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002

SEMIGROUP-THEORETICAL APPROACH TO HIGHER ORDER NONLINEAR EVOLUTION EQUATIONS*

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Abstract. We are concerned with application of the semigroup theory to the higher order nonlinear evolution equations. First, we show some necessary conditions for the accretivity of matrices of nonlinear operators in Banach spaces in relation to the underlying phase spaces and domains of operators. Then, we obtain a condition for a matrix of linear operators to generate an analytic semigroup. These results are, finally, applied to Cauchy problems of nonlinear and quasilinear evolution equations of higher order.

Key words. Accretivity of operator matrices, analytic semigroup, nonlinear higher order evolution equation, pseudo-hyperbolic system.

AMS subject classifications. 34G20, 35L82, 47D03, 47H20.

1. Introduction.

Linear abstract evolution equations have been studied widely (cf. [10], [12], [13], [16], [19], [20], [22], [27], [28], [29]), while nonlinear evolution equations are considered mainly within the scope of the 2-nd order (cf. [1], [3], [4], [7], [15], [18], [23]) and in some papers nonexistence of solution for higher order semilinear evolution equations was studied (cf. [2] and the references therein).

One of the conventional approach to the higher order equations is to reduce them to the first order systems in suitable phase spaces and then to use the operator semigroup theory (cf. the references in [12]). In [5] and [8], a wave equation and a linear parabolic equation of higher order in time, respectively, were reduced to the systems of order one and the analytic semigroup theory was applied for the obtained systems. However, it is generally difficult to find an ideal underlying phase space and the structure of the phase space may be complicated (see preface in [29]). Therefore, for the case of linear equations other techniques are widely used (cf. [29] and the references therein). But, many techniques utilized for linear equations are not applicable for nonlinear equations.

In this paper, we are concerned with the application of the semigroup theory to the higher order nonlinear evolution equations. To this end, we first show some necessary conditions on the phase spaces and operators acting on derivatives of unknown function for accretivity of the matrices of operators in the systems reduced from a simple higher order equations

$$u^{(n)} + A_{n-1}u^{(n-1)} + \cdots + A_0 u = f(t),\ n \geq 2. \tag{1.1}$$

And we get some sufficient conditions for matrices of operators to generate nonlinear one-parameter groups or semigroups. Also, we get a sufficient condition for a matrix of operators to generate an analytic semigroup, which is different from one in [5] and

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[8]. The results are applied to Cauchy problems of higher order nonlinear evolution equations. The results of this paper were published in 1990 as preprints of the Institute of Mathematics of the Academy of Sciences in DPR of Korea, but seem to be known to only a few. The main results of this paper are Theorems 2.1∼2.5 and 4.1.

This paper consists of four sections.

In Section 2 we study the equation (1.1) in a Banach space E, where $A_i,\ i = 0 \sim n-1$, are usually nonlinear and the derivatives are with respect to t. Long ago it was known that when a linear equation of order 2 in E is reduced to a system of order one, for well-posedness the product space $V \times E$ is useful instead of $E \times E$, where the norm of V is stronger than one of E (cf. [13]). On the other hand, the derivatives of function usually are in spaces weaker than one where the function belongs to.

Thus, concerning with $u^{(n)}$ in E, we naturally assume

ASSUMPTION 1.1. *The Banach spaces $V_i, i = 1 \sim n-1$, and E satisfy the following condition*

$$V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_{n-1} \hookrightarrow E, \tag{1.2}$$

where $\hookrightarrow$ denotes dense and continuous embedding.

And $\mathscr{D}(A_i)$, $i = 1 \sim n-1$, and $\mathscr{D}(A_0)$ are dense subsets of Banach spaces V_i and V_1, respectively.

Our first interest is to get necessary conditions for accretivity of the matrices of operators in the system reduced from (1.1). Our result shows that V_i, $i = 1 \sim n-1$, must be same (Theorem 2.1) and the operators A_{n-1} and A_{n-2} must satisfy some estimations (Theorems 2.3 ∼ 2.7).

In Section 3 considering the results in Section 2 and relying on the nonlinear semigroup theory and monotone operator theory, we study three kinds of Cauchy problems for higher order nonlinear evolution equations, which are equations with the perturbation operators of the main part A_{n-1} and A_{n-2}. To this end, we obtain some sufficient conditions for the matrices of operators to generate nonlinear semigroups or groups.

In Section 4 the semilinear and quasilinear equations which are not included in the scope of Section 3 are studied. To this end, first, a condition for a matrix of linear operators on Hilbert spaces to generate an analytic semigroup is obtained. Paying attention to role of the operator A_{n-1} and structure of the matrix, and using the space $V^{n-1} \times V^*$ as a representation of the dual space of V^n, we obtain the result without assumptions of self-adjoint property and positive-definiteness of operators (Theorem 4.1). Combining this result with results in [21] and [24], we study Cauchy problems of semilinear and quasilinear evolution equations of higher order. To compare with previous results, we apply our abstract results to the systems of pseudo-hyperbolic partial differential equations. Owing to Theorem 4.1, unlike [25] in study of the systems of pseudo-hyperbolic partial differential equations, symmetry of coefficient matrices is removed out.

We use the following notation.

When X is a space and X^* is its dual space, $(\cdot\,,\cdot)_X$ is inner product in X, $\langle\cdot,\cdot\rangle_X$ is duality product between X and its dual space X^* and $\|\cdot\|_X$ is the norm in X. Sometimes $\langle\cdot,\cdot\rangle_i$ means duality product when $X = V_i$, and so is it for norms. For $u_i^1,\, u_i^2 \in X$ let $u_i = u_i^1 - u_i^2$. For an operator A, $\mathscr{D}(A)$ is its domain and $\mathscr{R}(A)$ its range.

2. Necessary conditions for accretivity of operator matrices. Let $V_i, i = 1, \dots, n-1$, and E be real Banach spaces and (1.2) hold. Let nonlinear operators $A_i, i = 1, \dots, n-1$, in the Banach space E have domains $\mathscr{D}(A_i)$ and $\mathscr{D}(A_0)$ which are dense in V_i and V_1, respectively.

We consider the matrix of operators

$$\mathbb{A} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ & \cdots & & \cdots & \\ A_0 & A_1 & A_2 & \cdots & A_{n-1} \end{pmatrix}, \quad \mathscr{D}(\mathbb{A}) = \prod_{i=0}^{n-1} \mathscr{D}(A_i)$$

in the Banach space

$$\mathbb{X} = V_1 \times \cdots \times V_{n-1} \times E.$$

endowed with the norm $\| \cdot \|_{\mathbb{X}}^2 = \sum_{i=1}^{n-1} \| \cdot \|_{V_i}^2 + \| \cdot \|_E^2$.

Note that by the introduction of the matrix of operators $\mathbb{A}$ the abstract higher order evolution equation

$$u^{(n)} + A_{n-1}u^{(n-1)} + \cdots + A_0 u = f(t) \ (n \geq 2) \quad \text{in } E \tag{2.1}$$

is reduced to the first order system with a new unknown function $\mathbb{U} = (u, u_1, \dots, u_{n-1})^{\mathrm{T}}$:

$$\mathbb{U}' + \mathbb{A}\mathbb{U} = F(t) \quad \text{in } \ \mathbb{X}, \tag{2.2}$$

where $F(t) = (0, \dots, 0, f(t))^{\mathrm{T}}$.

Definition 2.1. *(cf. 13.2 in [9]) Let X be a real Banach space and $J : X \to 2^{X^*}$ its duality map. The operator $F : \mathscr{D}(F) \subset X \to X$ is said to be accretive if*

$$\max\{\langle Fx - Fy, j(x-y)\rangle_X : j(x-y) \in J(x-y)\} \geq 0 \quad \forall x, y \in \mathscr{D}(F).$$

If X^* is strictly convex, then J is one-value, and so it is said to be duality operator. (cf. Lemma 5.5, ch. 1 in [7])

Theorem 2.1. *Let Assumption 1.1 be valid, V_k, $k = 1, 2, \cdots, n-2$, be reflexive and V_k^* be strictly convex. If the operator $\mathbb{A} + \omega$ is accretive in $\mathbb{X}$ for some real number $\omega \in \mathcal{R}$, then the spaces V_i, $i = 1, 2, \cdots, n-1$, are the same with equivalent norms, that is, $\mathbb{X} = V^{n-1} \times E$, with a real Banach space V.*

Proof. Let $\mathbb{U} = \mathbb{U}^1 - \mathbb{U}^2$ for $\mathbb{U}^1, \mathbb{U}^2 \in \mathscr{D}(\mathbb{A})$. Let $V_n := E$. Note that if j_k, $k = 1, 2, \cdots, n$, are duality operators from V_k to V_k^*, then the operator $j = (j_1, \cdots, j_n)$ defined by $j\mathbb{U} = \{j_1 u_0, \cdots, j_n u_{n-1}\}$ is the duality operator from $\mathbb{X}$ to $\mathbb{X}^*$, where $u_0 = u$. Then, we have

$$\begin{aligned} I &\equiv \left\langle (\mathbb{A}+\omega)\mathbb{U}^1 - (\mathbb{A}+\omega)\mathbb{U}^2, j\mathbb{U}\right\rangle_{\mathbb{X}} = \\ &= \sum_{i=0}^{n-2} \left(-\langle u_{i+1}, j_{i+1}u_i\rangle_{i+1} + \omega\|u_i\|_{i+1}^2\right) + \sum_{i=0}^{n-1} \left\langle A_i u_i^1 - A_i u_i^2, j_n u_{n-1}\right\rangle_n + \omega\|u_{n-1}\|_n^2. \end{aligned} \tag{2.3}$$

Let us prove the equivalence of the norms of $V_k, \ k = 1, \dots, n-1$, by contradiction argument. Assume that for some $k \in \{1, \dots, n-2\}$ the norms of V_k and V_{k+1} are not equivalent. Then, putting $u_i^1 = u_i^2$ for $i \neq k-1$, k, in (2.3), we have that

$$I = -\left\langle u_k, j_k u_{k-1}\right\rangle_k + \omega\|u_{k-1}\|_k^2 + \omega\|u_k\|_{k+1}^2.$$

By the assumption, we have

$$\forall N \text{ (positive integer)}, \ \exists w \in V_k : \|w\|_k \geq N, \ \|w\|_{k+1} = 1. \tag{2.4}$$

Choose $u_k^1 \in \mathscr{D}(A_k) \subset V_k$ and let $\widetilde{u}_k^2 = u_k^1 - w (\in V_k)$. Since $\mathscr{D}(A_k)$ is dense in V_k, we have

$$\forall \varepsilon > 0, \quad \exists u_k^2 \in \mathscr{D}(A_k) : \|\widetilde{u}_k^2 - u_k^2\|_k \leq \varepsilon,$$

and, hence, $\|u_k - w\|_k \leq \varepsilon$, which together with (2.4) implies

$$-\|u_k\|_k \leq -\|w\|_k + \varepsilon \leq -N + \varepsilon. \tag{2.5}$$

By Assumption 1.1 we get that

$$\exists r > 0 : \ \|u_k - w\|_{k+1} \leq r\|u_k - w\|_k \leq r\varepsilon,$$

which together with (2.4) implies

$$\|u_k\|_{k+1} \leq \|w\|_{k+1} + r\varepsilon = 1 + r\varepsilon. \tag{2.6}$$

On the other hand,

$$\exists f \in V_k^* : \quad \langle u_k, f\rangle_k = \|u_k\|_k, \quad \|f\| = 1. \tag{2.7}$$

