HOLOMORPHIC MAPPINGS BETWEEN PSEUDOELLIPSOIDS IN DIFFERENT DIMENSIONS[*]

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Abstract. We give a necessary and sufficient condition for the existence of nondegenerate holomorphic mappings between pseudoellipsoidal real hypersurfaces, and provide an explicit parametrization for the collection of all such mappings (in the situations where they exist).

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1. Introduction. In recent years, much effort has been devoted to the ambitious program of classifying local holomorphic mappings H (or more generally CR mappings) sending a given real hypersurface $M \subset \mathbb{C}^{n+1}$ into a model hypersurface $M' \subset \mathbb{C}^{N+1}$. In the strictly pseudoconvex case, the natural model is the sphere $M' = \mathbb{S}^{2N+1} \subset \mathbb{C}^{N+1}$ (and, more generally, in the Levi nondegenerate case the model is the hyperquadric of signature l). Of particular interest is the case where the source manifold M is the model itself. There is a large body of work studying local holomorphic mappings H sending a piece of the sphere $\mathbb{S}^{2n+1}$ into $\mathbb{S}^{2N+1}$. The reader is referred to e.g. [29], [18], [19], [11], [10], [21], [22], [23], [12], [24] and the references therein; see also [15], [16] for the case of general strictly pseudoconvex source manifolds M. The classification of such holomorphic mappings in *low codimensions* $N - n$ is completely understood due to the works [18], [22], [23], [24]. In particular, it was shown in [18] that any local holomorphic mapping H sending a piece of $\mathbb{S}^{2n+1}$ into $\mathbb{S}^{2N+1}$ with $N - n < n$ is necessarily of the form $H = T \circ L$, where L denotes the standard linear embedding of $\mathbb{S}^{2n+1}$ into an $(n+1)$-dimensional complex subspace section of $\mathbb{S}^{2N+1}$ and T is an automorphism of $\mathbb{S}^{2N+1}$. (This is often referred to as *rigidity*.) In this note, we shall consider local (non-constant) holomorphic mappings sending a pseudoellipsoid in $\mathbb{C}^{n+1}$ into another pseudoellipsoid in $\mathbb{C}^{N+1}$ in low codimension. A pseudoellipsoid in $\mathbb{C}^{N+1}$ is a compact, algebraic real hypersurface of the form

$$(1) \qquad E_q^N = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \langle z^q, \bar{z}^q \rangle + |w|^2 = 1\}$$

where $q = (q_1, q_2, \dots q_N)$ is an N-tuple of integers with $q_j \geq 1$, $z^q = (z_1^{q_1}, \dots, z_N^{q_N})$ and $\langle \cdot, \cdot \rangle$ denotes the standard bilinear form in $\mathbb{C}^N$,

$$\langle u, v \rangle := \sum_{j=1}^{N} u_j v_j, \quad u, v \in \mathbb{C}^N.$$

We note that E_q^N is weakly pseudoconvex along those coordinate planes $z_j = 0$ for which $q_j \geq 2$, and in particular at the point $p_0 := (0, \dots, 0, 1) \in E_q^N$ (unless all $q_j = 1$

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and E_q^N is the sphere). The pseudoellipsoids are natural models (albeit not homogeneous in general, of course) for certain classes of weakly pseudoconvex hypersurfaces. The domains that they bound are the only ones, up to biholomorphic equivalence, with noncompact automorphism groups in a fairly general class of smoothly bounded pseudoconvex domains [8] (as with the case of the ball in the strictly pseudoconvex category [30]); see also [25]. Biholomorphic equivalence of pseudoellipsoids, their automorphism groups, as well as the existence and geometric properties of (non-constant) local holomorphic mappings between pseudoellipsoids in the equidimensional case (i.e. the source and target are both hypersurfaces in $\mathbb{C}^{n+1}$) have been investigated in e.g. [26], [14], [13], [27], [28]. In particular, the following result follows from these works:

THEOREM 0. *Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ be n-tuples of positive integers. Then there exists a non-constant local holomorphic mapping $H \colon (\mathbb{C}^{n+1}, p_0) \to (\mathbb{C}^{n+1}, p_0)$, with $p_0 := (0, \ldots, 0, 1) \in \mathbb{C}^{n+1}$, sending E_p^n into E_q^n if and only if there exists a permutation $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $q_k \mid p_{\sigma(k)}$ for all $k = 1, \ldots, n$.*

Moreover, if such mappings H exist, then the collection of all such H can be described as follows:

$$(2) \qquad H(z, w) = T \circ (z_{\sigma(1)}^{p_{\sigma(1)}/q_1}, \ldots, z_{\sigma(n)}^{p_{\sigma(n)}/q_n}, w),$$

where σ ranges over all permutations $\{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $q_k \mid p_{\sigma(k)}$, for $k = 1, \ldots, n$, and T over the automorphisms of E_q^n.

A complete and explicit description of the automorphism group of E_q^n also follows from the works mentioned above (see e.g. [26]). For the reader's convenience, we provide in Section 3 a description (or, more precisely, a decomposition into elementary mappings) of the stability group of E_q^n at p_0, i.e. the group of automorphisms of E_q^n preserving p_0.

The main purpose of this note is to extend Theorem 0 to the positive (but low) codimensional situation. We note, for reference, that the defining equation for E_q^N is plurisubharmonic and, hence, by a standard application of the Hopf boundary point lemma it follows that any nonconstant holomorphic mapping sending E_p^n into E_q^N is necessarily transversal to E_q^N. In what follows, we shall only consider holomorphic mappings that are transversal to their target manifolds.

