METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 20, No. 2, pp. 165–178, June 2013

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A FREE BOUNDARY MODEL FOR KORTEWEG FLUIDS AS A LIMIT OF BAROTROPIC COMPRESSIBLE NAVIER-STOKES EQUATIONS*

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Abstract. We consider the limit of some barotropic compressible fluid model with Korteweg forcing term, studied in [1], as the exponent of the barotropic law goes to infinity. This provides a free boundary problem model, with capillary effects, and therefore generalizes the free boundary model obtained by Lions and Masmoudi [5]. Our interest for such free boundary problem stems from a study of the Leidenfrost effect.

Key words. Compressible Navier-Stokes equations, Korteweg fluid, free boundary problem.

AMS subject classifications. 35Q30, 35R35, 76N10, 76T10.

1. Introduction.

1.1. The physical motivations.

In this article, we focus on the dynamic behavior of a fluid submitted to an evaporation process. In particular, this study is motivated by the understanding of the Leidenfrost effect. This effect is observed, for example, for droplets of fluid set on heated support whose temperature is much higher than the evaporation temperature of the fluid. Many applications could be cited, from the optimization of lab-on-chip to the understanding of heating transfers default in nuclear plants.

This modeling problem is rather difficult to address mathematically in its full complexity. In this article, we present a first step toward the comprehension of the full mechanism, where we removed all thermals effects. In this study, the simplified model will give, as described below, asymptotically a complete isothermal Leidenfrost model.

1.1.1. The Leidenfrost phenomenon.

The Leidenfrost effect can be observed in the behavior of a droplet of water set on a heating support. If the support temperature is sufficiently high (typically 200 Celsius degrees), the droplet will slide on the support much more rapidly and longer than in the case of a drop on a less heated supports. This mechanism is very important and is the key of several phenomenas, in particular, it can be observed in cooling circuit of nuclear plants. Typically, we consider the case of a droplet of fluid separated from an heated support by a thin film of vapor. Several modeling milestones can be cited

- management of the fluid state transition at the interface,
- understanding of the interface evolution submitted to evaporation,
- choice of the fluid models in the gas and in the liquid phase.

In [3], a model is proposed and numerical simulations via a level-set method are exposed. The main choices are

- fluid and vapor are incompressible,
- the interface is a thin zone of compressible melting of fluid and vapor,

This system, temperature dependent, models the phase transition under heat constraint via a relaxed divergence in the compressible zone (the limit case is given by a divergence whose support is the interface between gaz and fluid).

*Received October 23, 2012; accepted for publication March 14, 2013.
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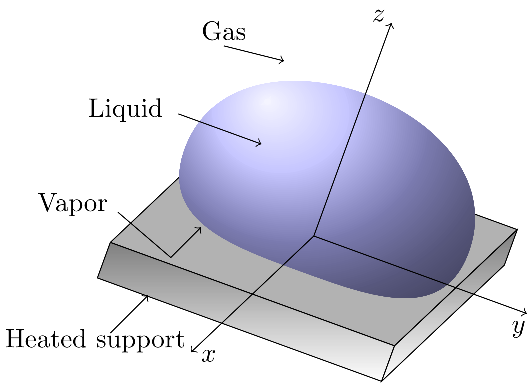

FIG. 1. *Leindenfrost effect.*

The size of the interface asymptotically has to vanish in order to obtain a Leidenfrost model. However we developed a numerical scheme based on the level-set method to the non sharp interface model. The model will induce very rich partial differential equations whose theoretical study is not directly reachable. In order to begin to understand the theoretical trends of the problem, we introduce a simplified model.

1.1.2. A simplified model of droplet. In this simplified model, only two species are considered (against three for the Leidenfrost relaxed system). Furthermore, the model does not take into consideration the temperature variations, in particular, the compressible zone does not manage the phase transition between liquid and gaz. The two species are an incompressible fluid and a compressible one. Using ideas introduced by Lions and Masmoudi [5], we prove existence of a solution to this model as a limit of a subsequence of solutions of a compressible Navier-Stokes-Korteweg equation which was studied by Bresch, Desjardins and Lin [1].

1.2. Goals of the study.

1.2.1. The Lions and Masmoudi free boundary problem. In [5], the authors studied in a very interesting way the existence of solutions for a free boundary problem between a compressible fluid and an incompressible one. The idea of the proof is to consider a barotropic compressible model and to pass to the limit when the pressure law exponent goes to infinity. If one chooses a suitable initial density, it is possible to obtain a compressible/incompressible coupling.

Let us recall more precisely this result. Let $T > 0$ and $\Omega = \mathbb{T}^3$ be the torus in $\mathbb{R}^3$. We consider a source term $f \in L^1(0,T;L^2(\Omega))$ and an intial density such that

$$\rho_n^0 \in L^1(\Omega), \quad 0 \le \rho_n^0 \le 1, \quad ⨍ \rho_n^0 \le M < 1.$$

Let $m_n^0 \in L^2(\Omega)$, and assume that (ρ_n^0, m_n^0) converges in some suitable space to some (ρ^0, m^0). For γ_n large enough, there exists (ρ_n, u_n) solution of the compressible Navier-Stokes problem:

$$\partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0 \qquad \text{in } \Omega \times (0,T) \qquad (1)$$

$$\partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n) - \nu \Delta u_n + \nabla \rho_n^{\gamma_n} = \rho_n f \qquad \text{in } \Omega \times (0,T) \qquad (2)$$

$$\rho_n = \rho_n^0, \quad \rho_n u_n = m_n^0 \qquad \text{in } \Omega \times \{0\} \qquad (3)$$

Lions and Masmoudi proved the convergence, when $\gamma_n \to \infty$ of (at least a subsequence of) (ρ_n, u_n) to a solution of the following free-boundary problem:

