METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 19, No. 3, pp. 289–312, September 2012

004

# ASYMPTOTIC BERNSTEIN TYPE INEQUALITIES AND ESTIMATION OF WAVELET COEFFICIENTS*

SUSANNA SPEKTOR† AND XIAOSHENG ZHUANG‡

**Abstract.** In this paper, we investigate the wavelet coefficients for function spaces $\mathcal{A}_k^p := \{f : \|(i\omega)^k \hat{f}(\omega)\|_p \leqslant 1\}$, $k \in \mathbb{N} \cup \{0\}$, $p \in (1, \infty)$ using an important quantity $C_{k,p}(\psi) := \sup\{\frac{|\langle f,\psi\rangle|}{\|\hat{\psi}\|_p} : f \in \mathcal{A}_k^{p'}\}$ with $1/p + 1/p' = 1$. In particular, Bernstein type inequalities associated with wavelets are established. We obtained an sharp inequality of Bernstein type for splines and a lower bound for the quantity $C_{k,p}(\psi)$ with $\psi$ being the semiorthogonal spline wavelets. We also study the asymptotic behavior of wavelet coefficients for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets. Comparison of these two families is done by using the quantity $C_{k,p}(\psi)$.

**Key words.** Wavelet coefficients, asymptotic estimation, Bernstein type inequalities, Daubechies orthonormal wavelets, semiorthogonal spline wavelets.

**AMS subject classifications.** 42C40, 41A05, 42C15, 65T60.

## 1. Introduction and motivations.

Almost any kind of practical sciences requires the analysis of data. Depending on the specific application, the collection of data may consist of measurements, signals, or images. In mathematical framework, all of those objects are functions. One way to analyze them is by representing them into wavelet decomposition. Such methods are not only used in mathematics, but also in physics, electrical engineering, and medical imaging [6, 7, 10, 12, 14, 15, 20]. Wavelets provides reconstruction (approximation) of the original function (the collection of data). In order to characterize the approximation class, one has to establish Bernstein inequality. We will first give some basic definitions before representing the importance and applications of the mentioned inequality.

We say that $\varphi : \mathbb{R} \to \mathbb{C}$ is a 2-*scaling function* if

$$\varphi = 2\sum_{\nu\in\mathbb{Z}} a(\nu)\varphi(2\cdot -\nu), \tag{1.1}$$

where $a : \mathbb{Z} \to \mathbb{C}$ is a finitely supported sequence of complex numbers on $\mathbb{Z}$, called the *mask (or low-pass filter )* for $\varphi$. In frequency domain, the refinement equation in (1.1) can be rewritten as

$$\hat{\varphi}(2\omega) = \hat{a}(\omega)\hat{\varphi}(\omega), \quad \omega \in \mathbb{R}, \tag{1.2}$$

where $\hat{a}$ is the *Fourier series* of $a$ given by

$$\hat{a}(\omega) := \sum_{\nu\in\mathbb{Z}} a(\nu)e^{-i\nu\omega}, \quad \omega \in \mathbb{R}. \tag{1.3}$$

The Fourier transform $\hat{f}$ of $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} f(x)e^{-i\omega x}dx$ and can be extended to square integrable functions and tempered distributions.

*Received March 27, 2012; accepted for publication November 30, 2012.
†Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton T6G 2G1, Canada (spektor@ualberta.ca).
‡Department of Mathematics, City University of Hong Kong, Hong Kong, China (xzhuang7@cityu.edu.hk).

Usually, a wavelet system is generated by some wavelet function $\psi$ from a 2-scaling function $\varphi$ as follows:

$$\psi = 2\sum_{\nu\in\mathbb{Z}} b(\nu)\varphi(2\cdot-\nu) \quad \text{or} \quad \hat{\psi}(2\cdot) = \hat{b}(\cdot)\hat{\varphi}(\cdot), \tag{1.4}$$

where $b:\mathbb{Z}\to\mathbb{C}$ is a finitely supported sequence of complex numbers on $\mathbb{Z}$, called the *mask (or band-pass filter)* for $\psi$. For a more general approach on obtaining wavelet functions from a $d$-scaling function, see [8, 17].

Many wavelet applications, for example, image/signal compression, denoising, inpainting, compressive sensing, and so on, are based on investigation of the wavelet coefficients $\langle f,\varphi_{j,\nu}\rangle$ and $\langle f,\psi_{j,\nu}\rangle$ for $j,\nu\in\mathbb{Z}$, where $\langle f,g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}dx$ and $\varphi_{j,\nu} := 2^{j/2}\varphi(2^j\cdot-\nu), \psi_{j,\nu} := 2^{j/2}\psi(2^j\cdot-\nu)$. The magnitude of the wavelet coefficients depends on both the smoothness of the function $f$ and the wavelet $\psi$. In this paper, we shall investigate the quantity

$$C_{k,p}(\psi) = \sup_{f\in\mathcal{A}_k^{p'}} \frac{|\langle f,\psi\rangle|}{\|\hat{\psi}\|_p}, \tag{1.5}$$

where $1<p,p'<\infty$, $1/p'+1/p=1$, $k\in\mathbb{N}\cup\{0\}$, and $\mathcal{A}_k^{p'} := \{f\in L_{p'}(\mathbb{R}) : \|(i\omega)^k\hat{f}(\omega)\|_{p'}\leqslant 1\}$. This quantity is closely related to Bernstein type inequality in wavelet analysis. The classical Bernstein inequality states that for any $\alpha\in\{\mathbb{N}\cup\{0\}\}^n$, one has $\|\partial^\alpha f\|_p\leqslant R^{|\alpha|}\|f\|_p$, where $f\in L_p(\mathbb{R}^n)$ is an arbitrary function whose Fourier transform $\hat{f}$ is supported in the ball $|\omega|\leqslant R$. The quantity $C_{k,p}(\psi)$ in (1.5) is the best possible constant in the following Bernstein type inequality

$$|\langle f,\psi_{j,\nu}\rangle| \leqslant C_{k,p}(\psi)2^{-j(k+1/p-1/2)}\|\hat{\psi}\|_p\|(i\omega)^k\hat{f}(\omega)\|_{p'}. \tag{1.6}$$

This inequality gives us a way of investigating the magnitude of the coefficients in wavelet decomposition of the function. The coefficients tell in what way the analyzing function needs to be modified in order to reconstruct the data (see [13]). On the other hand, bound of type (1.6) gives a-priori information on the size of wavelet coefficients which is important for such application as compression of data (see e.g. [6, 20]). Also, such types of inequalities play an important role in wavelet algorithms for the numerical solution of integral equations (see e.g. [5, 25]), where wavelet coefficients arise by applying an integral operator to a wavelet; and for the estimation of wavelet coefficients of the space of distributions with bounded variations derivatives (see [4]).

Note that

$$C_{k,p}(\psi) = \sup_{f\in\mathcal{A}_k^{p'}} \frac{|\langle f,\psi\rangle|}{\|\hat{\psi}\|_p} = \sup_{f\in\mathcal{A}_k^{p'}} \frac{|\langle \hat{f},\hat{\psi}\rangle|}{\|\hat{\psi}\|_p} = \frac{\|\widehat{{}_k\psi}\|_p}{\|\hat{\psi}\|_p}, \tag{1.7}$$

where for a function $f\in L_1(\mathbb{R})$, ${}_kf$ is defined to be the function such that

$$\widehat{{}_kf}(\omega) = (i\omega)^{-k}\hat{f}(\omega), \quad \omega\in\mathbb{R}. \tag{1.8}$$

For $\psi$ that is compactly supported, it is easily shown that the quantity $C_{k,p}(\psi)<\infty$ is equivalent to

$$\int_{\mathbb{R}} \psi(x)x^\nu dx = 0 \quad \text{or} \quad \frac{d^\nu}{dx^\nu}\hat{\psi}(0) =: \hat{\psi}^{(\nu)}(0) = 0 \tag{1.9}$$

for $\nu = 0, \ldots, m-1$. That is, $\psi$ has $m$ *vanishing moments*. Consequently, for a wavelet $\psi$ with $m$ vanishing moments, we can investigate the magnitude of the wavelet coefficients in the function spaces $\mathcal{A}_1^{p'}, \ldots, \mathcal{A}_m^{p'}$ for $1 < p' < \infty$ using the quantity $C_{k,p}(\psi)$.

On the other hand, a fundamental question in wavelet application is which type of wavelets one should choose for a specific purpose. In [18], Keinert used a constant $G_M$ in the following approximation for comparison of wavelets.

$$\int_{\mathbb{R}} f(x)\psi_{j,\nu}(x)dx \approx 2^{-(j+1)(M+1/2)}\frac{G_M}{M!}f^{(M)}(2^{-j}\nu), \tag{1.10}$$

where $f$ is sufficient smooth, $\psi$ has exactly $M$ vanishing moments, and $G_M$ depends only on $\psi$. Keinert presented numerical values of $G_M$ for some commonly used wavelets and provided constructions for wavelets with short support and minimal $G_M$, which lead to better compression in practical calculation. By considering the quantity $C_{k,p}(\psi)$, the "$\approx$" in (1.10) can be replaced by precise inequality. In [16], Ehrich investigate the quantity $C_{k,p}(\psi)$ for $p = 2$ and for two important families of wavelets, namely, Daubechies orthonormal wavelets and semiorthogonal wavelets. Precise asymptotic relations of quantities $C_{k,2}(\psi)$ are established in [16] showing that the quantity for the family of semiorthogonal spline wavelets is generally smaller than that for the family of Daubechies orthonormal wavelets.

In this paper, we shall investigate the quantity $C_{k,p}(\psi), p \in (1,\infty)$ mainly for the family of Daubechies orthonormal wavelets (see [10]) and the family of semiorthogonal spline wavelets (see [9]). We next give a brief introduction of these two families.

Let $m$ be a positive integer. Let $a_m^D$ and $b_m^D$ be two masks determined by:

$$|\widehat{a_m^D}(\omega)|^2 = \cos^{2m}(\omega/2)\sum_{\nu=0}^{m-1}\binom{m-1+\nu}{\nu}\sin^{2\nu}(\omega/2), \tag{1.11}$$

and

$$\widehat{b_m^D}(\omega) = e^{i\omega}\overline{\widehat{a_m^D}(\omega+\pi)}. \tag{1.12}$$

It is well-known that $|\widehat{a_m^D}(\omega)|^2$ is the Dubuc-Deslauriers interpolatory mask of order $m$ ([11]) and $a_m^D$ can be obtained by factoring (1.11) via Riesz Lemma ([10]). The Daubechies 2-scaling function $\varphi_m^D$ of order $m$ associated with mask $a_m^D$ and Daubechies orthonormal wavelet $\psi_m^D$ of order $m$ associated with mask $b_m^D$ are then given by

$$\widehat{\phi_m^D} = \frac{1}{\sqrt{2\pi}}\prod_{\ell=1}^{\infty}\widehat{a_m^D}(2^{-\ell}\cdot) \quad \text{and} \quad \widehat{\psi_m^D} = \widehat{b_m^D}(\cdot/2)\widehat{\phi_m^D}(\cdot/2).$$

The semiorthogonal spline wavelet $\psi_m^S$ of order $m$ is given by

$$\psi_m^S(x) = \sum_{\nu=0}^{2m-2}\frac{(-1)^\nu}{2^{m-1}}N_{2m}(\nu+1)N_{2m}^{(m)}(2x-\nu), \quad x \in \mathbb{R}, \tag{1.13}$$

where $N_m$ is the B-spline of order $m$. That is,

$$N_m(x) = \frac{1}{(m-1)!}\sum_{\nu=0}^{m}(-1)^\nu\binom{m}{\nu}(x-\nu)_+^{m-1}, \quad x \in \mathbb{R} \tag{1.14}$$

or equivalently,

$$\widehat{N_m}(\omega) = \frac{1}{\sqrt{2\pi}}\left(e^{-i\omega/2}\frac{\sin(\omega/2)}{\omega/2}\right)^m = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{i\omega}}{i\omega}\right)^m, \quad \omega \in \mathbb{R}. \tag{1.15}$$

Here for $k \geqslant 1$,

$$(y)_+^k = \begin{cases} y^k & y > 0, \\ 0, & y \leqslant 0, \end{cases} \quad \text{and} \quad (y)_+^0 = \begin{cases} 1 & y > 0, \\ \frac{1}{2}, & y = 0, \\ 0, & y < 0. \end{cases}$$

Note that $\psi_m^S$ is generated from the 2-scaling function $\varphi_m^S := N_m$ via (1.4) by some mask $b_m^S$ (cf. [7, 9]).