It is convenient to note that E_q^N minus a point is biholomorphically equivalent (via an algebraic transformation) to the real algebraic hypersurface given by

$$(3) \qquad P_q^N = \{(z, w) \in \mathbb{C}^N \times \mathbb{C} : \operatorname{Im} w = \langle z^q, \bar{z}^q \rangle\},$$

with the point $p_0 := (0, \ldots, 0, 1)$ on E_q^N corresponding to the origin on P_q^N. A transformation is given by a branch of

$$(z, w) \mapsto \left(\frac{z_1}{(1+w)^{1/q_1}}, \frac{z_2}{(1+w)^{1/q_2}}, \ldots, \frac{i(1-w)}{1+w} \right),$$

which is biholomorphic in a neighborhood of $E_q^N \setminus \{(0, 0, \ldots, -1)\}$ and sends $E_q^N \setminus \{(0, 0, \ldots, -1)\}$ onto P_q^N.

Our main result is a necessary and sufficient condition for the existence of local holomorphic mappings $H \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$ sending P_p^n transversally into P_q^N, as well as a description of the collection of all such mapping (when they exist). The latter description is also given in an explicit formula in Theorem 2.1 below.

THEOREM 1.1. *Consider $P_p^n \subset \mathbb{C}^{n+1}$ and $P_q^N \subset \mathbb{C}^{N+1}$, where $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_N)$ are an n-tuple and N-tuple, respectively, of positive integers, the latter arranged such that $q_1 = \ldots = q_s = 1$ and $q_k \geq 2$, $k = s+1, \ldots, N$, for some $s \geq 0$. Assume that*

$$(4) \qquad\qquad N - n < n.$$

The following are equivalent:
  (i) *There exist a subset $K \subset \{s+1, \ldots, N\}$ (possibly empty) and a map $\sigma \colon K \to \{1, \ldots, n\}$ such that $\#\sigma(K) \geq n - s$ and $q_k \,|\, p_{\sigma(k)}$ for all $k \in K$.*
  (ii) *There exists a local holomorphic mapping $H \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$ sending P_p^n into P_q^N, transversal to P_q^N at 0.*

Moreover, if (ii) *holds then the collection of all such mappings H can be described as follows. Let σ and K be as in* (i)*, and $W = (u_{ij})$ an $n \times N$ matrix such that*
(a) *$WW^* = I_{n \times n}$, and*
(b) *for every $j \in \{s+1, \ldots, N\}$, it holds that $u_{ij} \neq 0$ if and only if $j \in K$ and $\sigma(j) = i$.*
Then, the mapping $H_{\sigma,W}(z,w) = (F(z), w)$, where

$$(5) \qquad F_j(z) = \begin{cases} \sum_{i=1}^n u_{ij} z_i^{p_i}, & j = 1, \ldots, s, \\ \sqrt[q_j]{|u_{\sigma(j)j}|}\, z_{\sigma(j)}^{p_{\sigma(j)}/q_j}, & j \in K, \\ 0, & j \in \{s+1, \ldots, N\} \setminus K. \end{cases}$$

sends P_p^n transversally into P_q^N, and any mapping H as in (ii) *is of the form $H = T \circ H_{\sigma,W}$ for some σ (and K), W, and T, where T is an automorphism of P_q^N preserving the origin.*

The proof of Theorem 1.1 will be given in Section 3 below.

REMARK 1.2. In the equidimensional case $N = n$, we note that any subset K and mapping σ as in (i) in Theorem 1.1 must be such that $K = \{s+1, \ldots, n\}$ and such that σ can be extended to a permutation $\tilde{\sigma}$ on $\{1, \ldots, n\}$ with $q_k \,|\, p_{\tilde{\sigma}(k)}$ for all $k = 1, \ldots, n$, and vice versa, any such permutation $\tilde{\sigma}$ induces a mapping σ by taking $K := \{s+1, \ldots, n\}$ and $\sigma := \tilde{\sigma}|_K$. If we reorder the coordinates on the source side so that the permutation $\tilde{\sigma}$ becomes the identity, then any $n \times n$ matrix W satisfying (a) and (b) has the block form

$$(6) \qquad\qquad W = \begin{pmatrix} U & 0 \\ 0 & D \end{pmatrix},$$

where U is a unitary $s \times s$ matrix and D is a diagonal $(n-s) \times (n-s)$ matrix whose diagonal elements have modulus one. The corresponding mapping $H_{\sigma,W}$ (where now σ is the identity on $\{s+1, \ldots, n\}$) is then of the form $H_{\sigma,W} := T_W \circ H_0$, where

$$(7) \qquad\qquad H_0(z,w) := (z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n}, w)$$

and T_W is the automorphism of P_q^n given by

$$(8) \qquad\qquad T_W(z,w) = (z'U, z''D, w)$$

with $z' = (z_1, \ldots, z_s)$ and $z'' = (z_{s+1}, \ldots, z_n)$. Returning to the original ordering of the source coordinates, we recover the equidimensional result stated in Theorem

0 above. We notice a redundancy in the statement of the theorem in this case; the additional mappings afforded by the choice of W can be incorporated into the action of the stability group. In the general case, there may also be some redundancy in that $H_{\sigma,W}$ could equal $T \circ H_{\sigma,W'}$ for $W \neq W'$ and a suitable choice of automorphism T, but for $N > n$ there will be (in general) different choices of W such that the corresponding mappings are not related by an automorphism of P_q^N. This is explained more closely in the context of an example in Section 5.

A consequence of Theorem 1.1 (or, more precisely, a consequence of Theorem 2.1 below) is the following "localization principle" (c.f. [13], [27]).

