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TRANSMISSION PROBLEM FOR THE ELECTROMAGNETIC SCATTERING BY A DISSIPATIVE CHIRAL OBSTACLE*

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Abstract. Electromagnetic scattering by a dissipative chiral obstacle in achiral surrounding leads to a transmission problem. The transmission problem is reduced to a single integral equation over the boundary of the obstacle with one unknown tangential vector field. The equation is shown to be uniquely solvable except for some values of the material parameters and the frequency. The principal symbol of a boundary integral operator is also calculated.

Key words. Transmission problem, electromagnetic scattering, chiral media, boundary integral equation.

AMS subject classifications. 35Q61, 45B05, 78A45.

1. Introduction. In the early nineteenth century Arago and Biot discovered independently that the plane of polarization of linearly polarized light rotates in certain substances e.g. in sugar solution. In 1848 Pasteur proposed that this phenomenon is caused by the chirality at the molecular level. Chirality is an asymmetry; a molecule is called chiral if it cannot be superimposed onto its mirror image. Chiral media can be characterized by constitutive equations, where the electric and magnetic fields are coupled by a material parameter called chirality measure. There are different expressions for these equations [13, 10], and we will use the Drude-Born-Fedorov constitutive equations

$$\begin{cases} D = \varepsilon(E + \beta\nabla \times E) \\ B = \mu(H + \beta\nabla \times H), \end{cases} \tag{1}$$

where E is the electric field, H is the magnetic field, B is the magnetic flux density, D is the electric flux density, ε is the electric permittivity, μ is the magnetic permeability and β is the chirality measure.

In chiral media we have two wave numbers

$$\gamma_1 = \frac{k}{1 - k\beta} \quad \text{and} \quad \gamma_2 = \frac{k}{1 + k\beta}$$

for left-circularly (LCP) and right-circularly polarized (RCP) waves respectively. Here $k = \omega\sqrt{\varepsilon\mu}$ and $\omega > 0$ is the angular frequency. The real parts of the wave numbers are related to unequal phase velocities of RCP and LCP waves, which causes the rotation. Presence of the imaginary parts of γ_1 and γ_2 results in that RCP and LCP waves attenuate unequally. This phenomenon is called optical activity.

In this work we solve the transmission problem for the electromagnetic scattering by a chiral obstacle in an achiral medium by a single integral equation over the boundary. We consider the problem in the time-harmonic case. We assume that the obstacle and the surrounding are homogeneous and the material of the obstacle is defined by complex material parameters. Earlier the problem has been solved by different methods. In the article [2] Athanasiadis and Stratis consider this transmission problem via

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a weak solution. In [6] Athanasiadis, Martin and Stratis solve the problem by a pair of coupled boundary integral equations. In [4, 5] Athanasiadis, Costakis and Stratis assume in addition that the surrounding is chiral, and in [19] Ola assumes that all the material parameters are real. See also the book [21] for various electromagnetic problems in chiral media.

The single integral equation method is used for solving acoustic transmission problem by Kleinman and Martin in the article [12]. The simplest problem in electromagnetism, which can be written as one boundary integral equation, is the scattering by a perfectly conducting obstacle in free space [7]. The case when perfectly conducting body is in a chiral environment is studied in [3]. In [15] Martin and Ola solve the electromagnetic transmission problem for an achiral non-dissipative scatterer by a single integral equation. The same problem with complex wave numbers is reduced to a single integral equation by Costabel and Le Louër in [8]. They assume only Lipschitz continuity for the boundary of the obstacle whereas Martin and Ola consider smooth boundary. In [15] the operator is proved to be Fredholm with index zero by using pseudodifferential operators and in [8] by Grding inequality.

In this work we generalize a part of the article [15] for a chiral scatterer. The transmission problem can be reduced to a single integral equation by first choosing an ansatz (with one unknown tangential vector field) as a solution for the Maxwell's equations in the exterior domain and the representation formulas as a solution for the Drude-Born-Fedorov equations (2) in the interior domain. Then the tangential traces of both the solutions on the boundary and the transmission boundary conditions give one boundary integral equation with one unknown. By solving the boundary integral equation we get a solution for the transmission problem. Because some of the operators in the equation are not compact, we interpret the boundary integral operators as pseudodifferential operators and show that an operator is elliptic. For proving the ellipticity with chiral scatterer we have to choose the ansatz differently than in [15] (Remark 1) and calculate the principal symbol of a boundary integral operator. Because we use pseudodifferential operators and compute a symbol in local coordinates, we assume that the boundary of the obstacle is defined by C^∞-functions. However, by different method this transmission problem is shown to be uniquely solvable for C^2-domains [2]. Mainly the results in this paper were also proved in the dissertation [11] where we consider real and positive permittivities and permeabilities and real chirality. Now we show that the method works with complex parameters as well, without any new difficulties. Compared to other methods the single integral equation method seems to offer computational advantages.

In Section 2 we introduce the Drude-Born-Fedorov equations and formulate the transmission problem. In Section 3 we define appropriate Sobolev spaces and needed integral operators, their mapping properties and the representation formulas for Drude-Born-Fedorov equations. In the last section we write the transmission problem as a single integral equation and show that the equation is uniquely solvable with some restrictions for the material parameters and the frequency.

2. Statement of the problem. We will consider the time-harmonic Maxwell's equations

$$\begin{cases}\nabla \times E = \mathrm{i}\,\omega B\\ \nabla \times H = -\,\mathrm{i}\,\omega D.\end{cases}$$

By substituting the constitutive equations (1) we get the Drude-Born-Fedorov equations

$$\begin{cases} \nabla \times E = \gamma^2 \beta E + \mathrm{i}\,\omega\mu \left(\frac{\gamma}{k}\right)^2 H \\ \nabla \times H = \gamma^2 \beta H - \mathrm{i}\,\omega\varepsilon \left(\frac{\gamma}{k}\right)^2 E, \end{cases} \tag{2}$$

where $\gamma^2 = \gamma_1\gamma_2$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded and smooth penetrable chiral obstacle in achiral medium such that the exterior domain $\mathbb{R}^3\backslash\bar{\Omega}$ is simply-connected. Then the scattering of time-harmonic electromagnetic waves leads to the following problem:

Transmission Problem. Find vector fields $\{E_i, H_i\}$ and $\{E_{sc}, H_{sc}\}$ that satisfy the Maxwell's equations

$$\nabla \times E_{sc} - \mathrm{i}\,\omega\mu_e H_{sc} = 0 \quad \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega},$$
$$\nabla \times H_{sc} + \mathrm{i}\,\omega\varepsilon_e E_{sc} = 0 \quad \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega},$$

the Drude-Born-Fedorov equations

$$\nabla \times E_i - \gamma^2\beta E_i - \mathrm{i}\,\omega\mu_i \left(\frac{\gamma}{k_i}\right)^2 H_i = 0 \quad \text{in} \quad \Omega,$$
$$\nabla \times H_i - \gamma^2\beta H_i + \mathrm{i}\,\omega\varepsilon_i \left(\frac{\gamma}{k_i}\right)^2 E_i = 0 \quad \text{in} \quad \Omega,$$

and the transmission conditions

$$n \times E_e = n \times E_i \quad \text{and} \quad n \times H_e = n \times H_i \quad \text{on} \quad \partial\Omega, \tag{3}$$

where

$$E_e = E_{sc} + E_{inc}, \quad H_e = H_{sc} + H_{inc} \quad \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega},$$

and the incident wave $\{E_{inc}, H_{inc}\}$ is a whole space solution of the Maxwell's equation with $\mu = \mu_e$ and $\varepsilon = \varepsilon_e$. Furthermore, the scattered fields $\{E_{sc}, H_{sc}\}$ must satisfy one of the Silver-Müller radiation conditions

$$\frac{x}{|x|} \times H_{sc} + \sqrt{\frac{\varepsilon_e}{\mu_e}} E_{sc} = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{4}$$

or

$$\frac{x}{|x|} \times E_{sc} - \sqrt{\frac{\mu_e}{\varepsilon_e}} H_{sc} = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

uniformly for all directions $\frac{x}{|x|}$. We assume that the constants ε_e and μ_e are real and positive. The constants β, ε_i and μ_i can be complex, and the following restrictions must hold:

$$\operatorname{Im} k_i \geq 0, \quad \operatorname{Im} \gamma_1 \geq 0, \quad \operatorname{Im} \gamma_2 \geq 0 \quad \text{and} \quad \eta > 0, \tag{5}$$

where

$$k_i = \omega\sqrt{\varepsilon_i\mu_i}, \quad \gamma_1 = \frac{k_i}{1 - k_i\beta}, \quad \gamma_2 = \frac{k_i}{1 + k_i\beta} \quad \text{and} \quad \eta = \sqrt{\frac{\mu_i}{\varepsilon_i}}.$$

In the following we always assume that $|k_i\beta| < 1$. The next boundary value problem will be important in the uniqueness considerations.

