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# ON POSSIBLE ISOLATED BLOW-UP PHENOMENA AND REGULARITY CRITERION OF THE 3D NAVIER-STOKES EQUATION ALONG THE STREAMLINES*

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**Abstract.** The first goal of our paper is to give a new type of regularity criterion for solutions $u$ to Navier-Stokes equation in terms of some supercritical function space condition $u \in L^\infty(L^{\alpha,\infty})$ (with $\frac{3}{4}(17^{\frac{1}{2}} - 1) < \alpha < 3$) and some exponential control on the growth rate of $\operatorname{div}(\frac{u}{|u|})$ along the streamlines of u. This regularity criterion greatly improves a previous result of the first author. However, we also point out that totally new idea which involves the use of the new supercritical function space condition is necessary for the success of our new regularity criterion in this paper.

The second goal of our paper is to give a geometric description or characterization of a possible divergence free vector field $u$ within a flow-invariant tubular region with increasing twisting of streamlines towards one end of a bundle of streamlines. The increasing twisting of streamlines is controlled in such a way that the associated quantities $\|u\|_{L^p}$, for some fixed choice of $(2 < p < 3)$ and $\|\operatorname{div}(\frac{u}{|u|})\|_{L^6}$ blow up while preserving the finite energy property $u \in L^2$ at the same time. We also briefly mention how this construction is related to the regularity criterion proved in our paper.

**Key words.** Navier-Stokes equation, regularity criterion.

**AMS subject classifications.** 35B65, 76D03, 76D05.

## 1. Introduction.

The first goal of this paper is to give a new type of regularity criterion of solutions $u$ to the Navier-Stokes equation in terms of some weak $L^\alpha$ space condition on the velocity $u$ (with some $\frac{3}{4}(17^{\frac{1}{2}} - 1) < \alpha < 3$ ) and some exponential control of $\operatorname{div} \frac{u}{|u|}$ along the streamlines. The second goal of this paper is to give *possible* blow-up situations for 3D-Navier-Stokes equation through the construction of a finite energy divergence free velocity field $u$ with $u \notin L^\alpha$ with $2 < \alpha < 3$ and $\operatorname{div} u \notin L^6$. The Navier-Stokes equation on $\mathbb{R}^3$ is given by

$$\begin{cases} \partial_t u - \triangle u + \operatorname{div}(u \otimes u) + \nabla P = 0, \\ div(u) = 0, \quad u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

in which $u$ is a vector-valued function representing the velocity of the fluid, and $P$ is the pressure. The initial value problem of the above equation is endowed with the condition that $u(0, \cdot) = u_0 \in L^2(\mathbb{R}^3)$.

Modern regularity theory for solutions to equation (1.1) began with the works of Leray [12] and Hopf [6] in which they established, with respect to any given initial datum $u_0 \in L^2(\mathbb{R}^3)$ which is weakly divergence free, the existence of a weak solutions $u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ lying in the class of $L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^3))$ which satisfies the global energy inequality. Since the time of Leary and Hopf, any weak solution to equation (1.1) which satisfies the finite energy, finite dissipation, and global energy inequalities is called Leray-Hopf solutions to (1.1).

After the fundamental works of Leray and Hopf, progress in addressing the full regularity of Leray-Hopf solutions has been very slow. It was only in 1960 that

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significant progress was made by Prodi [14], Serrin [15], Ladyzhenskaya [11], and their joint efforts lead to the following famous Prodi-Serrin-Ladyzhenskaya criterion for Leray-Hopf solutions (see the introduction of [7] for more historical remarks about this).

THEOREM 1.1. *[Prodi, Serrin, Ladyzhenskaya] Let* $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ *be a Leray-Hopf weak solution to* (1.1)*, which also satisfies* $u \in L^p(0,\infty;L^q(\mathbb{R}^3))$*, for some* $p,q$ *satisfying* $\frac{2}{p}+\frac{3}{q}=1$*, with* $q>3$*. Then,* $u$ *is smooth on* $(0,T]\times\mathbb{R}^3$ *and is uniquely determined in the following sense*

- *suppose* $v \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ *is another Leray-Hopf weak solution such that* $u(0,\cdot)=v(0,\cdot)$*. Then, it follows that* $u=v$ *on* $(0,T]\times\mathbb{R}^3$*.*

The success of the Prodi-Serrin-Ladyzhenskaya criterion was based on the fact that the integral condition $u \in L^p(L^q)$ with $p,q$ satisfying $\frac{2}{p}+\frac{3}{q}=1$ *and* $q>3$ ensures that the Leray-Hopf solution $u$ behaves *like* a solution to a slightly pertubated heat equation. It is also worthwhile to mention that the exceptional case of $u \in L^\infty(L^3)$ was missed in the above regularity criterion of Prodi, Serrin, and Ladyzhenskaya, and it was not until very recently that the regularity of solutions in the exceptional case $u \in L^\infty(L^3)$ was finally established in the famous work [7] due to L. Escauriaza, G. Seregin, and V. Sverak.