THEOREM 1.3. *Consider $P_p^n \subset \mathbb{C}^{n+1}$ and $P_q^N \subset \mathbb{C}^{N+1}$, where $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_N)$ are an n-tuple and N-tuple, respectively, of positive integers, and assume that $N - n < n$. If a local holomorphic mapping $H \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$ sends P_p^n transversally into P_q^N, then H extends as an algebraic mapping which is holomorphic in a neighborhood of P_p^n.*

The (short) proof of Theorem 1.3 is given in Section 2.

REMARK 1.4. The extendability of H as an algebraic map, possibly singular and multi-valued, follows from previous results due to Huang [20] (see also [31]). In the setting of Theorem 1.3, it follows (see the proof) that, in fact, the $H_k^{q_k}$ are rational, and are (possibly) ramified along a complex hypersurface that does not meet P_p^n.

We shall conclude this introduction with a brief discussion of an analogous non-pseudoconvex situation. Consider the "positive signature" counterparts of P_q^N, i.e. the "pseudohyperboloids" given by

$$(9) \qquad P_{q,\ell}^N = \{(z, w) \in \mathbb{C}^N \times \mathbb{C} : \operatorname{Im} w = \langle z^q, \bar{z}^q \rangle_\ell\},$$

where $\langle \cdot, \cdot \rangle_\ell$ is the standard bilinear form of signature $\ell > 0$, i.e.,

$$(10) \qquad \langle u, v \rangle_\ell := -\sum_{j=1}^{\ell} u_j v_j + \sum_{j=\ell+1}^{N} u_j v_j, \quad u, v \in \mathbb{C}^N.$$

In the Levi nondegenerate case, i.e. $q = (1, \ldots, 1)$, the pseudohyperboloid $P_{q,\ell}^N$ coincides with the standard hyperquadric Q_ℓ^N of signature ℓ. For a local holomorphic mapping $H \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$ sending Q_ℓ^n into Q_ℓ^N, with $\ell > 0$, it is known by [5] that $H = T \circ L$, where L is a linear embedding of Q_ℓ^n into Q_ℓ^N and T an automorphism of Q_ℓ^N, *regardless* of the codimension $N - n$ (i.e. *super-rigidity* holds, in stark contrast to the pseudoconvex case, as in [18], where $N - n < n$ is necessary for rigidity to hold). By following the same arguments as in the pseudoconvex case (modulo replacing the use of the rigidity result in [18] by the super-rigidity result in [5]), one obtains analogous classification results to those in Theorems 1.1 and 2.1 for local mappings $P_{p,\ell}^n \to P_{q,\ell}^N$ with $\ell > 0$, the major difference being that the conditions (4) and (11) in Theorems 1.1 and 2.1, respectively, are no longer needed. In the positive signature case ($\ell > 0$), one needs, however, to distinguish between the coordinates that appear with a plus sign and those that appear with a minus sign in the hermitian form $\langle \cdot, \cdot \rangle_\ell$, which has as a consequence that there are different cases to consider and the results become more cumbersome to state. The diligent reader is invited to work out the details.

2. An explicit formula for mappings from P_p^n into P_q^N. In this section, we shall prove the following result, which is the main ingredient in the proof of Theorem 1.1.

THEOREM 2.1. *Consider $P_p^n \subset \mathbb{C}^{n+1}$ and $P_q^N \subset \mathbb{C}^{N+1}$, where $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_N)$ are an n-tuple and N-tuple, respectively, of positive integers, the latter arranged such that $q_1 = \ldots = q_s = 1$ and $q_k \geq 2$, $k = s+1, \ldots, N$, for some $s \geq 0$. Assume that*

$$(11) \qquad\qquad N - n < n.$$

If $H \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$ is a local holomorphic mapping sending P_p^n into P_q^N, transversal to P_q^N at 0, then there exists
(A) *a subset $K \subset \{s+1, \ldots, N\}$ (possibly empty) and a map $\sigma \colon K \to \{1, \ldots, n\}$ such that $\#\sigma(K) \geq n - s$ and $q_k \mid p_{\sigma(k)}$ for all $k \in K$,*
(B) *a unitary $N \times N$ matrix $U = (u_{ij})$ with the property that, for $j \in \{s+1, \ldots, N\}$, it holds that $u_{ij} \neq 0$ if and only if $j \in K$ and $\sigma(j) = i$, and*
(C) *$r \in \mathbb{R}$, $\lambda > 0$, and $b = (b', b'') \in \mathbb{C}^n \times \mathbb{C}^{N-n}$ with the property that $c := bU$ satisfies $c_k = 0$ for $k = s+1, \ldots, N$,*
such that H takes the following form:

$$(12) \qquad H_j(z, w) = \begin{cases} \lambda \left(\sum_{i=1}^n u_{ij} z_i^{p_i} + c_j w \right) / \delta(z, w), & j = 1, 2, \ldots s; \\ \left(\lambda u_{\sigma(j)j} z_{\sigma(j)}^{p_{\sigma(j)}} \right)^{1/q_j} / \delta(z, w)^{1/q_j}, & j \in K; \\ 0, & j \in \{s+1, \ldots N\} \setminus K; \\ \lambda^2 w / \delta(z, w), & j = N+1, \end{cases}$$

where $\delta(z, w) := 1 - 2i\langle z^p, \bar{b}' \rangle - (r + i|b|^2)w$, and $z^p := (z_1^{p_1}, \ldots z_n^{p_n})$.
Furthermore, if $s = 0$ then $b = 0$, and if $N = n$ then σ can be extended to a permutation $\tilde{\sigma}$ on $\{1, \ldots, n\}$ with $q_k \mid p_{\tilde{\sigma}(k)}$ for all $k = 1, \ldots, n$.