These two families are widely used in many applications. For example, see [1, 5, 12, 21, 23, 24, 25] for their applications on numerical solution of PDE and signal/image processing. Both of the Daubechies orthonormal wavelet $\psi_m^D$ and the semiorthogonal spline wavelet $\psi_m^S$ have vanishing moments of order $m$ and support length $2m-1$. The Daubechies orthonormal wavelet $\psi_m^D$ generates an orthonormal basis $\{2^{j/2}\psi_m^D(2^j\cdot -\nu) : j, \nu \in \mathbb{Z}\}$ for $L_2(\mathbb{R})$ (see [10]). However, the wavelet function $\psi_m^D$ is implicitly defined and the coefficients in the mask for $\psi_m^D$ are not rational numbers. Though the semiorthogonal spline wavelets generated by $\psi_m^S$ are not orthogonal in the same level $j$, they are orthogonal on different levels. And more importantly, the semiorthogonal spline wavelet $\psi_m^S$ is explicitly defined and the coefficients for its mask are indeed rational numbers, which is a very much desirable property in the implementation of fast wavelet algorithms. We shall see that these two families significantly differ with respect to the magnitude of their wavelet coefficients in terms of $C_{k,p}(\psi_m^D)$ and $C_{k,p}(\psi_m^S)$.

We are using several strategies to study asymptotic and non-asymptotic behavior of different kind wavelets. In Section 2, for $k, m \in \mathbb{N}$ fixed and $p \in (1, \infty)$, we shall investigate the quantity $C_{k,p}(\psi_m^S)$ in the Bernstein type inequality in (1.6) for the family of semiorthogonal spline wavelets. One of crucial new ingredients is Proposition 1, which is essential in the setting of non-asymptotic behaviour of semiorthogonal spline wavelets. In Section 3, we shall establish results on the asymptotic behaviors ($m \to \infty$) of the quantities $C_{k,p}(\varphi)$ and $C_{k,p}(\psi)$ for both the scaling function $\varphi$ and wavelet function $\psi$ and for both the two families of wavelets. We shall generalize our results to high-dimensional wavelets in Section 4. The last section are some technical proofs for some results in previous sections.

**2. Bernstein type inequalities for splines.** In this section, we shall first establish a sharp result on the Bernstein type inequality for splines and then present a lower bound for the quantity $C_{k,p}(\psi_m^S)$.

Recall that a function $s$ is a *spline of order $m$ of minimal defect with nodes* $\ell h, h > 0, \ell \in \mathbb{Z}$, if

(1) $s$ is a polynomial with real coefficients of the degree $< m$ on each interval $(h(\ell-1), h\ell)$, $\ell \in \mathbb{Z}$;
(2) $s \in C^{m-2}(\mathbb{R})$.

The collection of all such splines is denoted by $S_{m,h}$. It is well known that any spline $s \in S_{m,h}$ can be uniquely represented by

$$s(x) = \sum_{\nu \in \mathbb{Z}} c_\nu N_m(x/h - \nu), \qquad x \in \mathbb{R}. \tag{2.1}$$

Here $N_m$ is the B-spline of order $m$ given in (1.14). One can show that for $m \geqslant 2$,

$$N_m'(x) = N_{m-1}(x) - N_{m-1}(x-1), \qquad x \in \mathbb{R}. \tag{2.2}$$

The following result provides an exact upper bound for the Bernstein type inequality for any spline $s \in S_{m,h}$ (also cf. [3] for a special case $p = 2$).

THEOREM 1. *Let $k, m \in \mathbb{N} \cup \{0\}$, $0 \leqslant k < m$, and $h > 0$. Let $p \in (1, \infty)$. Then, for any spline function $s \in S_{m,h}$ such that $\hat{s} \in L_p(\mathbb{R})$, the following inequality holds:*

$$\|\widehat{s^{(k)}}\|_p \le K_{p,m,k} \left(\frac{2\pi}{h}\right)^k \|\hat{s}\|_p, \tag{2.3}$$

*where $K_{p,m,k}$ is a constant depending on $p, m$, and $k$ and is defined to be*

$$K_{p,m,k} := \max_{\omega \in [0,2\pi]} \left( \frac{\sum_{\ell \in \mathbb{Z}} |\frac{\omega}{2\pi} + \ell|^{-p(m-k)}}{\sum_{\ell \in \mathbb{Z}} |\frac{\omega}{2\pi} + \ell|^{-pm}} \right)^{1/p}. \tag{2.4}$$

*Moreover, the constant $K_{p,m,k}$ is sharp in the sense that there exists a sequence $s_j \in S_{m,h}$ such that*

$$\frac{\|\widehat{s_j^{(k)}}\|_p}{\|\widehat{s_j}\|_p} \to (2\pi/h)^k K_{p,m,k}, \quad j \to \infty.$$

*Proof.* We first show that (2.3) is true for $h = 1$.
Recursively applying (2.2), we can deduce that

$$\widehat{s^{(k)}}(\omega) = \sum_{\nu \in \mathbb{Z}} c_\nu e^{-i\nu\omega} (1 - e^{-i\omega})^k \widehat{N_{m-k}}(\omega), \quad \omega \in \mathbb{R}; 0 \leqslant k < m.$$

Consequently,

$$\begin{aligned}
\|\widehat{s^{(k)}}\|_p^p &= \int_{\mathbb{R}} \left| \sum_{\nu \in \mathbb{Z}} c_\nu e^{-i\nu\omega} \widehat{N_{m-k}}(\omega)(1 - e^{-i\omega})^k \right|^p d\omega \\
&= \int_0^{2\pi} \left|\widehat{a_s}(\omega)(1 - e^{-i\omega})^k\right|^p \sum_{\ell \in \mathbb{Z}} \left|\widehat{N_{m-k}}(\omega + 2\pi\ell)\right|^p d\omega \\
&= \int_0^{2\pi} \frac{\left|1 - e^{-i\omega}\right|^{kp} \sum_{\ell \in \mathbb{Z}} \left|\widehat{N_{m-k}}(\omega + 2\pi\ell)\right|^p}{\sum_{\ell \in \mathbb{Z}} \left|\widehat{N_m}(\omega + 2\pi\ell)\right|^p} |\widehat{a_s}(\omega)|^p \sum_{\ell \in \mathbb{Z}} \left|\widehat{N}_m(\omega + 2\pi\ell)\right|^p d\omega \\
&\le \max_{\omega \in [0,2\pi]} \frac{\left|1 - e^{-i\omega}\right|^{kp} \sum_{\ell \in \mathbb{Z}} \left|\widehat{N_{m-k}}(\omega + 2\pi\ell)\right|^p}{\sum_{\ell \in \mathbb{Z}} \left|\widehat{N_m}(\omega + 2\pi\ell)\right|^p} \|\hat{s}\|_p^p.
\end{aligned}$$

Here $\widehat{a_s}(\omega) = \sum_{\nu \in \mathbb{Z}} c_\nu e^{-i\nu\omega}$. Define

$$L(\omega) := \frac{\left|1 - e^{-i\omega}\right|^{kp} \sum_{\ell \in \mathbb{Z}} \left|\widehat{N_{m-k}}(\omega + 2\pi\ell)\right|^p}{\sum_{\ell \in \mathbb{Z}} \left|\widehat{N_m}(\omega + 2\pi\ell)\right|^p}, \quad \omega \in [0, 2\pi]. \tag{2.5}$$

Then, we obtain

$$\|\widehat{s^{(k)}}\|_p^p \leqslant \max_{\omega\in[0,2\pi]} L(\omega)\cdot\|\hat{s}\|_p^p.$$

Since $\widehat{N_m}(\omega) = \dfrac{1}{\sqrt{2\pi}}\left(\dfrac{1-e^{i\omega}}{i\omega}\right)^m$, we have

$$\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega+2\pi\ell)\right|^p = \frac{1}{(\sqrt{2\pi})^p}\left|1-e^{-i\omega}\right|^{pm}\sum_{\ell\in\mathbb{Z}}|\omega+2\pi\ell|^{-pm},$$

and, similarly,

$$\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p = \frac{1}{(\sqrt{2\pi})^p}\left|1-e^{-i\omega}\right|^{p(m-k)}\sum_{\ell\in\mathbb{Z}}|\omega+2\pi\ell|^{-p(m-k)}.$$

Hence,

$$\max_{\omega\in[0,2\pi]} L(\omega)^{1/p} = \max_{\omega\in[0,2\pi]}\left(\frac{\sum_{\ell\in\mathbb{Z}}|\omega+2\pi\ell|^{-p(m-k)}}{\sum_{l\in\mathbb{Z}}|\omega+2\pi\ell|^{-pm}}\right)^{1/p} = (2\pi)^k K_{p,m,k}.$$

Therefore

$$\|\widehat{s^{(k)}}\|_p \leq (2\pi)^k K_{p,m,k}\|\hat{s}\|_p.$$

Next, for any $h>0$ and $s\in S_{m,h}$, we have $s=\sum_{\nu\in\mathbb{Z}}c_\nu N_m(\cdot/h+\nu)$. Let $s_1 := s(h\cdot)$. Then $s_1\in S_{m,1}$ and it is easy to deduce that

$$\hat{s}(\omega) = h\widehat{s_1}(h\omega) \quad\text{and}\quad \widehat{s^{(k)}}(\omega) = h^{-k+1}\widehat{s_1^{(k)}}(h\omega). \tag{2.6}$$

By what we have been proved, we get

$$\|\widehat{s_1^{(k)}}\|_p \leq (2\pi)^k K_{p,m,k}\|\widehat{s_1}\|_p.$$

Now, it is straightforward to deduce (2.3) from above inequality using (2.6).

Finally, we show that the constant in (2.3) is the best possible one.

