{L^{p'}(D,d\lambda)}+1\Big)\cdot\eta\end{aligned}$$

Since $\eta > 0$ is arbitrary, this implies that $\|C_\phi(f_s)\|_{B^p(D)} \to 0$ as $s \to \infty$. Therefore, C_ϕ is compact on $B^p(D)$ proof of (6) Theorem 1.2 is complete, and so is the proof of Theorem 1.2. □

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