Proof. We introduce the map $\tilde{\phi}_q(\tilde{z}, \tilde{w}) = (\tilde{z}_1^{q_1}, \ldots \tilde{z}_N^{q_N}, \tilde{w})$, which is a holomorphic mapping sending P_q^N into the (Heisenberg) sphere $\mathbb{H}^N := P_{(1,\ldots,1)}^N$. Thus, $\tilde{\phi}_q \circ H$ is a non-constant mapping from a neighborhood of 0 in P_p^n into $\mathbb{H}^N$. We also introduce $\phi_p(z, w) = (z_1^{p_1}, \ldots z_n^{p_n}, w)$, which is a holomorphic mapping sending P_p^n into $\mathbb{H}^n$. Let $a \in P_p^n$ be some point near 0 whose coordinate components do not vanish, and U a neighborhood of a such that $U \cap \{z_j = 0\} = \emptyset$ for all $j = 1, \ldots, n$. We can also choose U small so that ϕ_p is biholomorphic on U.

Now, let $\tau \in \mathrm{Aut}(\mathbb{H}^n)$ be such that $\tau(\phi_p(a)) = 0$ and $T \in \mathrm{Aut}(\mathbb{H}^N)$ such that $T(0) = \phi_q(H(a))$. Consider the following mapping defined on $\hat{U} = \tau(\phi_p(U))$

$$(13) \qquad\qquad \hat{H} = T^{-1} \circ \tilde{\phi}_q \circ H \circ \phi_p^{-1} \circ \tau^{-1}.$$

Clearly, $\hat{H}(\hat{U} \cap \mathbb{H}^n) \subset \mathbb{H}^N$ and $\hat{H}(0) = 0$. Since $N - n < n$, we can apply the rigidity theorem in [18], mentioned in the introduction, to conclude that there is an automorphism $\hat{T} \in \mathrm{Aut}(\mathbb{H}^N, 0)$ such that $\hat{H} = \hat{T} \circ L$ with $L(z, w) = (z, 0, w)$. This implies that the following holds on U and, by analytic continuation, in any connected open set containing a where H is defined (in particular, in an open neighborhood of 0):

$$(14) \qquad\qquad \tilde{\phi}_q \circ H = T \circ \hat{T} \circ L \circ \tau \circ \phi_p.$$

Since the non-constant mapping $T \circ \hat{T} \circ L \circ \tau$ sends $\mathbb{H}^n$ into $\mathbb{H}^N$ and 0 into 0, it follows that $T \circ \hat{T} \circ L \circ \tau = T' \circ L$ for some $T' \in \mathrm{Aut}(H^N, 0)$, and hence:

$$(15) \qquad \tilde{\phi}_q \circ H = T' \circ L \circ \phi_p.$$

It follows from the explicit description of $\mathrm{Aut}(\mathbb{H}^N, 0)$ (see, e.g., [9] or [2]) that there are $\lambda > 0$, $r \in \mathbb{R}$, $b \in \mathbb{C}^N$ and a unitary $N \times N$-matrix $U = (u_{ij})$ (i.e. $UU^* = U^*U = I$) such that

$$(16) \qquad H_{N+1}(z, w) = \lambda^2 w / \delta(z, w)$$

$$(17) \qquad H_j^{q_j}(z, w) = \lambda \left(\sum_{i=1}^n u_{ij} z_i^{p_i} + c_j w \right) / \delta(z, w), \quad \text{for} \quad j = 1, 2, \ldots N,$$

where

$$(18) \qquad c_j = \sum_{i=1}^N u_{ij} b_i \quad \text{for} \quad j = 1, 2, \ldots N$$

$$(19) \qquad \delta(z, w) = 1 - 2i \langle z^p, \bar{b}' \rangle - (r + i|b|^2)w, \quad b = (b', b'') \in \mathbb{C}^n \times \mathbb{C}^{N-n}.$$

Recall that $q_1 = \ldots q_s = 1$ and $2 \le q_{s+1} \le \cdots \le q_N$. By setting $z = 0$ in (17), we have

$$(20) \qquad \left(1 - (r + i|b|^2)w \right) H_j^{q_j}(0, w) = \lambda c_j w.$$

If $c_j \ne 0$, then $H_j^{q_j}(0, w)$ divides w in $\mathbb{C}\{w\}$. This is impossible if $q_j > 1$. Thus, $c_j = 0$ for $j = s + 1, \ldots N$. Let us define

$$(21) \qquad K = \{k \ge s + 1 \mid u_{tk} \ne 0 \text{ for some } 1 \le t \le n\}.$$