Find (ρ, u, Π) solution of

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad \text{in } \Omega \times (0,T) \tag{4}$$

$$0 \leq \rho \leq 1 \qquad \text{in } \Omega \times (0,T) \tag{5}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \nu \Delta u + \nabla \Pi = \rho f \qquad \text{in } \Omega \times (0,T) \tag{6}$$

$$\operatorname{div} u = 0 \qquad \text{a. e. in } \{\rho = 1\} \tag{7}$$

$$\Pi = 0 \qquad \text{a. e. in } \{\rho < 1\} \tag{8}$$

$$\Pi \geq 0 \qquad \text{a. e. in } \{\rho = 1\} \tag{9}$$

$$\rho = \rho^0, \quad \rho u = m^0 \qquad \text{in } \Omega \times \{0\} \tag{10}$$

The compressible part of the model is obtained with a zero pressure law. More general pressure law in the approximating compressible model could converge to a more general pressure law at the limit, but we will not study this generalization in the present work. Let us recall which weak meaning of solutions is understood in the above problem statement:

- Regularity of solutions: the following regularity was obtained for the above problem. $\forall p \in [1, +\infty)$,

$$\rho \in L^\infty(0,T; L^\infty(\Omega)) \cap \mathcal{C}(0,T; L^p(\Omega)),\ \nabla u \in L^2(0,T; L^2(\Omega)),$$
$$u \in L^2(0,T; H^1(\Omega)),\ \Pi \in \mathcal{M}(0,T; L^1(\Omega)).$$

- Initial conditions: $\rho u(0) = m^0 \in L^2(\Omega)$ and $\rho(0) = \rho^0 \in L^1(\Omega)$, $0 \leq \rho^0 \leq 1$ and $⨍ \rho^0 < 1$.
- Equation (4) is considered to hold almost everywhere.
- Condition (6) and equations (3-4) are compatible from the following:

  Proposition 1 ([5], Lemma 2.1). *Let $u \in L^2(0,T; H^1(\Omega))$ and $\rho \in L^2(\Omega \times (0,T))$ such that $\partial_t \rho + \operatorname{div}(\rho u) = 0$ and $\rho(0) = \rho^0$. Then the following two assertions are equivalent:*
  1. $\operatorname{div} u = 0$ *on* $\rho \geq 1$ *and* $0 \leq \rho^0 \leq 1$,
  2. $0 \leq \rho \leq 1$.

- Next, equations (3) and (5) are considered in the distributional sense.
- At last, the condition (7) and (8) are rewritten as

$$\rho \Pi = \Pi \geq 0$$

  where the product ρ times Π was given a meaning using the regularity results.

1.2.2. The Korteweg case: what we study. We are aiming at generalizing this existence result and limiting process to the case where the source term is a Korteweg type term, and the viscosity is proportional to density. This is of interest for our study since this would take into account the capillary energy on the interface between vapor and liquid.

We consider the following free-boundary problem: find (ρ, u) solution of

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \Omega \times (0,T) \tag{11}$$
$$0 \le \rho \le 1 \quad \text{in } \Omega \times (0,T) \tag{12}$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \nu \operatorname{div}(\rho D(u)) + \nabla \Pi = \kappa \rho \nabla(\Delta \rho) \quad \text{in } \Omega \times (0,T) \tag{13}$$
$$\operatorname{div} u = 0 \quad \text{a. e. in } \{\rho = 1\} \tag{14}$$
$$\Pi = 0 \quad \text{a. e. in } \{\rho < 1\} \tag{15}$$
$$\Pi \ge 0 \quad \text{a. e. in } \{\rho = 1\} \tag{16}$$
$$\rho = \rho^0, \quad \rho u = m^0 \quad \text{in } \Omega \times \{0\} \tag{17}$$

where κ and ν are positive constants. This problem will be shown to be the limit, when $\gamma_n \to \infty$ of a compressible Korteweg model studied by Bresch, Desjardins and Lin [1], in its weak formulation: see Theorem 1, section 2.2.

2. From a compressible Korteweg-Navier-Stokes problem to the free boundary problem.

2.1. A Korteweg barotropic Navier-Stokes model from Bresch, Desjardins and Lin. Let us recall the existence result obtained in [1]. Let P a pressure law such that $P(s) \ge 0$, $P'(s) \ge 0$. Let

$$\Xi(s) = \int_0^s \tau P'(\tau) d\tau, \qquad \Pi(s) = s \int_0^s \frac{P(\tau)}{\tau^2} d\tau.$$

Assume there exists $A > 0$, $\eta < +\infty$ when $d = 2$ (resp. $\eta < 4$ when $d = 3$) such that for s large enough,

$$\Xi(s) \le A s^\eta \Pi(s). \tag{18}$$

Let $m^0 \in L^2(\Omega)$ and $\rho^0 \in L^1(\Omega)$ verifying suitable energy estimates. Then there exists a global weak solution of

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \Omega \times (0,T) \tag{19}$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \nu \operatorname{div}(\rho D(u)) + \nabla P(\rho) = \kappa \rho \nabla(\Delta \rho) \quad \text{in } \Omega \times (0,T) \tag{20}$$
$$\rho = \rho^0, \quad \rho u = m^0 \quad \text{in } \Omega \times \{0\} \tag{21}$$

The existence of solutions of compressible Navier-Stokes equations with Korteweg term is obtained for this particular form of the viscous stress tensor, i.e. with a viscosity coefficient proportional to the density. As we got interested into a free boundary problem for fluids with surface tension (in relation with a phase change problem we are investigating), the fact that this viscosity coefficient vanishes in the void regions was relevant for our application. Due to this particular form, however, the notion of solution developed by Lions [4] has to be modified. The weak formulation indeed incorporates the density as a weight function, as test functions are now of the form $\rho\phi$.