Associated Problem. Find fields $\{E, H\}$ that satisfy the Maxwell's equations

$$\nabla \times E - \mathrm{i}\,\omega\mu H = 0 \quad \text{and} \quad \nabla \times H + \mathrm{i}\,\omega\varepsilon E = 0 \quad \text{in} \quad \Omega$$

and the boundary condition

$$a(n \times E) + b(n \times H) = 0 \quad \text{on} \quad \partial\Omega, \tag{6}$$

where a and b are (complex) constants.

If this problem has a non-trivial solution, the constant $k^2 = \omega^2\mu\varepsilon$ is called an eigenvalue of the associated interior Maxwell problem.

3. Function spaces and integral operators. Let us assume that Ω is a bounded smooth obstacle in $\mathbb{R}^3$. We denote by $H^s(\Omega)$ (or $H^s(\partial\Omega)$) the usual L^2-based Sobolev space in Ω (or on $\partial\Omega$), where $s \in \mathbb{R}$. In the exterior domain the space $\overline{H}^s_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\Omega})$ consists of all such distributions $u \in \mathcal{D}'(\mathbb{R}^3\backslash\overline{\Omega})$ that

$$u|_{B\cap(\mathbb{R}^3\backslash\overline{\Omega})} \in H^s(B \cap (\mathbb{R}^3\backslash\overline{\Omega}))$$

for all open balls B for which $\overline{\Omega} \subset B$. In this space

$$\|u\|_B^2 = \|u|_{B\cap(\mathbb{R}^3\backslash\overline{\Omega})}\|^2_{H^s(B\cap(\mathbb{R}^3\backslash\overline{\Omega}))}, \quad B \text{ an open ball, } \overline{\Omega} \subset B,$$

are seminorms.

The aim is to find a solution $\{E_i, H_i\} \in H^1_{\mathrm{Div}}(\Omega)$ and $\{E_{sc}, H_{sc}\} \in \overline{H}^1_{\mathrm{loc,Div}}(\mathbb{R}^3\backslash\overline{\Omega})$ for the transmission problem, where

$$H^1_{\mathrm{Div}}(\Omega) := \{u \in \left(H^1(\Omega)\right)^3 : \mathrm{Div}(n \times u|^i_{\partial\Omega}) \in H^{1/2}(\partial\Omega)\}$$

and

$$\overline{H}^1_{\mathrm{loc,Div}}(\mathbb{R}^3\backslash\overline{\Omega}) := \{u \in \left(\overline{H}^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\Omega})\right)^3 : \mathrm{Div}(n \times u|^e_{\partial\Omega}) \in H^{1/2}(\partial\Omega)\}.$$

Here Div is the surface divergence, n is an outward unit normal to $\partial\Omega$ and the notation $|^\nu_{\partial\Omega}$, $\nu = i, e$, means the trace from the interior and the exterior domain, respectively. The space $H^1_{\mathrm{Div}}(\Omega)$ is equipped with the norm

$$\|u\|^2_{H^1_{\mathrm{Div}}(\Omega)} = \|u\|^2_{(H^1(\Omega))^3} + \|\,\mathrm{Div}(n \times u|^i_{\partial\Omega})\|^2_{H^{1/2}(\partial\Omega)}$$

and $\overline{H}^1_{\mathrm{loc,Div}}(\mathbb{R}^3\backslash\overline{\Omega})$ with the family of seminorms

$$\|u\|^2_{B,\mathrm{Div}} = \|u\|^2_B + \|\,\mathrm{Div}(n \times u|^e_{\partial\Omega})\|^2_{H^{1/2}(\partial\Omega)}.$$

On the boundary we define the spaces

$$TH^s(\partial\Omega) := \{u \in (H^s(\partial\Omega))^3 : n \cdot u = 0\}, \quad s \in \mathbb{R},$$

and

$$TH^{1/2}_{\mathrm{Div}}(\partial\Omega) := \{u \in TH^{1/2}(\partial\Omega) : \mathrm{Div}\, u \in H^{1/2}(\partial\Omega)\}.$$

The space $TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$ is equipped with the norm

$$\|u\|^2_{TH^{1/2}_{\mathrm{Div}}(\partial\Omega)} = \|u\|^2_{\left(H^{1/2}(\partial\Omega)\right)^3} + \|\operatorname{Div} u\|^2_{H^{1/2}(\partial\Omega)}.$$

The same spaces have been used e.g. in [20, 3, 4, 1, 11].

In the following, we will denote by $\mathbf{x}$ and $\mathbf{y}$ the points in $\Omega \cup (\mathbb{R}^3 \backslash \overline{\Omega})$ and by x and y the points on $\partial\Omega$. Further, we will use the subindex υ to denote in which domain $\mathbf{x}$ belongs so that

$$\upsilon = \begin{cases} i, & \text{if } \ \mathbf{x} \in \Omega, \\ e, & \text{if } \ \mathbf{x} \in \mathbb{R}^3 \backslash \overline{\Omega}. \end{cases}$$

The single-layer potential is defined by

$$\mathbf{S}_{\alpha,\upsilon} f(\mathbf{x}) = \int_{\partial\Omega} \Phi_\alpha(\mathbf{x} - y) f(y)\, d\sigma(y),$$

where f is a scalar or vector-valued function and

$$\Phi_\alpha(\mathbf{x} - y) = \frac{e^{\mathrm{i}\,\alpha|\mathbf{x}-y|}}{4\pi|\mathbf{x} - y|}, \quad \alpha \in \mathbb{C},$$

is a fundamental solution of the Helmholtz operator $-\Delta - \alpha^2 I$. For a vector-valued function u we define the operators

$$\mathbf{C}_{\alpha,\upsilon} u = \nabla \times \{\mathbf{S}_{\alpha,\upsilon} u\} \quad \text{and} \quad \mathbf{F}_{\alpha,\upsilon} u = \nabla \times \nabla \times \{\mathbf{S}_{\alpha,\upsilon} u\}.$$

It is known [20] that the mappings

$$\mathbf{C}_{\alpha,i}, \mathbf{F}_{\alpha,i} : TH^{1/2}_{\mathrm{Div}}(\partial\Omega) \to H^1_{\mathrm{Div}}(\Omega) \tag{7}$$

and

$$\mathbf{C}_{\alpha,e}, \mathbf{F}_{\alpha,e} : TH^{1/2}_{\mathrm{Div}}(\partial\Omega) \to \overline{H}^1_{\mathrm{loc,Div}}(\mathbb{R}^3 \backslash \overline{\Omega}) \tag{8}$$

are continuous. On the boundary we will need the tangential components of $\mathbf{C}_{\alpha,\upsilon}$ and $\mathbf{F}_{\alpha,\upsilon}$ which we denote by

$$M_\alpha u(x) = n(x) \times (\nabla \times \{S_\alpha u(x)\})$$

and

$$P_\alpha u(x) = n(x) \times (\nabla \times \nabla \times \{S_\alpha u(x)\}),$$

where

$$S_\alpha u(x) = \int_{\partial\Omega} \Phi_\alpha(x - y) u(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

If $u \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$, we have the jump relations [7]

$$M_\alpha u = \frac{1}{2} u + n \times (\mathbf{C}_{\alpha,i} u)|^i_{\partial\Omega} = -\frac{1}{2} u + n \times (\mathbf{C}_{\alpha,e} u)|^e_{\partial\Omega} \tag{9}$$

and

$$P_\alpha u = n \times (\mathbf{F}_{\alpha,i} u)|^i_{\partial\Omega} = n \times (\mathbf{F}_{\alpha,e} u)|^e_{\partial\Omega}. \tag{10}$$

It is known [20, 14, 15, 11] that

$$M_\alpha, P_\alpha : TH^{1/2}_{\mathrm{Div}}(\partial\Omega) \to TH^{1/2}_{\mathrm{Div}}(\partial\Omega), \tag{11}$$

where M_α is compact and P_α is continuous, and

$$M_\alpha : TH^s(\partial\Omega) \to TH^{s+1}(\partial\Omega), \quad s \in \mathbb{R}, \tag{12}$$

$$P_\alpha : TH^s(\partial\Omega) \to TH^{s-1}(\partial\Omega), \quad s \in \mathbb{R}, \tag{13}$$

and

$$P_{\alpha'} - P_\alpha : TH^s(\partial\Omega) \to TH^{s+1}(\partial\Omega), \quad s \in \mathbb{R}, \tag{14}$$

are continuous. Moreover, the operators M_α and P_α, $\alpha \in \mathbb{C}$, satisfy the equalities [17, 18]

$$\frac{1}{\alpha} P_\alpha^2 = \frac{1}{4}\alpha I - \alpha M_\alpha^2 \quad \text{and} \quad M_\alpha P_\alpha = -P_\alpha M_\alpha \tag{15}$$

on $TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$.