Let $|\widehat{a_s}(\omega)|^p := \frac{1}{2\pi}\Phi_j(\omega-\omega_0)$ and $\hat{s}(\omega) := \widehat{a_s}(\omega)\widehat{N_m}(\omega)$, $\omega\in\mathbb{R}$, where $\Phi_j(\omega)$ is a Feyer's kernel of order $j$ and $\omega_0$ is the point which realizes the maximum of the function $L(\omega)$ on $[0,2\pi]$. Note, $\frac{1}{2\pi}\int_0^{2\pi}\Phi_j(\omega)d\omega = 1$. Then,

$$\begin{aligned}
\|\widehat{s^{(k)}}\|_p^p &= \int_{\mathbb{R}}\left|\sum_{\nu\in\mathbb{Z}}c_\nu e^{-i\nu x}\widehat{N_{m-k}}(\omega)(1-e^{-i\omega})^k\right|^p d\omega\\
&= \int_0^{2\pi}|\widehat{a_s}(\omega)(1-e^{-i\omega})^k|^p\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{|1-e^{-i\omega}|^{kp}\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega+2\pi\ell)\right|^p}|\Phi_j(\omega-\omega_0)|\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega+2\pi\ell)\right|^p d\omega\\
&\to \frac{|1-e^{-i\omega_0}|^{kp}\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_{m-k}}(\omega_0+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega_0+2\pi\ell)\right|^p}\|\hat{s}\|_p^p, \qquad j\to\infty.
\end{aligned}$$

Consequently,

$$\frac{\|\widehat{s^{(k)}}\|_p}{\|\hat{s}\|_p} \to \max_{\omega\in[0,2\pi]} \left( \frac{|1-e^{-i\omega}|^{kp} \sum_{\ell\in\mathbb{Z}} \left|\widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}} \left|\widehat{N_m}(\omega+2\pi\ell)\right|^p} \right)^{1/p} = (2\pi)^k K_{p,m,k}, \quad j\to\infty$$

which completes the proof. ◻

We remark that for $p=2$, the function $L(\omega), \omega\in[0,2\pi]$ defined in (2.5) assumes its maximal value at $\omega=\pi$ and the constant $K_{2,m,k}$ can be obtained explicitly as follows:

$$K_{2,m,k} = \pi^2 \frac{\sum_{\ell\in\mathbb{Z}} |1+2\ell|^{-2(m-k)}}{\sum_{\ell\in\mathbb{Z}} |1+2\ell|^{-(2m)}},$$

which is related to the Favard's constant (see [3]). For general $p$, $L$ can be expressed as

$$L(\omega) = (2\pi)^{kp} \frac{\zeta(p(m-k), \frac{\omega}{2\pi}) + \zeta(p(m-k), -\frac{\omega}{2\pi}) - \left(-\frac{\omega}{2\pi}\right)^{-p(m-k)}}{\zeta(pm, \frac{\omega}{2\pi}) + \zeta(pm, -\frac{\omega}{2\pi}) - \left(-\frac{\omega}{2\pi}\right)^{-pm}}, \quad \omega\in[0,2\pi],$$

where $\zeta(x,y) := \sum_{\ell=0}^{\infty} (\ell+y)^{-x}$ is the Hurwitz zeta function.

For $s\in S_{m,h}$. Let $f :=_k s$. Then $f^{(k)} = s$, which implies $f\in S_{m+k,h}$. By Theorem 1 and the definition of $C_{k,p}(f)$ in (1.7), we have,

$$C_{k,p}(s) = \frac{\|\widehat{{}_k s}\|_p}{\|\hat{s}\|_p} = \frac{\|\hat{f}\|_p}{\|\widehat{f^{(k)}}\|_p} \geqslant \left(\frac{h}{2\pi}\right)^k \frac{1}{K_{p,m+k,k}}.$$

Moreover, by the definition of $\psi_m^S$ in (1.13), we have the following result.

PROPOSITION 1. *Let $\psi_m^S$ be the semiorthogonal spline wavelet of order $m$ defined in* (1.13). *Let $k$ be an nonnegative integer such that $k\leqslant m$. Then*

$$C_{k,p}(\psi_m^S) \geqslant \left(\frac{1}{4\pi}\right)^k \frac{1}{K_{p,m+k,k}}.$$

*Proof.* Let $f :=_k \psi_m^S$. Then $\widehat{f^{(k)}} = \widehat{\psi_m^S}$. By (1.13),

$$f(x) =_k \psi_m^S(x) = \sum_{\nu=0}^{2m-2} \frac{(-1)^\nu}{2^{m+k-1}} N_{2m}(\nu+1) N_{2m}^{(m-k)}(2x-\nu).$$

Consequently, $f\in S_{m+k,1/2}$. In view of Theorem 1, we have

$$\frac{\|\widehat{f^{(k)}}\|_p}{\|\hat{f}\|_p} \leqslant \left(\frac{2\pi}{\frac{1}{2}}\right)^k K_{p,m+k,k} = (4\pi)^k K_{p,m+k,k}.$$

Now, by that $C_{k,p}(\psi_m^S) = \frac{\|\hat{f}\|_p}{\|\widehat{f^{(k)}}\|_p}$, we are done. ◻

From Proposition 1, when $m$ is large enough, we see that $C_{k,p}(\psi_m^S) \approx (4\pi)^{-k}$. In next section, we shall study the exact asymptotic behavior of these types of quantities as $m\to\infty$ for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets.

**3. Asymptotic estimation of wavelet coefficients.** In this section, we shall study the asymptotic behavior of wavelet coefficients for both Daubechies orthonormal wavelets and semiorthogonal spline wavelets (also see [19] for the asymptotic behavior of Battle-Lemari wavelet family). We shall discuss the asymptotic behavior of the wavelet coefficients for Daubechies orthonormal wavelets in the first subsection. In the second subsection, we shall investigate the asymptotic behavior of the wavelet coefficients for semiorthogonal spline wavelets. In the last subsection, we shall compare the asymptotic behaviors of wavelet coefficients for these two families based on the quantities obtained in previous two subsections.

**3.1. Wavelet coefficients of Daubechies orthonormal wavelets.** In this subsection, we shall discuss the asymptotic behavior of the following quantities: $\|\widehat{{}_{-k}\varphi_m^D}\|_p$, $\|\widehat{{}_k\psi_m^D}\|_p$, and $\|\widehat{{}_m\psi_m^D}\|_p$, $p \in (1,\infty)$.

To facilitate our investigation on the asymptotic behavior of Daubechies orthonormal wavelets, let us rewrite the mask $a_m^D$ in another equivalent form. Let $H_m(t)$ be a $2\pi$-periodic trigonometric function defined by

$$H_m(t) = \sum_{\nu=0}^{L} h_\nu e^{-i\nu t}, \quad |H_m(t)|^2 = 1 - c_m \int_0^t \sin^{2m-1}\omega d\omega, \tag{3.1}$$

where $c_m = \left(\int_0^\pi \sin^{2m-1}\omega d\omega\right)^{-1} = \dfrac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}\Gamma(m)} \sim \sqrt{\dfrac{m}{\pi}}$. Then, we have $|H_m(\cdot)|^2 = |\widehat{a_m^D}(\cdot)|^2$. Hence, $H_m$ is the Daubechies orthonormal mask of order $m$ (cf. [16]).

To compare with the semiorthogonal spline wavelets, we need the following result for the Daubechies scaling function $\varphi_m^D$.

THEOREM 2. *Let $\varphi_m^D$ be the Daubechies orthonormal scaling function of order $m$, i.e., $\widehat{\varphi_m^D}(\omega) = \frac{1}{\sqrt{2\pi}}\prod_{\ell=1}^\infty H_m(2^{-\ell}\omega)$. Then, for $p \in (1,\infty)$,*

$$\lim_{m\to\infty} \|\widehat{{}_{-k}\varphi_m^D}\|_p = \pi^k \frac{(2\pi)^{1/p-1/2}}{(1+pk)^{1/p}}, \quad k \in \mathbb{N}\cup\{0\}. \tag{3.2}$$

*Proof.* Let $\Phi := \dfrac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}$. We have

$$\|\widehat{{}_{-k}\varphi_m^D}\|_p^p = \int_{\mathbb{R}} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega)\right|^p d\omega = \int_{\mathbb{R}} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega) - \Phi(\omega) + \Phi(\omega)\right|^p d\omega.$$

Note that

$$\int_{\mathbb{R}} |\omega|^{pk} |\Phi(\omega)|^p d\omega = \pi^{pk} \frac{(2\pi)^{1-p/2}}{1+pk}.$$

We next prove that

$$I := \int_{\mathbb{R}} |\omega|^{pk} \left|\widehat{\varphi_m^D} - \Phi(\omega)\right|^p d\omega \to 0, \quad \text{as } m \to \infty.$$

In fact,

$$I = \int_{|\omega|>\pi} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega)\right|^p d\omega + \int_{|\omega|\leqslant\pi} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega) - \Phi(\omega)\right|^p d\omega =: I_1 + I_2.$$

By the regularity of $\varphi_m^D$, i.e., $|\widehat{\varphi_m^D}(\omega)| \leqslant C_1|\omega|^{-C_2\log(m)}$ (see [10]), obviously, $I_1 \to 0$ as $m \to \infty$. For $I_2$, let $I := [-\pi, \pi]$, $\delta > 0$ be fixed, and $I_\delta := [-\pi+\delta, \pi-\delta]$. Then

$$I_2 = \int_{I_\delta} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega) - \Phi(\omega)\right|^p d\omega + \int_{I\setminus I_\delta} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega) - \Phi(\omega)\right|^p d\omega := I_{21} + I_{22}.$$

For $I_{22}$, we have $I_{22} \leqslant C\delta$ for some $C$ depending only on $p, k$, since $\widehat{\varphi_m^D}$ and $\Phi$ are both bounded. For $I_{21}$, we have

$$\begin{aligned} I_{21} &\leqslant \int_{I_\delta} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega) - \frac{1}{\sqrt{2\pi}} H_m(\omega/2)\right|^p d\omega + \int_{I_\delta} |\omega|^{pk} \left|\frac{1}{\sqrt{2\pi}} H_m(\omega/2) - \Phi(\omega)\right|^p d\omega \\ &=: I_{31} + I_{32}. \end{aligned}$$

$I_{32} \to 0$ as $m \to \infty$ since $\frac{1}{\sqrt{2\pi}} H_m(\omega/2)$ converges to $\Phi$ uniformly in $I_\delta$. To see that $I_{31} \to 0$ as $m \to \infty$, by the definition of $H_m$, for $\omega \in [0, \pi]$, we have

$$\begin{aligned} \left|H_m\left(\frac{\omega}{4}\right)\right|^2 &\geqslant 1 - c_m \frac{\omega}{4} \sin^{2m-1}\left(\frac{\omega}{4}\right) \\ &\geqslant 1 - c_m \frac{\pi}{4} \sin^{2m-1}\left(\frac{\pi}{4}\right) \\ &\geqslant 1 - \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(m)} \left(\frac{\pi}{4}\right)^{2m} \end{aligned} \tag{3.3}$$

and

$$\left|H_m\left(\frac{\omega}{8}\right)\right|^2 \geqslant 1 - c_m \left(\frac{\pi}{4}\right)^{2m}. \tag{3.4}$$