The map j_k is demicontinuous, d-monotone and coercive (cf. Lemma 5.6, ch. 1 and Remark 1.4, ch. 3 in [7]). Therefore, there exists $j_k^{-1} \in (V_k^* \to V_k)$, and so

$$\exists v \in V_k : \qquad j_k v = f, \tag{2.8}$$

where $\|v\|_k = \|f\| = 1$. Choose $u_{k-1}^1 \in \mathscr{D}(A_{k-1}) \subset V_k$ and let $\widetilde{u}_{k-1}^2 = u_{k-1}^1 - v \in V_k$. Since $D(A_{k-1})$ is dense in V_k, we can take u_{k-1}^2 such that $\|\widetilde{u}_{k-1}^2 - u_{k-1}^2\|_k = \|u_{k-1} - v\|_k$ is arbitrarily small. Therefore, demicontinuity of j_k implies that

$$\begin{aligned} \forall \varepsilon > 0, \exists u_{k-1}^1, u_{k-1}^2 \in \mathscr{D}(A_{k-1}) : &\|u_{k-1} - v\|_k \leq \varepsilon, \\ &|\langle u_k, j_k u_{k-1}\rangle_k - \langle u_k, j_k v\rangle_k| \leq \varepsilon. \end{aligned} \tag{2.9}$$

Using (2.9) we have

$$I \leq -\langle u_k, j_k v\rangle_k + \varepsilon + |\omega|(\|v\|_k + \varepsilon)^2 + |\omega|\|u_k\|_{k+1}^2. \tag{2.10}$$

Consequently, (2.5)$\sim$(2.10) imply

$$I \leq -N + 2\varepsilon + |\omega|(1+\varepsilon)^2 + |\omega|(1 + r\varepsilon)^2. \tag{2.11}$$

In (2.11) r is a constant and for the fixed ε and ω arbitrarily large integer N can be taken, which is contradictory to accretivity of $\mathbb{A} + \omega$. Therefore, the spaces $V_k, k = 1 \sim n-1$, are equivalent each other. □

REMARK 2.1. *Every reflexive Banach space V and its dual V^* may be strictly convex by changing the norm of V with a proper equivalent norm. However, since the duality map is variable according to the norms, we assume the strict convexity of V^* in Theorem 2.1.*

REMARK 2.2. *Accretivity of operator is not a necessary condition to generate a linear one-parameter semigroup and is a part of the sufficient conditions to generate a nonlinear semigroup by Crandall-Liggett (cf. [6]). Thus, in fact, there exists an example of linear semigroup in which under Assumption 1.1 the spaces V_i, $i = 1 \sim n-1$, are not equivalent (cf. Proposition 1.6 in [8]).*

Under consideration of the result above, in Sections 2 and 3 we take the following

ASSUMPTION 2.1. *Banach spaces V and E are real and $V \hookrightarrow E$. The sets $\mathscr{D}(A_i)$, $i = 0 \sim n-1$, are dense in V.*

THEOREM 2.2. *Let the space V be reflexive and V^* be strictly convex. Assume the following inequalities:*

$$\left\|A_{n-2}u^1_{n-2} - A_{n-2}u^2_{n-2}\right\|_E \le K\|u_{n-2}\|_V \quad \forall u^1_{n-2}, u^2_{n-2} \in \mathscr{D}(A_{n-2}), \tag{2.12}$$

$$\left|\left\langle A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, j_E u_{n-1}\right\rangle_E\right| \le K\|u_{n-1}\|^2_E \quad \forall u^1_{n-1}, u^2_{n-1} \in \mathscr{D}(A_{n-1}), \tag{2.13}$$

where j_E is the duality operator from E to E^*.

Then, the equivalence of the spaces V and E, that is, $\mathbb{X} = E^n$ is a necessary condition for the existence of a real number ω such that the operator $\mathbb{A}+\omega$ is accretive in the space $\mathbb{X} = V^{n-1} \times E$.

Proof. Putting $u^1_k = u^2_k$ for $k = 0, 1, \cdots, n-3$ in (2.3) and taking into account the conditions of theorem, we have

$$I \le -\langle u_{n-1}, j_V u_{n-2}\rangle_V + \omega\|u_{n-2}\|^2_V + K_1\|u_{n-2}\|_E\|u_{n-1}\|_E + (K+\omega)\|u_{n-1}\|^2_E,$$

where j_V is the duality operator from V to V^*.

Thus, as before we come to the asserted conclusion. □

THEOREM 2.3. *Assume that the space E is reflexive, E^* is strictly convex and formula* (2.13) *holds.*

If the operator $\mathbb{A}+\omega$ is accretive in $\mathbb{X} = V^{n-1} \times E$ for some $\omega \in \mathcal{R}$, then we have the followings:

1) $|\langle A_k u^1_k - A_k u^2_k, j_E u_k\rangle_E| \le M_k\|u_k\|^2_V, k = 0, 1, \cdots, n-2,\ \forall u^1_k,\ u^2_k \in \mathscr{D}(A_k) \cap \mathscr{D}(A_{n-1})$,
where M_k are constants;

2) Linear operators among $A_k, k = 0, 1, \cdots, n-3$, are the restrictions of operators belonging to $\mathcal{BL}(V, E)$;

3) If the set $\mathscr{D}(A_{n-1})$ is linear, then

$$\left\langle A_{n-2}u^1_{n-2} - A_{n-2}u^2_{n-2}, j_E u_{n-2}\right\rangle \ge a\|u_{n-2}\|^2_V + b\|u_{n-2}\|^2_E \qquad \forall u^1_{n-2}, u^2_{n-2} \in \mathscr{D}(A_{n-2}) \cap \mathscr{D}(A_{n-1}),$$

where $a > 0$ and b is a real number.

Proof of 1). If the dual space of a reflexive Banach space Y is strictly convex, then it is easy to prove that j_Y is homogeneous, i.e., $\forall\lambda:\ j_Y(\lambda x) = \lambda j_Y(x)$.

Putting $u^1_i = u^2_i, i \ne k, n-1$, in (2.3) and using accretivity of $\mathbb{A}+\omega$, we have

$$\omega\|u_k\|^2_V + \langle A_k u^1_k - A_k u^2_k, j_E u_{n-1}\rangle + \langle A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, j_E u_{n-1}\rangle + \omega\|u_{n-1}\|^2_E \ge 0.$$

Putting $u^i_{n-1} = u^i_k$ in the inequality above, in view of that $V \hookrightarrow E$ which means $\|w\|_E \leq r\|w\|_V \ \forall w \in V$, we have

$$\left\langle A_k u^1_k - A_k u^2_k, j_E u_k \right\rangle_E \geq -(Kr^2 + \omega + r^2|\omega|)\|u_k\|^2_V.$$

On the other hand, putting $u^1_{n-1} = u^2_k, u^2_{n-1} = u^1_k$, we have $u_{n-1} = -u_k$. Using homogeneity of j_E, we get

$$\left\langle A_k u^1_k - A_k u^2_k, j_E u_k \right\rangle_E \leq (Kr^2 + \omega + r^2|\omega|)\|u_k\|^2_V.$$

Above two estimates imply the asserted conclusion.

Proof of 2). Suppose that the conclusion is not true. Then, since $\mathscr{D}(A_k)$ is dense in V, we have

$$\forall N,\ \exists v_k \in \mathscr{D}(A_k)(\|v_k\|_V = 1):\ \|A_k v_k\|_E > N. \tag{2.14}$$

Putting $u^1_i = u^2_i, i \neq k, n-1$, in (2.3) and using the fact that A_k are linear, we get

$$I = \omega\|u_k\|_V + \langle A_k u_k, j_E u_{n-1}\rangle + \left\langle A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, j_E u_{n-1}\right\rangle_E + \omega\|u_{n-1}\|^2_E.$$

Let $u_k = v_k$. Then by (2.13) and (2.14),

$$I \leq \omega - \langle -A_k u_k, j_E u_{n-1}\rangle_E + (K + \omega)\|u_{n-1}\|^2_E. \tag{2.15}$$

On the other hand, there exists $f \in E^*$ such that

$$\langle -A_k u_k, f\rangle_E = \| - A_k u_k\|_E,\ \|f\| = 1. \tag{2.16}$$

As mentioned in the proof of Theorem 2.1, there exists j_E^{-1}, and so there exists $v \in E$ such that $j_E v = f,\ \|v\|_E = 1$. Since $\mathscr{D}(A_{n-1})$ is dense in E and j_E is demicontinuous, using (2.16) and arguing as the proof of Theorem 2.1, we have

$$\begin{aligned}\forall \varepsilon > 0,\ \exists u^1_{n-1}, u^2_{n-1} \in \mathscr{D}(A_{n-1}) : \|u_{n-1} - v\|_E \leq \varepsilon,&\\ \langle -A_k u_k, j_E u_{n-1}\rangle \geq \|A_k u_k\|_E - \varepsilon.&\end{aligned} \tag{2.17}$$

From (2.14), (2.15) and (2.17) we have

$$\begin{aligned} I &\leq \omega - \|A_k u_k\|_E + \varepsilon + (K + |\omega|)(\|v\|_E + \varepsilon)^2 \\ &\leq \omega - N + \varepsilon + (K + |\omega|)(1 + \varepsilon)^2. \end{aligned}$$

This shows contradiction to accretivity of $\mathbb{A} + \omega$, and so we come to the asserted conclusion.

Proof of 3). Let $u^1_i = u^2_i, i = 0, 1, \cdots, n-3$, in (2.3). Taking into account linearity of the set $\mathscr{D}(A_{n-1})$, for $\lambda > 0$ let $u^i_{n-1} = \lambda u^i_{n-2}\ \forall u^i_{n-2} \in \mathscr{D}(A_{n-2}) \cap \mathscr{D}(A_{n-1})$, $i = 1, 2$. Then, the accretivity of $\mathbb{A} + \omega$ implies that

$$\begin{aligned} &- \langle \lambda u_{n-2}, j_V u_{n-2}\rangle_V + \omega\|u_{n-2}\|^2_V + \left\langle A_{n-2}u^1_{n-2} - A_{n-2}u^2_{n-2}, j_E(\lambda u_{n-2})\right\rangle_E \\ &\qquad + \left\langle A_{n-1}(\lambda u^1_{n-2}) - A_{n-1}(\lambda u^2_{n-2}), j_E(\lambda u_{n-2})\right\rangle_E + \omega\|\lambda u_{n-2}\|^2_E \geq 0. \end{aligned}$$

Using homogeneity of j_E and (2.13), we have

$$\left\langle A_{n-2}u^1_{n-2} - A_{n-2}u^2_{n-2}, j_E u_{n-2}\right\rangle_E \geq \left(1 - \frac{\omega}{\lambda}\right)\|u_{n-2}\|^2_V - (K\lambda + \omega\lambda)\|u_{n-2}\|^2_E.$$

Now, taking $\lambda > 0$ such that $|\frac{\omega}{\lambda}| < 1$, we come to the asserted conclusion. □

THEOREM 2.4. *Let E be a Hilbert space. Assume that*

$$\left\|A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-2}\right\|_{V^*} \le K\|u_{n-1}\|_V \ \forall u^1_{n-1},\, u^2_{n-1} \in \mathscr{D}(A_{n-1}).$$

If $\mathbb{A}+\omega$ is accretive in $\mathbb{X} = V^{n-1}\times E$ with some $\omega \in \mathcal{R}$, then linear operators among $A_k, k = 0, 1, \cdots, n-3$, are the restrictions of bounded linear operators from V to V^*.