If the fields $\{E, H\}$ satisfy the Drude-Born-Fedorov equations in Ω, then E and H have the representations [1]

$$-2E(\mathbf{x}) = \mathbf{K}_i(n \times E)(\mathbf{x}) + \mathrm{i}\sqrt{\frac{\mu}{\varepsilon}}\mathbf{D}_i(n \times H)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{16}$$

and

$$-2H(\mathbf{x}) = \mathbf{K}_i(n \times H)(\mathbf{x}) - \mathrm{i}\sqrt{\frac{\varepsilon}{\mu}}\mathbf{D}_i(n \times E)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{17}$$

where

$$\mathbf{K}_i = \mathbf{C}_{\gamma_1,i} + \mathbf{C}_{\gamma_2,i} + \frac{1}{\gamma^2}(\gamma_2 \mathbf{F}_{\gamma_1,i} - \gamma_1 \mathbf{F}_{\gamma_2,i})$$

and

$$\mathbf{D}_i = \mathbf{C}_{\gamma_1,i} - \mathbf{C}_{\gamma_2,i} + \frac{1}{\gamma^2}(\gamma_2 \mathbf{F}_{\gamma_1,i} + \gamma_1 \mathbf{F}_{\gamma_2,i}).$$

If we choose $\beta = 0$, we get the representation formulas for the usual Maxwell's equations [7]

$$-2E(\mathbf{x}) = \mathbf{C}_{k,i}(n \times E)(\mathbf{x}) + \frac{\mathrm{i}}{\omega\varepsilon}\mathbf{F}_{k,i}(n \times H)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{18}$$

and

$$-2H(\mathbf{x}) = \mathbf{C}_{k,i}(n \times H)(\mathbf{x}) - \frac{\mathrm{i}}{\omega\mu}\mathbf{F}_{k,i}(n \times E)(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{19}$$

When we denote

$$K = M_{\gamma_1} + M_{\gamma_2} + \frac{1}{\gamma^2}(\gamma_2 P_{\gamma_1} - \gamma_1 P_{\gamma_2})$$

and

$$D = M_{\gamma_1} - M_{\gamma_2} + \frac{1}{\gamma^2}(\gamma_2 P_{\gamma_1} + \gamma_1 P_{\gamma_2}),$$

we get by (9) and (10) that

$$Ku = u + n \times (\mathbf{K}_i u)|^i_{\partial\Omega} = -u + n \times (\mathbf{K}_e u)|^e_{\partial\Omega}, \tag{20}$$

and

$$Du = n \times (\mathbf{D}_i u)|^i_{\partial\Omega} = n \times (\mathbf{D}_e u)|^e_{\partial\Omega}. \tag{21}$$

4. Single integral equations via representation formulas. In this section we reduce the transmission problem to a boundary integral equation with one unknown tangential vector field. We use an ansatz with one unknown tangential vector field in the exterior domain and the representation formulas in the interior domain for constructing the equation.

In achiral surrounding we choose that

$$E_{sc}(\mathbf{x}) = a\frac{\mathrm{i}}{\omega\varepsilon_e}\mathbf{F}_{k_e,e}j(\mathbf{x}) - b\mathbf{C}_{k_e,e}j(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{\Omega}, \tag{22}$$

and

$$H_{sc}(\mathbf{x}) = a\mathbf{C}_{k_e,e}j(\mathbf{x}) + b\frac{\mathrm{i}}{\omega\mu_e}\mathbf{F}_{k_e,e}j(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{\Omega}, \tag{23}$$

where $j \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$, $k_e = \omega\sqrt{\varepsilon_e\mu_e}$ and a and b are complex constants. The constants a and b will be chosen later. The fields E_{sc} and H_{sc} satisfy the Maxwell's equations in the exterior domain and the Silver-Müller radiation condition (4) at infinity. By (9) and (10) the tangential components on the boundary are

$$n \times E_{sc} = a\frac{\mathrm{i}}{\omega\varepsilon_e}P_{k_e}j - b\left(\frac{1}{2}I + M_{k_e}\right)j := T_e j \tag{24}$$

and

$$n \times H_{sc} = a\left(\frac{1}{2}I + M_{k_e}\right)j + b\frac{\mathrm{i}}{\omega\mu_e}P_{k_e}j := L_e j. \tag{25}$$

In the interior domain we use the representation formulas

$$-2E_i(\mathbf{x}) = \mathbf{K}_i(n \times E_i)(\mathbf{x}) + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}}\mathbf{D}_i(n \times H_i)(\mathbf{x}), \ \mathbf{x} \in \Omega,$$

and

$$-2H_i(\mathbf{x}) = \mathbf{K}_i(n \times H_i)(\mathbf{x}) - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}}\mathbf{D}_i(n \times E_i)(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

By the jump relations (20) and (21) and the boundary condition (3) we get that

$$-2n \times E_e = (-I + K)(n \times E_e) + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}} D(n \times H_e) \tag{26}$$

and

$$-2n \times H_e = (-I + K)(n \times H_e) - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}} D(n \times E_e). \tag{27}$$

Next we substitute the equalities

$$n \times E_e = n \times E_{sc} + n \times E_{inc} = T_e j + n \times E_{inc} \tag{28}$$

and

$$n \times H_e = n \times H_{sc} + n \times H_{inc} = L_e j + n \times H_{inc} \tag{29}$$

into the equation (26) and obtain

$$(I + K)T_e j + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}} DL_e j = f, \tag{30}$$

where

$$f = -(I + K)(n \times E_{inc}) - \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}} D(n \times H_{inc}). \tag{31}$$

In the same way, the equation (27) gives that

$$(I + K)L_e j - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}} DT_e j = g, \tag{32}$$

where

$$g = -(I + K)(n \times H_{inc}) + \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}} D(n \times E_{inc}). \tag{33}$$

Notice that the functions f and g belong to $TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$ because $\{n \times E_{inc}, n \times H_{inc}\} \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$ and the operators K and D map continuously on this space.

The boundary integral equations (30) and (32) are the single integral equations which we are going to consider. If we solve j from either one of them, then the exterior field $\{E_{sc}, H_{sc}\}$ is given by the representations (22) and (23), and the interior field is given by the representation formulas with the boundary conditions (28) and (29) i.e.

$$-2E_i(\mathbf{x}) = \mathbf{K}_i(n \times E_{inc} + T_e j)(\mathbf{x}) + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}}\mathbf{D}_i(n \times H_{inc} + L_e j)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{34}$$

and

$$-2H_i(\mathbf{x}) = \mathbf{K}_i(n \times H_{inc} + L_e j)(\mathbf{x}) - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}}\mathbf{D}_i(n \times E_{inc} + T_e j)(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{35}$$

Next we prove the connection between the equations (30) and (32) and the transmission problem. The proof is similar to the proof of [15, Theorem 7.1].

THEOREM 4.1. *If* $j \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$ *solves* (30) *or* (32), *then* $\{E_{sc}, H_{sc}\}$ *and* $\{E_i, H_i\}$ *given by* (22), (23), (34) *and* (35) *solve the transmission problem.*

Proof. Because by the mapping properties (7) and (8) the fields $\{E_i, H_i\} \in H^1_{\mathrm{Div}}(\Omega)$ and $\{E_{sc}, H_{sc}\} \in \overline{H}^1_{\mathrm{loc,Div}}(\mathbb{R}^3\backslash\overline{\Omega})$, and they satisfy the required equations, it is enough to show that the transmission boundary conditions hold.