Moreover, by (cf. [16, Lemma 2])

$$\begin{aligned} \prod_{\ell=1}^{\infty} \left|H_m\left(2^{-\ell-3}\omega\right)\right|^2 &\geqslant \prod_{\ell=1}^{\infty} \left|1 - c_m \left(2^{-\ell-3}\omega\right)^{2m}\right| \\ &\geqslant \prod_{\ell=1}^{\infty} \left|1 - c_m \left(\frac{\pi}{4}\right)^{2m} \left(2^{-2m}\right)^{\ell}\right| \\ &\geqslant \prod_{\ell=1}^{\infty} \left|1 - \left(2^{-2m}\right)^{\ell}\right| \geqslant \left(1 - 2^{-2m}\right)^{1/(1-2^{-2m})}. \end{aligned} \tag{3.5}$$

In view of (3.3) – (3.5), we have $1 \geqslant \left|\prod_{\ell=1}^{\infty} H_m\left(2^{-l-1}\omega\right)\right| \geqslant 1 - o(1)$. Consequently,

$$\begin{aligned} I_{31} &= \int_{I_\delta} |\omega|^{pk} \left|\widehat{\varphi_m^D}(\omega) - \frac{1}{\sqrt{2\pi}} H_m(\omega/2)\right|^p d\omega \\ &\leqslant \int_{I_\delta} |\omega|^{pk} \left|\frac{1}{\sqrt{2\pi}} H_m(\omega/2) \left(\prod_{\ell=1}^{\infty} H_m(2^{-l-1}\omega) - 1\right)\right|^p d\omega \\ &\leqslant \int_{I_\delta} |\omega|^{pk} \left|\frac{1}{\sqrt{2\pi}} \left(\prod_{\ell=1}^{\infty} H_m(2^{-l-1}\omega) - 1\right)\right|^p d\omega \to 0, \quad m \to \infty. \end{aligned}$$

Therefore, we obtain

$$\lim_{m\to\infty} \|\widehat{{}_{-k}\varphi_m^D}\|_p = \pi^k \frac{(2\pi)^{1/p-1/2}}{(1+pk)^{1/p}}, \quad k \in \mathbb{N} \cup \{0\}. \qquad \square$$

More generally, one can show that for $\alpha \in \mathbb{R}$ such that $1 - p\alpha > 0$,

$$\lim_{m\to\infty} \|\widehat{{}_\alpha\varphi_m^D}\|_p = \pi^{-\alpha}\frac{(2\pi)^{1/p-1/2}}{(1-p\alpha)^{1/p}}, \tag{3.6}$$

where for a real number $\alpha \in \mathbb{R}$, the function ${}_\alpha\varphi_m^D$ is similarly defined as in (1.8). However, when $1 - p\alpha \leqslant 0$, i.e., $\alpha \geqslant 1/p$, the constant $\|\widehat{{}_\alpha\varphi_m^D}\|_p \to \infty$ as $m \to \infty$.

When $k$ is fixed and $m \to \infty$, Babenko and Spektor ([2]) show that, for the Daubechies orthonormal wavelet function $\psi_m^D$ with $m$ vanishing moments, one has

$$\lim_{m\to\infty} \|\widehat{{}_k\psi_m^D}\|_p = \frac{(2\pi)^{1/p-1/2}}{\pi^k}\left(\frac{1-2^{1-pk}}{pk-1}\right)^{1/p}, \quad k \in \mathbb{N}. \tag{3.7}$$

For $k = m$ and $m \to \infty$, we can deduce the following estimation, which in turn gives rise to the asymptotic behavior of the constant $(C_{m,p}(\psi_m^D))^{1/m}$ (see Subsection 3.3).

Theorem 3. *Let $\psi_m^D$ be the Daubechies wavelet with $m$ vanishing moments, i.e., $\widehat{\psi_m^D}(\omega) = \frac{1}{\sqrt{2\pi}} H_m(\omega/2+\pi)\prod_{\ell=1}^{\infty} H_m(2^{-l-1}\omega)$. Then, for $p \in (1,\infty)$,*

$$\|\widehat{{}_m\psi_m^D}\|_p = C\cdot\frac{2^{1/p}}{\sqrt{2\pi}}\cdot\frac{2^{-m}\cdot A(m)}{(\sqrt{mp/2})^{1/p}}\cdot(1+\mathcal{O}(m^{-1/2})), \tag{3.8}$$

*where $C$ is a positive constant independent of $m, p$ and $\sqrt{\frac{c_m}{2m}} \leqslant A(m) \leqslant \sqrt{\frac{1}{2}}$, where $c_m = \frac{\Gamma(m+\frac12)}{\sqrt{\pi}\Gamma(m)} \sim \sqrt{\frac{m}{\pi}}$.*

*Proof.* By definition,

$$\begin{aligned}\|\widehat{{}_m\psi_m^D}\|_p^p &= \int_{\mathbb{R}} |\omega|^{-mp}\left|\widehat{\psi_m^D}(\omega)\right|^p d\omega \\ &= \int_{|\omega|\leqslant\pi} |\omega|^{-mp}\left|\widehat{\psi_m^D}(\omega)\right|^p d\omega + \int_{|\omega|>\pi} |\omega|^{-mp}\left|\widehat{\psi_m^D}(\omega)\right|^p d\omega =: I_1 + I_2.\end{aligned}$$

We first estimate $I_2$. Since $|H_m(t)| \leqslant 1$,

$$I_2 \leqslant \frac{2}{(\sqrt{2\pi})^p}\int_\pi^\infty \omega^{-mp}d\omega \leqslant \frac{2}{(\sqrt{2\pi})^p}\cdot\frac{1}{mp-1}\left(\frac{1}{\pi}\right)^{mp-1}, \quad mp > 1.$$

Next, we show that $I_1 \sim C\cdot c_m^{p/2}\cdot(\sqrt{mp/2})^{-1}\cdot 2^{-mp}$. Again, by (3.3) – (3.5),

$$\begin{aligned} I_1 = \frac{1}{(\sqrt{2\pi})^p}\int_{|\omega|\leqslant\pi} |\omega|^{-mp}\Bigg[&\left|H_m(\omega/2+\pi)\right|^2\left|H_m(\omega/4)\right|^2\left|H_m(\omega/8)\right|^2 \\ &\times\prod_{\ell=1}^{\infty}\left|H_m\left(2^{-l-3}\omega\right)\right|^2\Bigg]^{p/2} d\omega \\ \geqslant (1-o(1))\,\frac{1}{(\sqrt{2\pi})^p}&\int_{|\omega|\leqslant\pi} |\omega|^{-mp}\left|H_m(\omega/2+\pi)\right|^p d\omega. \end{aligned}$$

Obviously,

$$I_1 \leqslant \frac{1}{(\sqrt{2\pi})^p} \int_{|\omega|\leqslant\pi} |\omega|^{-mp} \, |H_m(\omega/2+\pi)|^p \, d\omega.$$

Now, we use the property of $H_m$ to deduce the asymptotic behavior of

$$I_{11} := \int_{|\omega|\leqslant\pi} |\omega|^{-mp} \, |H_m(\omega/2+\pi)|^p \, d\omega.$$

Let $u = \dfrac{\sin^2 t}{\sin^2(\omega/2)}$. We have

$$\begin{aligned} |H_m(\omega/2+\pi)|^2 &= c_m \int_0^{\omega/2} \sin^{2m-1} t dt \\ &= \frac{c_m}{2} \sin^{2m}(\omega/2) \int_0^1 u^{m-1}(1-u\sin^2(\omega/2))^{-1/2} du. \end{aligned}$$

Since

$$\frac{1}{m} = \int_0^1 u^{m-1} du \leqslant \int_0^1 u^{m-1}(1-u\sin^2(\omega/2))^{-1/2} du \leqslant \int_0^1 u^{m-1}(1-u)^{-1/2} du = c_m^{-1}$$

and

$$I_{11} = 2\int_0^\pi |\omega|^{-mp} \cdot \left[\frac{c_m}{2} \sin^{2m}(\omega/2) \int_0^1 u^{m-1}(1-u\sin^2(\omega/2))^{-1/2} du\right]^{p/2} d\omega,$$

we obtain

$$\left(\frac{c_m}{2m}\right)^{p/2} \cdot 2^{-mp} \cdot \int_0^\pi \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{mp} d\omega \leqslant \frac{1}{2} I_{11} \leqslant \left(\frac{1}{2}\right)^{-p/2} \cdot 2^{-mp} \cdot \int_0^\pi \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{mp} d\omega.$$

Now using that $\int_{-\pi}^{\pi} \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{2\cdot mp/2} d\omega = C(\sqrt{mp/2})^{-1}(1+\mathcal{O}(m^{-1/2}))$ and $\frac{1}{\pi} < \frac{1}{2}$, we conclude

$$\|\widehat{{}_m\psi_m^D}\|_p^p = C \cdot \frac{2}{(\sqrt{2\pi})^p} \cdot \frac{2^{-mp} \cdot A(m)^p}{\sqrt{mp/2}} \cdot (1+\mathcal{O}(m^{-1/2})),$$

which completes our proof. □

**3.2. Wavelet coefficients of semiorthogonal spline wavelets.** In this subsection, we mainly focus on the asymptotic behavior of wavelet coefficients for the semiorthogonal spline wavelets. Next three theorems present the asymptotic estimations of the following quantities: $\|\widehat{{}_k\varphi_m^S}\|_p$, $\|\widehat{{}_k\psi_m^S}\|_p$, and $\|\widehat{{}_m\psi_m^S}\|_p$, $p \in (1,\infty)$. Since the proofs of the main results in this subsection share the similar idea of proofs in previous subsection but with more technical treatment, we therefore postpone their proofs to the last section.

For the scaling function $\varphi_m^S$, which is the B-spline $N_m$ of order $m$, we have the following result gives the asymptotic estimate of $\|\widehat{{}_k\varphi_m^S}\|_p$.