Proof. Suppose that the conclusion is not true. Then, since $\mathscr{D}(A_k)$ is dense in V and A_k is linear, it follows that

$$\forall N, \exists v_k \in \mathscr{D}(A_k)\ (\|v_k\|_V = 1), \exists w \in V\ (\|w\|_V = 1): \\ \langle A_k v_k, w\rangle_V = (A_k v_k, w)_E \ge N. \tag{2.18}$$

Setting $u^1_i = u^2_i$, $i \ne k$, $n-1$, in (2.3) we have

$$I = \omega\|u_k\|^2_V + (A_k u_k, u_{n-1})_E + \left(A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, u_{n-1}\right)_E + \omega\|u_{n-1}\|^2_E.$$

Putting $u_k = -v_k$, from this we have that

$$\begin{aligned} I &\le \omega\|u_k\|^2_V - (A_k v_k, u_{n-1})_E \\ &\qquad + \left\|A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}\right\|_{V*}\cdot\|u_{n-1}\|_V + \omega\|u_{n-1}\|^2_E \\ &\le \omega\|u_k\|^2_V - (A_k v_k, u_{n-1})_E + K\|u_{n-1}\|^2_V + \omega\|u_{n-1}\|^2_E. \end{aligned} \tag{2.19}$$

On the other hand, since $\mathscr{D}(A_{n-1})$ is dense in V, arguing as the proof of Theorem 2.1, we have

$$\forall\varepsilon > 0, \exists u^1_{n-1},\, u^2_{n-1} \in \mathscr{D}(A_{n-1}) : \|w - u_{n-1}\|_V \le \varepsilon, \\ -(A_k v_k, u_{n-1})_E \le (A_k v_k, w)_E + \varepsilon. \tag{2.20}$$

Then, by (2.18), (2.20) and Assumption 2.1, from (2.19) we have that

$$\begin{aligned} I &\le \omega\|u_k\|^2_V - (A_k v_k, w)_E + \varepsilon + K\|u_{n-1}\|^2_V + \omega\|u_{n-1}\|^2_E \\ &\le \omega - N + \varepsilon + K(1+\varepsilon)^2 + |\omega|r^2(1+\varepsilon)^2, \end{aligned}$$

where r is the number in the proof of 1) of Theorem 2.3. Therefore, we come to a contradiction to accretivity of $\mathbb{A}+\omega$, and so the asserted conclusion is proved. □

THEOREM 2.5. *Let the space V be reflexive, V^* be strictly convex and $A_{n-2} \in \mathcal{L}ip\,(V, E)$. If the operator $\mathbb{A}+\omega$ is accretive in $\mathbb{X} = V^{n-1}\times E$ with some $\omega \in \mathcal{R}$, then*

$$\left\langle A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, j_E u_{n-1}\right\rangle \ge a\|u_{n-1}\|^2_V + b\|u_{n-1}\|^2_E \quad \forall u^1_{n-1}, u^2_{n-1} \in \mathscr{D}(A_{n-1}),$$

where $a > 0$ and b is a real number.

Proof. Put $u^1_i = u^2_i$ for $i \ne n-1$, $n-2$. Taking into account $\mathscr{D}(A_{n-2}) = V$, let $u^i_{n-2} = \lambda u^i_{n-1}$ for $\lambda > 0$ and u^1_{n-1}, $u^2_{n-1} \in \mathscr{D}(A_{n-1})$. Then, by the accretivity of $\mathbb{A}+\omega$ from (2.3) it follows that

$$\begin{aligned} &-\langle u_{n-1}, j_V(\lambda u_{n-1})\rangle_V + \omega\|\lambda u_{n-1}\|^2_V \\ &+ \left\langle A_{n-2}(\lambda u^1_{n-1}) - A_{n-2}(\lambda u^2_{n-1}), j_E u_{n-1}\right\rangle_E \\ &+ \left\langle A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, j_E u_{n-1}\right\rangle_E + \omega\|u_{n-1}\|^2_E \ge 0. \end{aligned} \tag{2.21}$$

By virtue of homogeneity of j_E and the condition of theorem, from (2.21) it follows that for $\varepsilon > 0$

$$\begin{aligned}\big\langle A_{n-1}u_{n-1}^1 - A_{n-1}u_{n-1}^2, j_E u_{n-1}\big\rangle_E \\ &\geq (\lambda - \omega\lambda^2)\|u_{n-1}\|^2 - \lambda K\|u_{n-1}\|_V \cdot \|u_{n-1}\|_E - \omega\|u_{n-1}\|_E^2 \\ &\geq \big(\lambda - \omega\lambda^2 - \lambda K\frac{\varepsilon}{2}\big)\|u_{n-1}\|_V^2 - \big(\omega + \lambda K\frac{1}{2\varepsilon}\big)\|u_{n-1}\|_E^2,\end{aligned}$$

which shows the asserted conclusion. ◻

Similarly, we have the following two Theorems.

THEOREM 2.6. *If the operator $\mathbb{A} + \omega$ is accretive in $\mathbb{X} = V^{n-1} \times E$ with some $\omega \in \mathcal{R}$, then the following inequality is valid.*

$$\begin{aligned}&\forall u_{n-1}^1, u_{n-1}^2 \in \mathscr{D}(A_{n-1}), \forall u_{n-2}^1, u_{n-2}^2 \in \mathscr{D}(A_{n-2}), \exists j_E u_{n-1}, j_V u_{n-2} : \\ &\quad - \langle u_{n-1}, j_V u_{n-2}\rangle_V + \omega\| \ u_{n-2}\|_V^2 + \big\langle A_{n-2}u_{n-2}^1 - A_{n-2}u_{n-2}^2, j_E u_{n-1}\big\rangle_E \\ &\qquad + \big\langle A_{n-1}u_{n-1}^1 - A_{n-1}u_{n-1}^2, j_E u_{n-1}\big\rangle_E + \omega\|u_{n-1}\|_E^2 \geq 0.\end{aligned} \tag{2.22}$$

If

$$\|A_k u_k^1 - A_k u_k^2\|_E \leq \|u_k\|_V, \ k = 0, \cdots, n-3, \tag{2.23}$$

then, (2.22) *is sufficient for the existence of $\omega_1 \in \mathcal{R}$ such that $\mathbb{A} + \omega_1$ is accretive.*

REMARK 2.3. *Let $V = H_0^1(\Omega)$, $E = L_2(\Omega)$ and $\mathscr{D}(A_{n-1}) = \mathscr{D}(A_{n-2}) = H_0^1(\Omega) \cap H^2(\Omega)$. Define the operators A_{n-1} and A_{n-2}, respectively, by*

$$A_{n-1}u_{n-1} = \Sigma_i \frac{\partial}{\partial x_i} a_i\Big(x, \cdots, \frac{\partial u_{n-1}}{\partial x_j}, \cdots\Big) + b\Big(x, u_{n-1}, \cdots, \frac{\partial u_{n-1}}{\partial x_j}, \cdots\Big),$$

and

$$A_{n-2}u_{n-2} = \sum_i \frac{\partial}{\partial x_i} c\Big(x, u_{n-2}, \cdots, \frac{\partial u_{n-2}}{\partial x_j}, \cdots\Big).$$

Let the matrix $\{a_{ij}(x, \cdots, y_j, \cdots)\}$, where $a_{ij}(x, \cdots, y_j, \cdots) = \frac{\partial}{\partial y_j} a_i(x, \cdots, y_j, \cdots)$, be positive-definite at a.a. $x \in \Omega$ and $a_{ij}(\cdot, y_0, \cdots, y_j, \cdots) \in L_\infty(\Omega)$ for any $y_i \in \mathcal{R}$; $b(x, y_0, \cdots, y_n)$, $c(x, y_0, \cdots, y_n)$ be Lipschitz continuous with respect to y_j, $j = 0, \cdots, n$. Then, the operators A_{n-1} and A_{n-2} satisfy (2.22).

THEOREM 2.7. *For the existence of a real number $\omega \in \mathcal{R}$ such that both $\mathbb{A} + \omega$ and $\mathbb{A} - \omega$ are accretive in $\mathbb{X} = V^{n-1} \times E$ it is necessary that the following inequalities*

$$\begin{aligned}&\big|-\langle u_{n-1}, j_V u_{n-2}\rangle_V + \big\langle A_{n-2}u_{n-2}^1 - A_{n-2}u_{n-2}^2, \ j_E u_{n-1}\big\rangle_E\big| \\ &\qquad\qquad \leq K_1\|u_{n-2}\|_V^2 + K_2\|u_{n-1}\|_E^2, \\ &\big|\big\langle A_{n-1}u_{n-1}^1 - A_{n-1}u_{n-1}^2, \ j_E u_{n-1}\big\rangle_E\big| \leq K_3\|u_{n-1}\|_E^2.\end{aligned} \tag{2.24}$$

are valid.

If (2.23) *is valid, then* (2.24) *is sufficient for the existence of such an ω.*

REMARK 2.4. *Let $a_{ij}(x) = a_{ji}(x)$, $a_{ij}(x) \in W^1_\infty(\Omega)$ and the matrix $\{a_{ij}(x)\}$ be positive-definite at a.a. $x \in \Omega$. Let $V = H^1_0(\Omega)$, where an inner product is one by $(u,v) = \sum_{ij} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{x_j}\, dx$, $E = L_2(\Omega), \mathscr{D}(A_{n-1}) = H^1_0(\Omega)$ and $\mathscr{D}(A_{n-2}) = H^1_0(\Omega) \cap H^2(\Omega)$. Define the operators A_{n-1} and A_{n-2}, respectively, by*

$$A_{n-1}u_{n-1} = \sum_i a_i(x) \frac{\partial u_{n-2}}{\partial x_i} + b(x, u_{n-1}),$$

and

$$A_{n-2}u_{n-2} = \sum_{ij} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u_{n-2}}{\partial x_j}\Big) + c\Big(x, u_{n-2}, \cdots, \frac{\partial u_{n-2}}{\partial x_j}, \cdots\Big),$$

where $b(x, y_0)$, $c(x, y_0, \cdots, y_n)$ are Lipschitz continuous with respect to y_j, $j = 0, \cdots, n$. Then, the operators A_{n-1} and A_{n-2} satisfy (2.24).

3. Cauchy problems of some nonlinear equations.

DEFINITION 3.1. *If $u \in C^{n-2}([T_0,T]; V)$, $u^{(n-1)} \in C([T_0,T]; E)$ and*

$$(*) \qquad u^{(n)} + A(u, \cdots, u^{(n-1)}) = f(t) \quad \text{for a.a. } t \in [T_0, T] \text{ in } E,$$

then, u is called a strong solution to $(*)$ on $[T_0, T]$.

The results above show that accretivity of $\mathbb{A} + \omega$ depends mainly on characters of the operators acting on $u^{(n-1)}$ and $u^{(n-2)}$. Therefore, we will consider three kinds of equations in view of the operators acting on $u^{(n-1)}$ and $u^{(n-2)}$.

First, we study a Cauchy problem

$$\begin{aligned} &u^{(n)} + A_{n-1}u^{(n-1)} + A_{n-2}u^{(n-2)} + A(u, \cdots, u^{(n-1)}) = f(t), \\ &u(0) = u_0,\ u'(0) = u_0', \cdots, u^{(n-1)}(0) = u_0^{(n-1)}. \end{aligned} \tag{3.1}$$

LEMMA 3.1. *Let E be a real Hilbert space and V be a real reflexive Banach space. Suppose that*
1) $A \in \mathcal{L}ip\,(V^{n-1} \times E, E)$;
2) $A_{n-1} \in \mathcal{L}ip\,(V, E)$ and

$$\big|\big(A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, u_{n-1}\big)_E\big| \le K\|u_{n-1}\|^2_E; \tag{3.2}$$

3) A_{n-2} is a restriction on E of a radially continuous (cf. Definition 1.1, ch. 3 in [7]) operator $\widetilde{A}_{n-2}$ from V to V^* such that

$$\big\langle \widetilde{A}_{n-2}u^1_{n-2} - \widetilde{A}_{n-2}u^2_{n-2},\, u_{n-2}\big\rangle_V \ge a\|u_{n-2}\|^2_V + b\|u_{n-2}\|^2_E, \tag{3.3}$$

where $a > 0$ and b is a real number.