Assume that j solves (30). By the definition (24), the representation (34) and the jump relations (20) and (21) we get that

$$\begin{aligned}
&2(n \times E_{sc} + n \times E_{inc} - n \times E_i)\\
&= 2(T_e j + n \times E_{inc}) - T_e j - n \times E_{inc} + K(n \times E_{inc} + T_e j)\\
&\quad + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}} D(n \times H_{inc} + L_e j)\\
&= (I + K)T_e j + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}} D L_e j - f
\end{aligned}$$

on the boundary. By the assumption it follows that $n \times E_e = n \times E_i$.

We still have to show that $n \times H_e = n \times H_i$. To prove this we construct the fields

$$2\tilde{E}_{sc}(\mathbf{x}) = \mathbf{K}_e(n \times E_{inc} + T_e j)(\mathbf{x}) + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}}\mathbf{D}_e(n \times H_{inc} + L_e j)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{\Omega},$$

and

$$2\tilde{H}_{sc}(\mathbf{x}) = \mathbf{K}_e(n \times H_{inc} + L_e j)(\mathbf{x}) - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}}\mathbf{D}_e(n \times E_{inc} + T_e j)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{\Omega},$$

that satisfy the Drude-Born-Fedorov-equations in the exterior domain, which has the same material parameters as the interior domain in the transmission problem. They also satisfy the Silver-Müller radiation condition (4) which can be checked by using [7, Theorem 4.4]. Since j solves (30), we get by (20) and (21) that $n \times \tilde{E}_{sc} = 0$ on the boundary. Because the exterior boundary value problem for the Drude-Born-Fedorov equations has at most one solution (see [3, Lemma 1]), the fields $\tilde{E}_{sc} = \tilde{H}_{sc} = 0$ in $\mathbb{R}^3\backslash\overline{\Omega}$. Since

$$2n \times \tilde{H}_{sc} = (I + K)(n \times H_{inc} + L_e j) - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}} D(n \times E_{inc} + T_e j) = 0,$$

we get that

$$2(n \times H_{sc} + n \times H_{inc} - n \times H_i) = (I + K)L_e j - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}} D T_e j - g = 0.$$

Therefore $n \times H_e = n \times H_i$. The proof for the equation (32) is similar. □

4.1. Uniqueness. Next we study the uniqueness of solutions to the boundary integral equations (30) and (32). It will be shown that the uniqueness depends on the eigenvalues of the associated interior Maxwell problem. If the incident fields are zero, we have the homogeneous boundary integral equations

$$\left((I + K)T_e + \mathrm{i}\sqrt{\frac{\mu_i}{\varepsilon_i}} D L_e\right) j_0 = 0 \tag{36}$$

and

$$\left((I+K)L_e - \mathrm{i}\sqrt{\frac{\varepsilon_i}{\mu_i}}DT_e\right) j_0 = 0. \tag{37}$$

THEOREM 4.2. *The equations* (36) *and* (37) *have a non-trivial solution* $j_0 \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$ *if and only if* k_e^2 *is an eigenvalue of the associated interior Maxwell problem.*

Proof. We will prove this claim in the similar manner as [15, Theorem 7.2]. First we assume that $j_0 \not\equiv 0$ solves (36) or (37). We denote by $\{E_\upsilon^0, H_\upsilon^0\}$, $\upsilon = i, sc$, such fields (22), (23), (34) and (35) that j is replaced by j_0 and $n \times E_{inc} = n \times H_{inc} = 0$. By Theorem 4.1 these fields solve the homogeneous transmission problem which has only the trivial solution by [2, Theorem 1]. It means that the fields $\{E_\upsilon^0, H_\upsilon^0\}$, $\upsilon = i, sc$, vanish identically.

Next we construct the fields

$$\tilde{E}_i(\mathbf{x}) = a\frac{\mathrm{i}}{\omega\varepsilon_e}\mathbf{F}_{k_e,i}j_0(\mathbf{x}) - b\mathbf{C}_{k_e,i}j_0(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

and

$$\tilde{H}_i(\mathbf{x}) = a\mathbf{C}_{k_e,i}j_0(\mathbf{x}) + b\frac{\mathrm{i}}{\omega\mu_e}\mathbf{F}_{k_e,i}j_0(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

which satisfy the Maxwell's equations with the material parameters ε_e and μ_e. Because the field $\{E_{sc}^0, H_{sc}^0\}$ vanishes, we get by the jump relations (9) and (10) that

$$n \times \tilde{E}_i = n \times E_{sc}^0 + bj_0 = bj_0 \tag{38}$$

and

$$n \times \tilde{H}_i = n \times H_{sc}^0 - aj_0 = -aj_0 \tag{39}$$

on the boundary. These equalities give us that

$$a(n \times \tilde{E}_i) + b(n \times \tilde{H}_i) = 0 \quad \text{on} \quad \partial\Omega.$$

Because $j_0 \not\equiv 0$, then also $\tilde{H}_i \not\equiv 0$ and $\tilde{E}_i \not\equiv 0$. Hence, $k_e^2 = \omega^2\varepsilon_e\mu_e$ is an eigenvalue of the associated interior Maxwell problem.

Next we assume that k_e^2 is an eigenvalue of the associated interior Maxwell problem. Then the problem has a non-trivial solution which is given by the representation formulas

$$-\tilde{E}_i(\mathbf{x}) = \mathbf{C}_{k_e,i}(n \times \tilde{E}_i)(\mathbf{x}) + \frac{\mathrm{i}}{\omega\varepsilon_e}\mathbf{F}_{k_e,i}(n \times \tilde{H}_i)(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

and

$$-\tilde{H}_i(\mathbf{x}) = \mathbf{C}_{k_e,i}(n \times \tilde{H}_i)(\mathbf{x}) - \frac{\mathrm{i}}{\omega\mu_e}\mathbf{F}_{k_e,i}(n \times \tilde{E}_i)(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

By the jump relations we get the equalities

$$\left(\frac{1}{2}I + M_{k_e}\right)(n \times \tilde{E}_i) + \frac{\mathrm{i}}{\omega\varepsilon_e}P_{k_e}(n \times \tilde{H}_i) = 0 \tag{40}$$

and

$$\left(\frac{1}{2}I + M_{k_e}\right)(n \times \tilde{H}_i) - \frac{\mathrm{i}}{\omega\mu_e} P_{k_e}(n \times \tilde{E}_i) = 0 \tag{41}$$

on the boundary. If $a \neq 0$, we can solve $n \times \tilde{E}_i$ from the boundary condition (6) and substitute it into the above equalities. We get that $T_e(n\times\tilde{H}_i) = 0$ and $L_e(n\times\tilde{H}_i) = 0$, and therefore $n \times \tilde{H}_i$ is a non-trivial solution of the equations (36) and (37). In the case $a = 0$ the boundary condition is equal to $n \times \tilde{H}_i = 0$. Then the equations (40) and (41) give that $T_e(n \times \tilde{E}_i) = 0$ and $L_e(n \times \tilde{E}_i) = 0$, and the field $n \times \tilde{E}_i$ is a non-trivial solution of (36) and (37). ◻

4.2. Solvability. Let us denote

$$\tau = \frac{\varepsilon_e}{\varepsilon_i} \quad \text{and} \quad \rho = \frac{\mu_i}{\mu_e}.$$

When we substitute (24) and (25) into the equation (30) and separate the a dependent and b dependent terms, the equation (30) can be written in the form

$$\left(a\frac{\mathrm{i}}{\omega\varepsilon_e}\mathcal{B}_{\tau,\beta} - b\mathcal{A}_{\rho,\beta}\right) j = f,$$

where

$$\begin{aligned}\mathcal{B}_{\tau,\beta} =& \left(I + M_{\gamma_1} + M_{\gamma_2} + \frac{1}{\gamma_1}P_{\gamma_1} - \frac{1}{\gamma_2}P_{\gamma_2}\right)P_{k_e} \\ &+ k_i\tau\left(M_{\gamma_1} - M_{\gamma_2} + \frac{1}{\gamma_1}P_{\gamma_1} + \frac{1}{\gamma_2}P_{\gamma_2}\right)\left(\frac{1}{2}I + M_{k_e}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_{\rho,\beta} =& \left(I + M_{\gamma_1} + M_{\gamma_2} + \frac{1}{\gamma_1}P_{\gamma_1} - \frac{1}{\gamma_2}P_{\gamma_2}\right)\left(\frac{1}{2}I + M_{k_e}\right) \\ &+ \frac{\rho}{k_i}\left(M_{\gamma_1} - M_{\gamma_2} + \frac{1}{\gamma_1}P_{\gamma_1} + \frac{1}{\gamma_2}P_{\gamma_2}\right)P_{k_e}.\end{aligned}$$