THEOREM 4. *Let $\varphi_m^S := N_m$ be the B-Spline of order $m$. Let $k \geqslant 0$ be a nonnegative integer. Then, for $p \in (1, \infty)$,*

$$\|\widehat{{}_k\varphi_m^S}\|_p = \frac{4^{1/p}}{(\sqrt{2\pi})^{1-1/p}} \cdot \frac{1}{(\sqrt{\Lambda_1 mp})^{1/p}} \cdot (2\xi_1)^{-k} \cdot (\lambda_1/\xi_1)^{m/2} \cdot (1 + \mathcal{O}(m^{-1/2})), \quad (3.9)$$

*where*

$$\begin{aligned} \lambda_1 &= \frac{\sin^2(\xi_1)}{\xi_1} \approx 0.72461, \\ \Lambda_1 &= -\frac{1}{2}\frac{d^2}{d\omega^2} \ln \frac{\sin^2(\xi_1 - \omega)}{\xi_1 - \omega}\Big|_{\omega=0} \approx 0.81597, \end{aligned} \quad (3.10)$$

*and $\xi_1 \approx 1.1655$ is the unique solution of the transcendental equation $\xi_1 - 2\cot(\xi_1) = 0$ in the interval $(0, \pi)$.*

Similarly, for the spline wavelet function $\psi_m^S$, we have the following theorem:

THEOREM 5. *Let $k \geqslant 0$ be a nonnegative integer. Let $\psi_m^S$ be the semiorthogonal spline wavelet of order $m$. Then, for $p \in (1, \infty)$,*

$$\|\widehat{{}_k\psi_m^S}\|_p = \frac{2^{3/p}}{(\sqrt{2\pi})^{1-1/p}} \cdot \frac{(2\pi - 4\xi_2)^{-k}}{(\sqrt{2\Lambda_2 mp})^{1/p}} \cdot \lambda_2^m \cdot (1 + \mathcal{O}(m^{-1/2})). \quad (3.11)$$

*where*

$$\begin{aligned} \lambda_2 &= \frac{\sin^2(\xi_2 - \pi/2)\sin^2(\xi_2)}{(\pi/2 - \xi_2)\xi_2^2} \approx 0.69706 \\ \Lambda_2 &= -\frac{1}{2}\frac{d^2}{du^2} \ln \frac{\sin^2(u - \pi/2)\sin^2(u)}{(\pi/2 - u)u^2}\Big|_{u=\xi_2} \approx 1.2229, \end{aligned} \quad (3.12)$$

*and $\xi_2 = 0.2853...$ is the unique solution of the transcendental equation*

$$(2\pi\xi - 4\xi^2)\cos(2\xi) + (3\xi - \pi)\sin(2\xi) = 0, \quad \xi \in (0, \pi/2).$$

Finally, to compare with the wavelet case for $k = m$, we also provide the following estimation for the spline case with $k = m$:

THEOREM 6. *Let $\psi_m^S$ be the semiorthogonal spline wavelet of order $m$. Then, for $p \in (1, \infty)$,*

$$\|\widehat{{}_m\psi_m^S}\|_p = \frac{2^{1/p}}{(\sqrt{2\pi})^{1-1/p}} \cdot \left(\frac{\pi}{\sqrt{\pi^2 - 8}}\right)^{1/p} \cdot \frac{1}{(\sqrt{2mp})^{1/p}} \cdot \left(\frac{16}{\pi^4}\right)^m \cdot (1 + \mathcal{O}(m^{-1/2})). \quad (3.13)$$

**3.3. Comparison of Daubechies orthonormal wavelets and semiorthogonal wavelets.** Now, by the results we obtained in the above two subsections, we can compare the Daubechies orthonormal wavelets and the semiorthogonal spline wavelets using the constants $C_{k,p}(f)$. Note that both Daubechies orthonormal wavelets and the semiorthogonal spline wavelets have the same support length and number of vanishing moments, thereby a comparison is possible in this respect.

We first consider the situation when $k$ is fixed and let $m \to \infty$. For Daubechies family, by Theorem 2 and (3.7), we can deduce the following result.

Corollary 1. *Let $\varphi_m^D$ and $\psi_m^D$ be the Daubechies orthonormal scaling function and wavelet function of order $m$, respectively. Let $k \geqslant 0$ be a nonnegative integer. Then, for $p \in (1, \infty)$,*

$$\lim_{m\to\infty} C_{-k,p}(\varphi_m^D) = \lim_{m\to\infty} \frac{\|\widehat{{}_{-k}\varphi_m^D}\|_p}{\|\widehat{\varphi_m^D}\|_p} = \frac{\pi^k}{(1+pk)^{1/p}} \tag{3.14}$$

*and*

$$\lim_{m\to\infty} C_{k,p}(\psi_m^D) = \lim_{m\to\infty} \frac{\|\widehat{{}_k\psi_m^D}\|_p}{\|\widehat{\psi_m^D}\|_p} = \pi^{-k}\left(\frac{1-2^{1-pk}}{pk-1}\right)^{1/p}. \tag{3.15}$$

For the semiorthogonal spline wavelet family, by Theorems 4 and 5, similarly, we have the following result.

Corollary 2. *Let $\varphi_m^S$ and $\psi_m^S$ be the semiorthogonal spline wavelet of order $m$, respectively. Let $k \geqslant 0$ be an integer. Then, for $p \in (1, \infty)$,*

$$\lim_{m\to\infty} C_{k,p}(\varphi_m^S) = \lim_{m\to\infty} \frac{\|\widehat{{}_k\varphi_m^S}\|_p}{\|\widehat{\varphi_m^S}\|_p} = (2\xi_1)^{-k} \approx (2.331)^{-k} \tag{3.16}$$

*and*

$$\lim_{m\to\infty} C_{k,p}(\psi_m^S) = \lim_{m\to\infty} \frac{\|\widehat{{}_k\psi_m^S}\|_p}{\|\widehat{\psi_m^S}\|_p} = (2\pi - 4\xi_2)^{-k} \approx (5.1419)^{-k}, \tag{3.17}$$

*where $\xi_1 \approx 1.1655$ and $\xi_2 \approx 0.2853$ are constants given in Theorems 4 and 5.*

Comparing Corollaries 1 and 2, we see that for every $k \in \mathbb{N} \cup \{0\}$, the semiorthogonal spline wavelets are better than the Daubechies orthonormal wavelets in the sense of asymptotically smaller constants. More precisely, we have

Corollary 3. *Let $\psi_m^D$ and $\psi_m^S$ be Daubechies orthonormal wavelet and the semiorthogonal spline wavelet of order $m$, respectively. Then, for $p \in (1, \infty)$,*

$$\lim_{k\to\infty} \lim_{m\to\infty} \left(\frac{C_{k,p}(\psi_m^S)}{C_{k,p}(\psi_m^D)}\right)^{1/k} = \frac{\pi}{2\pi - 4\xi_2}. \tag{3.18}$$

Note that $\frac{\pi}{2\pi-4\xi_2} \approx 0.61098 < 1$. In other words, (3.18) shows that the semiorthogonal spline wavelet constant $C_k(\psi_m^S)$ is exponentially better than Daubechies orthonormal wavelet constant $C_k(\psi_m^D)$ for increasing $k$.

Since the number of vanishing moments increases with $m$, it is natural to consider the behavior of the constants $C_k(\psi_m^S)$ with $k = k(m) = m$. In this situation, from Theorems 3 and 6, we have the following result, which shows that for smooth functions, the ratio in (3.18) when $k = m$ is even more in favor of the semiorthogonal spline wavelets.

Corollary 4. *Let $\psi_m^D$ and $\psi_m^S$ be Daubechies orthonormal wavelet and the semiorthogonal spline wavelet of order $m$, respectively. Then*

$$\lim_{m\to\infty} \left(C_{m,p}(\psi_m^D)\right)^{1/m} = \frac{1}{2}, \quad \lim_{m\to\infty} \left(C_{m,p}(\psi_m^S)\right)^{1/m} = \frac{16}{\lambda_2 \pi^4}, \tag{3.19}$$

*and*

$$\lim_{m\to\infty}\left(\frac{C_{m,p}(\psi_m^S)}{C_{m,p}(\psi_m^D)}\right)^{1/m}=\frac{32}{\lambda_2\pi^4}. \tag{3.20}$$

We end this section by comparing the asymptotic behaviors between the scaling function $\varphi$ and the wavelet function $\psi$ for both the Daubechies orthonormal wavelets and semiorthogonal spline wavelets.

For the Daubechies orthonormal wavelets, again, by Theorems 2 and (3.7), we have the following result.

Corollary 5. *Let $\varphi_m^D$ and $\psi_m^D$ be the Daubechies orthonormal scaling function and wavelet function of order $m$, respectively. Let $k_1, k_2 \geqslant 0$ be nonnegative integers. Then, for $p \in (1,\infty)$*

$$\lim_{m\to\infty}\frac{\|\widehat{{}_{-k_1}\varphi_m^D}\|_p}{\|\widehat{{}_{k_2}\psi_m^D}\|_p}=\frac{\pi^{k_1+k_2}}{(1-2^{1-pk_2})^{1/p}}\left(\frac{pk_1+1}{pk_2-1}\right)^{1/p}. \tag{3.21}$$

For the semiorthogonal wavelets, similarly, using the results of Theorems 4 and 5, we have

Corollary 6. *Let $\varphi_m^S$ and $\psi_m^S$ be the semiorthogonal spline scaling function and wavelet function of order $m$, respectively. Let $k_1, k_2 \geqslant 0$ be nonnegative integers. Then, for $p \in (1,\infty)$*

$$\lim_{m\to\infty}\left(\frac{\|\widehat{{}_{k_1}\varphi_m^S}\|_p}{\|\widehat{{}_{k_2}\psi_m^S}\|_p}\right)^{1/m}=\sqrt{\frac{\lambda_1}{\xi_1\lambda_2^2}}. \tag{3.22}$$

**4. High-dimensional wavelet coefficients.** One of the simple ways to construct high-dimensional wavelets is using tensor product. In this section, we discuss the wavelet coefficients for high-dimensional tensor product wavelets. We shall mainly focus on dimension two while results of higher dimensions can be similarly obtained using the properties of tensor product.

Let $\varphi, \psi$ be the one-dimensional scaling function and wavelet function that generates a wavelet basis in $L_2(\mathbb{R})$. Then, in two-dimensional case, the scaling function $\mathbf{\Phi}(x_1,x_2)=\varphi(x_1)\varphi(x_2)$ and we have three wavelets instead of one,

$$\begin{aligned}\mathbf{\Psi}^1(x_1,x_2)&:=\psi(x_1)\varphi(x_2),\\ \mathbf{\Psi}^2(x_1,x_2)&:=\varphi(x_1)\psi(x_2),\\ \mathbf{\Psi}^3(x_1,x_2)&:=\psi(x_1)\psi(x_2).\end{aligned} \tag{4.1}$$

Let $k=(k_1,k_2)\in\mathbb{Z}^2$ be a two-dimensional index. Then, for a two-dimensional wavelet function $\mathbf{\Psi}$, we can define $C_{k,p}(\mathbf{\Psi})$ similar to (1.5) by

$$C_{k,p}(\mathbf{\Psi})=\sup_{f\in\mathcal{A}_k^{p'}}\frac{|\langle f,\mathbf{\Psi}\rangle|}{\|\widehat{\mathbf{\Psi}}\|_p}=\frac{\|{}_k\widehat{\mathbf{\Psi}}\|_p}{\|\widehat{\mathbf{\Psi}}\|_p}, \tag{4.2}$$

where $1<p,p'<\infty$, $1/p'+1/p=1$ and $\mathcal{A}_k^{p'}:=\{f\in L_{p'}(\mathbb{R}^2):\|(i\omega)^k\hat{f}(\omega)\|_{p'}\leqslant 1\}$. Here, for $x=(x_1,x_2)\in\mathbb{R}^2, k=(k_1,k_2)\in\mathbb{Z}^2$, $x^k:=x_1^{k_1}x_2^{k_2}$. And for a function $f \in L_1(\mathbb{R}^2)$, ${}_kf$ is defined to be a function such that $\widehat{{}_kf}(\omega) = (i\omega)^k \widehat{f}$, $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$. In particular, when $\mathbf{\Psi}(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$, one can easily show that $C_{k,p}(\mathbf{\Psi}) = C_{k_1,p}(\psi_1)C_{k_2,p}(\psi_2)$.

In two-dimensional case, the semiorthogonal wavelets can be represented by

$$\begin{aligned}
\mathbf{\Psi}_m^{S,1}(x_1, x_2) &:= \psi_m^S(x_1)\varphi_m^S(x_2) = \psi_m^S(x_1)N_m(x_2),\\
\mathbf{\Psi}_m^{S,2}(x_1, x_2) &:= \varphi_m^S(x_1)\psi_m^S(x_2) = N_m(x_1)\psi_m^S(x_2),\\
\mathbf{\Psi}_m^{S,3}(x_1, x_2) &:= \psi_m^S(x_1)\psi_m^S(x_2).
\end{aligned} \tag{4.3}$$

We can obtain that following corollaries using results in previous sections and the properties of tensor product.