Then,

$$\exists \lambda_0 > 0,\, \forall \lambda\, (0 < |\lambda| \le \lambda_0) : \mathscr{R}(I + \lambda\mathbb{A}) = \mathbb{X},$$

where I is the unit operator on $\mathbb{X}$ and

$$\mathbb{A}\mathbb{U} = \begin{pmatrix} -u_1 \\ -u_2 \\ \vdots \\ A_{n-1}u_{n-1} + A_{n-2}u_{n-2} + A(u, \cdots, u_{n-1}) \end{pmatrix},\ \mathscr{D}(\mathbb{A}) = \begin{pmatrix} V \\ \vdots \\ \mathscr{D}(A_{n-2}) \\ V \end{pmatrix}. \tag{3.4}$$

Proof. The asserted conclusion is equivalent to the existence of a solution to the following problem

$$(\lambda+\mathbb{A})\mathbb{U}=F,\ F=\begin{pmatrix} f_0\\ \vdots \\ f_{n-1}\end{pmatrix}\in\mathbb{X}\qquad \forall\lambda(|\lambda|\geq\widetilde{\lambda}_0), \tag{3.5}$$

where $\widetilde{\lambda}_0$ is a constant. Equation (3.5) is equivalent to the following system

$$\begin{aligned}
&u_{n-1}=\lambda u_{n-2}-f_{n-2},\\
&u_{n-3}=\frac{f_{n-3}+u_{n-2}}{\lambda},\\
&\cdots\cdots\cdots,\\
&u_i=\sum_{k=i}^{n-3}\frac{f_k}{\lambda^{k-i+1}}+\frac{u_{n-2}}{\lambda^{n-2-i}},\\
&\cdots\cdots\cdots,\\
&u=\sum_{k=0}^{n-3}\frac{f_k}{\lambda^{k+1}}+\frac{u_{n-2}}{\lambda^{n-2}},\\
&A\Big(\sum_{k=0}^{n-3}\frac{f_k}{\lambda^{k+1}}+\frac{u_{n-2}}{\lambda^{n-2}},\cdots,\sum_{k=i}^{n-3}\frac{f_k}{\lambda^{k-i+1}}+\frac{u_{n-2}}{\lambda^{n-2-i}},\cdots,u_{n-2},\,\lambda u_{n-2}-f_{n-2}\Big)\\
&\qquad\qquad+A_{n-2}u_{n-2}+A_{n-1}(\lambda u_{n-2}-f_{n-2})+\lambda^2u_{n-2}-\lambda f_{n-2}=f_{n-1}.
\end{aligned}\tag{3.6}$$

If there exists a solution u_{n-2} to the last equation of (3.6), substituting it into other we have a solution to (3.5).

Let us prove the existence of a solution to the last equation of (3.6). Replacing A_{n-2} with $\widetilde{A}_{n-2}$ and denoting the left-hand side of the last one in (3.6) by $\widetilde{A}_\lambda$, we have the equation

$$\widetilde{A}_\lambda u=f_{n-1}. \tag{3.7}$$

Since an arbitrarily large $\overline{\lambda}_0$ can be taken, we can assume that $|\lambda|>1$ without loss of generality. Using the conditions of theorem and the fact that $\langle v,u\rangle_V=(v,u)_E$ for $u\in V,v\in E$, we have

$$\begin{aligned}
&\big\langle\widetilde{A}_\lambda u^1-\widetilde{A}_\lambda u^2,u\big\rangle\\
&\geq -M\sum_{i=0}^{n-3}\frac{\|u\|_V\cdot\|u\|_E}{|\lambda|^{n-2-i}}-M\|u\|_V\cdot\|u\|_E-M|\lambda|\|u\|_E^2+\big\langle\widetilde{A}_{n-2}u^1-\widetilde{A}_{n-2}u^2,u\big\rangle_V\\
&\quad+\frac{1}{\lambda}\big(A_{n-1}(\lambda u^1-f_{n-2})-A_{n-1}(\lambda u^2-f_{n-2}),\lambda u\big)+\lambda^2\|u\|_E^2\\
&\geq-\frac{M}{|\lambda|-1}\|u\|_V\|u\|_E-M\|u\|_V\cdot\|u\|_E-M|\lambda|\|u\|_E^2\\
&\quad+a\|u\|_V^2+b\|u\|_E^2-\frac{K}{|\lambda|}\|\lambda u\|_E^2+\lambda^2\|u\|_E^2\\
&\geq\Big(a-\frac{M}{2(|\lambda|-1)}-\frac{\varepsilon M}{2}\Big)\|u\|_V^2+\Big(\lambda^2-M|\lambda|-K|\lambda|+b-\frac{M}{2(|\lambda|-1)}-\frac{M}{2\varepsilon}\Big)\|u\|_E^2\\
&\qquad\qquad\qquad\forall u^1,u^2\in V.
\end{aligned}$$

Taking $\varepsilon = \frac{a}{2M}$, we can see that $\widetilde{A}_\lambda$ is strongly monotone for all $\lambda(|\lambda| > \overline{\lambda}_0)$ provided $\overline{\lambda}_0$ is large enough. On the other hand, by the conditions of theorem the operator $\widetilde{A}_\lambda \in (V \to V^*)$ is radially continuous. Hence, there exists a solution $\overline{u} \in V$ to equation (3.7). (cf. Theorem 2.1 of ch. 3 in [7]) From (3.7) we get

$$\begin{aligned}\widetilde{A}_{n-2}\overline{u} = f_{n-1} - A\Big(\cdots, \sum_{k=i}^{n-3}\big(\frac{f_k}{\lambda^{k-i+1}} + \frac{\overline{u}}{\lambda^{n-2-i}}\big), \cdots\Big)\\ - A_{n-1}(\lambda\overline{u} - f_{n-2}) - \lambda^2\overline{u} + \lambda f_{n-2}.\end{aligned} \tag{3.8}$$

Since $f_0, \cdots, f_{n-2} \in V, f_{n-1} \in E$ and $\overline{u} \in V$, the right hand side of (3.8) is an element of E. Thus, $\widetilde{A}_{n-2}\overline{u} \in E$, which shows that $\overline{u} \in \mathscr{D}(A_{n-2})$. Thus, we proved the existence of a solution to the last equation of (3.6). □

LEMMA 3.2. *Suppose that in a Banach space for some $\omega \in R$ both operators $B + \omega$ and $-B + \omega$ are accretive and both B and $-B$ generate, respectively, nonlinear semigroups $T_+(t)$ and $T_-(t)$, $t \in [0,\infty)$, on $\overline{\mathscr{D}(B)}$. If $T_+(t)x_0$ and $T_-(t)x_0$ for $x_0 \in \mathscr{D}(B)$ are, respectively, unique strong solutions to the equations $\dot{x} + Bx = 0$ and $\dot{x} - Bx = 0,\ t \in [0,\infty)$, then the operator B generates one-parameter nonlinear group on $\overline{\mathscr{D}(B)}$ $U(t)$, $t \in (-\infty,\infty)$, and*

$$U(t + r) = T_{\mathrm{sign(t)}}(|t|) \cdot T_{\mathrm{sign(r)}}(|r|)\ \forall t \in (-\infty,\infty).$$

Proof. Under consideration of the uniqueness of a strong solution to $\dot{x} + Bx = 0$, $t \in (-\infty,\infty)$, we can prove it. □

THEOREM 3.3. *Let V and E be real Hilbert spaces such that $V \hookrightarrow E$. Assume that the conditions 1) and 2) in Lemma 3.1 hold.*

If A_{n-2} is the restriction on E of a self-adjoint and strongly monotone operator $\widetilde{A}_{n-2} \in \mathcal{BL}(V,V^*)$, then the operator $\mathbb{A}$ in (3.4) generates a nonlinear group on $\mathbb{X} = V^{n-1} \times E$. And so, when $\mathbb{U}_0 \equiv (u_0, u_0', \cdots, u_0^{(n-1)}) \in \mathscr{D}(\mathbb{A})$ and $f \in \mathcal{BV}([-T,T];E), T > 0$ (the space of functions with bounded variation), there exists a unique strong solution to (3.1) on $[-T,T]$.

Proof. A new inner product in V can be introduced by $\langle \widetilde{A}_{n-2}u, v\rangle_V$. Denote by $\widetilde{V}$ the new Hilbert space with this inner product. Then,

$$-(u_{n-1}, u_{n-2})_{\widetilde{V}} + (A_{n-2}u_{n-2}, u_{n-1})_E = 0 \quad \forall u^1_{n-2}, u^2_{n-2} \in \mathscr{D}(A_{n-2}). \tag{3.9}$$

Let us prove that the operator $\mathbb{A} + \omega$ for an $\omega \in \mathcal{R}$ is accretive in the space $\widetilde{\mathbb{X}} = \widetilde{V}^{n-1} \times E$. Taking into account (3.9) and the fact that $A \in \mathcal{L}ip\,(\widetilde{V}^{n-1} \times E, E)$, we have that

$$\begin{aligned}I &\equiv \big(\mathbb{A}\mathbb{U}^1 - \mathbb{A}\mathbb{U}^2, \mathbb{U}\big)_{\widetilde{\mathbb{X}}} =\\ &= -\sum_{i=0}^{n-2}(u_{i+1}, u_i)_{\widetilde{V}} + \big(A_{n-1}u^1_{n-1} - A_{n-1}u^2_{n-1}, u_{n-1}\big)_E + (A_{n-2}u_{n-2}, u_{n-1})_E\\ &\quad + \big(A(u^1, u^1_1, \cdots, u^1_{n-2}, u^1_{n-1}) - A(u^2, u^2_1, \cdots, u^2_{n-2}, u^2_{n-1}), u_{n-1}\big)_E\\ &\geq -\sum_{i=0}^{n-3}\left(\frac{\|u_{i+1}\|^2_{\widetilde{V}}}{2} + \frac{\|u_i\|^2_{\widetilde{V}}}{2}\right) - \frac{M^2}{2}\sum_{i=0}^{n-2}\|u_i\|^2_{\widetilde{V}} - \frac{1}{2}\|u_{n-1}\|^2_E - \frac{M^2}{2}\|u_{n-1}\|^2_E\\ &\geq -\frac{M^2+2}{2}\sum_{i=0}^{n-3}\|u_i\|^2_{\widetilde{V}} - \frac{M^2+1}{2}\|u_{n-2}\|^2_{\widetilde{V}} - \frac{M^2+1}{2}\|u_{n-1}\|^2_E,\end{aligned}$$

where as before $u_0 = u$. Thus, putting $\omega = \frac{M^2+2}{2}$, we see that $\mathbb{A} + \omega$ is accretive in the space $\widetilde{\widetilde{\mathbb{X}}} = \widetilde{V}^{n-1} \times E$. In the same way we can prove that $-\mathbb{A} + \omega$ is also accretive in $\widetilde{\mathbb{X}}$.

By Lemma 3.1, we obtain that $\mathscr{R}(I + \lambda\mathbb{A}) = \widetilde{\widetilde{\mathbb{X}}}$, $\mathscr{R}(I - \lambda\mathbb{A}) = \widetilde{\widetilde{\mathbb{X}}}$ for any λ small enough. We can see that $\mathscr{D}(\mathbb{A}) = \mathscr{D}(-\mathbb{A})$ is dense in $\mathbb{X}$. Thus, $\mathbb{A}$ and $-\mathbb{A}$ generate, respectively, nonlinear semigroups on $\mathbb{X}$, and so by Lemma 3.2 the operator $\mathbb{A}$ generates a nonlinear group. Existence of a unique strong solution to (3.1) follows by the semigroup theory (cf. [14]). □

Second, we study the following Cauchy problem

$$\begin{aligned} &u^{(n)} + A_{n-1}u^{(n-1)} + A(u, \cdots, u^{(n-1)}) = f(t), \\ &u(0) = u_0,\, u'(0) = u_0', \cdots, u^{(n-1)}(0) = u_0^{(n-1)}. \end{aligned} \tag{3.10}$$

For (3.10) Definition 3.1 is used with $V \equiv E$.

Theorem 3.4. *Suppose that $A \in \mathcal{L}ip\,(E^n, E)$, $\mathscr{D}(A_{n-1})$ is dense in E, formula* (2.13) *holds and*

$$\exists\lambda_0 > 0,\, \forall\lambda(|\lambda| \geq \lambda_0) : \mathscr{R}(\lambda + A_{n-1}) = E. \tag{3.11}$$

Then, both $\mathbb{A}$ and $-\mathbb{A}$ generate nonlinear semigroups on $\mathbb{X} = E^n$, where

$$\mathscr{D}(\mathbb{A}) = \begin{pmatrix} E \\ \vdots \\ E \\ \mathscr{D}(A_{n-1}) \end{pmatrix}, \ \mathbb{A}\mathbb{U} = \begin{pmatrix} -u_1 \\ -u_2 \\ \vdots \\ A_{n-1}u_{n-1} + A(u, \cdots, u_{n-1}) \end{pmatrix}.$$

If E is reflexive, then $\mathbb{A}$ generates a nonlinear group. If E is reflexive, $\mathbb{U}_0 \in \mathscr{D}(\mathbb{A})$ and $f(t) \in \mathcal{BV}([-T,T]; E)$, then there exists a unique strong solution to (3.10) *on* $[-T, T]$.