For $k \in K$, we claim that there is a unique $t^* \in \{1, 2, \ldots n\}$ such that $u_{t^* k} \ne 0$ and $u_{tk} = 0$ for all $1 \le t \le n$, $t \ne t^*$. To prove the claim, suppose that there are two indices t, say $t = 1, 2$, such that $u_{tk} \ne 0$. Setting $z_3 = \cdots = z_n = w = 0$ in equation (17) we would obtain

$$(22) \qquad (1 - 2ib_1 z_1^{p_1} - 2ib_2 z_2^{p_2}) H_k^{q_k}(z_1, z_2, 0) = \lambda(u_{1k} z_1^{p_1} + u_{2k} z_2^{p_2}),$$

which is impossible. Indeed, by differentiating both sides of (22) with respect to z_1 we note that $H_k^{q_k - 1}(z_1, z_2, 0) \mid z_1^{p_1 - 1}$. The same argument with z_1 replaced by z_2 shows that $H_k^{q_k - 1}(z_1, z_2, 0) \mid z_2^{p_2 - 1}$, which would lead to a contradiction since $q_k - 1 \ge 1$ and H_k is not an unit. The claim follows. We now define a map $\sigma \colon K \to \{1, 2, \ldots n\}$ by $\sigma(k) = t^*$. Thus, we note that $H_k \equiv 0$ for $k \in \{s + 1, \ldots, N\} \setminus K$ and

$$(23) \qquad H_k^{q_k}(z, w) = \lambda u_{\sigma(k)k} z_{\sigma(k)}^{p_{\sigma(k)}} / \delta(z, w), \quad \text{for} \quad k \in K.$$

From this it readily follows that $q_k \mid p_{\sigma(k)}$. We conclude that

$$(24) \qquad H_k(z, w) = \lambda v_{\sigma(k)} z_{\sigma(k)}^{p_{\sigma(k)}/q_k} / \delta(z, w)^{1/q_k}, \quad \text{for} \quad k \in K.$$

where $v_{\sigma(k)}^{q_k} = u_{\sigma(k)k}$.

To show that $\#\sigma(K) \ge n - s$, we shall need the following lemma, whose proof is deferred to Section 4.

LEMMA 2.2. *Let $H\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{N+1},0)$ be a holomorphic mapping sending P_p^n to P_q^N, transversal to P_q^N at 0. Then H is finite at 0, i.e. the ideal $\mathcal{I}(H)$ generated by the components of H has finite codimension in $\mathbb{C}\{z,w\}$.*

We observe from (16) that H is transversal to P_q^N at 0, which is well known to be equivalent to $w \in \mathcal{I}(H)$ (see e.g. [17] for a general discussion). Thus, by Lemma 2.2, $\mathcal{I}(H)$ has finite codimension in $\mathbb{C}\{z,w\}$. Since $w \in \mathcal{I}(H)$, we also have

$$\dim \mathbb{C}\{z,w\}/(H) = \dim \mathbb{C}\{z\}/\mathcal{I}(h), \tag{25}$$

where $h(z) := H(z,0)$. Since $h_k \equiv 0$ for $j \in \{s+1, \ldots N\} \setminus K$, $h_{N+1} \equiv 0$, and $h_k(z)$, for $k \in K$, differs from $z_{\sigma(k)}^{p_{\sigma(k)}/q_k}$ only by a unit in $\mathbb{C}\{z\}$, we conclude that $\mathcal{I}(h)$ is contained in the ideal generated by $h_1, \ldots, h_s$ and $z_{\sigma(k)}$ for $k \in K$. Since the latter ideal must have finite codimension, it follows that $s + \#\sigma(K) \geq n$, or equivalently, $\#\sigma(K) \geq n - s$. This proves the existence of the subset $K \subset \{s+1, \ldots, N\}$ and the mapping $\sigma \colon K \to \{1, \ldots, n\}$ possessing the properties claimed in (A) of Theorem 2.1. The existence of $r \in \mathbb{R}$, $\lambda > 0$, $b \in \mathbb{C}^N$ and a unitary matrix $U = (u_{ij})$ was established above. The property in (B) of U and that in (C) of b were also established.

If $s = 0$, then we have $bU = 0$, and since U is invertible, we deduce that $b = 0$. If $N = n$, then (A) immediately implies that $K = \{s+1, \ldots, n\}$ and $\#\sigma(K) = n - s$, which in turn implies that σ is injective. It is clear that σ can be extended to a permutation. $\square$

Proof of Theorem 1.3. It follows immediately from Theorem 2.1 that H is algebraic; in fact, H^{q_k} for $k = 1, \ldots, N$ and H_{N+1} are all rational with poles along the complex hypersurface $\delta(z,w) = 0$ (unless $b = 0$ and $r = 0$, in which case H is a polynomial mapping). To complete the proof of Theorem 1.3 it remains to verify that $\delta(z,w)$ does not vanish along P_p^n (i.e. for $w = u + i\langle z^p, \bar{z}^p\rangle$). This is straightforward and left to the reader. $\square$

3. Stability group of P_q^N at the origin. A special case of Theorem 2.1 is when $P_p^n = P_q^N$. In this case, Theorem 2.1 describes the stability group of P_q^N at 0 (i.e. the group of automorphisms of P_q^N preserving the origin), denoted by $\mathrm{Aut}(P_q^N,0)$. This description is previously known due to work mentioned in the introduction. In this section, however, we shall (for the reader's convenience) use the formulae in Theorem 2.1 to provide a decomposition of the automorphisms of P_q^N fixing the origin into simpler ones. First, we note that when $P_p^n = P_q^N$ the subset K in (A) must equal $\{s+1, \ldots, N\}$ and σ is a permutation of K such that $q_{\sigma(k)} = q_k$ for all $k \in K$. Also, the unitary matrix U in (B) must have the block form

$$W = \begin{pmatrix} \tilde{U} & 0 \\ 0 & E \end{pmatrix}, \tag{26}$$

where $\tilde{U}$ is a unitary $s \times s$ matrix and E is a unitary $(N-s) \times (N-s)$ matrix (such that after reordering the coordinates $(z_{s+1}, \ldots, z_N)$ on the source side according to the permutation σ, the matrix E becomes diagonal with diagonal elements of modulus one). It then also follows that $b \in \mathbb{C}^N$ in (C) is of the form $b = (\beta, 0) \in \mathbb{C}^s \times \mathbb{C}^{N-s}$. (Recall that if $s = 0$, then $b = 0$.) For each permutation σ of $K = \{s+1, \ldots N\}$ such that $q_{\sigma(k)} = q_k$ for all $k \in K$, we define

$$\Sigma_\sigma(z,w) = (z_1, \ldots, z_s, z_{\sigma(s+1)}, \ldots z_{\sigma(N)}, w). \tag{27}$$