Corollary 7. *Let $\mathbf{\Psi}_m^{S,1}$, $\mathbf{\Psi}_m^{S,2}$, and $\mathbf{\Psi}_m^{S,3}$ be defined in* (4.3). *Let $k = (k_1, k_2) \in \mathbb{N}^2$. Then,*

$$\begin{aligned}
C_{k,p}(\mathbf{\Psi}_m^{S,1}) &\geqslant \frac{1}{2^{k_1}}\left(\frac{1}{2\pi}\right)^{k_1+k_2} \frac{1}{K_{p,m+k_1,k_1}K_{p,m+k_2,k_2}},\\
C_{k,p}(\mathbf{\Psi}_m^{S,2}) &\geqslant \frac{1}{2^{k_2}}\left(\frac{1}{2\pi}\right)^{k_1+k_2} \frac{1}{K_{p,m+k_1,k_1}K_{p,m+k_2,k_2}},\\
C_{k,p}(\mathbf{\Psi}_m^{S,3}) &\geqslant \left(\frac{1}{4\pi}\right)^{k_1+k_2} \frac{1}{K_{p,m+k_1,k_1}K_{p,m+k_2,k_2}},
\end{aligned} \tag{4.4}$$

*where $K_{p,m,k}$'s are constants defined by* (2.4). *Moreover,*

$$\begin{aligned}
\lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{S,1}) &= (2\pi - 4\xi_2)^{-k_1}(2\xi_1)^{-k_2},\\
\lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{S,2}) &= (2\pi - 4\xi_2)^{-k_2}(2\xi_1)^{-k_1},\\
\lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{S,3}) &= (2\pi - 4\xi_2)^{-k_1-k_2},
\end{aligned} \tag{4.5}$$

*where $\xi_1, \xi_2$ are constants given in Corollary 2.*

In two-dimensional case, Daubechies wavelets can be represented by

$$\begin{aligned}
\mathbf{\Psi}_m^{D,1}(x_1, x_2) &:= \psi_m^D(x_1)\varphi_m^D(x_2),\\
\mathbf{\Psi}_m^{D,2}(x_1, x_2) &:= \varphi_m^D(x_1)\psi_m^D(x_2),\\
\mathbf{\Psi}_m^{D,3}(x_1, x_2) &:= \psi_m^D(x_1)\psi_m^D(x_2).
\end{aligned}$$

Similarly, we have the following result.

Corollary 8. *Let $\mathbf{\Psi}_m^{D,1}$, $\mathbf{\Psi}_m^{D,2}$, and $\mathbf{\Psi}_m^{D,3}$ be defined in* (4.5). *Then,*

$$\begin{aligned}
\lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{D,1}) &= \pi^{-k_1}\left(\frac{1-2^{1-pk_1}}{pk_1-1}\right)^{1/p} \frac{\pi^{-k_2}}{(1-pk_2)^{1/p}},\ k_2 \leqslant 1/p, k_1 \in \mathbb{N},\\
\lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{D,2}) &= \pi^{-k_2}\left(\frac{1-2^{1-pk_2}}{pk_2-1}\right)^{1/p} \frac{\pi^{-k_1}}{(1-pk_1)^{1/p}},\ k_1 \leqslant 1/p, k_2 \in \mathbb{N},\\
\lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{D,3}) &= \pi^{-k_1-k_2}\left(\frac{1-2^{1-pk_1}}{pk_1-1}\cdot\frac{1-2^{1-pk_2}}{pk_2-1}\right)^{1/p},\ (k_1, k_2) \in \{\mathbb{N}\cup\{0\}\}^2.
\end{aligned} \tag{4.6}$$

**5. Proofs of theorems 4, 5, and 6.** In this section, we give the proofs of Theorem 4, Theorem 5, and Theorem 6.

*Proof of Theorem 4.* By (1.14),

$$\begin{aligned}
\|\widehat{{}_k\varphi^S_m}\|_p^p &= \int_{\mathbb{R}} |\omega|^{-kp} \cdot \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{mp} d\omega = \frac{2^{1-kp}}{(\sqrt{2\pi})^p} \int_R |\omega|^{-kp} \cdot \left|\frac{\sin(\omega)}{\omega}\right|^{mp} d\omega \\
&= \frac{2^{2-kp}}{(\sqrt{2\pi})^p} \int_0^\infty \omega^{-p(m/2+k)} \cdot \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} d\omega \\
&= \frac{2^{2-kp}}{(\sqrt{2\pi})^p} \Big( \int_0^\pi \omega^{-p(m/2+k)} \cdot \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} d\omega + \int_\pi^\infty \omega^{-p(m/2+k)} \cdot \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} d\omega \Big) \\
&=: \frac{2^{2-kp}}{(\sqrt{2\pi})^p} (I_1 + I_2).
\end{aligned}$$

For $I_2$ with $mp > 1$, we have

$$I_2 \leqslant \int_\pi^\infty \omega^{-mp} d\omega = \frac{1}{mp-1} \left(\frac{1}{\pi}\right)^{mp-1}.$$

To estimate $I_1$, we use the same technique as in the proof of [16, Lemma 4]. Let $\xi_1$ be the point where $\sin^2(\omega)/\omega$ takes its maximum value $\lambda_1$ in $(0, \pi)$, i.e., $\xi_1 \approx 1.1655$ is the root of the transcendental equation $\xi_1^{-1} - 2\cot(\xi_1) = 0$ and $\lambda_1 = \dfrac{\sin^2(\xi_1)}{\xi_1} \approx 0.72461$ . Separate $I_1$ to two parts as follows

$$I_1 = \int_0^{\xi_1} \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} \cdot \omega^{-p(m/2+k)} d\omega + \int_{\xi_1}^\pi \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} \cdot \omega^{-p(m/2+k)} d\omega =: I_{11} + I_{12}.$$

We first estimate $I_{11}$. Let

$$t = t(\omega) = \ln \frac{\xi_1 - \omega}{\sin^2(\xi_1 - \omega)} - \ln \frac{\xi_1}{\sin^2(\xi_1)} = \ln \frac{\lambda_1(\xi_1 - \omega)}{\sin^2(\xi_1 - \omega)}, \quad \omega \in (0, \xi_1).$$

Then, we have

$$t(\omega) \sim a_2\omega^2 + a_3\omega^3 + \cdots \sim a_2\omega^2 \left(1 + \frac{a_3}{a_2}\omega + \cdots\right), \quad \omega \to 0,$$

where

$$a_2 = \Lambda_1 = -\frac{1}{2}\frac{d^2}{d\omega^2} \ln \frac{\sin^2(\xi_1 - \omega)}{\xi_1 - \omega}\Big|_{\omega=0} \approx 0.81597\,.$$

Then, similar to the proof of [16, Lemma 4], we can obtain

$$\begin{aligned}
\omega = \omega(t) &\sim (\Lambda_1)^{-1/2}\sqrt{t}(1 + c_1 t^{1/2} + c_2 t + \cdots), \\
\frac{d\omega}{dt} &\sim \frac{1}{2\sqrt{\Lambda_1 t}}(1 + d_1 t^{1/2} + d_2 t + \cdots), \\
\xi_1 - \omega(t) &\sim \xi_1(1 - e_1 t^{1/2} - e_2 t - \cdots),
\end{aligned}$$

for $t \to 0$. Changing the variable of $I_{11}$, we have

$$\begin{aligned} I_{11} &= \int_0^{\xi_1} \left( \frac{\sin^2(\omega)}{\omega} \right)^{mp/2} \cdot \omega^{-p(m/2+k)} d\omega \\ &= \int_0^{\xi_1} \left( \frac{\sin^2(\xi_1-\omega)}{\xi_1-\omega} \right)^{mp/2} \cdot (\xi_1-\omega)^{-p(m/2+k)} d\omega \\ &= \lambda_1^{mp/2} \int_0^{\infty} e^{-\frac{mp}{2}t} q(t) dt, \end{aligned}$$

where

$$q(t) \sim \left( \xi_1^{p(m/2+k)} 2\sqrt{\Lambda_1 t} \right)^{-1} (1 + f_1 t^{1/2} + f_2 t + \cdots).$$

Now by Watson's lemma, i.e.,

$$\int_0^T e^{-xt} t^s f(t) dt \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\Gamma(s+n+1)}{n! x^{s+n+1}}, x \to \infty \tag{5.1}$$

for function $f$ having an infinite number of derivatives in the neighborhood of $t = 0$ (c.f. [22, Theorem 3.1]), we have

$$\begin{aligned} I_{11} &= \lambda_1^{mp/2} \cdot \left( \xi_1^{p(m/2+k)} 2\sqrt{\Lambda_1} \right)^{-1} \cdot \frac{\sqrt{\pi}}{\sqrt{mp/2}} \cdot (1 + \mathcal{O}(m^{-1/2})) \\ &= \frac{\sqrt{2\pi}}{2\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})). \end{aligned}$$

For $I_{12}$, we use

$$t = t(\omega) = \ln \frac{\xi_1 + \omega}{\sin^2(\xi_1+\omega)} - \ln \frac{\xi_1}{\sin^2(\xi_1)} = \ln \frac{\lambda_1(\xi_1+\omega)}{\sin^2(\xi_1+\omega)}, \quad \omega \in (0, \pi - \xi_1).$$

Similarly, we have $I_{12} = \frac{\sqrt{2\pi}}{2\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2}))$. Consequently,

$$I_1 = \frac{\sqrt{2\pi}}{\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Noting that $\frac{1}{\pi} \approx 0.31830 < \left( \frac{\lambda_1}{\xi_1} \right)^{1/2} \approx 0.78846$, we conclude

$$\begin{aligned} \|\widehat{{}_k\varphi_m^S}\|_p^p &= \frac{2^{2-kp}}{(\sqrt{2\pi})^p} \cdot \frac{2\sqrt{2\pi}}{\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})) \\ &= \frac{4}{(\sqrt{2\pi})^{p-1}} \cdot \frac{1}{\sqrt{\Lambda_1 mp}} \cdot (2\xi_1)^{-kp} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})), \end{aligned}$$

which completes our proof. $\square$

*Proof of Theorem 5.* Using the Fourier transform of the B-spline and the definition of Euler-Frobenius polynomial $E_{2m-1}(z)$ for $z = e^{i\omega}$:

$$\frac{E_{2m-1}(z)}{(2m-1)!} = \sum_{\nu=0}^{2m-2} N_{2m}(\nu+1) z^{\nu} = e^{-i(m-1)\omega} (2\sin(\omega/2))^{2m} \sum_{\ell=-\infty}^{\infty} \frac{1}{(\omega + 2\pi\ell)^{2m}}, \tag{5.2}$$

we can derive (c.f. [16, Lemma 4])