We now consider the case where $P_n(s) = s^{\gamma_n}$, with $\gamma_n > 0$. Therefore the function Ξ_n is given by

$$\Xi_n(s) = \frac{\gamma_n}{\gamma_n + 1} s^{\gamma_n + 1}.$$

We take the initial density $\rho_n^0 \geq 0$ a.e. and bounded in $L^1(\Omega)$ and such that

$$\rho_n^0 \in L^{\gamma_n}(\Omega), \qquad \exists C > 0, \int (\rho_n^0)^{\gamma_n} dx \leq C\gamma_n.$$

and

$$⨍ \rho_n^0 = M_n \text{ for some } M_n \text{ such that } 0 < M_n \leq M < 1, \text{ and } M_n \to M.$$

The initial momentum $m_n^0 \in L^{\frac{2\gamma_n}{\gamma_n+1}}(\Omega)$, and set $u_n^0 = \frac{m_n^0}{\rho_n^0}$ on $\{\rho_n^0 > 0\}$, and zero elsewhere. This initial velocity is assumed to be such that $\rho_n^0|u_n^0|^2$ is bounded in $L^1(\Omega)$. The initial conditions are

$$\rho_n u_n(0) = m_n^0, \quad \rho_n(0) = \rho_n^0 \tag{22}$$

and we assume that $\rho_n^0 u_n^0$ converges weakly in $L^2(\Omega)$ to some m^0 and ρ_n^0 weakly in $L^1(\Omega)$ to some ρ^0. Moreover, the following uniform (in n) energy estimates are assumed on ρ_n^0 and u_n^0:

$$\exists C > 0, \forall n, \quad \int_\Omega \kappa \frac{|\nabla \rho_n^0|^2}{2} + \Pi(\rho_n^0) + \rho_n^0 \frac{|u_n^0|^2}{2} dx \leq C, \quad \int_\Omega |\nabla \sqrt{\rho_n^0}|^2 dx \leq C. \tag{23}$$

From [1] the existence of (ρ_n, u_n) verifying (22-23) and

$$\partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0 \qquad \text{in } \Omega \times (0,T) \tag{24}$$

$$\partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n) - \nu \operatorname{div}(\rho_n D(u_n)) + \nabla \rho_n^{\gamma_n} = \kappa \rho_n \nabla(\Delta \rho_n) \qquad \text{in } \Omega \times (0,T) \tag{25}$$

is proved, for any $\gamma_n > 0$. The following notion of solution was introduced: (ρ_n, u_n) is solution to the compressible Navier Stokes with Korteweg term provided that the following regularity holds,

$$\rho_n \in L^2(0,T; H^2(\Omega)), \quad \nabla \rho_n \text{ and } \nabla\sqrt{\rho_n} \in L^\infty(0,T; L^2(\Omega)^d),$$
$$\sqrt{\rho_n} u_n \in L^\infty(0,T; L^2(\Omega)^d), \quad \sqrt{\rho_n} D(u_n) \in L^2(0,T; L^2(\Omega)^{d\times d})$$

and the following equations are fulfilled:

$$\partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0 \quad \text{in } \mathcal{D}((0,T)\times\Omega), \qquad \rho_n(0,\cdot) = \rho_n^0 \quad \text{in } \mathcal{D}'(\Omega) \tag{26}$$

and for all $v \in \mathcal{C}^\infty([0,T]\times\Omega)^d$ with $v(T,\cdot) = 0$, there holds

$$\begin{aligned}\int_\Omega \rho_n^0 u_n^0 \cdot \rho_n^0 v(0,\cdot) + \int_0^T \int_\Omega (\rho_n^2 u_n \cdot \partial_t v + (\rho_n u_n \otimes \rho_n u_n) : D(v) \\ - \rho_n^2 (u_n \cdot v) \operatorname{div} u_n - \nu \rho_n D(u_n) : \rho_n D(v) - \nu \rho_n D(u_n) : (v \otimes \nabla \rho_n) \\ + \Xi_n(\rho_n) \operatorname{div} v - \kappa \rho_n^2 \Delta \rho_n \operatorname{div} v - 2\kappa \rho_n (v\cdot\nabla\rho_n)\Delta\rho_n) dx dt = 0.\end{aligned} \tag{27}$$

What we aim to do is to prove compactness as $\gamma_n \to +\infty$ of this sequence of solutions, in order to recover a free boundary problem with Korteweg source term.

Remark 1. *The Korteweg term brings H^2 space-regularity thanks to the density dependence of viscosity. But the latter kills the space regularity of u for vanishing ρ. A special notion of weak solution was therefore necessary.*

2.2. The limit problem: a Korteweg free boundary problem in weak formulation. In the remaining of this article, we will prove and comment the following compactness result:

THEOREM 1. *At least a subsequence of* $(\rho_n, u_n, \Xi_n(\rho_n))$ *converges toward* (ρ, u, Ξ), *such that*

$$\rho \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \quad \nabla\rho \text{ and } \nabla\sqrt{\rho} \in L^\infty(0,T;L^2(\Omega)^d),$$
$$\sqrt{\rho}u \in L^\infty(0,T;L^2(\Omega)^d), \quad \sqrt{\rho}D(u) \in L^2(0,T;L^2(\Omega)^{d\times d}), \quad \Xi \in \mathcal{M}((0,T)\times\Omega),$$

verifying the following weak formulation of (11-17):