Also, for each $\lambda > 0$, we define the (non-isotropic) dilation Δ_λ,

$$(28) \qquad \Delta_\lambda(z, w) = (\lambda z_1, \dots, \lambda z_s, \lambda^{1/q_{s+1}} z_{s+1} \dots, \lambda^{1/q_N} z_N, \lambda^2 w),$$

and, for each $b = (\beta, 0) \in \mathbb{C}^s \times \mathbb{C}^{N-s}$ (with $b = 0$ if $s = 0$) and $r > 0$, we define
(29)

$$\Psi_{b,r}(z, w) = \left(\frac{z_1 + \beta_1 w}{\delta(z, w)}, \dots, \frac{z_s + \beta_s w}{\delta(z, w)}, \frac{z_{s+1}}{\delta(z, w)^{1/q_{s+1}}}, \dots, \frac{z_N}{\delta(z, w)^{1/q_N}}, \frac{w}{\delta(z, w)} \right),$$

where as above $\delta(z, w) = 1 - 2i\langle z^q, \bar{b} \rangle - (r + i|b|^2)w$. Finally, for each unitary $s \times s$ matrix $\tilde{U}$ and $\theta_{s+1}, \dots, \theta_N \in \mathbb{R}$, we define

$$(30) \qquad \Lambda_{\tilde{U}, \theta}(z, w) = ((z_1, \dots z_s)\tilde{U}, e^{i\theta_{s+1}} z_{s+1}, \dots, e^{i\theta_N} z_N, w).$$

It is readily seen from Theorem 2.1 that $\Sigma_\sigma, \Delta_\lambda, \psi_{b,r}, \Lambda_{\tilde{U}, \theta} \in \mathrm{Aut}(P_q^N, 0)$, and it is straightforward (and left to the reader) to check, using Theorem 2.1, that these elementary mappings generate $\mathrm{Aut}(P_q^N, 0)$ via compositions; we mention here that $\mathrm{Aut}(P_q^N, 0)$ is a finite dimensional Lie group (see [6]).

THEOREM 3.1. *The stability group of P_q^N at 0 consists of mapping of the form*

$$(31) \qquad T = \Delta_\lambda \circ \Lambda_{\tilde{U}, \theta} \circ \Psi_{b,r} \circ \Sigma_\sigma,$$

for $\tilde{U}, \theta, b, r, \lambda, \sigma$ as described above. Furthermore, the identity component $\mathrm{Aut}_{\mathrm{Id}}(P_q^N, 0)$ consists of mapping of form (31) in which $\sigma = \mathrm{Id}$. In fact, each choice of σ gives rise to a connected component of $\mathrm{Aut}(P_q^N, 0)$.

REMARK 3.2. A similar decomposition for CR mappings between connected pieces of generalized pseudoellipsoids was also given is the recent paper [28] using a very different method.

We shall now give a proof of Theorem 1.1.

Proof of Theorem 1.1. The implication (ii) $\Longrightarrow$ (i) follows from Theorem 2.1 (A). Moreover, any mapping H as in (ii) is of the form described in Theorem 2.1. It is straightforward (and left to the reader) to check that there are σ and W as in Theorem 1.1, automorphisms $\Delta_\lambda, \Lambda_{I_{s \times s}, \theta}$ and $\Psi_{c,r}$ as in Theorem 3.1, with the parameters λ, c and r as in Theorem 2.1, such that

$$(32) \qquad H = \Delta_\lambda \circ \Lambda_{I_{s \times s}, \theta} \circ \Psi_{c,r} \circ H_{W, \sigma}.$$

Next, assume that (i) holds. We shall construct a transversal map $H : P_p^n \to P_q^N$ as follows. If $K = \emptyset$, then necessarily $s \geq n$. In this case, we can simply take

$$(33) \qquad H(z, w) = (z_1^{p_1}, \dots z_n^{p_n}, 0, \dots 0, w).$$

Suppose now that K is nonempty. Since $\#\sigma(K) \geq n - s$, we can write

$$\{1, 2, \dots, n\} \setminus \sigma(K) = \{t_1, t_2, \dots t_r\}$$

for some $r \leq s$ (with $r = 0$ if $\sigma(K) = \{1, \dots, n\}$). We define a transversal (to P_q^N at 0) map $H(z, w) = (F(z), w)$ with

$$(34) \qquad F_k(z) = \begin{cases} z_{t_k}^{p_{t_k}}, & k = 1, 2, \dots r, \\ v_k z_{\sigma(k)}^{p_{\sigma(k)}/q_k}, & k \in K \\ 0 & \text{otherwise,} \end{cases}$$

where the coefficients v_k are chosen such that, for every $l \in \sigma(K)$,

$$(35) \qquad\qquad \sum_{k \in \sigma^{-1}(l)} |v_k|^{2q_l} = 1.$$

It is easy to check that H sends P_p^n into P_q^N. The proof is complete. $\square$

4. Proof of Lemma 2.2. Recall (see [3]) that given a real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ and $p \in M$, there are so-called normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, vanshing at p, such that M is given by