$$
\begin{aligned}
|\widehat{{}_k\psi_m^S}(\omega)| &= \frac{2^{-2k}}{\sqrt{2\pi}}\Big|\frac{\sin^2(\omega/4)}{\omega/4}\Big|^m\Big|\frac{\omega}{4}\Big|^{-k}\Big|\frac{E_{2m-1}(\tilde z)}{(2m-1)!}\Big| \\
&= \frac{2^{-2k}}{\sqrt{2\pi}}\Big|\frac{\sin^2(\omega/4)}{\omega/4}\Big|^m\Big|\frac{\omega}{4}\Big|^{-k}\Big|2\sin(\tilde\omega/2)\Big|^{2m}\Big|\sum_{\ell=-\infty}^{\infty}\frac{1}{(\tilde\omega+2\pi\ell)^{2m}}\Big|,
\end{aligned}
$$

where $\tilde z = e^{i\tilde\omega}$ and $\tilde\omega = \pi-\omega/2$. Then, we obtain

$$
\begin{aligned}
&\|\widehat{{}_k\psi_m^S}\|_p^p = \frac{2^{-2kp}}{(\sqrt{2\pi})^p}\int_{\mathbb{R}}\Big|\frac{\sin^2(\omega/4)}{\omega/4}\Big|^{mp}\Big|\frac{\omega}{4}\Big|^{-kp}\Big|2\sin(\tilde\omega/2)\Big|^{2mp}\Big|\sum_{\ell=-\infty}^{\infty}\frac{1}{(\tilde\omega+2\pi\ell)^{2m}}\Big|^p d\omega \\
&= \frac{2^{-2kp}}{(\sqrt{2\pi})^p}\int_{\mathbb{R}}\left[\Big(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\Big)^{2m}(u-\pi/2)^{-2k}\,(2\sin(u))^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(2u+2\pi\ell)^{2m}}\right)^2\right]^{p/2} 4du \\
&= \frac{4\cdot 2^{-2kp}}{(\sqrt{2\pi})^p}\Big(\int_{-\infty}^{-\pi/2}\left[\Big(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\Big)^{2m}(u-\pi/2)^{-2k}\,(\sin(u))^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du \\
&+\int_{-\pi/2}^{\xi_2}\left[\Big(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\Big)^{2m}(u-\pi/2)^{-2k}\,(\sin(u))^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du \\
&+\int_{\xi_2}^{\pi/2}\left[\Big(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\Big)^{2m}(u-\pi/2)^{-2k}\,(\sin(u))^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du \\
&+\int_{\pi/2}^{3\pi/2}+\int_{3\pi/2}^{\infty}\left[\Big(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\Big)^{2m}(u-\pi/2)^{-2k}\,(\sin(u))^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du\Big) \\
&=: \frac{4\cdot 2^{-2kp}}{(\sqrt{2\pi})^p}(I_1+I_2+I_3+I_4+I_5).
\end{aligned}
$$

Here, $\xi_2$ is the point where the function

$$
g(u) := \frac{\sin^2(u-\pi/2)\sin^2(u)}{(\pi/2-u)u^2}
$$

takes its maximum value in $(0,\pi/2)$, i.e. point at which $g'(u)=0$. One can show that $\xi_2 \approx 0.28532$ is the root of the transcendental equation

$$
h(u) := (2\pi u-4u^2)\cos(2u)+(3u-\pi)\sin(2u).
$$

Note that $g'(u) = \frac{\sin(2u)}{4(\pi/2-u)^2u^4}\cdot h(u)$ and $\lambda_2 = g(\xi_2) \approx 0.69706$.

We first estimate $I_2$. By [16, Lemma 3], we have

$$
I_2 = \int_{-\pi/2}^{\xi_2}\big[g(u)^{2m}(u-\pi/2)^{-2k}(1+R_1+r(u))^2\big]^{p/2}\,du =: I_{21}+\tilde R,
$$

where $|R_1| \leqslant (2m-1)^{-1}$,

$$
r(u) = \begin{cases} \Big(\frac{u}{\pi+u}\Big)^{2m}, & -\pi/2 < u \leqslant 0, \\ \Big(\frac{u}{\pi-u}\Big)^{2m}, & 0 \leqslant u < \xi_2. \end{cases}
$$

$$I_{21} := \int_{-\pi/2}^{\xi_2} \left[g(u)^{2m}(u-\pi/2)^{-2k}(1+R_2(u))^2\right]^{p/2} du,$$

where

$$R_2(u) = \begin{cases} R_1 + r(u), & -\pi/2+\delta < u < \xi_2, \\ R_1, & -\pi/2 < u < -\pi/2+\delta. \end{cases}$$

$0 < \delta < \pi/2 - \xi_2$ is fixed. Hence

$$|R_2(u)| \leqslant \frac{1}{2m-1} + \left(\frac{\pi/2-\delta}{\pi/2+\delta}\right)^{2m},$$

and

$$\begin{aligned}\tilde{R} &= \int_{-\pi/2}^{-\pi/2+\delta} \left[g(u)^{2m}(u-\pi/2)^{-2k}\right]^{p/2} \cdot \left[(1+R_1+r(u))^p - (1+R_1)^p\right] du \\ &\leqslant (p2^p + o(1)) \int_{-\pi/2}^{-\pi/2+\delta} \left[g(u)^{2m}(u-\pi/2)^{-2k}\right]^{p/2} du \\ &\leqslant (p2^p+o(1))\delta \cdot \left[\frac{\sin^{4m}\delta}{(\pi-\delta)^{2m+2k}}\right]^{p/2} \leqslant (p2^p+o(1))\delta \cdot \frac{\sin^{2mp}\delta}{(\pi-\delta)^{p(m+k)}}.\end{aligned}$$

For the estimation of $I_{21}$, we shall employ the Watson's lemma. We introduce

$$t = t(v) := \ln g(\xi_2) - \ln g(\xi_2 - v) = \ln \frac{\lambda_2}{g(\xi_2-v)}, \quad \frac{dt}{dv} = \frac{g'(\xi-v)}{g(\xi-v)},$$

for $v \in [0, \pi/2+\xi_2]$. We have $t \to 0$ as $v \to 0$ and $t$ goes from 0 to $\infty$ monotonically as $v$ increases from 0 to $\pi/2+\xi_2$. We can state the asymptotic expansion of $t(v)$ near $v=0$ as follows:

$$t(v) \sim a_2v^2 + a_3v^3 + \cdots \sim a_2v^2(1 + a_3/a_2 v + \cdots),$$

where

$$a_2 = \Lambda_2 = -\frac{1}{2}\frac{d^2}{dv^2} \ln g(\xi_2 - v)\Big|_{v=0} = -\frac{h'(\xi_2)}{2\xi_2(\pi/2-\xi_2)\sin(2\xi_2)} \approx 1.2229.$$

Let $s = \sqrt{t}$. Then

$$s(v) \sim \sqrt{\Lambda_2}v(1 + b_1v + \cdots), \quad v \to 0.$$

Now $s'(v) \neq 0$, we can reverse this expansion,

$$v = v(t) \sim \Lambda_2^{-1/2} s(1 + c_1s + c_2s^2 + \cdots) \sim \Lambda_2^{-1/2} t^{1/2}(1 + c_1t^{-1/2} + c_2t + \cdots).$$

Also,

$$\frac{dv}{dt} = \frac{(\pi/2+v-\xi_2)(\xi_2-v)\sin 2(\xi_2-v)}{h(\xi_2-v)}$$

Asymptotic expansion of numerator and denominator at $v=0$ and division yields

$$\begin{aligned}\frac{dv}{dt} &\sim \frac{(\pi/2-\xi_2)\xi_2\sin(2\xi_2)}{-h'(\xi_2)v(t)}(1 + d_1v(t)^2 + \cdots) \\ &\sim \frac{1}{2\Lambda_2v(t)}(1 + d_1v(t)^2 + \cdots) \\ &\sim \frac{1}{2\sqrt{\Lambda_2 t}}(1 + e_1t^{1/2} + e_2t + \cdots).\end{aligned}$$

Now changing the variable in $I_{21}$ and noting $g(\xi_2 - v) = \lambda_2 e^{-t}$, we have

$$\begin{aligned} I_{21} &\sim \int_{-\pi/2}^{\xi_2} [g(u)^{2m}(u-\pi/2)^{-2k}]^{p/2} du \\ &= \lambda_2^{mp} \int_0^{\xi_2+\pi/2} [(g(\xi_2 - v)/\lambda_2)^{2m}(\xi_2 - v - \pi/2)^{-2k}]^{p/2} dv \\ &= \lambda_2^{mp} \int_0^{\infty} e^{-mpt} q(t) dt, \end{aligned}$$

where

$$\begin{aligned} q(t) &= (\pi/2 + v(t) - \xi_2)^{-kp} \cdot \frac{dv}{dt} \\ &\sim \frac{(\pi/2-\xi_2)^{-kp}}{2\sqrt{\Lambda_2 t}} (1 + f_1 t^{1/2} + f_2 t + \cdots)^{-kp} (1 + e_1 t^{1/2} + e_2 t + \cdots) \\ &\sim \frac{(\pi/2-\xi_2)^{-kp}}{2\sqrt{\Lambda_2 t}} (1 + g_1 t^{1/2} + g_2 t + \cdots). \end{aligned}$$

By Watson's lemma and choosing $\delta$ such that $\sin^2(\delta/(\pi-\delta)) < \lambda_2$, we conclude that

$$I_2 \sim I_{21} \sim \lambda_2^{mp} \cdot \frac{(\pi/2-\xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Similarly, we can estimate the asymptotic behavior of $I_3$. We use

$$t = t(v) = \ln g(\xi_2) - \ln g(\xi + v) = \ln \frac{\lambda_2}{g(\xi+v)}, \quad v \in (0, \pi/2 - \xi_2).$$

Same technique implies

$$I_3 \sim \lambda_2^{mp} \cdot \frac{(\pi/2-\xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Next, for $I_4$, observing the period of $\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}}$ is $\pi$, we have

$$\begin{aligned} I_4 &= \int_{\pi/2}^{3\pi/2} \left[ \left( \frac{\sin^2(u-\pi/2)\sin^2(u)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \left( \sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du \\ &\overset{u\to\pi-u}{=} \int_{-\pi/2}^{\pi/2} \left[ \left( \frac{\sin^2(u-\pi/2)\sin^2(u)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \left( \sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du \\ &= I_2 + I_3. \end{aligned}$$

Consequently,

$$I_4 \sim 2\lambda_2^{mp} \cdot \frac{(\pi/2-\xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Next, we estimate $I_5$. By $E_{2m-1}(z) = (2m-1)! \sum_{\nu=0}^{2m-2} N_{2m}(\nu+1) z^\nu$, we derive that $|E_{2m-1}(z)| \leqslant (2m-1)!$ for $|z| = 1$ and

$$\begin{aligned} I_5 &= \int_{3\pi/2}^{\infty} \left[ \left( \frac{\sin^2(u-\pi/2)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \frac{|E_{2m-1}(e^{2iu})|}{(2m-1)!} \right]^{p/2} du \\ &\leqslant \int_{\pi}^{\infty} \left[ \left( \frac{\sin^2(u)}{u} \right)^{2m} u^{-2k} \right]^{p/2} du \leqslant \frac{1}{\pi^{2k}} \int_{\pi}^{\infty} u^{-mp} du \\ &\leqslant \frac{1}{(mp-1)\pi^{2k}} \left( \frac{1}{\pi} \right)^{mp-1}, \quad mp-1 > 0. \end{aligned}$$