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \qquad \text{in } \mathcal{D}'((0,T)\times\Omega), \qquad \text{with } \rho(x,0) = \rho^0 \in \mathcal{D}'(\Omega),$$

and for all $\phi \in \mathcal{C}^\infty((0,T)\times\Omega)^d$ *such that* $\phi(\cdot,T) = 0$, *one has:*

$$\int_\Omega \rho^0 u^0 \cdot \rho^0\phi(\cdot,0)dx + \int_0^T\int_\Omega \big[\rho^2 u\cdot\partial_t\phi + \rho u\otimes\rho u : D(\phi)$$
$$- \rho^2(u\cdot\phi)\operatorname{div} u - \nu\rho D(u) : \rho D(\phi) - \nu\rho D(u) : \phi\otimes\nabla\rho$$
$$+\Xi\operatorname{div}\phi - \kappa\rho^2\Delta\rho\operatorname{div}\phi - 2\kappa\rho(\phi\cdot\nabla\rho)\Delta\rho\big]\, dxdt = 0 \quad (28)$$

Moreover, there holds $0 \le \rho \le 1$ *and* $\rho\Xi = \Xi \ge 0$ *a.e.*

REMARK 2. *This model therefore corresponds to the interaction of an incompressible fluid with a compressible pressureless fluid with possibly void areas. The incompressible fluid is located on* $\{\rho = 1\}$, *the function* Ξ *being determined by incompressibility condition. Note that* Ξ *physically represents the product of the density times a pressure. The latter seems not defined as such in this model. However remark that the identity* $\rho\Xi = \Xi$ *means that* Ξ *itself can be thought as a pressure, and for* $\rho > 0$ *we formally recover (11-17). The compressible fluid is located elsewhere (*$0 \le \rho < 1$*), where* $\rho\Xi = \Xi$ *means that the* Ξ *is zero. The model therefore provides no information on the velocity in the void area* $\{\rho = 0\}$.

REMARK 3. *Let us explain why we considered this limit model by a formal argument. The pressure term before the passing to the limit is* $P_n(\rho) = \rho^{\gamma_n}$. *One has*

$$\int \Xi_n(\rho_n)\operatorname{div}(\phi) = -\int \nabla\Xi_n(\rho_n)\cdot\phi = -\int \rho_n P_n'(\rho_n)\nabla\rho_n\cdot\phi$$
$$= -\int \rho_n\nabla P_n(\rho_n)\cdot\phi = \int P_n(\rho_n)\operatorname{div}(\rho_n\phi) \to \int \Pi\operatorname{div}(\rho\phi) = \int \rho\Pi\operatorname{div}(\phi) + \Pi\nabla\rho\cdot\phi$$

Since $\Pi \neq 0$ *only when* $\rho = 1$, $\Pi\nabla\rho$ *is formally taken to be zero. Therefore a first guess would have been to write* $\rho\Pi$ *instead of* Ξ *in the above formulation. However, we were unable to prove that (a subsequence of)* $\Xi_n(\rho_n)$ *converges toward* $\rho\Pi$. *In contrast, we were able to prove that* $\Xi_n(\rho_n)$ *converges, and that* $\rho_n\Xi_n(\rho_n)$ *converges to the same limit.*

Some remarks are in order for the energy estimates and limiting process:

- We start from the weak formulation of [1] and then pass to the limit on the exponent of pressure law. The proof of compactness used in [1] was valid for a fixed exponent in the pressure law and pressure estimate is of course not uniform with respect to that exponent.

- Hopefully, estimates on (ρ_n, u_n) from [1] do not make use of pressure induced bounds which permits to re-use them.
- We adapt the steps from [5] to our setting. The first step amounts to prove that the limit density remains between 0 and 1, and to control the regions where ρ_n exceeds 1, as $n \to +\infty$.
- In step 2 one seeks an extra integrability (uniform in γ_n) for the pressure term. The presence of Korteweg term brings space regularity, and therefore compactness, but the presence of a viscosity coefficient proportional to the density introduces difficulties to control the velocity. Therefore the special weak form (28) of [1] has to be used.
- The last step is the passing to the limit in this weak formulation.
- For now on, we consider the case $f = 0$ for simplicity. Extension to e.g. $f \in L^1(0,T;L^2(\Omega))$ is indeed straightforward.

2.3. Energy estimates. The sequence of functions $(\Xi_n(\rho_n))$, does verify assumptions (18) but not uniformly in n when $\gamma_n \to +\infty$. This estimate was used in [1] in order to obtain uniform integrability on the pressure. This means that we will have to use techniques from [5] to gain this extra uniform integrability of ρ_n, which are is given directly by the estimates. On another hand, some estimates of [1] are indeed uniform in n, as far as the condition (18) on the pressure law is not involved:

$$\|\nabla\rho_n\|_{L^\infty(0,T;L^2(\Omega)^d)} \le C, \quad \|\nabla\sqrt{\rho_n}\|_{L^\infty(0,T;L^2(\Omega)^d)} \le C, \tag{29}$$

$$\|\rho_n\|_{L^\infty(0,T;H^1(\Omega))} \le C, \quad \|\rho_n\|_{L^2(0,T;H^2(\Omega))} \le C, \tag{30}$$

$$\|\sqrt{\rho_n}D(u_n)\|_{L^2(0,T;L^2(\Omega)^{d\times d})} \le C, \quad \|D(\rho_n^{\frac{3}{2}}u_n)\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega)^{d\times d})} \le C, \tag{31}$$

$$\|\rho_n^{\frac{3}{2}}u_n\|_{L^2(0,T;L^3(\Omega)^d)} \le C \ (n \le 3), \quad \|\sqrt{\rho_n}u_n\|_{L^\infty(0,T;L^2(\Omega)^d)} \le C \tag{32}$$