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w),$$

where $\phi(z, 0, s) = \phi(0, \chi, s) \equiv 0$, or in complex form

$$(36) \qquad\qquad w = Q(z, \bar{z}, \bar{w}),$$

where Q satisfies $Q(0, \chi, \tau) \equiv Q(z, 0, \tau) \equiv \tau$ and the reality condition

$$(37) \qquad\qquad Q(z, \bar{z}, \bar{Q}(\bar{z}, z, w)) \equiv w.$$

We note that P_p^n and P_q^N are already presented in normal coordinates. It is convenient to use the complex defining equation (36) to define the notions of essential finiteness and essential type as follows. We replace $\bar{z}, \bar{w}$ by independent variables χ, τ and write

$$Q(z, \chi, 0) = \sum_{I \in \mathbb{N}^n} q_I(z) \chi^I.$$

Let $\mathcal{I}_M$ be the ideal in $\mathbb{C}[[z]]$ generated by $\{q_I(z)\}_{I \in \mathbb{N}^n}$. Following Baouendi, Jacobowitz and Treves (see [3]), we shall say that M is *essentially finite* at p if $\mathcal{I}_M$ is of finite codimension in $\mathbb{C}[[z]]$. The dimension $\dim_{\mathbb{C}} \mathbb{C}[[z]]/\mathcal{I}_M$ is a biholomorphic invariant of M and is called the *essential type* of M at p, denoted by $\operatorname{ess type}_p M$. We note that e.g. $\mathcal{I}_{P_p^n}$ is generated by $z_1^{p_1}, \dots, z_n^{p_n}$ and therefore P_p^n is essentially finite at 0. Recall also that a germ of a holomorphic mapping $H \colon (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{N+1}, p')$ is said to be *finite* at p if the ideal $\mathcal{I}(H)$ generated by the components of H in the ring $\mathcal{O}_p$ of germs of holomorphic functions at p is of finite codimension. In this case, we shall refer to this codimension as the *multiplicity* of H at p,

$$\operatorname{mult}_p H := \dim_{\mathbb{C}} \mathcal{O}_p / \mathcal{I}(H).$$

It is well known (see e.g. [1]) that if H is finite at p, then for every q close p the number of preimages $m := H^{-1}(H(q))$ is finite and $m \leq \operatorname{mult}_p H$. (In the equidimensional case $N = n$, the generic number of preimages equals $\operatorname{mult}_p H$, but in general we only have the inequality). Now we can state and prove the following result, which in view of the above comments regarding P_p^n proves Lemma 2.2. We remark that the equidimensional case in the proposition below follows from [7, Theorem 3].

PROPOSITION 4.1. *Let M and M' be real-analytic hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$ respectively and let $p \in M$, $p' \in M'$. Suppose that $H : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{N+1}, p')$ is a germ of holomorphic mapping sending (M, p) into M'. If M is essentially finite at p and H is transversal to M' at $p' = H(p)$, then H is finite and*

$$(38) \qquad\qquad \operatorname{mult}_p H \leq \operatorname{ess type}_p M.$$

Proof. Suppose that M and M' are given in normal coordinates $Z = (z, w)$ and $Z' = (z', w')$, vanishing at p and p' respectively, by complex defining functions ρ and ρ' of the forms:

$$\rho(z, w, \bar{z}, \bar{w}) = w - Q(z, \bar{z}, \bar{w}), \quad \rho'(z', w', \bar{z}', \bar{w}') = w' - Q'(z', \bar{z}', \bar{w}').$$

Since H sends M into M', the following holds for some real-analytic function $a(Z, \xi)$.

$$(39) \qquad G(Z) - Q'(F(Z), \bar{H}(\xi)) = a(Z, \xi)(w - Q(z, \xi)).$$

Here, $H = (F, G)$ with $F = (F_1, \ldots, F_N)$. By setting $\xi = 0$, taking into account that $\bar{H}(0) = 0$, $Q(z, 0, 0) \equiv 0$ and $Q'(z', 0, 0) \equiv 0$ we deduce that

$$(40) \qquad G(Z) = a(Z, 0)\, w.$$

Setting $w = \tau = 0$ and observing from (40) that $G(z, 0) \equiv 0$ and $\bar{G}(\chi, 0) \equiv 0$, we get

$$(41) \qquad Q'(F(z, 0), \bar{F}(\chi, 0), 0) = a(z, 0, \chi, 0) \cdot Q(z, \chi, 0).$$

Since H is transversal, we have $a(0) \neq 0$ (see e.g. [4]). Therefore, $a(z, 0, \chi, 0)$ is non-vanishing for (z, χ) close to zero and hence

$$(42) \qquad a(z, 0, \chi, 0)^{-1} \cdot Q'(F(z, 0), \bar{F}(\chi, 0), 0) = Q(z, \chi, 0).$$

We expand

$$(43) \qquad Q(z, \chi, 0) = \sum_I q_I(z)\, \chi^I.$$

Let $\mathcal{I}_M$ and $\mathcal{I}(F)$ be the ideals in $\mathbb{C}[[z]]$ generated by $\{q_I(z) \colon I \in \mathbb{N}^n\}$ and $\{F_j(z, 0) \colon j = 1, \ldots N\}$, respectively. We claim that

$$(44) \qquad \mathcal{I}_M \subset \mathcal{I}(F).$$

Indeed, for each multi-index $I \in \mathbb{N}^n$, one has from (42) that

$$(45) \qquad q_I(z) = \frac{1}{I!} \frac{\partial^I}{\partial \chi^I} \left(a(z, 0, \chi, 0)^{-1} \cdot Q'(F(z, 0), \bar{F}(\chi, 0), 0) \right) \Big|_{\chi = 0}.$$