After the appearance of the Prodi-Serrin-Ladyzhenskaya criterion, many different regularity criteria of solutions to (1.1) was established by researchers working in the regularity theory of (1.1). Among these, for instance, Beirão da Veiga established in [2] a regularity criterion in terms of the integral condition $\nabla u \in L^p(0,\infty;(L^q(\mathbb{R}^3)))$ with $\frac{2}{p}+\frac{3}{q}=2$ (and $1<p<\infty$) imposed on $\nabla u$. In the same spirit of [2], Beale, Kato and Majda [1] gave a regularity criterion for solutions $u$ to (1.1) in terms of the condition $\omega \in L^1(0,\infty;L^\infty(\mathbb{R}^3))$ imposed on the vorticity $\omega = \operatorname{curl} u$ associated to $u$. This regularity criterion was further improved by Kozono and Taniuchi in [10] (see also [13]). Besides these, other important works such as [5] and [9], in which type I blow up was excluded for axisymmetric solutions to (1.1), are attracting a lot of attentions. Due to the limitation of space and the vast literature in the regularity theory for solutions to (1.1), we do not try to do a complete survey here.

However, we would like to mention an interesting regularity criterion in [16] due to Vasseur, since it is related to the main result of this paper and also to the previous partial result [4] by the first author. [16] gave a regularity criterion for solutions $u$ to (1.1) in terms of the integral condition $\operatorname{div}(\frac{u}{|u|}) \in L^p(0,\infty;L^q(\mathbb{R}^3))$ with $\frac{2}{p}+\frac{3}{q}\leqslant\frac{1}{2}$ imposed on the scalar quantity $F=\operatorname{div}(\frac{u}{|u|})$.

One of the main purposes of this paper, however, is to establish the following regularity criterion for solutions $u$ to (1.1) in terms of some exponential control on the rate of change of $F=\operatorname{div}(\frac{u}{|u|})$ along the streamlines of $u$ and some *weak* $L^\alpha$ *space* condition imposed on $u$.

THEOREM 1.2. *Let* $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ *be a Leray Hopf solution to* (1.1) *which is smooth up to a possible blow up time* $T$ *with* $u_0 \in \mathcal{S}(\mathbb{R}^3)$*. Let us assume that* $u$*,* $F=\operatorname{div}(\frac{u}{|u|})$*, and the pressure* $P$ *satisfy the following conditions.*

- $u \in L^\infty(0,T;L^{\alpha,\infty}(\mathbb{R}^3))$*, for some given* $\alpha \in (2,3)$ *which satisfies* $1+2(\frac{\alpha}{3}-\frac{3}{\alpha})>0$*.*
- *There exists some* $r_0>0$ *and* $M_0>0$ *such that* $|u|\leqslant M_0$ *is valid on the region* $[0,T)\times\{x\in\mathbb{R}^3:|x|\geqslant r_0\}$

- *For some given constants $A > 0$ and $L > 0$, the property $|\frac{u\cdot\nabla F}{|u|}| \leq A|F|$ is valid on $\{(t,x) \in [0,T) \times B(r_0) : |F(t,x)| \geqslant L\}$ (Here, $B(r_0) = \{x \in \mathbb{R}^3 : |x| < r_0\}$).*
- $P \in L^{\frac{5}{3}}((0,T) \times \mathbb{R}^3)$.

*Then the smoothness of $u$ can be extended beyond the time $T$.*

Here, we give a few remarks which illustrate the significance of Theorem 1.2. We start with the third condition in Theorem 1.2 in which we see the condition $|\frac{u\cdot\nabla F}{|u|}| \leq A|F|$ imposed on the region $[0,T) \times \{x \in \mathbb{R}^3 : |x| \geqslant r_0\} \cap \{|F| \geqslant L\}$. We can see the geometric meaning of the constraint $|\frac{u\cdot\nabla F}{|u|}| \leq A|F|$ on $[0,T) \times \{x \in \mathbb{R}^3 : |x| \geqslant r_0\} \cap \{|F| \geqslant L\}$ if we recast it in the following geometric language.

- For any time slice $t \in [0,T)$, and any streamline $\gamma : [0,S) \to \mathbb{R}^3$ of the velocity profile $u(t,\cdot)$ which is *parameterized by arclength* (that is, $\frac{d\gamma}{ds} = \frac{u}{|u|}(\gamma(s))$) and with image $\gamma([0,S))$ lying in the region $\{x \in \mathbb{R}^3 : |x| \geqslant r_0, |F(t,x)| \geqslant L\}$, we have $|\frac{d}{ds}(F(\gamma(s)))| \leqslant A \cdot |F(\gamma(s))|$, for any $0 \leqslant s \leqslant S$.