The proof is similar to the case of Theorem 3.3 and is omitted. □

Remark 3.1. *An example of linear partial differential operator satisfying the conditions* (2.13) *and* (3.11) *is shown in 3.5 in [26].*

Finally, we consider the following Cauchy problem

$$\begin{aligned} &u^{(n)} + A(u, u', \cdots, u^{(n-1)}) = f(t), \\ &u(0) = u_0,\, u'(0) = u_0', \cdots, u^{(n-1)}(0) = u_0^{(n-1)},\ \mathbb{U}_0 \equiv (u_0, u_0', \cdots, u_0^{(n-1)}). \end{aligned} \tag{3.12}$$

Theorem 3.5. *Let V and E be real Hilbert spaces such that $V \hookrightarrow E$. Suppose that $A \in \mathcal{L}ip(V^n, V^*)$ and*

$$\begin{aligned} &\left\langle A\left(u, u_1, \cdots, u_{n-2}, u_{n-1}^1\right) - A\left(u, u_1, \cdots, u_{n-2}, u_{n-1}^2\right), u_{n-1}\right\rangle_V \\ &\qquad \geq a\|u_{n-1}\|_V^2 + b\|u_{n-1}\|_E^2 \quad \forall u, u_1, \cdots, u_{n-2}, u_{n-1}^1, u_{n-1}^2 \in V, \end{aligned}$$

where $a > 0$ and b is real number.

Then, the restriction of $\mathbb{A}$ *to* $\mathbb{X} = V^{n-1} \times E$ *generates a nonlinear semigroup, where*

$$\mathbb{A}\mathbb{U} = \begin{pmatrix} -u_1 \\ \vdots \\ -u_{n-1} \\ A(u, u_1, \cdots, u_{n-1}) \end{pmatrix},$$

and so there exists a unique strong solution to (3.12) *on* $[0, T]$ *for* $\mathbb{U}_0 \in \mathscr{D}(\mathbb{A})$ *and* $f(t) \in \mathcal{BV}([0, T], E)$. *If* $\mathbb{U}_0 \in \mathbb{X}$ *and* $f(t) \in L^2(0, T; V^*)$, *then there exists a unique solution* $u \in C^{n-2}([0, T]; V)$ *such that* $u^{(n-1)} \in L^2(0, T; V) \cap C([0, T]; E)$ *and* $u^{(n)} \in L_2(0, T; V^*)$.

Proof. Let $\mathbb{Y} = V^n$. According to Riesz theorem, let us identify every component of $(V^*)^n$ with the exception of the last one, with V. Then, we can regard $\mathbb{Y}^* = V^{n-1} \times V^*$ as a representation of the dual space of $\mathbb{Y}$, and $\mathbb{A}$ is an operator from $\mathscr{D}(\mathbb{A}) = \mathbb{Y}$ to $\mathbb{Y}^*$. If the space $\mathbb{X} = V^{n-1} \times E$ is identified with $\mathbb{X}^*$, then we get

$$\mathbb{Y} \hookrightarrow \mathbb{X} \hookrightarrow \mathbb{Y}^*. \tag{3.13}$$

Thus, we have

$$\begin{aligned}
&\forall \mathbb{U}^1, \mathbb{U}^2 \in \mathbb{Y} : \langle \mathbb{A}\mathbb{U}^1 - \mathbb{A}\mathbb{U}^2, \mathbb{U}\rangle_{\mathbb{Y}} = \\
&= -\sum_{i=0}^{n-2} (u_{i+1}, u_i)_V + \big\langle A(u^1, u_1^1, \cdots, u_{n-2}^1, u_{n-1}^1) - A(u^1, u_1^1, \cdots, u_{n-2}^1, u_{n-1}^2), u_{n-1}\big\rangle_V \\
&\qquad\qquad + \big\langle A(u^1, u_1^1, \cdots, u_{n-2}^1, u_{n-1}^2) - A(u^2, u_1^2, \cdots, u_{n-2}^2, u_{n-1}^2), u_{n-1}\big\rangle_V \\
&\geq -\sum_{i=0}^{n-3} \left(\frac{\|u_{i+1}\|_V^2}{2} + \frac{\|u_i\|_V^2}{2} \right) - \|u_{n-1}\|_V \cdot \|u_{n-2}\|_V \\
&\qquad\qquad + a\|u_{n-1}\|_V^2 + b\|u_{n-1}\|_E^2 - M \sum_{i=0}^{n-2} \|u_i\|_V \cdot \|u_{n-1}\|_V \\
&\geq -\sum_{i=0}^{n-3} \left(\frac{\|u_{i+1}\|_V^2}{2} + \frac{\|u_i\|_V^2}{2} \right) - \|u_{n-1}\|_V \cdot \|u_{n-2}\|_V \\
&\qquad\qquad - M \sum_{i=0}^{n-2} \|u_i\|_V \cdot \|u_{n-1}\|_V + a\|u_{n-1}\|_V^2 + b\|u_{n-1}\|_E^2 \\
&\geq \Big[a - \frac{\varepsilon}{2}\big(M(n-1) + 1\big) \Big] \|u_{n-1}\|_V^2 - \left(\frac{M+1}{2\varepsilon} + 1 \right) \|u_{n-2}\|_V^2 \\
&\qquad\qquad - \left(\frac{M}{2\varepsilon} + 1 \right) \sum_{i=0}^{n-3} \|u_i\|_V^2 + b\|u_{n-1}\|_E^2,
\end{aligned}$$

where $\varepsilon > 0$, M is the Lipschitz constant of A, $\mathbb{U} = \mathbb{U}^1 - \mathbb{U}^2$ and $u_0 = u$.

Let $\varepsilon = \frac{a}{M(n-1)+1}$, $\omega = max\{\frac{M+1}{2\varepsilon} + 1, -b\} + \delta$, $\delta > 0$. Then $\mathbb{A} + \omega I$ is a strongly monotone Lipschitz operator, where I is embedding operator from $\mathbb{Y}$ into $\mathbb{Y}^*$ by (3.13). Therefore, by the semigroup theory (cf. [14]), the operator $\mathbb{A}$ generates a nonlinear semigroup on $\mathbb{X}$. Also, by Theorem 1.3, ch. 6 in [7], we come to the last conclusion. ∎

Relying on Theorem 1.1 of ch. 6 in [7], in the same way we have

THEOREM 3.6. *Let A be a Volterra operator such that $A \in \mathcal{L}ip\big(L_2(0,T;V)^n \to L_2(0,T;V^*)\big)$, and*

$$\begin{aligned}\big\langle A(u,\cdots,u_{n-2},u_{n-1}^1) - A(u,\cdots,u_{n-2},u_{n-1}^2), u_{n-1}\big\rangle_{L_2(0,T;V)} \\ \geq a\|u_{n-1}\|^2_{L_2(0,T;V)} + b\|u_{n-1}\|^2_{L_2(0,T;E)} \\ \forall u,\cdots,u_{n-2},u_{n-1}^1,u_{n-1}^2 \in L_2(0,T;V),\end{aligned}$$

where $a > 0$ and b is a real number. Also, let $\mathbb{U}_0 \in V^{n-1} \times E$ and $f \in L_2(0,T;V^*)$.

Then, there exists a unique solution to (3.12) such that $u \in C^{n-2}([0,T];V)$, $u^{(n-1)} \in C([0,T];H) \cap L_2(0,T;V)$ and $u^{(n)} \in L_2(0,T;V^*)$.

REMARK 3.2. *Theorems 3.3 and 3.4 have applications in semilinear hyperbolic differential equations, and Theorems 3.5 and 3.6 do it in pseudo-hyperbolic differential equations. But here we omit the concrete examples for application .*

The assertion of generation of nonlinear group in Theorems 3.3 and 3.4 may be obtained by perturbation of a generator of linear operator semigroup with a Lipschitz operator.

Theorem 3.5 can not be applied to PDE with coefficients variable in time t, but Theorem 3.6 is useful for such cases.

4. Analytic semigroup and semilinear and quasilinear equations. In this section, first, we get a sufficient condition for a matrix of operators to generate an analytic semigroup.

THEOREM 4.1. *Let V and E be Hilbert spaces such that $V \hookrightarrow E$. Suppose that $A_i \in \mathcal{BL}(V,V^*)$, $i = 0,1,\cdots,n-1$, and there exists a constant $m > 0$ and a real number k such that*

$$\mathrm{Re}\langle A_{n-1}u,u\rangle_V \geq m\|u\|_V^2 + k\|u\|_E^2 \quad \forall u \in V,$$

where V^* is the space of all antilinear continuous functionals on V.

Then, the operator $\mathbb{A}$ generates an analytic semigroups on $V^{n-1} \times V^*$ and so does the restriction of $\mathbb{A}$ on $V^{n-1} \times E$, where

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{n-1} \end{pmatrix}.$$

Proof. Let $\mathbb{Y} = V^n$, $\mathbb{X} = V^{n-1} \times E$. Then, as the proof of Theorem 3.5 we can write $\mathbb{Y}^* = V^{n-1} \times V^*$ (cf. 2.2 in [26]) and $\mathbb{A}$ is an operator from $\mathscr{D}(\mathbb{A}) = \mathbb{Y}$ to $\mathbb{Y}^*$. Define a functional on $\mathbb{Y} \times \mathbb{Y}$ by

$$a(\mathbb{U},\mathbb{V}) \equiv \langle(-\mathbb{A}+\omega I)\mathbb{U},\mathbb{V}\rangle_Y \quad \text{for} \quad \mathbb{U},\mathbb{V} \in \mathbb{Y},$$

where ω is a real number determined later. Let $\varepsilon > 0$ and $M = \max_{0\le i\le n-2}\{\|A_i\|_{V\to V^*}\}$. Then, we have

$$\begin{aligned}
&\operatorname{Re} a(\mathbb{U},\mathbb{U})\\
&= \operatorname{Re}\left(-\sum_{i=0}^{n-2}(u_{i+1},u_i)_V + \sum_{k=0}^{n-1}\langle A_k u_k, u_{n-1}\rangle_V\right) + \omega\sum_{i=0}^{n-2}\|u_i\|_V^2 + \omega\|u_{n-1}\|_E^2\\
&\ge -\sum_{i=0}^{n-3}\left(\frac{\|u_{i+1}\|_V^2}{2}+\frac{\|u_i\|_V^2}{2}\right) - \|u_{n-1}\|_V\cdot\|u_{n-2}\|_V + m\|u_{n-1}\|_V^2 + k\|u_{n-1}\|_E^2\\
&\qquad - M\sum_{i=0}^{n-2}\|u_i\|_V\|u_{n-1}\|_V + \omega\sum_{i=0}^{n-2}\|u_i\|_V^2 + \omega\|u_{n-1}\|_E^2\\
&\ge -\sum_{i=0}^{n-3}\|u_i\|_V^2 + \frac{\|u_0\|_V^2}{2} - \frac{\|u_{n-2}\|_V^2}{2} - \frac{1}{2\varepsilon}\|u_{n-2}\|_V^2 - \frac{\varepsilon}{2}\|u_{n-1}\|_V^2 - M\sum_{i=0}^{n-2}\frac{1}{2\varepsilon}\|u_i\|_V^2\\
&\qquad - \frac{M\varepsilon}{2}(n-1)\|u_{n-1}\|_V^2 + m\|u_{n-1}\|_V^2 + k\|u_{n-1}\|_E^2 + \omega\sum_{i=0}^{n-2}\|u_i\|_V^2 + \omega\|u_{n-1}\|_E^2\\
&\ge [m - \frac{\varepsilon}{2}(M(n-1)+1)]\|u_{n-1}\|_V^2 - (\frac{M}{2\varepsilon}+1)\sum_{i=0}^{n-3}\|u_i\|_V^2\\
&\qquad - (\frac{M+1}{2\varepsilon}+\frac{1}{2})\|u_{n-2}\|_V^2 + k\|u_{n-1}\|_E^2 + \omega\sum_{i=0}^{n-2}\|u_i\|_V^2 + \omega\|u_{n-1}\|_E^2.
\end{aligned}$$