Similarly,

$$\begin{aligned} I_1 &= \int_{-\infty}^{-\pi/2} \left[ \left( \frac{\sin^2(u-\pi/2)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \frac{|E_{2m-1}(e^{2iu})|}{(2m-1)!} \right]^{p/2} du \\ &\leqslant \int_{-\infty}^{-\pi} \left[ \left( \frac{\sin^2(u)}{u} \right)^{2m} u^{-2k} \right]^{p/2} du \leqslant \frac{1}{(mp-1)\pi^{2k}} \left( \frac{1}{\pi} \right)^{mp-1}, \quad mp-1 > 0. \end{aligned}$$

In summary, we have

$$I_1 \sim I_5 \leqslant \frac{1}{(mp-1)\pi^{2k}} \left( \frac{1}{\pi} \right)^{mp-1}$$

and

$$I_2 \sim I_3 \sim \frac{1}{2} I_4 \sim \lambda_2^{mp} \cdot \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Due to $\dfrac{1}{\pi} \approx 0.31830 \leqslant \lambda_2 \approx 0.69706$, we conclude that

$$\begin{aligned} \|\widehat{{}_k\psi_m^S}\|_p^p &= \frac{4 \cdot 2^{-2kp}}{\sqrt{2\pi}^p} \cdot 4\lambda_2^{mp} \cdot \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})) \\ &= \frac{8}{\sqrt{2\pi}^{p-1}} \cdot \frac{(2\pi - 4\xi_2)^{-kp}}{\sqrt{2\Lambda_2 mp}} \cdot \lambda_2^{mp} \cdot (1 + \mathcal{O}(m^{-1/2})), \end{aligned}$$

which completes our proof. ▯

*Proof of Theorem 6.* By definition,

$${}_m\psi_m^S(x) = \sum_{\nu=0}^{2m-2} \frac{(-1)^\nu}{2^{m-1}} N_{2m}(\nu+1) N_{2m}(2x-\nu).$$

Hence,

$$\begin{aligned} |\widehat{{}_m\psi_m^S}(\omega)| &= \frac{2^{-2m}}{\sqrt{2\pi}} \left( \frac{\sin(\omega/4)}{\omega/4} \right)^{2m} \frac{|E_{2m-1}(\tilde{z})|}{(2m-1)!} \\ &= \frac{2^{-2m}}{\sqrt{2\pi}} \left( \frac{\sin(\omega/4)}{\omega/4} \right)^{2m} (2\sin(\tilde{\omega}/2))^{2m} \left| \sum_{l=-\infty}^{\infty} \frac{1}{(\tilde{\omega} + 2\pi l)^{2m}} \right|, \end{aligned}$$

where $E_{2m-1}$ is the Euler-Frobenius polynomial, $\tilde{z} = e^{i\tilde{\omega}}$, and $\tilde{\omega} = \pi - \omega/2$. Setting $u = \tilde{\omega}/2 = \pi/2 - \omega/4$, we obtain

$$
\begin{aligned}
\|\widehat{{}_m\psi_m^S}\|_p^p &= \frac{4\cdot 2^{-2mp}}{(\sqrt{2\pi})^p}\int_{\mathbb{R}}\left[\left(\frac{\sin(u-\pi/2)}{u-\pi/2}\right)^{4m}\cdot(\sin(u))^{4m}\cdot\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2}du\\
&= \frac{4\cdot 2^{-2mp}}{(\sqrt{2\pi})^p}\Big(\int_{-\infty}^{-\pi/2}\left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2}\right)^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2}du\\
&+\int_{-\pi/2}^{\pi/4}\left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2}\right)^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2}du\\
&+\int_{\pi/4}^{\pi}\left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2}\right)^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2}du\\
&+\int_{\pi}^{\infty}\left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2}\right)^{4m}\left(\sum_{\ell=-\infty}^{\infty}\frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2}du\Big)\\
&=: \frac{4\cdot 2^{-2mp}}{(\sqrt{2\pi})^p}(I_1+I_2+I_3+I_4).
\end{aligned}
$$

Let

$$g(u) := \left(\frac{\sin(u-\pi/2)\sin(u)}{(u-\pi/2)u}\right)^2.$$

Then $g$ is symmetric about $u = \pi/4$ and $g(u) \leqslant g(\pi/4) = 64/\pi^4$. Similarly, using [16, Lemma 3], we have

$$I_2 \sim \int_{-\pi/2}^{\pi/4}(g(u))^{mp}du.$$

Introducing

$$t = t(v) = \ln\frac{g(\pi/4)}{g(\pi/4-v)}, \quad v \in [0, \frac{3}{4}\pi],$$

we can derive

$$q(t) := \frac{dv}{dt} \sim \frac{\pi}{4}(\pi^2-8)^{-1/2}t^{-1/2}(1+e_1t^{1/2}+e_2t+\cdots).$$

Changing the variable $u \to \pi/4 - v$ in $I_2$ and using Watson's lemma, we deduce

$$
\begin{aligned}
\frac{4\cdot 2^{-2mp}}{(\sqrt{2\pi})^p}I_2 &\sim \frac{4\cdot 2^{-2mp}}{(\sqrt{2\pi})^p}[g(\pi/4)]^{mp}\int_0^{\infty}e^{-mpt}q(t)dt\\
&\sim \frac{1}{\sqrt{2\pi}^{p-1}}\cdot\frac{\pi}{\sqrt{\pi^2-8}}\left(\frac{16}{\pi^4}\right)^{mp}\cdot\frac{1}{\sqrt{2mp}}\cdot(1+\mathcal{O}(m^{-1/2}))
\end{aligned}
$$

It is easily seen that $I_3 = I_2$ due to the symmetry of $g(u)$. Also, by the symmetry, we have $I_1 = I_4$. Using the fact that $|E_{2m-1}(z)| \leqslant (2m-1)!$ for $|z| = 1$, we have

$$\begin{aligned}\frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} I_4 &= \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_{\pi}^{\infty} \left(\frac{\sin(u-\pi/2)}{u-\pi/2}\right)^{2mp} \left(\frac{|E_{2m-1}(\tilde{z})|}{(2m-1)!}\right)^p d\omega \\ &\leqslant \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_{\pi}^{\infty} \left(\frac{\sin(u-\pi/2)}{u-\pi/2}\right)^{2mp} d\omega \leqslant \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_{\pi/2}^{\infty} \left(\frac{1}{\omega}\right)^{2mp} d\omega \\ &\leqslant \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \frac{1}{2mp-1} \left(\frac{2}{\pi}\right)^{2mp-1} = \frac{1}{2mp-1} \frac{1}{(\sqrt{2\pi})^{p-2}} \left(\frac{1}{\pi^2}\right)^{mp}.\end{aligned}$$

Noting that $1/\pi^2 \leqslant 16/\pi^4$, we conclude

$$\|\widehat{{}_m\psi_m^S}\|_p^p = \frac{2}{\sqrt{2\pi}^{p-1}} \cdot \frac{\pi}{\sqrt{\pi^2-8}} \cdot \frac{1}{\sqrt{2mp}} \cdot \left(\frac{16}{\pi^4}\right)^{mp} \cdot (1 + \mathcal{O}(m^{-1/2})),$$

which completes our proof. ◻

**Acknowledgement.** The authors would like to thank the anonymous reviewers for many helpful suggestions on improving the presentation of the paper.

## REFERENCES


[1] L. Alvarez Diaz, V. Vampa, and M. T. Martin, *The construction of plate finite elements using wavelet basis functions*, Revista Investigacion Operacional. Vol. 30, 3 (2009), pp. 193–204.

[2] V. F. Babenko and S. A. Spektor, *Estimates for wavelet coefficients on some classes of functions*, Ukrainian Mathematical Journal. Vol. 59, 12 (2007), pp. 1791–1799.

[3] V .F. Babenko and S .A. Spektor, *Inequalities similar to those of Bernstein for non-periodic splines in $L_2$ space*, Vestnik DNU. Vol. 1, 6 (2008), pp. 21–29.

[4] P. Bechler, *Wavelet approximation of distributions with bounded variation derivatives*, J. Fourier Anal. Appl., 15 (2009), pp. 31–57.

[5] G. Beylkin, R. Coifman, and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I*, Comm. Pure Appl. Math., 44 (1991), pp. 141–183.

[6] E. Candes and M. B. Wakin, *An introduction to compressive sampling*, IEEE Signal Proc. Magazine, 21, March (2008).

[7] C .K. Chui, *An Introduction to Wavelets*, Wavelet Analysis and Its Applications, Vol. 1, Academic Press 1992.

[8] C. K. Chui, B. Han, and X. Zhuang, *A dual-chain approach for bottom-up construction of wavelet filters with any integer dilation*, Appl. Comput. Harmon. Anal., 33 (2012), pp. 204–225.

[9] C. K. Chui and J. Wang, *On compactly supported spline wavelets and duality principle*, Trans. Amer. Math. Soc., 330 (1992), pp. 903–915.

[10] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, CBMS Series, 1992.

[11] G. Deslauriers and S. Dubuc, *Symmetric iterative interpolation processes*, Constr. Approx., 5 (1989), pp. 49–68.

[12] P. L. Dragotti, *Sparse sampling of structured information and its application to compression*, MMSP, October (2009), pp. 5–7.

[13] M. Ehler, *Nonlinear approximation schemes associated with nonseparable wavelet bi-frames*, J. of Approx. Theory., 161 (2009), pp. 292–313.

[14] M. Ehler, *The multiresolution structure of pairs of dual wavelet frams for a pair of Sobolev spaces*, Jaen J. Approx., 2 (2) (2010), pp. 193–214.

[15] M. Ehler, *The minimal degree of solutions to polynomial equations used for the construction of refinable functions*, Sampl. Theory Signal Image Process., 9 (1-3) (2010), pp. 155–165.

[16] S. Ehrich, *On the estimate of wavelet coefficients*, Adv. Comput. Math., 13 (2000), pp. 105–129.

[17] B. Han and X. Zhuang, *Matrix extension with symmetry and its application to symmetric orthonormal multiwavelets*, SIAM J. Math. Anal., 42 (5) (2010), pp. 2297–2317.

[18] F. Keinert, *Biorthogonal wavelets for fast matrix computations*, Appl. Comput. Harm. Anal., 1 (1994), pp. 147–156.
[19] H. O. Kim, R. Y. Kim, and J. S. Ku, *On asymptotic behavior of BattleLemari scaling functions and wavelets*, Applied Mathematics Letters, 20 (4) (2007), pp. 376–381.
[20] Y. Kim, M. S. Nadar, and A. Bilgin, *Exploiting wavelet-domain dependencies in compressed sensing*, DCC, March (2010), pp. 24–26.
[21] N. P. Kornejchuk, *Splines in the Approximation Theory*, Moscow., 1984.
[22] F. W. J. Olver, *Asymptotic and special functions*, Academic press, New York, 1974.
[23] T. Petersdorff and C. Schwab, *Wavelet approximations for first kind boundary integral equations on polygons*, Numer. Math., 74 (1996), pp. 479–519.
[24] A. Rathsfeld, *A wavelet algorithm for the solution of the double layer potential equation over polygonal boundaries*, J. Integral Equation Appl., 7 (1995), pp. 47–97.
[25] M. Unser, *Ten good reasons for using spline wavelets*, in: Wavelet Applications in Signal and Image Processing V, Proc. SPIE, 3169 (1997), pp. 422–432.