The first step first relies on an estimate on the density. The energy estimate (5) from [1], states that

$$\frac{d}{dt}\int_\Omega \frac{\kappa|\nabla\rho_n|^2}{2} + \Pi_n(\rho_n) + \rho_n\frac{|u_n|^2}{2}dx + \nu\int_\Omega 2\rho_n D(u_n) : D(u_n)dx = 0 \tag{33}$$

with $\Pi_n(s) = \frac{1}{\gamma_n - 1}s^{\gamma_n}$. Thus this energy equality still gives the same estimate as in [5] and the first step is identical. We give a sketch of it for the reader's convenience. The above estimate leads to

$$\exists C > 0,\ \forall n,\ \|\rho_n\|_{L^\infty(0,T;L^{\gamma_n})} \le C\gamma_n.$$

By Holder inequality we have for any $1 < p < \infty$, and n such that $\gamma_n > p$,

$$\|\rho_n\|_{L^\infty(0,T;L^p)} \le \|\rho_n\|^{\theta_n}_{L^\infty(0,T;L^1)}\|\rho_n\|^{1-\theta_n}_{L^\infty(0,T;L^{\gamma_n})}$$

for θ_n defined by $\frac{1}{p} = \theta_n + \frac{1-\theta_n}{\gamma_n}$. From the mass conservation and assumptions made on ρ_n^0 we have

$$\|\rho_n\|_{L^\infty(0,T;L^p)} \le M_n^{\theta_n}(C\gamma_n)^{\frac{1-\theta_n}{\gamma_n}}.$$

And passing to the lim sup in p and n gives:

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \le 1.$$

Moreover, $(\rho_n - 1)_+$ goes to zero uniformly in t in all L^p.

2.4. Uniform bound on pressure terms. The next step aims at proving a L^1 bound on $\Xi_n(\rho_n)$. Consider (27) in the distributional sense:

$$\partial_t(\rho_n^2 u_n) + \operatorname{div}(\rho_n u_n \otimes \rho_n u_n) + \rho_n^2(\operatorname{div} u_n)u_n - \nu \operatorname{div}(\rho_n^2 D(u_n)) \\ + \nu\rho_n D(u_n)\nabla\rho_n + \nabla\Xi_n(\rho_n) - \kappa\nabla(\rho_n^2\Delta\rho_n) + 2\kappa\rho_n\Delta\rho_n\nabla\rho_n = 0 \quad (34)$$

and apply $(-\Delta)^{-1}\operatorname{div}$ with periodic boundary conditions: we get an extra ρ_n compared to [5]:

$$\Xi_n(\rho_n) - \fint \Xi_n(\rho_n) = \partial_t[(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)] + R_iR_j(\rho_n^2 u_n^i u_n^j) \\ + (-\Delta)^{-1}\operatorname{div}(\rho_n^2(\operatorname{div} u_n)u_n) - \nu R_iR_j[\rho_n^2 D(u_n)] + \nu(-\Delta)^{-1}\operatorname{div}(\rho_n D(u_n)\nabla\rho_n) \\ + \kappa\rho_n^2\Delta\rho_n + 2\kappa(-\Delta)^{-1}\operatorname{div}[\rho_n\Delta\rho_n\nabla\rho_n] \quad (35)$$

where $R_i := (-\Delta)^{-\frac{1}{2}}\partial_i$ is the Riesz transform (which is bounded on L^r, $1 < r < \infty$, [2]). Integrating would cancel everything, hence we multiply by ρ_n, to get

$$\rho_n\Xi_n(\rho_n) - \rho_n \fint \Xi_n(\rho_n) = \rho_n\partial_t[(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)] \\ + \rho_n R_iR_j(\rho_n^2 u_n^i u_n^j) + \rho_n(-\Delta)^{-1}\operatorname{div}[\rho_n^2 u_n \operatorname{div} u_n] \\ - \nu\rho_n R_iR_j[\rho_n^2 D(u_n)] + \nu\rho_n(-\Delta)^{-1}\operatorname{div}[\rho_n D(u_n)\nabla\rho_n] \\ + \kappa\rho_n^3\Delta\rho_n + 2\kappa\rho_n(-\Delta)^{-1}\operatorname{div}[\rho_n\Delta\rho_n\nabla\rho_n] \quad (36)$$

We compute

$$\rho_n\partial_t[(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)] = \partial_t[\rho_n(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)] - \operatorname{div}(\rho_n u_n)(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)$$

and as

$$\operatorname{div}(\rho_n u_n)(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n) = \operatorname{div}(\rho_n u_n(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)) - \rho_n u_n^i R_iR_j(\rho_n^2 u_n^j)$$

we are reduced to bound all these terms.

Arguing as in [5], equation (39), p. 384, but with the extra ρ_n, we have a bound on the time derivative given by

$$2\|\rho_n(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)\|_{L^\infty(0,T;L^1(\Omega))}.$$

This quantity is indeed bounded, since we have bounds on:

$$\rho_n \text{ in } L^\infty(0,T;L^6(\Omega)), \quad \sqrt{\rho_n} \text{ in } L^\infty(0,T;L^{12}(\Omega)), \quad \sqrt{\rho_n}u_n \text{ in } L^\infty(0,T;L^2(\Omega)).$$

This gives a bound on:

$$\rho_n^2 u_n \text{ in } L^\infty(0,T;L^{\frac{4}{3}}(\Omega)).$$

Then $(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n)$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for $\frac{1}{q} = \frac{3}{4} - \frac{1}{N}$, that is $q = \frac{12}{5}$ since $N = 3$. This can be multiplied by ρ_n bounded in $L^\infty(0,T;L^6(\Omega))$ to give

$$\rho_n(-\Delta)^{-1}\operatorname{div}(\rho_n^2 u_n) \text{ bounded in } L^\infty(0,T;L^{\frac{12}{7}}(\Omega)).$$

Next, from the boundness of operators R_i, the term $\rho_n u_n^i R_iR_j(\rho_n^2 u_n^j)$ behaves like $\rho_n R_iR_j(\rho_n^2 u_n^i u_n^j)$, which we consider now.