Now, putting

$$\varepsilon = \frac{m}{M(n-1)+1}, \quad \omega = \max\left\{\frac{M+1}{2\varepsilon}+1, -k\right\}+\delta, \quad \delta > 0, \tag{4.1}$$

we have

$$Re\langle(-\mathbb{A}+\omega I)\mathbb{U},\mathbb{U}\rangle_{\mathbb{Y}} \ge \frac{m}{2}\|u_{n-1}\|_V^2 + \delta\sum_{i=0}^{n-2}\|u_i\|_V^2 \ge \min\{\frac{m}{2},\delta\}\cdot\|\mathbb{U}\|_{\mathbb{Y}}^2. \tag{4.2}$$

On the other hand, from the condition of theorem it follows that

$$\exists M_1 > 0: \quad |a(\mathbb{U},\mathbb{V})| \le M_1\cdot\|\mathbb{U}\|_{\mathbb{Y}}\cdot\|\mathbb{V}\|_{\mathbb{Y}} \quad \forall\,\mathbb{U},\mathbb{V}\in\mathbb{Y}.$$

Therefore, when V and E are complex Hilbert spaces, by Theorem 3.6.1 in [26] the operator $\mathbb{A}-\omega I$ and its restriction generate analytic semigroups, respectively, on the space $V^{n-1}\times V^*$ and $V^{n-1}\times E$. Thus, in the case of complex spaces the theorem is proved, because the value of ω is no mater. (cf. Remark 3.3.2 of ch. 3 in [26]) When V and E are real Hilbert spaces, the theorem is proved by complexification. □

Remark 4.1. *For the pseudo-hyperbolic systems, the condition in Theorem 4.1 is more useful than one in [5], because it is not required that A_i are self-adjoint and positive-definite.*

Let us study an initial value problem

$$u^{(n)} + Au^{(n-1)} + f(t,u,\cdots,u^{(n-1)}) = 0, \tag{4.3}$$

$$u(0) = u_0,\, u'(0) = u_0', \cdots, u^{(n-1)}(0) = u_0^{(n-1)}, \tag{4.4}$$

where V and E are Hilbert spaces such that $V \hookrightarrow E$. Let $V_\theta^* = [V, V^*]_{1-\theta},\ 0 \le \theta \le 1$ (complex interpolation space, see [17]).

Lemma 4.2. *Suppose that the conditions of Theorem 4.1 are satisfied and $\mathscr{D}\big((-\mathbb{A} + \omega I)^\theta\big),\ 0 < \theta \le 1$, is the Banach space with a norm equivalent to the graph norm of $(-\mathbb{A} + \omega I)^\theta$, where ω is the number in* (4.1).

Then, the space $\mathscr{D}\big((-\mathbb{A}+\omega I)^\theta\big)$ is continuously imbedded into the space $V^{n-1} \times V_{\theta'}^*$, $0 \le \theta' \le \theta \le 1$.

Proof. Let $\mathbb{Y} = V^n$, $\mathbb{Y}^* = V^{n-1} \times V^*$ and $J \in (\mathbb{Y} \to \mathbb{Y}^*)$ be the duality operator. Then, the interpolation space $[\mathbb{Y}, \mathbb{Y}^*]_{1-\theta}$ is the space $\mathscr{D}(J^\theta)$ with a norm equivalent to $\|J^\theta \mathbb{U}\|_{\mathbb{Y}^*}$ and $[\mathbb{Y}, \mathbb{Y}^*]_{1-\theta} = V^{n-1} \times [V, V^*]_{1-\theta}$. By proposition 2.3, ch. 1 in [17],

$$\|\mathbb{U}\|_{\mathscr{D}(J^\theta)} \le C\|\mathbb{U}\|_{\mathbb{Y}}^\theta \cdot \|\mathbb{U}\|_{\mathbb{Y}^*}^{1-\theta} \quad \forall \mathbb{U} \in \mathbb{Y}. \tag{4.5}$$

On the other hand, by (4.2) we have

$$\exists \omega_1 > 0 : \|(-\mathbb{A} + \omega I)\mathbb{U}\|_{\mathbb{Y}^*} \ge \omega_1 \|\mathbb{U}\|_{\mathbb{Y}}. \tag{4.6}$$

By virtue of (4.5) and (4.6), we get

$$\|\mathbb{U}\|_{\mathscr{D}(J^\theta)} \le C_1 \|(-\mathbb{A} + \omega I)\mathbb{U}\|_{\mathbb{Y}^*}^\theta \cdot \|\mathbb{U}\|_{\mathbb{Y}^*}^{1-\theta} \quad \forall \mathbb{U} \in \mathbb{Y}. \tag{4.7}$$

By Theorem 4.1, the operator $\mathbb{A} - \omega I$ is a generator of an analytic semigroup in $\mathbb{Y}^*$, and so $-\mathbb{A} + \omega I$ is sectorial. Thus, by (4.7) (cf. Exercise 11, section 4, ch.1, in [11])

$$\|J^\theta \mathbb{U}\|_{\mathbb{Y}^*} \le C\|\mathbb{U}\|_{\mathscr{D}((-\mathbb{A}+\omega I)^\theta)} \quad \forall \mathbb{U} \in \mathscr{D}\big((-\mathbb{A} + \omega I)^\theta\big),$$

which means

$$\mathscr{D}\big((-\mathbb{A} + \omega I)^\theta\big) \hookrightarrow \mathscr{D}(J^\theta) = V^{n-1} \times [V, V^*]_{1-\theta} \hookrightarrow V^{n-1} \times [V, V^*]_{1-\theta'} = V^{n-1} \times V_{\theta'}^*.$$

□

Theorem 4.3. *Suppose the following conditions are satisfied:*

1) $A \in \mathcal{BL}(V, V^*)$ and there exists $m > 0$ and a real number k such that

$$Re\langle Au, u\rangle_V \ge m\|u\|_V^2 + k\|u\|_E^2 \quad \forall\, u \in V;$$

2) $f \in ([0,T] \times V^{n-1} \times V_\theta^* \to V^*)$ and
$\forall\, t,\, \tau \in [0,T], \forall r > 0, \forall \mathbb{U}, \mathbb{V} \in \mathscr{O}_r$:

$$\|f(t, \mathbb{U}) - f(\tau, \mathbb{V})\|_{V^*} \le K(r)\big(|t - \tau|^\varepsilon + \|\mathbb{U} - \mathbb{V}\|_{V^{n-1} \times V_\theta^*}\big),\ 0 \le \varepsilon \le 1,\ 0 \le \theta < 1,$$

where $\mathscr{O}_r$ is the r-neighborhood of zero element of $V^{n-1} \times V_\theta^*$;

3) $\mathbb{U}_0 \equiv (u_0, u_0', \cdots, u_0^{(n-1)}) \in V^{n-1} \times V_\theta^*$.

Then, the initial value problem (4.3), (4.4) *has a unique local solution* $u \in C^{n-2}([0, t_0); V) \cap C^{n-1}((0, t_0); V) \cap C^{n-1}([0, t_0); V^*) \cap C^n((0, t_0); V^*)$, *where* $t_0 > 0$.

If, in addition, $K(r)$ *is independent of* r *and* $T = \infty$, *then there exists a unique solution on* $[0, \infty)$.

Proof. The first conclusion is equivalent to the existence of a unique solution $\mathbb{U} \in C([0, t_0);$
$V^{n-1} \times V^*) \cap C^1((0, t_0); V^{n-1} \times V^*)$ to problem

$$\begin{aligned} &\dot{\mathbb{U}} - (\mathbb{A} - \omega I)\mathbb{U} = F(t, \mathbb{U}), \\ &\mathbb{U}(0) = \mathbb{U}_0, \end{aligned} \tag{4.8}$$

where

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & -A \end{pmatrix}, \quad F(t, \mathbb{U}) = \begin{pmatrix} \omega u \\ \omega u_1 \\ \vdots \\ -f(t, \mathbb{U}) + \omega u_{n-1} \end{pmatrix}$$

and ω is the number in (4.1) with $A_i = 0\,(i = 0 \sim n-1)$, $A_{n-1} = A$. From the condition 2) of theorem, we have
(4.9)

$$\begin{aligned} \|F(t, \mathbb{U}) - F(\tau, \mathbb{V})\|_{\mathbb{Y}^*} &\le \Big\{\omega^2 \sum_{i=0}^{n-2} \|u_i - v_i\|_V^2 \\ &+ \big[K(r)(|t-\tau|^\varepsilon + \|\mathbb{U} - \mathbb{V}\|_{V^{(n-1)} \times V_\theta^*}) + \omega \|u_{n-1} - v_{n-1}\|_{V^*} + K|t-\tau|^\varepsilon\big]^2\Big\}^{\frac{1}{2}} \\ &\le K'(r)\big[|t - z|^\varepsilon + \|\mathbb{U} - \mathbb{V}\|_{V^{n-1} \times V_\theta^*}\big] \\ &\qquad \forall\, t, \tau \in [0, T],\ \forall\, \mathbb{U}, \mathbb{V} \in \mathscr{O}_r. \end{aligned}$$

From (4.2) it follows that if $0 \le \theta \le \theta' < 1$, then the graph norm of $\mathbb{U}$ in $\mathscr{D}((-\mathbb{A} + \omega I)^{\theta'})$ is equivalent to $\|(-\mathbb{A} + \omega I)^{\theta'} \mathbb{U}\|_{\mathbb{Y}^*}$. And by Lemma 4.2, $\mathscr{D}((-\mathbb{A} + \omega I)^{\theta'})$ is continuously embedded into $V^{(n-1)} \times V_{\theta'}^*$. Thus, we have

$$\begin{aligned} \|\mathbb{U}\|_{V^{n-1} \times V_\theta^*} \le K_1 \|U\|_{\mathscr{D}((-\mathbb{A}+\omega I)^{\theta'})} &\le K_2 \|(-\mathbb{A} + \omega I)^{\theta'})\mathbb{U}\|_{\mathbb{Y}^*} \\ &\forall\, \mathbb{U} \in \mathscr{D}((-\mathbb{A} + \omega I)^{\theta'}), \end{aligned}$$

from which it follows that if $\|(-\mathbb{A} + \omega I)^{\theta'} \mathbb{U}\|_{\mathbb{Y}^*}$, $\|(-\mathbb{A} + \omega I)^{\theta'} \mathbb{V}\|_{\mathbb{Y}^*} < r_1 \equiv \frac{r}{K_2}$, then $\mathbb{U}, \mathbb{V} \in \mathscr{O}_r$. Therefore, from (4.9) we have that if $t, \tau \in [0, T]$, $\mathbb{U}, \mathbb{V} \in \mathscr{D}((-\mathbb{A} + \omega I)^{\theta'})$ and $\|(-\mathbb{A} + \omega I)^{\theta'} \mathbb{U}\|_{\mathbb{Y}^*}$, $\|(-\mathbb{A} + \omega I)^{\theta'} \mathbb{V}\|_{\mathbb{Y}^*} < r_1$, then

$$\|F(t, \mathbb{U}) - F(\tau, \mathbb{V})\|_{\mathbb{Y}^*} \le K''(r)(|t - \tau|^\varepsilon + \|\mathbb{U} - \mathbb{V}\|_{\mathscr{D}((-\mathbb{A}+\omega I)^{\theta'})}).$$

Consequently, by Theorem 3.1 of ch. 6 in [21], there exists a unique local solution to problem (4.8).

Let us prove the second conclusion.