We will first consider this equation on the space $TH^{1/2}(\partial\Omega)$. In general, the operator $a\frac{\mathrm{i}}{\omega\varepsilon_e}\mathcal{B}_{\tau,\beta} - b\mathcal{A}_{\rho,\beta}$ is not continuous on $TH^{1/2}(\partial\Omega)$. But, if we choose

$$a = -\mathrm{i}\,\frac{\beta\omega\varepsilon_e}{\tau+1} \quad \text{and} \quad b = -1, \tag{42}$$

we get the operator

$$\mathcal{T}_\beta := \frac{\beta}{\tau+1}\mathcal{B}_{\tau,\beta} + \mathcal{A}_{\rho,\beta} : TH^{1/2}(\partial\Omega) \to TH^{1/2}(\partial\Omega),$$

which is continuous. The operator is

$$\begin{aligned}\mathcal{T}_\beta =& \frac{1}{2}I + \frac{\beta}{\tau+1}\left(P_{k_e} - \frac{1}{2}(P_{\gamma_1} + P_{\gamma_2})\right) + \left[\frac{1}{2k_i} - \frac{\tau k_i\beta^2}{2(\tau+1)}\right](P_{\gamma_1} - P_{\gamma_2}) + M_{k_e} \\ &+ \frac{1}{2}a_+M_{\gamma_1} - \frac{1}{2}a_-M_{\gamma_2} + [a_+M_{\gamma_1} - a_-M_{\gamma_2}]M_{k_e} + [b_+M_{\gamma_1} + b_-M_{\gamma_2}]P_{k_e} \\ &+ \left(\frac{a_+}{\gamma_1}P_{\gamma_1} + \frac{a_-}{\gamma_2}P_{\gamma_2}\right)M_{k_e} + \frac{b_+}{\gamma_1}P_{\gamma_1}P_{k_e} - \frac{b_-}{\gamma_2}P_{\gamma_2}P_{k_e},\end{aligned}$$

where

$$a_{\pm} = \frac{k_i\tau\beta \pm (1+\tau)}{\tau+1} \text{and} \quad b_{\pm} = \frac{k_i\beta \pm \rho(1+\tau)}{k_i(\tau+1)},$$

and it is continuous because

$$P_{\gamma_j}P_{k_e} = P_{\gamma_j}(P_{k_e} - P_{\gamma_j}) + \frac{1}{4}\gamma_j^2(I - M_{\gamma_j}^2), \quad j = 1, 2, \tag{43}$$

which holds by the equality (15). From now on, we will only consider the equation (30). The equation (32) could be handled in a similar manner.

We will prove the unique solvability of the equation

$$\mathcal{T}_\beta j = f \tag{44}$$

in $TH^{1/2}(\partial\Omega)$ by using the Fredholm alternative [16, Theorem 2.27], and therefore we need to show that $\mathcal{T}_\beta$ is a Fredholm operator with index zero. We get the Fredholm property by showing that $\mathcal{T}_\beta$ is an elliptic pseudodifferential operator (see [22]). A classical pseudodifferential operator is said to be elliptic if the determinant of the principal symbol is non-zero. All the operators we are considering in this work are classical pseudodifferential operators.

REMARK 1. In achiral case the operator $\mathcal{A}_{\rho,0}$ is elliptic and we can choose $a = 0$ and $b = -1$. In chiral case we need the combination of the operators $\mathcal{A}_{\rho,\beta}$ and $\mathcal{B}_{\tau,\beta}$ in order to get an elliptic operator. Because of this we chose the ansatz (22) and (23) differently than Martin and Ola in [15] (we replaced $n \times j$ by j). Furthermore, the choice of the ansatz is related to the boundary condition of the associated interior Maxwell problem (Theorem 4.2). Therefore, our associated interior Maxwell problem differs from the impedance problem which was used in [15].

The principal symbols for P_α and $P_\alpha P_{\alpha'}$ are known [15] but for

$$M_\alpha u(x) = n(x) \times \left(\nabla_x \times \int_{\partial\Omega} \Phi_\alpha(x-y)u(y)\, d\sigma(y)\right), \quad x \in \partial\Omega,$$

where u is a tangential density, we need to calculate it. If we calculate the symbol for the corresponding operator in halfspace, we will get zero matrix (see [9]). This is not enough for us because we need M_α as an operator of order -1. Therefore we must take into account the boundary of the domain. In the symbol calculations we will use the following lemmas.

LEMMA 4.1. *Let $V_{x'}$ be a neighborhood of $x' = (x_1, x_2) \in \mathbb{R}^2$ and let $\phi : V_{x'} \to \mathbb{R}$ be a C^∞-function such that $\nabla\phi(x') = 0$. Then*

$$\partial_{x_l}\Phi_\alpha(x'-y', \phi(x')-\phi(y')) = -\frac{x_l - y_l}{4\pi|x'-y'|^3} + \mathcal{O}(1), \quad l = 1, 2,$$

as $|x'-y'| \to 0$.

Proof. Since $\nabla\phi(x') = 0$, we get that

$$\partial_{x_l}\Phi_\alpha(x'-y', \phi(x')-\phi(y')) = -\frac{x_l - y_l}{4\pi|x'-y'|^3}a(x', y') + \mathcal{O}(1),$$

where

$$a(x',y') = \left(1 + \frac{|\phi(x') - \phi(y')|^2}{|x'-y'|^2}\right)^{-3/2}.$$

Furthermore, by the Taylor's formula

$$\begin{aligned}\phi(y') - \phi(x') =& \frac{1}{2}\big((y_1-x_1)^2\partial_1^2\phi(x') + 2(y_1-x_1)(y_2-x_2)\partial_1\partial_2\phi(x') \\ &+ (y_2-x_2)^2\partial_2^2\phi(x')\big) + \mathcal{O}(|x'-y'|^3).\end{aligned} \tag{45}$$

The claim follows by the binomial series because

$$\begin{aligned}a(x',y') &= 1 - \frac{3}{2}\frac{|\phi(x') - \phi(y')|^2}{|x'-y'|^2} + \mathcal{O}\left(\frac{|\phi(x') - \phi(y')|^4}{|x'-y'|^4}\right) \\ &= 1 + \mathcal{O}(|x'-y'|^2).\end{aligned}$$

□

In the following we will need the Fourier transform

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^2} e^{-\mathrm{i}\,x'\cdot\xi} u(x')\,dx', \quad \xi \in \mathbb{R}^2,$$

and the inverse Fourier transform

$$\mathcal{F}^{-1}u(x') = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}\,x'\cdot\xi} u(\xi)\,d\xi, \quad x' \in \mathbb{R}^2.$$

Lemma 4.2. *Let $x' = (x_1,x_2) \in \mathbb{R}^2$ and $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$. Then*

$$\mathcal{F}\left(\frac{x_1^2}{4\pi|x'|^3}\right)(\xi) = \frac{\xi_2^2}{2|\xi|^3},$$

$$\mathcal{F}\left(\frac{x_1x_2}{4\pi|x'|^3}\right)(\xi) = -\frac{\xi_1\xi_2}{2|\xi|^3} \quad \text{and} \quad \mathcal{F}\left(\frac{x_2^2}{4\pi|x'|^3}\right)(\xi) = \frac{\xi_1^2}{2|\xi|^3}.$$

Proof. The first claim follows from

$$\mathcal{F}\left(\frac{x_1^2}{4\pi|x'|^3}\right)(\xi) = \frac{1}{4\pi}\mathcal{F}\left(x_1\left(-\partial_1\frac{1}{|x|}\right)\right)(\xi) = \partial_1\left(\frac{\xi_1}{2|\xi|}\right) = \frac{\xi_2^2}{2|\xi|^3}.$$

The other claims can be proved in a similar manner. □

In the next theorem it is essential to assume that the density is tangential. If the normal component is involved, the symbol will be of order zero.