As $\rho_n^{\frac{3}{2}} u_n$ is bounded in $L^2(0,T;L^3(\Omega))$ and $\sqrt{\rho_n} u_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$, we get:

$$\rho_n^2 u_n^i u_n^j \text{ bounded in } L^2(0,T;L^{\frac{6}{5}}(\Omega)).$$

As the R_i are bounded on all L^r, $1 < r < \infty$, and ρ_n is bounded in $L^\infty(0,T;L^6(\Omega))$ this gives:

$$\rho_n R_i R_j(\rho_n^2 u_n^i u_n^j) \text{ bounded in } L^2(0,T;L^1(\Omega)).$$

Let us turn to $\rho_n(-\Delta)^{-1} \operatorname{div}[\rho_n^2 u_n \operatorname{div} u_n]$. From the bound on $\sqrt{\rho_n} D(u_n)$ in $L^2(0,T;L^2(\Omega))$, and since $\rho_n^{\frac{3}{2}}$ is bounded in

$$L^2(0,T;L^3(\Omega)) \cap L^\infty(0,T;L^{\frac{3}{2}}(\Omega)) \subset L^{\frac{4}{3}}(0,T;L^1(\Omega)),$$

the $(-\Delta)^{-1}\operatorname{div}$ operator and Sobolev embedding $W^{1,1}(\Omega) \subset L^{\frac{3}{2}}(\Omega)$ (in dimension 3), gives

$$(-\Delta)^{-1} \operatorname{div}[\rho_n^2 u_n \operatorname{div} u_n] \text{ bounded in } L^{\frac{4}{3}}(0,T;L^{\frac{3}{2}}(\Omega)).$$

Upon multiplication by ρ_n, which is bounded in $L^\infty(0,T;L^6(\Omega))$ this gives a

$$\rho_n(-\Delta)^{-1} \operatorname{div}[\rho_n^2 u_n \operatorname{div} u_n] \text{ bounded in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)).$$

As for the term $\rho_n R_i R_j(\rho_n^2 D(u_n))$, $\sqrt{\rho_n} D(u_n)$ is bounded in $L^2(0,T;L^2(\Omega))$ and ρ_n is bounded in:

$$L^\infty(0,T;L^6(\Omega)) \cap L^2(0,T;L^\infty(\Omega)) \subset L^8(0,T;L^8(\Omega)).$$

From the boundness of Riesz operators, this amounts to study $\rho_n^{\frac{5}{2}} D(u_n)$, and $\rho_n^{\frac{5}{2}}$ is bounded in $L^{\frac{16}{5}}(0,T;L^{\frac{16}{5}}(\Omega))$, which gives

$$\rho_n R_i R_j(\rho_n^2 D(u_n)) \text{ bounded in } L^{\frac{16}{13}}(0,T;L^{\frac{16}{13}}(\Omega)).$$

The last momentum term is $\rho_n(-\Delta)^{-1} \operatorname{div}[\rho_n D(u_n)\nabla\rho_n]$. First $\nabla\rho_n$ is bounded in:

$$L^2(0,T;L^6(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \subset L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}(\Omega)),$$

and $\sqrt{\rho_n} D(u_n)$ in $L^2(0,T;L^2(\Omega))$. Recall that ρ_n is bounded in $L^8(0,T;L^8(\Omega))$, thus we obtain

$$\rho_n D(u_n)\nabla\rho_n \text{ bounded in } L^{\frac{80}{69}}(0,T;L^{\frac{80}{69}}(\Omega)).$$

Note that $\frac{69}{80} + \frac{1}{8} < 1$, so the smoothing operator $(-\Delta)^{-1}\operatorname{div}$ and the $L^8(0,T;L^8(\Omega))$ bound on ρ_n give:

$$\rho_n(-\Delta)^{-1} \operatorname{div}[\rho_n D(u_n)\nabla\rho_n] \text{ bounded in } L^p(0,T;L^q(\Omega)) \text{ for some } p,q > 1.$$

Let us turn to the Korteweg terms, the first being $\rho_n^3 \Delta\rho_n$, we note that $\Delta\rho_n$ is bounded in $L^2(0,T;L^2(\Omega))$ and ρ_n^3 is bounded in $L^{\frac{8}{3}}(0,T;L^{\frac{8}{3}}(\Omega))$ which leads to:

$$\rho_n^3 \Delta\rho_n \text{ bounded in } L^{\frac{8}{7}}(0,T;L^{\frac{8}{7}}(\Omega)).$$

Next, we look after the second term $\rho_n(-\Delta)^{-1}\operatorname{div}(\rho_n\Delta\rho_n\nabla\rho_n)$, which is handled since $\Delta\rho_n$ is bounded in $L^2(0,T;L^2(\Omega))$ and $\nabla\rho_n$ in $L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}(\Omega))$. As ρ_n is bounded in $L^8(0,T;L^8(\Omega))$, we get a bound of

$$\rho_n\Delta\rho_n\nabla\rho_n \text{ in } L^{\frac{40}{37}}(0,T;L^{\frac{40}{37}}(\Omega)).$$

Applying $(-\Delta)^{-1}\operatorname{div}$ gives a bound in $L^{\frac{40}{37}}(0,T;W^{1,\frac{40}{37}}(\Omega)) \subset L^{\frac{40}{37}}(0,T;L^{\frac{120}{71}}(\Omega))$ by Sobolev embeddings. Noting that one also has ρ_n bounded in

$$L^\infty(0,T;L^6(\Omega)) \cap L^2(0,T;L^\infty(\Omega)) \subset L^{14}(0,T;L^7(\Omega)),$$

as $\frac{1}{14}+\frac{37}{40}<1$ and $\frac{71}{120}+\frac{1}{7}<1$ we get a bound in a $L^p(0,T;L^q(\Omega))$ with $p,q>1$.