By (4.2) the number 0 belongs to the resolvent set $\rho((\mathbb{A} - \omega I))$ and the analytic semigroup $T(t)$ generated by the operator $\mathbb{A} - \omega I$ is bounded on $[0, \infty)$. And if $t \in [0, \infty)$ and $\mathbb{U} \in \mathscr{D}((-\mathbb{A} + \omega I)^{\theta'}), 0 \le \theta < \theta' < 1$, then

$$\begin{aligned} \|F(t, \mathbb{U})\|_{\mathbb{Y}^*} &\le \|F(t, \mathbb{U}) - F(0, 0_{\mathbb{Y}})\|_{\mathbb{Y}^*} + \|F(0, 0_{\mathbb{Y}})\|_{\mathbb{Y}^*} \\ &\le K\big(t^\varepsilon + \|\mathbb{U}\|_{V^{n-1} \times V_\theta^*}\big) + \|F(0, 0_{\mathbb{Y}})\|_{\mathbb{Y}^*}. \end{aligned}$$

Define

$$k(t) = \begin{cases} K + \|F(0, 0_{\mathbb{Y}})\|_{\mathbb{Y}^*} & t \in [0,1], \\ Kt^{\varepsilon} + \|F(0, 0_{\mathbb{Y}})\|_{\mathbb{Y}^*} & t > 1, \end{cases}$$

which is a continuous nondecreasing function. Then, taking into account Lemma 4.2, we have

$$\|F(t,\mathbb{U})\|_{\mathbb{Y}^*} \le k(t)\big(1 + \|\mathbb{U}\|_{V^{n-1}\times V_\theta^*}\big) \le Kk(t)\big(1 + \|\mathbb{U}\|_{\mathscr{D}((-\mathbb{A}+\omega I)^{\theta'})}\big).$$

Thus, by Theorem 3.3 of ch. 6 in [21], we come to the second conclusion. □

Let us study another initial value problem

$$(4.10) \qquad \begin{aligned} &u^{(n)} + A_{n-1}(t,u,\cdots,u^{(n-1)})u^{(n-1)} \\ &\quad + \cdots + A_0(t,u,\cdots,u^{(n-1)})u = f(t,u,\cdots,u^{(n-1)}), \end{aligned}$$

$$(4.11) \qquad u(0) = u_0, u'(0) = u_0', \cdots, u^{(n-1)}(0) = u_0^{(n-1)},$$

where $A_i(t,v,\cdots,v_{n-1}) \in \mathcal{BL}(V,V^*)$ for every fixed $(t,v,\cdots,v_{n-1}) \in [0,T]\times V^{n-1}\times V_\theta^*$, $f \in ([0,T]\times V^{n-1}\times V_\theta^* \to V^*)$ and $0 \le \theta < 1$.

First, we study an equation

$$(4.12) \qquad u^{(n)} + A_{n-1}(t,u,\cdots,u^{(n-1)})u^{(n-1)} = f(t,u,\cdots,u^{(n-1)}).$$

THEOREM 4.4. *Suppose that*

1) There exists $m > 0$ and a real number k such that

$$Re\langle A_{n-1}(0,\mathbb{U}_0)u,u\rangle_V \ge m\|u\|_V^2 + k\|u\|_E^2 \quad \forall u \in V;$$

2) For $\mathbb{U}_1, \mathbb{U}_2 \in \mathscr{O}_r$ and $t,\tau \in [0,T]$,

$$\|A_{n-1}(t,\mathbb{U}_1) - A_{n-1}(\tau,\mathbb{U}_2)\|_{V\to V^*} \le K(r)\big(|t-\tau|^{\varepsilon} + \|\mathbb{U}_1 - \mathbb{U}_2\|_{V^{n-1}\times V_\theta^*}\big)$$

and

$$\|f(t,\mathbb{U}_1) - f(\tau,\mathbb{U}_2)\|_{V^*} \le K(r)\big(|t-\tau|^{\varepsilon} + \|\mathbb{U}_1 - \mathbb{U}_2\|_{V^{n-1}\times V_\theta^*}\big),$$

where $0 \le \varepsilon \le 1$, $\mathscr{O}_r$ is the same as in Theorem 4.2 and $\|\cdot\|_{V\to V^*}$ means the norm of the space $\mathcal{BL}(V,V^*)$,

3) $\mathbb{U}_0 \equiv (u_0, u_0', \cdots, u_0^{(n-1)}) \in V^n$.

Then, the initial value problem (4.11),(4.12) *has a unique local solution on* $[0,t_0)$, $t_0 > 0$.

Proof. The conclusion is equivalent to the existence of a unique solution

$$\mathbb{U} \in C([0,t_0);V^{n-1}\times V^*) \cap C^1((0,t_0);V^{n-1}\times V^*)$$

to problem

$$\begin{aligned} &\dot{\mathbb{U}} - \mathbb{A}_1(t,\mathbb{U})\mathbb{U} = F_1(t,\mathbb{U}), \\ &\mathbb{U}(0) = \mathbb{U}_0, \end{aligned}$$

where

$$\mathbb{A}_1(t,\mathbb{U}) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -A_{n-1}(t,\mathbb{U}) \end{pmatrix} - \omega I, \ \ F_1(t,\mathbb{U}) = \begin{pmatrix} \omega u \\ \omega u_1 \\ \vdots \\ f(t,\mathbb{U}) + \omega u_{n-1} \end{pmatrix},$$

and ω is the number in (4.8) with $A = A_{n-1}(0,\mathbb{U}_0)$.

Using (3.51) of Lemma 3.6.1 in [26], we obtain

$$\|[\mathbb{A}_1(0,\mathbb{U}_0)+\lambda]^{-1}\|_{\mathbb{Y}^*\to\mathbb{Y}^*} \le \frac{M_1}{|\lambda|} \quad \text{for} \operatorname{Re}\lambda \ge 0. \tag{4.13}$$

On the other hand, from (4.2) it follows that there exists $\delta_1 > 0$ such that

$$\|\mathbb{A}_1(0,\mathbb{U}_0)^{-1}\mathbb{U}\|_{\mathbb{Y}} \le \delta_1\|\mathbb{U}\|_{\mathbb{Y}^*} \ \forall \mathbb{U} \in \mathbb{Y}^*, \tag{4.14}$$

where $\mathbb{Y}^*$ is the same as in Theorem 4.1. Inequality (4.14) implies that the number 0 belongs to the resolvent set $\rho(\mathbb{A}_1(0,\mathbb{U}_0))$. This fact together with (4.13) implies that

$$\|[\mathbb{A}_1(0,\mathbb{U}_0)+\lambda]^{-1}\|_{\mathbb{Y}^*\to\mathbb{Y}^*} \le \frac{M_2}{1+|\lambda|} \quad \text{for} \operatorname{Re}\lambda \ge 0.$$

In view of (4.14) and the condition 2) of theorem, we know that

$$\begin{aligned} &\|[\mathbb{A}_1(t,\mathbb{U}_1) - \mathbb{A}_1(\tau,\mathbb{U}_2)]\cdot\mathbb{A}_1(0,\mathbb{U}_0)^{-1}\|_{\mathbb{Y}^*\to\mathbb{Y}^*} \\ &\qquad \le \|A_{n-1}(t,\mathbb{U}_1) - A_{n-1}(\tau,\mathbb{U}_2)\|_{V\to V^*}\cdot\delta_1 \\ &\qquad \le K'(r)\big(|t-\tau|^{\varepsilon} + \|\mathbb{U}_1-\mathbb{U}_2\|_{V^{n-1}\times V_\theta^*}\big) \\ &\qquad\qquad \forall\, t,\tau\in[0,T],\ \forall\,\mathbb{U}_1,\mathbb{U}_2\in\mathscr{O}_r. \end{aligned} \tag{4.15}$$

Moreover, as the proof of Theorem 4.3 we have that

$$\begin{aligned} \|F_1(t,\mathbb{U}_1) - F_1(\tau,\mathbb{U}_2)\|_{\mathbb{Y}^*} &\le K'(r)\big(|t-\tau|^{\varepsilon} + \|\mathbb{U}_1-\mathbb{U}_2\|_{V^{n-1}\times V_\theta^*}\big) \\ &\qquad \forall\, t,\tau\in[0,T],\ \forall\,\mathbb{U}_1,\mathbb{U}_2\in\mathscr{O}_r. \end{aligned} \tag{4.16}$$

Let us take θ' such that $0 \le \theta < \theta' < 1$. Since $\mathbb{U}_0 \in \mathbb{Y}$ and $\mathbb{A}_1(0,\mathbb{U}_0) \in \mathcal{BL}(\mathbb{Y},\mathbb{Y}^*)$, there exists a number $r_1 > 0$ such that

$$\|\mathbb{A}_1^{\theta'}(0,\mathbb{U}_0)\cdot\mathbb{U}_0\|_{\mathbb{Y}^*} \le K\|\mathbb{A}_1(0,\mathbb{U}_0)\cdot\mathbb{U}_0\|_{\mathbb{Y}^*} < r_1. \tag{4.17}$$

On the other hand, by Lemma 4.2 we have

$$\begin{aligned} \|\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\cdot\mathbb{U}\|_{V^{n-1}\times V_\theta^*} &\le K_1\|\mathbb{A}_1^{\theta'}(0,\mathbb{U}_0)\cdot\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\cdot\mathbb{U}\|_{\mathbb{Y}^*} \\ &\le K_1\|\mathbb{U}\|_{\mathbb{Y}^*} \quad \forall\,\mathbb{U}\in\mathbb{Y}^*. \end{aligned}$$

Thus, if $\|\mathbb{U}\|_{\mathbb{Y}^*} \le r_2 \equiv \frac{r}{K_1}$, then $\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\mathbb{U}$ belongs to $\mathscr{O}_r$. Therefore, substituting $\mathbb{U}_i$ with $\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\mathbb{U}_i$ in (4.15), (4.16) and applying Lemma 4.2, we have that if $\|\mathbb{U}_1\|_{\mathbb{Y}^*}$, $\|\mathbb{U}_2\|_{\mathbb{Y}^*} \le r_2$, then

$$\begin{aligned} &\|[\mathbb{A}_1(t,\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\mathbb{U}_1) - \mathbb{A}_1(\tau,\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\mathbb{U}_2)]\mathbb{A}_1(0,\mathbb{U}_0)^{-1}\|_{\mathbb{Y}^*\to\mathbb{Y}^*} \\ &\qquad\qquad \le K_2(r)(|t-\tau|^{\varepsilon} + \|\mathbb{U}_1-\mathbb{U}_2\|_{\mathbb{Y}^*}), \\ &\|F_1(t,\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\mathbb{U}_1) - F_1(\tau,\mathbb{A}_1^{-\theta'}(0,\mathbb{U}_0)\mathbb{U}_2)\|_{\mathbb{Y}^*} \\ &\qquad\qquad \le K_3(r)\big(|t-\tau|^{\varepsilon} + \|\mathbb{U}_1-\mathbb{U}_2\|_{\mathbb{Y}^*}\big). \end{aligned} \tag{4.18}$$

Consequently, by Theorem 7 of [24], from (4.17), (4.18) we get the conclusion. □

COROLLARY 4.5. *Assume that the conditions in Theorem 4.4 are satisfied. If condition 2) for* A_{n-1} *is valid for* $A_i, i = 0, 1, \cdots, n-2,$ *then problem* (4.10), (4.11) *has a unique local solution on* $[0, t_0), t_0 > 0$.

Proof. When $A_i(t, u, \cdots, u^{(n-1)})u^{(i)},\ i = 0, 1, \cdots, n-2,$ are included into f, problem (4.10), (4.11) is reduced to problem (4.11), (4.12). Thus, Theorem 4.4 implies the corollary. □

REMARK 4.2. *If condition 1) in Theorem 4.4 is satisfied for any* $\mathbb{U}_0 \in V^n$ *and* $K(r)$ *is independent of* r, *we obtain existence of a global solution.*

Applying the abstract results above, we can get unique existence of the solutions to systems of pseudo-hyperbolic equations, which are given below in Theorems 4.6 and 4.7.

Let Ω be a bounded domain of R^n of class C^∞ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, where α_i are nonnegative integers, $|\alpha| = \sum \alpha_i$ and $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Let $u = (u_1, \cdots, u_l)^T$, where u_i are real functions on Ω.