Theorem 4.3. *Let ϕ be a C^∞-function which graph defines the boundary $\partial\Omega$ in a neighborhood of $x' \in \mathbb{R}^2$. Then the operator M_α with tangential density has the principal symbol of order -1 of the form*

$$\frac{1}{4|\xi|^3}\begin{pmatrix} \partial_1^2\phi(x')\xi_2^2 - \partial_2^2\phi(x')\xi_1^2 & 2\partial_1\partial_2\phi(x')\xi_2^2 - 2\partial_2^2\phi(x')\xi_1\xi_2 \\ 2\partial_1\partial_2\phi(x')\xi_1^2 - 2\partial_1^2\phi(x')\xi_1\xi_2 & \partial_2^2\phi(x')\xi_1^2 - \partial_1^2\phi(x')\xi_2^2 \end{pmatrix}.$$

This statement holds in such a local coordinate system that $\nabla\phi(x') = 0$.

Proof. Let u be a smooth tangential vector field on the boundary. Then the operator M_α can be written as

$$M_\alpha u(x) = \int_{\partial\Omega} [\nabla_x \Phi_\alpha(x-y)][n(x)\cdot u(y)]\,d\sigma(y) - \int_{\partial\Omega} [n(x)\cdot \nabla_x \Phi_\alpha(x-y)]u(y)\,d\sigma(y).$$

Let U_x be a neighborhood of $x \in \mathbb{R}^3$. Then it is enough to calculate the symbol in the domain $U_x \cap \partial\Omega$, $x \in \partial\Omega$, because the kernel Φ_α is smooth if $x \neq y$. Let us denote

$$(I_1 - I_2)u(x) := \int_{U_x\cap\partial\Omega} [\nabla_x \Phi_\alpha(x-y)][n(x)\cdot u(y)]\,d\sigma(y) - \int_{U_x\cap\partial\Omega} [n(x)\cdot \nabla_x \Phi_\alpha(x-y)]u(y)\,d\sigma(y). \tag{46}$$

We assume that ϕ is defined in a neighborhood $V_{x'}$ of x'. In addition, we choose such a local coordinate system that at a fixed point x' we have $\nabla\phi(x') = 0$ which we get by applying rotation in $\mathbb{R}^3$. Because the outward unit normal vector for $y \in \partial\Omega$ is of the form

$$n(y) = \frac{(-\nabla\phi(y'), 1)}{\sqrt{1+|\nabla\phi(y')|^2}},$$

$\nabla\phi(x') = 0$ and $n\cdot u = 0$, then $n(x) = (0,0,1)$,

$$u_3(y) = (\partial_1\phi(y'))u_1(y) + (\partial_2\phi(y'))u_2(y),$$

and further, $\nabla_x\Phi_\alpha(x) = \nabla_{(x',\phi(x'))}\Phi_\alpha(x',\phi(x'))$. By these equalities and the change of variables $y = (y', \phi(y'))$ we have in a neighborhood $V_{x'}$ that

$$\begin{aligned}
&I_1 u(x',\phi(x')) \\
&= \int [\nabla_{(x',\phi(x'))}\Phi_\alpha(x'-y', \phi(x')-\phi(y'))]u_3(y',\phi(y'))\sqrt{1+|\nabla\phi(y')|^2}\,dy' \\
&= \int [\nabla_{(x',\phi(x'))}\Phi_\alpha(x'-y', \phi(x')-\phi(y'))][(\partial_1\phi(y')-\partial_1\phi(x'))u_1(y',\phi(y')) \\
&\quad +(\partial_2\phi(y')-\partial_2\phi(x'))u_2(y',\phi(y'))]\sqrt{1+|\nabla\phi(y')|^2}\,dy'.
\end{aligned}$$

By the binomial series and the Taylor's formula

$$\begin{aligned}
(1+|\nabla\phi(y')|^2)^{1/2} &= (1+|\nabla\phi(y')-\nabla\phi(x')|^2)^{1/2} \\
&= 1 + \frac{1}{2}|\nabla\phi(y')-\nabla\phi(x')|^2 + \cdots \\
&= 1 + \mathcal{O}(|x'-y'|^2).
\end{aligned}$$

Next we simplify the notations and denote

$$\widetilde{\Phi}_\alpha(x',y') := \Phi_\alpha(x'-y', \phi(x')-\phi(y'))$$

and

$$\tilde{u}_l(y') := u_l(y', \phi(y')), \quad l = 1,2. \tag{47}$$

Then by the Taylor's formula the first component of the vector I_1u is equal to

$$\begin{aligned}
I_1^1u(x',\phi(x')) := &-\int[\partial_{x_1}\widetilde{\Phi}_\alpha(x',y')][\nabla\partial_1\phi(x')\cdot(x'-y')\tilde{u}_1(y') \\
&+\nabla\partial_2\phi(x')\cdot(x'-y')\tilde{u}_2(y')]\,dy' + R(x',y') \\
= &-\partial_1^2\phi(x')\int[\partial_{x_1}\widetilde{\Phi}_\alpha(x',y')](x_1-y_1)\tilde{u}_1(y')\,dy' \\
&-\partial_2\partial_1\phi(x')\int[\partial_{x_1}\widetilde{\Phi}_\alpha(x',y')](x_2-y_2)\tilde{u}_1(y')\,dy' \\
&-\partial_1\partial_2\phi(x')\int[\partial_{x_1}\widetilde{\Phi}_\alpha(x',y')](x_1-y_1)\tilde{u}_2(y')\,dy' \\
&-\partial_2^2\phi(x')\int[\partial_{x_1}\widetilde{\Phi}_\alpha(x',y')](x_2-y_2)\tilde{u}_2(y')\,dy' + R(x',y'),
\end{aligned}$$

where

$$R(x',y') = -\int[\partial_{x_1}\widetilde{\Phi}_\alpha(x',y')]k(x',y')[\tilde{u}_1(y')+\tilde{u}_2(y')]\,dy',$$

and $k(x',y') = \mathcal{O}(|x'-y'|^2)$. Then by the power series of the exponential function and the equality (45) the first component of the integral I_2u is

$$\begin{aligned}
I_2^1u(x',\phi(x')) &:= \int\partial_{\phi(x')}\Phi_\alpha(x'-y',\phi(x')-\phi(y'))\tilde{u}_1(y')\,dy' \\
&= \frac{1}{8\pi}\int\sum_{|\alpha'|=2}\partial_{x'}^{\alpha'}\phi(x')(y'-x')^{\alpha'}|x'-y'|^{-3}\tilde{u}_1(y')\,dy' \\
&\quad+\int\tilde{k}(x',y')\tilde{u}_1(y')\,dy',
\end{aligned}$$

where $\tilde{k}(x',y') = \mathcal{O}(1)$. By Lemma 4.1, the equality

$$\int v(x'-y')u(y')\,dy' = \mathcal{F}^{-1}\mathcal{F}(v*u)(x') = \mathcal{F}^{-1}(\mathcal{F}v\mathcal{F}u)(x')$$

and Lemma 4.2 we obtain that the principal part of the first component of the operator

M_α, i.e. the principal part of $I_1^1 - I_2^1$, is

$$\begin{aligned}
m_1 u(x', \phi(x')) := {} & \partial_1^2 \phi(x') \int \frac{(x_1 - y_1)^2}{4\pi |x' - y'|^3} \tilde{u}_1(y')\, dy' \\
& + \partial_2 \partial_1 \phi(x') \int \frac{(x_1 - y_1)(x_2 - y_2)}{4\pi |x' - y'|^3} \tilde{u}_1(y')\, dy' \\
& + \partial_1 \partial_2 \phi(x') \int \frac{(x_1 - y_1)^2}{4\pi |x' - y'|^3} \tilde{u}_2(y')\, dy' \\
& + \partial_2^2 \phi(x') \int \frac{(x_1 - y_1)(x_2 - y_2)}{4\pi |x' - y'|^3} \tilde{u}_2(y')\, dy' \\
& - \frac{1}{8\pi} \sum_{|\alpha'|=2} \partial_{x'}^{\alpha'} \phi(x') \int (x' - y')^{\alpha'} |x' - y'|^{-3} \tilde{u}_1(y')\, dy' \\
= {} & \frac{1}{2} \partial_1^2 \phi(x') \int \frac{(x_1 - y_1)^2}{4\pi |x' - y'|^3} \tilde{u}_1(y')\, dy' \\
& - \frac{1}{2} \partial_2^2 \phi(x') \int \frac{(x_2 - y_2)^2}{4\pi |x' - y'|^3} \tilde{u}_1(y')\, dy' \\
& + \partial_1 \partial_2 \phi(x') \int \frac{(x_1 - y_1)^2}{4\pi |x' - y'|^3} \tilde{u}_2(y')\, dy' \\
& + \partial_2^2 \phi(x') \int \frac{(x_1 - y_1)(x_2 - y_2)}{4\pi |x' - y'|^3} \tilde{u}_2(y')\, dy' \\
= {} & \frac{1}{4(2\pi)^2} \partial_1^2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_2^2}{|\xi|^3} \mathcal{F}\tilde{u}_1(\xi) \right) d\xi \\
& - \frac{1}{4(2\pi)^2} \partial_2^2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_1^2}{|\xi|^3} \mathcal{F}\tilde{u}_1(\xi) \right) d\xi \\
& + \frac{1}{2(2\pi)^2} \partial_1 \partial_2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_2^2}{|\xi|^3} \mathcal{F}\tilde{u}_2(\xi) \right) d\xi \\
& - \frac{1}{2(2\pi)^2} \partial_2^2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_1 \xi_2}{|\xi|^3} \mathcal{F}\tilde{u}_2(\xi) \right) d\xi.
\end{aligned}$$