If we expand

$$(46) \qquad Q'(z', \chi', 0) = \sum_J q'_J(z)\, (\chi')^J,$$

then it is clear from (45) that $q_I(z)$ belongs to the ideal generated by the $q'_J(F(z, 0))$, $J \in \mathbb{N}^N$, which in turn belongs to the ideal $\mathcal{I}(F)$ (since the ideal $\mathcal{I}_{M'}$, generated by the $q'_J(z')$, of course is contained in the maximal ideal). Therefore, we obtain (44). Furthermore, since M is essentially finite, $\mathcal{I}_M$ is of finite codimension in $\mathbb{C}[[z]]$ and so is $\mathcal{I}(F)$, by (44), and hence $F(z, 0)$ is finite. Moreover,

$$(47) \qquad \mathrm{mult}_0(F(\cdot, 0)) = \dim_{\mathbb{C}} \mathbb{C}[[z]]/\mathcal{I}(F) \leq \dim_{\mathbb{C}} \mathbb{C}[[z]]/\mathcal{I}_M = \mathrm{ess\, type}_0(M).$$

On the other hand, it follows from (40) and the invertibility of $a(Z, 0)$ that $w \in \mathcal{I}(H)$ and, hence, H is also finite and

$$(48) \qquad \mathrm{mult}_0(H) = \mathrm{mult}_0(F(\cdot, 0)).$$

From (47) and (48), we obtain (38). $\square$

5. An example. As mentioned in Remark 1.2, there is some redundancy in general in Theorem 1.1; it may happen that $H_{\sigma,W} = T \circ H_{\sigma',W'}$ for $W \neq W'$, $\sigma \neq \sigma'$, and a suitable automorphism T. In the equidimensional case, the collection of all possible maps $H_{\sigma,W}$, for a given σ, is formed by the single orbit of one such map H_{σ,W_0} under the action of the identity component $\mathrm{Aut}_{\mathrm{Id}}(P_q^N, 0)$ of the stability group, i.e. any $H_{\sigma,W}$ is of the form $T \circ H_{\sigma,W_0}$ for some $T \in \mathrm{Aut}_{\mathrm{Id}}(P_q^N, 0)$. (The orbit under the action of the full stability group has, potentially, several components corresponding to different permutations ρ of $\{s+1, \ldots, N\}$ such that $q_{\rho(k)} = q_k$; see Section 3.)

In this section, we shall give an example illustrating (hopefully) the general principle behind why both parameters σ and W in Theorem 1.1 are needed in general. In particular, for a given σ, the orbit of single H_{σ,W_0} under the action of the stability group $\mathrm{Aut}(P_q^N, 0)$ is in general "smaller" (in fact, lower dimensional) than the collection of all maps $H_{\sigma,W}$.

EXAMPLE 5.1. Let $M \subset \mathbb{C}^4$ and $M' \in \mathbb{C}^6$ be the hypersurface given by
(49)
$$M = \{\mathrm{Im}\, w = |z_1|^4 + |z_2|^8 + |z_3|^{12}\}, \quad M' = \{\mathrm{Im}\, w' = |z_1'|^2 + |z_2'|^2 + |z_3'|^2 + |z_4'|^4 + |z_5'|^4\}.$$

Thus, in this example $n = 3$, $N = 5$, and $s = 3$. In particular, $N - n = 2 < 3 = n$ and hence Theorem 1.1 applies.

(a) Consider $K = \{4, 5\}$ and $\sigma(4) = \sigma(5) = 3$, and so $\#\sigma(K) = 1 > 0 = n - s$. For $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 1$, consider the following $n \times N$ matrix

(50)
$$W_{a,b,c} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & a & b \end{pmatrix},$$

which satisfies the requirements (a) and (b) in Theorem 1.1. The corresponding mapping is of the form

$$H_{\sigma,W_{a,b,c}}(z, w) = (z_1^2, z_2^4, cz_3^6, \sqrt{a}z_3^3, \sqrt{b}z_3^3, w).$$

It is easy to check, using Theorem 3.1, that the orbits of $H_{\sigma,W_{a,b,c}}$ are disjoint for distinct values of (a, b, c). Indeed, if $H' = T \circ H_{\sigma,W_{a,b,c}}(z, w)$ for some $T \in \mathrm{Aut}(M', 0)$ then

$$\left| \frac{H_4'(z, w)}{H_5'(z, w)} \right| = \sqrt{\frac{a}{b}} \quad \text{or} \quad \sqrt{\frac{b}{a}}.$$

Thus, for any two choices of values (a, b, c) such that the (un-ordered) pairs of values $\left\{\sqrt{a/b}, \sqrt{b/a}\right\}$ are different, the orbits must be disjoint.

(b) Consider the two mappings $H_{\sigma,W}$ and $H_{\sigma',W'}$, where $K = \{4, 5\}$,

$$\sigma(4) = 1, \ \sigma(5) = 2, \qquad \sigma'(4) = 2, \ \sigma'(5) = 3,$$

and

(51)
$$W := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad W' := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The corresponding mappings are of the form

(52)
$$H_{\sigma,W}(z, w) = (z_3^6, 0, 0, z_1, z_2^2, w)$$
$$H_{\sigma',W'}(z, w) = (z_1^2, 0, 0, z_2^2, z_3^3, w)$$

Observe that any mapping H in the orbit of $H_{\sigma,W}$ has the property that $H_4 H_5$ vanishes identically on the hyperplane $\{z_1 = 0\}$, while none in the orbit of $H_{\sigma',W'}$ has this property.

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