As a conclusion, we obtained an uniform bound of

$$\rho_n\Xi_n(\rho_n) - \rho_n \fint \Xi_n(\rho_n) \text{ in } L^p(0,T;L^1(\Omega)) \text{ for some } p>1.$$

This estimation gives in turn a bound on $\Xi_n(\rho_n)$. Indeed integrating the bounded expression (recall that $\Xi_n(\rho_n) = \frac{\gamma_n}{\gamma_n+1}\rho_n^{\gamma_n+1}$) we get for n large enough,

$$\int_0^T\int_\Omega \rho_n^{\gamma_n+2} - M_n\int_0^T\int_\Omega \rho_n^{\gamma_n+1} \le C(1+\frac{1}{\gamma_n}) \le 2C \tag{37}$$

with $M_n \le M < 1$, which in turn implies, using $s^a \le s^{a+1}+s$ for $a\ge 1$ and $s\ge 0$,

$$\int_0^T\int_\Omega \rho_n^{\gamma_n+1} \le \int_0^T\int_\Omega \rho_n^{\gamma_n+2}+\rho_n \le 2C+M\int_0^T\int_\Omega \rho_n^{\gamma_n+1} \Rightarrow \int_0^T\int_\Omega \rho_n^{\gamma_n+1} \le \frac{2C}{1-M}. \tag{38}$$

This estimate induces an L^1 bound on the equivalent expression $\Xi_n(\rho_n)$ and then from (37) on $\rho_n\Xi_n(\rho_n)$. Moreover the same trick applied to $\rho_n^{\gamma_n}$ gives

$$\int_0^T\int_\Omega \rho_n^{\gamma_n} \le \int_0^T\int_\Omega \rho_n^{\gamma_n+1}+\rho_n \le \frac{2C}{1-M}+M\int_0^T\int_\Omega \rho_n^{\gamma_n} \Rightarrow \int_0^T\int_\Omega \rho_n^{\gamma_n} \le \frac{2C}{(1-M)^2}. \tag{39}$$

3. Passing to the limit.

3.1. Compactness. From (37-39) we deduce that up to the extraction of a subsequence there exists $\Pi, \Xi, \Lambda \in \mathcal{M}((0,T)\times\Omega)$ such that

$$(\rho_n)^{\gamma_n} \rightharpoonup \Pi, \quad \Xi_n(\rho_n) \rightharpoonup \Xi, \quad \rho_n\Xi_n(\rho_n) \rightharpoonup \Lambda. \tag{40}$$

Next, from the conservation law

$$\partial_t\rho_n = -\operatorname{div}(\rho_n u_n) \text{ is bounded in } L^2(0,T;H^{-1}(\Omega))$$

and from (30), ρ_n is bounded in $L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$, therefore using classic compactness results [6] we have up to subsequence, the following strong convergence of densities:

$$\rho_n \to \rho \text{ in } L^{\frac{2}{s}}(0,T;H^{1+s}(\Omega)) \cap \mathcal{C}([0,T];H^s(\Omega)) \tag{41}$$

for all $s \in (0,1)$. The limit function $\rho \in L^2(0,T;H^2(\Omega))$, thanks to (29), is such that $\nabla\rho$ and $\nabla\sqrt{\rho}$ belong to $L^\infty(0,T;L^2(\Omega)^d)$. Then arguing as in [1], using the bound (32) on $\sqrt{\rho_n}u_n$, which converges weakly to some g, we can define a limit velocity $u = \frac{g}{\sqrt{\rho}}\chi_{\{\rho>0\}}$ and prove that $\rho_n u_n$ converges weakly to $\sqrt{\rho}g = \rho u$, such that

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \qquad \text{in } \mathcal{D}'(\Omega\times(0,T)), \qquad \text{with } \rho(x,0) = \rho^0 \in \mathcal{D}'(\Omega).$$

Next, we recall the weak form,

3.2. Weak limits and nonlinear terms.

$$\begin{aligned}\int_\Omega \rho_n^0 u_n^0 \cdot \rho_n^0 \phi(\cdot,0)dx + \int_0^T\int_\Omega \big[\rho_n^2 u_n\cdot\partial_t\phi + \rho_n u_n\otimes\rho_n u_n : D(\phi) \\ - \rho_n^2(u_n\cdot\phi)\operatorname{div} u_n - \nu\rho_n D(u_n) : \rho_n D(\phi) - \nu\rho_n D(u_n) : \phi\otimes\nabla\rho_n \\ + \Xi_n(\rho_n)\operatorname{div}\phi - \kappa\rho_n^2\Delta\rho_n\operatorname{div}\phi - 2\kappa\rho_n(\phi\cdot\nabla\rho_n)\Delta\rho_n\big]\,dxdt = 0 \quad (42)\end{aligned}$$

and consider the different nonlinear terms for which we justify the limiting process.

The initial term $(\rho_n^0)^2 u_n^0$ is handled using the assumptions on initial data: indeed from the energy estimate (23) we get the strong convergence in $L^2(\Omega)$ of a subsequence of ρ_n^0, which shows that for this subsequence, $(\rho_n^0)^2 u_n^0$ converges toward $(\rho^0)^2 u^0$.