In the same way, we get the principal part of the second component

$$\begin{aligned}
m_2 u(x', \phi(x')) = {} & \frac{1}{4(2\pi)^2} \partial_2^2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_1^2}{|\xi|^3} \mathcal{F}\tilde{u}_2(\xi) \right) d\xi \\
& - \frac{1}{4(2\pi)^2} \partial_1^2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_2^2}{|\xi|^3} \mathcal{F}\tilde{u}_2(\xi) \right) d\xi \\
& + \frac{1}{2(2\pi)^2} \partial_1 \partial_2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_1^2}{|\xi|^3} \mathcal{F}\tilde{u}_1(\xi) \right) d\xi \\
& - \frac{1}{2(2\pi)^2} \partial_1^2 \phi(x') \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \left(\frac{\xi_1 \xi_2}{|\xi|^3} \mathcal{F}\tilde{u}_1(\xi) \right) d\xi.
\end{aligned}$$

These are all we need to calculate because the third component is equal to zero. When we combine m_1 and m_2, we get that the principal part of the operator M_α is

$$mu(x', \phi(x')) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathrm{i}\, x' \cdot \xi} \begin{pmatrix} m_{11}(x', \xi) & m_{12}(x', \xi) \\ m_{21}(x', \xi) & m_{22}(x', \xi) \end{pmatrix} \begin{pmatrix} \mathcal{F}\tilde{u}_1(\xi) \\ \mathcal{F}\tilde{u}_2(\xi) \end{pmatrix} d\xi,$$

where

$$
\begin{aligned}
m_{11}(x',\xi) &= \frac{1}{4|\xi|^3}(\partial_1^2\phi(x')\xi_2^2 - \partial_2^2\phi(x')\xi_1^2),\\
m_{12}(x',\xi) &= \frac{1}{2|\xi|^3}(\partial_1\partial_2\phi(x')\xi_2^2 - \partial_2^2\phi(x')\xi_1\xi_2),\\
m_{21}(x',\xi) &= \frac{1}{2|\xi|^3}(\partial_1\partial_2\phi(x')\xi_1^2 - \partial_1^2\phi(x')\xi_1\xi_2) \quad \text{and}\\
m_{22}(x',\xi) &= \frac{1}{4|\xi|^3}(\partial_2^2\phi(x')\xi_1^2 - \partial_1^2\phi(x')\xi_2^2).
\end{aligned}
$$

Because the functions m_{11}, m_{12}, m_{21} and m_{22} belong to $C^\infty(V_{x'} \times (\mathbb{R}^2\backslash\{0\}))$ and are positively homogeneous of degree -1 in ξ, then

$$
\begin{pmatrix} m_{11}(x',\xi) & m_{12}(x',\xi) \\ m_{21}(x',\xi) & m_{22}(x',\xi) \end{pmatrix} \tag{48}
$$

is a classical symbol and the operator m is a classical pseudodifferential operator of order -1 in $\mathbb{R}^2$. The operator m is the principal part of M_α, and therefore (48) is the principal symbol for M_α. ◻

We will denote the principal symbol of an operator T by $\sigma_m(T)$, where the subindex m indicates the order of the operator T.

LEMMA 4.3. *The principal symbol of the operator $P_\alpha M_{\alpha'}$ for tangential fields is equal to*

$$
\sigma_0(P_\alpha M_{\alpha'}) = \begin{pmatrix} a_{11}(x',\xi) & a_{12}(x',\xi) \\ a_{21}(x',\xi) & a_{22}(x',\xi) \end{pmatrix},
$$

where

$$
\begin{aligned}
a_{11}(x',\xi) &= \frac{1}{8|\xi|^4}\left(-\partial_1^2\phi(x')\xi_1\xi_2^3 - \partial_2^2\phi(x')\xi_1^3\xi_2 + 2\partial_2\partial_1\phi(x')\xi_1^2\xi_2^2\right),\\
a_{12}(x',\xi) &= \frac{1}{8|\xi|^4}\left(2\partial_1\partial_2\phi(x')\xi_1\xi_2^3 - \partial_2^2\phi(x')\xi_1^2\xi_2^2 - \partial_1^2\phi(x')\xi_2^4\right),\\
a_{21}(x',\xi) &= \frac{1}{8|\xi|^4}\left(\partial_1^2\phi(x')\xi_1^2\xi_2^2 + \partial_2^2\phi(x')\xi_1^4 - 2\partial_2\partial_1\phi(x')\xi_1^3\xi_2\right),\\
a_{22}(x',\xi) &= \frac{1}{8|\xi|^4}\left(-2\partial_1\partial_2\phi(x')\xi_1^2\xi_2^2 + \partial_2^2\phi(x')\xi_1^3\xi_2 + \partial_1^2\phi(x')\xi_1\xi_2^3\right).
\end{aligned}
$$

Proof. Because the principal symbol of the product $P_\alpha M_{\alpha'}$ is $\sigma_0(P_\alpha M_{\alpha'}) = \sigma_1(P_\alpha)\sigma_{-1}(M_{\alpha'})$ (see [22]), and [15]

$$
\sigma_1(P_\alpha) = \frac{1}{2|\xi|}\begin{pmatrix} \xi_1\xi_2 & \xi_2^2 \\ -\xi_1^2 & -\xi_1\xi_2 \end{pmatrix},
$$

we get the symbol $\sigma_0(P_\alpha M_{\alpha'})$ by Theorem 4.3 and straightforward calculations. ◻

LEMMA 4.4. *The operator $\mathcal{T}_\beta$ (for tangential fields) is elliptic if $(k_i\beta)^2 \neq 1$,*

$$
\tau \neq -1, \quad (k_i\beta)^2 \neq \frac{\rho}{\tau}(\tau+1)^2 \quad \textit{and} \quad (k_i\beta)^2 \neq \frac{(1+\tau)(1+\rho)}{\tau}. \tag{49}
$$

Proof. Since $M_\alpha P_\alpha = -P_\alpha M_\alpha$ by (15), and the principal symbols of the operators M_α and P_α do not depend on α, we have that

$$\begin{aligned}\sigma_0(\mathcal{T}_\beta) &= \sigma_0\bigg(\frac{1}{2}I + [b_+ M_{\gamma_1} + b_- M_{\gamma_2}]P_{k_e} + \left(\frac{a_+}{\gamma_1}P_{\gamma_1} + \frac{a_-}{\gamma_2}P_{\gamma_2}\right)M_{k_e} \\ &\quad + \frac{b_+}{\gamma_1}P_{\gamma_1}P_{k_e} - \frac{b_-}{\gamma_2}P_{\gamma_2}P_{k_e}\bigg) \\ &= \frac{1}{2}I - \frac{4\beta}{\tau+1}\sigma_0(P_\alpha M_{\alpha'}) + \frac{b_+}{\gamma_1}\sigma_0(P_{\gamma_1}P_{k_e}) - \frac{b_-}{\gamma_2}\sigma_0(P_{\gamma_2}P_{k_e}).\end{aligned}$$