First, let us consider the following mixed problem of a system of pseudo-hyperbolic equations

(4.19)
$$u_{tt} + (-1)^m \sum_{|\alpha|=m} \frac{\partial^\alpha}{\partial x^\alpha}\big[A_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}u_t\big] + \sum_{|\beta|\le m}\frac{\partial^\beta}{\partial x^\beta}A'_\beta\big(t, x, u_t, \cdots, \frac{\partial^\gamma}{\partial x^\gamma}u_t, \cdots\big)$$
$$+ \sum_{|\alpha|\le m}\frac{\partial^\alpha}{\partial x^\alpha}B_\alpha\big(t, x, u, \cdots, \frac{\partial^\delta}{\partial x^\delta}u, \cdots, u_t, \cdots, \frac{\partial^\sigma}{\partial x^\sigma}u_t, \cdots\big) = \sum_{|\alpha|\le m}\frac{\partial^\alpha}{\partial x^\alpha}f_\alpha(t, x),$$

$$u(0, x) = u_0(x) \in H_0^m(\Omega)^l, \quad u_t(0, x) = u'_0(x) \in H_0^{m-1}(\Omega)^l, \tag{4.20}$$

$$\frac{\partial^\alpha u(t, x)}{\partial \nu^\alpha}\ |_{\partial\Omega} = 0, \quad |\alpha| \le m-1, \tag{4.21}$$

where $|\gamma|, |\sigma| \le m-1$, $|\delta| \le m,$ and ν is outward normal unit vector on $\partial\Omega$.

THEOREM 4.6. *Suppose the following conditions hold:*

1) Matrices $A_\alpha(x),\ |\alpha| = m$, *are positive-definite at a.a.* $x \in \Omega$ *and their elements belong to* $L_\infty(\Omega)$;

2) $A'_\beta \in ([0, \infty) \times \Omega \times R^N \to R^l)$ *and*

$$\|A'_\beta(t, x, \xi^1) - A'_\beta(\tau, x, \xi^2)\|_{R^l} \le K\big(|t-\tau|^\varepsilon + \sum_{i=1}^N |\xi_i^1 - \xi_i^2|\big),\ 0 \le \varepsilon \le 1,\ \forall \xi_1, \xi_2 \in R^N;$$

3) $B_\alpha \in ([0, \infty) \times \Omega \times R^{N_1} \to R^l)$ *and*

$$\|B_\alpha(t, x, \eta^1) - B_\alpha(\tau, x, \eta^2)\|_{R^l} \le K_1\big(|t-\tau|^\varepsilon + \sum_{i=1}^{N_1} |\eta_i^1 - \eta_i^2|\big)\ \forall \eta^1, \eta^2 \in R^{N_1};$$

4) $f_\alpha(t, \cdot) \in L_2(\Omega)^l$ *and* $\|f_\alpha(t, x) - f_\alpha(\tau, x)\|_{L_2(\Omega)^l} \le K|t-\tau|^\varepsilon$.

Then, problem (4.19)∼(4.21) *has a unique solution*

$$u(t,x) \in C([0,\infty); H_0^m(\Omega)^l) \cap C^1((0,\infty); H_0^m(\Omega)^l) \cap C^1([0,\infty); H^{-m}(\Omega)^l) \cap C^2((0,\infty); H^{-m}(\Omega)^l).$$

Next, let us consider the following mixed problem

$$\begin{aligned} &u_{tt} + (-1)^m \sum_{|\alpha|=m} \frac{\partial^\alpha}{\partial x^\alpha}\big[A_\alpha\big(t,x,u,\cdots,\frac{\partial^\beta u}{\partial x^\beta},\cdots,u_t,\cdots,\frac{\partial^\gamma u_t}{\partial x^\gamma},\cdots\big)\frac{\partial^\alpha u_t}{\partial x^\alpha}\big] \\ &\quad + \sum_{|\alpha|\le m, |\delta|\le m-1} \frac{\partial^\alpha}{\partial x^\alpha}\big[A_{\alpha\delta}\big(t,x,u,\cdots,\frac{\partial^\beta u}{\partial x^\beta},\cdots,u_t,\cdots,\frac{\partial^\gamma u_t}{\partial x^\gamma},\cdots\big)\frac{\partial^\delta u_t}{\partial x^\delta}\big] \\ &\quad + \sum_{|\alpha|\le m, |\rho|\le m} \frac{\partial^\alpha}{\partial x^\alpha}\big[B_{\alpha\rho}\big(t,x,u,\cdots,\frac{\partial^\beta u}{\partial x^\beta},\cdots,u_t,\cdots,\frac{\partial^\gamma u_t}{\partial x^\gamma},\cdots\big)\frac{\partial^\rho u}{\partial x^\rho}\big] \\ &\quad + \sum_{|\alpha|\le m} \frac{\partial^\alpha}{\partial x^\alpha} F_\alpha\big(t,x,u,\cdots,\frac{\partial^\sigma u}{\partial x^\sigma},\cdots,u_t,\cdots,\frac{\partial^q u_t}{\partial x^q},\cdots\big) \\ &\quad = \sum_{|\alpha|\le m} \frac{\partial^\alpha}{\partial x^\alpha} G_\alpha(t,x), \end{aligned} \tag{4.22}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_0'(x) \in H_0^m(\Omega)^l, \tag{4.23}$$

$$\frac{\partial^\alpha u(t,x)}{\partial \gamma^\alpha}\ |_{\partial\Omega} = 0, \quad |\alpha| \le m-1, \tag{4.24}$$

where $|\sigma| \le m,\ |q| \le m-1,\ |\beta|, |\gamma| < m - n/2$.

THEOREM 4.7. *Suppose the following conditions hold:*

1) The matrices $A_\alpha(0,x,u_0(x),\cdots,\frac{\partial^\beta u_0(x)}{\partial x^\beta},\cdots,u_0'(x),\cdots,\frac{\partial^\gamma u_0'(x)}{\partial x^\gamma},\cdots)$ *are positive definite at a.a.* $x \in \Omega$;

2) As a function of $(t,x,\xi) \in [0,T] \times \Omega \times R^N$, *every component* $a(t,x,\xi)$ *of* $A_\alpha, A_{\alpha\delta}, B_{\alpha\rho}$ *and* F_α *is bounded and measurable with respect to* x *at every fixed* (t,ξ), *and continuous with respect to* (t,ξ) *at a.a.* $x \in \Omega$;

3) $|a(t,x,\xi) - a(\tau,x,\xi^1)| \le K(r)(|t-\tau|^\varepsilon + \sum_{n=1}^N |\xi_i - \xi_i^1|)$ *for* $t,\tau \in [0,T]$ *and* $\xi,\ \xi^1 \in R^N(|\xi|_{R^N},\ |\xi^1|_{R^N} < r)$;

4) $G_\alpha(t,\cdot) \in L_2(\Omega)^l$ *and* $\|G_\alpha(t,x) - G_\alpha(\tau,x)\|_{L^2(\Omega)} \le K|t-\tau|^\varepsilon$,
where integer N *depends on* $A_\alpha, A_{\alpha\delta}, B_{\alpha\rho}$ *and* F, *and* $0 \le \varepsilon \le 1$.

Then, there exists a unique local solution

$$u(t,x) \in C([0,t_0); H_0^m(\Omega)^l) \cap C^1((0,t_0); H_0^m(\Omega)^l) \cap C^1([0,t_0); H^{-m}(\Omega)^l) \cap C^2((0,t_0); H^{-m}(\Omega)^l)$$

to problem (4.22)∼(4.24).

REMARK 4.3. *For equation (2) of [25], it was assumed that* B_α *are independent of* u_t *and their derivatives with respect to* x, $l = 1$, $B_\alpha(|\alpha| = m)$ *are linear and matrices* A_α, $B_\alpha(|\alpha| = m)$ *are symmetric. For linear equation (7) of [25], it was assumed that* A_α, $B_\alpha(|\alpha| = m)$ *are symmetric and* $l = 1$.

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REFERENCES

[1] N. C. Apreutesei, *Nonlinear second order evolution equations of monotone type and applications*, Pushpa Publishing House, Allahabad, 2007.
[2] H. Assa and M. Hesaaraki, *Nonexistence of solution for higher order evolution equations and inequalities*, Methods and Applications of Analysis, 12:1 (2005), pp. 1–18.
[3] P. Aviles and J. Sandefur, *Nonlinear second order equations with applications to partial differential equations*, J. Diff. Equat., 58:3 (1985), pp. 404–427.
[4] G. Chen, H. Guo, and H. Zhang, *Global existence of solutions of Cauchy problem for generalized system of nonlinear evolution equations arising from DNA*, J. Math. Phys., 50, 083514 (2009), pp. 1–23.
[5] S. Chen and R. Triggiani, *Proof of extension of two conjectures on structural damping for elastic system*, Pacific J. Math., 136:1 (1989), pp. 15–55.
[6] M. G. Crandall and T. M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, American Journal of Mathematics, 93:2 (1971), pp. 265–298.
[7] H. Gajewski, K. Gröger, and K. Zacharias, *Nichtlineare operatorgleichungen und operatordifferentialgleichungen*, Akademie-Verlag, 1974 (Russian version, Mir, 1978).
[8] D. Guidetti, *On boundary value problems for parabolic equations of higher order in time*, J. Diff. Equat., 124:1 (1996), pp. 1–26.
[9] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, 1985
[10] Yu. A. Dubinskii, *On some differential-operator equations of arbitrary order*, Math. USSR Sbornik, 19:1 (1973), pp. 1-21.
[11] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, 840, Springer-Verlag, 1981.
[12] N. D. Kopachevsky, R. Mennicken, Ju. S. Pashkova and C. Tretter, *Complete second order linear differential operator equations in Hilbert space and applications in hydrodynamics*, Trans. Amer. Math. Soc., 356 (2004), pp. 4737–4766.
[13] S. G. Krein, *Linear differential equations in Banach spaces*, AMS, Providence, 1971.
[14] S. G. Krein and M. I. Hazian, *Differential equations in Banach spaces*, Itogi Nauki i Tehniki, Mathematical Analyis, 21 (1983), pp. 130–263 (in Russian).
[15] L. I. Kuchminskaya, *Nonlinear differential-operator equations and their applications*, Dokl. Akad. Nauk USSR, 278:1 (1984), pp. 27–30.
[16] J. Liang and T. Xiao, *Well-posedness results for certain classes of Higher order abstract Cauchy problems connected with integrated semigroups*, Semigroup Forum, 56:1 (1998), pp. 84-103.
[17] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, I, Springer-Verlag, 1972.
[18] M. Necula, M. Popescu, and I. I. Vrabie, *Nonlinear evolution equations on locally closed graphs*, RACSAM, 104:1 (2010), pp. 97–114.
[19] F. Neubrander, *Well-posedness of higher order abstract Cauchy problems*, Trans. American Math. Society, 295:1 (1986), pp. 257–290.
[20] E. Obrecht, *The Cauchy problem for time-dependent abstract parabolic equations of higher order*, J. Math. Anal. Appl., 125:2 (1987), pp. 508–530.
[21] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.
[22] J. Sandefur, *Higher order abstract Cauchy problem*, J. Math. Anal. Appl., 60:3 (1977), pp. 728–742.
[23] J. Sandefur, *Existence and uniqueness of solutions of second order nonlinear differential equations*, SIAM J. Math. Anal., 14:3 (1983), pp. 477–487.
[24] P. E. Sobolevskii, *Equations of parabolic type in a Banach space*, Amer. Math. Soc. Transl, 49:2 (1965), pp. 1–62.
[25] H. Sun, *A nonlinear pseudo-hyperbolic system*, Science in China, A, 32:1 (1989), pp. 1–19.
[26] H. Tanabe, *Equations of evolution*, Pitman, 1979.
[27] T. Xiao and J. Liang, *Entire solutions of higher order abstract Cauchy problems*, J. Math. Anal. Appl., 208:2 (1997), pp. 298–310.

[28] T. Xiao and J. Liang, *Exponential stability of solutions for higher order abstract Cauchy problems*, Ibid., 215:2 (1997), pp. 485–498.

[29] T. Xiao and J. Liang, *The Cauchy problem for higher-order abstract differential equations*, Lecture Notes in Mathematics 1701, Springer, 1998.