The momentum term $\rho_n^2 u_n$, up to a subsequence, weakly converges to $\rho^2 u$, from the strong convergence of a subsequence of ρ_n in $\mathcal{C}(0,T;L^3(\Omega))$ and the weak convergence of a subsequence of $\rho_n u_n$ in $L^2(\Omega)$.

The convergence of inertial term $\rho_n u_n\otimes\rho_n u_n$ amounts to prove the strong convergence of $\rho_n u_n$ to ρu in $L^2(0,T;L^2(\Omega)^d)$. This is ensured using (31) and (32) on $\rho_n^{\frac{3}{2}}u_n$ and the identity $\rho_n^2 u_n^2 = \rho_n^{\frac{3}{2}}u_n\cdot\sqrt{\rho_n}u_n$. Then the strong convergence of $\rho_n^{\frac{3}{2}}u_n$ to $\rho^{\frac{3}{2}}u$ is proved as in [1].

The viscous term $\rho_n^2 D(u_n)$ can be rewrittten as $\rho_n^{\frac{3}{2}}\sqrt{\rho_n}D(u_n)$, upon which a classical strong-weak argument can be used, from (41) and (31).

Likewise, the term $\rho_n\nabla\rho_n D(u_n)$ which can also be written as $\sqrt{\rho_n}\sqrt{\rho_n}D(u_n)\nabla\rho_n$ as a product of (sub-)sequences converging respectively strongly, weakly, and strongly from (41) and (31).

The first Korteweg term $\rho_n^2\Delta\rho_n$ is easy to handle due to the strong convergence of ρ_n in (41) and the weak of $\Delta\rho_n$ from (30). The second Korteweg term $\rho_n\Delta\rho_n\nabla\rho_n$ amounts to consider respectively strong, weak, strong convergences obtained from (41) and (30).

At last, the term $\Xi_n(\rho_n)$ which is equivalent to $\rho_n^{\gamma_n+1}$, converges in measure to Ξ, $\rho_n^{\gamma_n}$ toward Π, and $\rho_n\Xi_n(\rho_n)$ toward Λ from (40). The strong convergence of ρ_n in $\mathcal{C}([0,T];H^s(\Omega))$ to ρ, is not sufficient to derive $\Xi = \rho\Pi$ and $\Lambda = \rho\Xi$. By the way we were not able to prove $\Xi = \rho\Pi$, since our weak formulation has an extra ρ. However with similar arguments as [5] and [4], we can pass to the limit in (35) and subsequently multiply by ρ to get

$$\begin{aligned}\rho\Xi - \rho⨍\Xi = \rho\partial_t[(-\Delta)^{-1}\operatorname{div}(\rho^2 u)] + \rho R_i R_j(\rho^2 u^i u^j) + \rho(-\Delta)^{-1}\operatorname{div}(\rho^2(\operatorname{div} u)u) \\ - \nu\rho R_i R_j[\rho^2 D(u)] + \rho\nu(-\Delta)^{-1}\operatorname{div}(\rho D(u)\nabla\rho) + \kappa\rho^3\Delta\rho + 2\kappa\rho(-\Delta)^{-1}\operatorname{div}[\rho\Delta\rho\nabla\rho] \\ (43)\end{aligned}$$

and in (36) to get

$$\Lambda - \rho \fint \Xi = \rho\partial_t[(-\Delta)^{-1}\operatorname{div}(\rho u)] + \rho R_i R_j(\rho^2 u^i u^j) + \rho(-\Delta)^{-1}\operatorname{div}[\rho^2 u \operatorname{div} u]$$
$$- \nu\rho R_i R_j[\rho^2 D(u)] + \nu\rho(-\Delta)^{-1}\operatorname{div}[\rho D(u)\nabla\rho] + \kappa\rho^3\Delta\rho + 2\kappa\rho(-\Delta)^{-1}\operatorname{div}[\rho\Delta\rho\nabla\rho] \tag{44}$$

which gives $\Lambda = \rho\Xi$.

3.3. Pressure identity. It remains to prove the equality on Ξ and density which degenerate to the null pseudo pressure law in the compressible zone, to get our free boundary model. Recall that $\Xi_n(\rho_n) = \frac{\gamma_n}{\gamma_n+1}\rho_n^{\gamma_n+1}$. Let $\varepsilon > 0$, and n_0 large enough so that for $n \geq n_0$ and for all $x \geq 0$, there holds

$$x^{\gamma_n+2} \geq x^{\gamma_n+1} - \varepsilon.$$

Then we get

$$\rho_n\Xi_n(\rho_n) \geq \Xi_n(\rho_n) - \frac{\gamma_n}{\gamma_n+1}\varepsilon$$

so that using the above limits,

$$\rho\Xi \geq \Xi - \varepsilon$$

which gives, letting $\varepsilon \to 0$,

$$\rho\Xi \geq \Xi.$$

At last, since we proved that $0 \leq \rho \leq 1$, the reverse equality obviously holds: $\rho\Xi \leq \Xi$. We therefore proved that

$$\rho\Xi = \Xi$$

which achieves the proof of theorem 1.

4. Conclusion. From this model, we could ask which extensions could be developed in order to have a finer modelling of the Leidenfrost phenomenon, which is our ultimate goal. For instance, the pressure free result in the incompressible zone could be improved into a more general pressure law. This modification would be a first step toward the model introduced in [3]. As in the model developed in [5], we could tune $\Pi_n(\rho_n)$ in such a way that it would converge to $\Pi(\rho)$ when n goes to ∞. Another possibility would be to introduce, in the compressible part of the model, a mass transfer mechanism dependent of temperature.

Acknowledgment. The authors warmly thank D. Bresch for bringing [5] to their knowledge and for various discussions and advices around this work.

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