Since the principal symbol of $P_{\gamma_l}P_{k_e}$, $l = 1, 2$, is (see [15])

$$\sigma_0(P_{\gamma_l}P_{k_e}) = \frac{1}{4|\xi|^2}\begin{pmatrix} \gamma_l^2\xi_1^2 + k_e^2\xi_2^2 & (\gamma_l^2 - k_e^2)\xi_1\xi_2 \\ (\gamma_l^2 - k_e^2)\xi_1\xi_2 & k_e^2\xi_1^2 + \gamma_l^2\xi_2^2 \end{pmatrix},$$

by Lemma 4.3 the principal symbol of the operator $\mathcal{T}_\beta$ is equal to

$$\sigma_0(\mathcal{T}_\beta) = \begin{pmatrix} t_{11}(x',\xi) & t_{12}(x',\xi) \\ t_{21}(x',\xi) & t_{22}(x',\xi) \end{pmatrix},$$

where

$$\begin{aligned}t_{11}(x',\xi) &= \frac{1}{2} - \frac{4\beta}{(\tau+1)}a_{11}(x',\xi) + \frac{b_+}{4\gamma_1|\xi|^2}(\gamma_1^2\xi_1^2 + k_e^2\xi_2^2) \\ &\quad - \frac{b_-}{4\gamma_2|\xi|^2}(\gamma_2^2\xi_1^2 + k_e^2\xi_2^2), \\ t_{12}(x',\xi) &= -\frac{4\beta}{(\tau+1)}a_{12}(x',\xi) + \frac{b_+}{4\gamma_1|\xi|^2}(\gamma_1^2 - k_e^2)\xi_1\xi_2 \\ &\quad - \frac{b_-}{4\gamma_2|\xi|^2}(\gamma_2^2 - k_e^2)\xi_1\xi_2, \\ t_{21}(x',\xi) &= -\frac{4\beta}{(\tau+1)}a_{21}(x',\xi) + \frac{b_+}{4\gamma_1|\xi|^2}(\gamma_1^2 - k_e^2)\xi_1\xi_2 \\ &\quad - \frac{b_-}{4\gamma_2|\xi|^2}(\gamma_2^2 - k_e^2)\xi_1\xi_2, \\ t_{22}(x',\xi) &= \frac{1}{2} - \frac{4\beta}{(\tau+1)}a_{22}(x',\xi) + \frac{b_+}{4\gamma_1|\xi|^2}(\gamma_1^2\xi_2^2 + k_e^2\xi_1^2) \\ &\quad - \frac{b_-}{4\gamma_2|\xi|^2}(\gamma_2^2\xi_2^2 + k_e^2\xi_1^2),\end{aligned}$$

where a_{ij}, $i, j = 1, 2$ are given in Lemma 4.3. We get the determinant of this symbol by using the equalities

$$\frac{1}{4|\xi|^2}\left(\frac{b_+}{\gamma_1}(\gamma_1^2\xi_2^2 + k_e^2\xi_1^2) - \frac{b_-}{\gamma_2}(\gamma_2^2\xi_2^2 + k_e^2\xi_1^2)\right) = \frac{1}{2|\xi|^2}(A_1\xi_2^2 + A_2\xi_1^2),$$

$$\frac{1}{4|\xi|^2}\left(\frac{b_+}{\gamma_1}(\gamma_1^2\xi_1^2 + k_e^2\xi_2^2) - \frac{b_-}{\gamma_2}(\gamma_2^2\xi_1^2 + k_e^2\xi_2^2)\right) = \frac{1}{2|\xi|^2}(A_1\xi_1^2 + A_2\xi_2^2),$$

$$\frac{1}{4|\xi|^2}\left(\frac{b_+}{\gamma_1}(\gamma_1^2 - k_e^2) - \frac{b_-}{\gamma_2}(\gamma_2^2 - k_e^2)\right)\xi_1\xi_2 = \frac{1}{2|\xi|^2}(A_1 - A_2)\xi_1\xi_2,$$

$$a_{11} = -a_{22}, \quad a_{11}a_{22} - a_{12}a_{21} = 0,$$

$$a_{11}\xi_2^2 + a_{22}\xi_1^2 - a_{12}\xi_1\xi_2 - a_{21}\xi_1\xi_2 = 0$$

and

$$a_{11}\xi_1^2 + a_{22}\xi_2^2 + a_{12}\xi_1\xi_2 + a_{12}\xi_1\xi_2 = 0,$$

where

$$A_1 = \frac{\rho(1+\tau) + (k_i\beta)^2}{(\tau+1)(1-(k_i\beta)^2)} \quad \text{and} \quad A_2 = \frac{k_e^2(\rho(1+\tau) - (k_i\beta)^2)}{k_i^2(\tau+1)}.$$

Then the determinant is equal to

$$\det[\sigma_0(\mathcal{T}_\beta)] = \frac{1}{4}(A_1+1)(A_2+1),$$

and it is non-zero when the given assumptions hold. □

THEOREM 4.4. *The operator $\mathcal{T}_\beta$ is a Fredholm operator with index zero on $TH^{1/2}(\partial\Omega)$ if the restrictions* (49) *and $|k_i\beta| < 1$ hold.*

Proof. With the given restrictions the operator $\mathcal{T}_\beta$ is Fredholm because of the ellipticity. Thus, it is enough to show that the index is zero.

If $|k_i\beta| < 1$, the operator $\mathcal{T}_{\beta t}$ is continuous with respect to $t \in [0,1]$ in the operator norm $TH^{1/2}(\partial\Omega) \to TH^{1/2}(\partial\Omega)$. Then it follows from [22, Proposition 8.1] that

$$\operatorname{index} \mathcal{T}_\beta = \operatorname{index} \mathcal{T}_0 = \operatorname{index} \mathcal{A}_{\rho,0}.$$

In the same way,

$$\operatorname{index} \mathcal{A}_{\rho,0} = \operatorname{index} \mathcal{A}_{0,0} = \operatorname{index} \frac{1}{2}(I + 2M_{k_e})(I + 2M_{k_i}) = 0$$

since M_{k_e} and M_{k_i} are compact. □

The following theorem is the main result in this paper.

THEOREM 4.5. *Assume that $|k_i\beta| < 1$ and the restrictions* (49) *hold. If k_e^2 is not an eigenvalue of the associated interior Maxwell problem with the boundary condition*

$$\mathrm{i}\,\frac{\beta\omega\varepsilon_e}{\tau+1}(n \times E) + (n \times H) = 0,$$

the equation (44) *is uniquely solvable in $TH^{1/2}(\partial\Omega)$. In addition, if $f \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$, then $j \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$.*

Proof. The first claim holds by Theorem 4.2, Lemma 4.4 and the Fredholm alternative. So it remains to prove the second part of the theorem. First we use the equality (43) for the terms $P_{\gamma_1}P_{k_e}$ and $P_{\gamma_2}P_{k_e}$ in the operator $\mathcal{T}_\beta$. Then we show that all the operators or their combinations in $\mathcal{T}_\beta$ map from $TH^{1/2}(\partial\Omega)$ to $TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$, except the identity operator I. First, by the mapping properties (12), (14) and (11) we get that

$$P_\alpha(P_{k_e} - P_\alpha), P_\alpha M_{k_e} : TH^{1/2}(\partial\Omega) \to TH^{1/2}_{\mathrm{Div}}(\partial\Omega).$$

Because the equality (15) gives us that

$$M_\alpha P_{k_e} = M_\alpha(P_{k_e} - P_\alpha) + M_\alpha P_\alpha = M_\alpha(P_{k_e} - P_\alpha) - P_\alpha M_\alpha,$$

then

$$M_\alpha P_{k_e} : TH^{1/2}(\partial\Omega) \to TH^{1/2}_{\mathrm{Div}}(\partial\Omega).$$

Secondly, it is clear that

$$M_\alpha, M_\alpha^2, P_\alpha - P_{\alpha'} : TH^{1/2}(\partial\Omega) \to TH^{1/2}_{\mathrm{Div}}(\partial\Omega),$$

and hence

$$\mathcal{T}_\beta - cI : TH^{1/2}(\partial\Omega) \to TH^{1/2}_{\mathrm{Div}}(\partial\Omega),$$

where c is the coefficient of I in the operator $\mathcal{T}_\beta$. The constant c is equal to

$$\frac{\tau(1-(k_i\beta)^2) + \rho(1+\tau) + 1}{2(1-(k_i\beta)^2)(\tau+1)},$$

which is non-zero by the assumptions. Finally, if $f \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$, then

$$cIj + (\mathcal{T}_\beta - cI)j = f$$

if and only if

$$j = \frac{1}{c}[f - (\mathcal{T}_\beta - cI)j] \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega).$$

□

Since the transmission problem has at most one solution [2], we get the following corollary.

COROLLARY. *If the same assumptions as in Theorem 4.5 hold, then the transmission problem has a unique solution.*

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