The condition $|\frac{d}{ds}(F(\gamma(s)))| \leqslant A \cdot |F(\gamma(s))|$ gives some exponential control on $F$ *along each streamline* of the fluid within the space region on which both $u$ and $F = \operatorname{div}(\frac{u}{|u|})$ are large. Our *original motivation was* to prove that the smoothness of the solution $u : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ to (1.1) can be extended beyond the possible blow up time $T$ under the third condition of Theorem 1.2 and the Leray-Hopf property $u \in L^\infty(L^2) \cap L^2(\dot{H}^1)$ of the solution. But our experience told us that this cannot be so easily achieved *without* the involvement of the following additional condition (which is the first condition of Theorem 1.2 ).

- $u \in L^\infty(0,T : L^{\alpha,\infty}(\mathbb{R}^3))$, for some given $\alpha \in (2,3)$ with $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$.

To clarify the *necessity* of the condition $u \in L^\infty(L^{\alpha,\infty})$ with some $\alpha \in (2,3)$ satisfying $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$, let us mention a piece of work [4] by the first author in which smoothness of a Leray-Hopf solution $u : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ is established beyond the possible blow up time $T$ under the following condition.

- (condition in the regularity criterion of [4]) $|\frac{u\cdot\nabla F}{|u|^\delta}| \leqslant A|F|$ is valid on $[0,T) \times \mathbb{R}^3$, with $A > 0$ to be a given constant and $\delta$ to be a given constant with $0 < \delta < \frac{1}{3}$.

The above mentioned regularity criterion based on the condition $|\frac{u\cdot\nabla F}{|u|^\delta}| \leqslant A|F|$ (with $0 < \delta < \frac{1}{3}$) was established in [4] through applying the De Giorgi's method as developed by A. Vasseur in [17]. The main idea of the De Giorgi's method in [17] is based on the establishment of the following nonlinear recurrence relation of the energy $U_k$ of a truncated function $v_k = [|u| - R(1 - \frac{1}{2^k})]_+$ of the solution $u$ to (1.1) over a certain space time region (for a precise definition of $U_k$, see section 3 of this paper, or alternatively [17] or [4]),

$$U_k \leq \frac{C_0^k}{R^\lambda} U_{k-1}^\beta. \tag{1.2}$$

According to the idea in [17], for a given solution $u$ to (1.1) on $[0,T) \times \mathbb{R}^3$ with possible blow up time $T$, the $L^\infty$-boundedness *conclusion* $|u| \leqslant R$ over $[\frac{T}{2},T) \times \mathbb{R}^3$ (for some sufficiently large $R$) can be drawn from relation (1.2) provided one can ensure that $\beta > 1$ and $\lambda > 0$ are valid *simultaneously.* Roughly speaking, $\lambda > 0$

ensures the smallness of the energy $U_1$ of the first truncated function $v_1$, due to the fact that $\frac{1}{R^\lambda}$ will become small as $R$ is sufficiently large. The smallness of $U_1$ will trigger the nonlinear recurrence effect of relation (1.2) which eventually causes the very fast decay of $U_k$ to 0 (see Lemma 3.2 which originally appeared in [17] ). This resulting decay of $U_k$ to 0 then implies the desired boundedness conclusion $|u| \leqslant R$ over $[\frac{T}{2}, T) \times \mathbb{R}^3$, which in turn extends the smoothness of $u$ beyond the possible blow up time $T$. However, it was illustrated in [4] that the requirement that $\beta > 1$ *and* $\lambda > 0$ have to hold simultaneously *prevents* us to push the constant $\delta$ (in the condition $|\frac{u\cdot\nabla F}{|u|^\delta}| \leqslant A|F|$) to go beyond the range $(0, \frac{1}{3})$. This limitation of the De Giorgi method of [17] basically comes from the fact that the index $\beta$ in relation (1.2) is typically $\frac{5}{3}$ or $\frac{4}{3}$, which is *too large* for the survival of the condition $\lambda > 0$ in the same relation (1.2).

As a result, the use of the extra condition $u \in L^\infty(L^{\alpha,\infty})$, with $\alpha \in (2,3)$ satisfying $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$ can help us to *lower* the index $\beta$ of relation (1.2) from the typical $\frac{5}{3}$ or $\frac{4}{3}$ to become *as close to* 1 *as possible*, and this in turn ensures the survival of $\lambda > 0$ in the same relation (1.2). On the other hand, we have to address the question of whether the condition of $u \in L^\infty(L^{\alpha,\infty})$, with $\alpha \in (2,3)$ satisfying $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$ is too strong as an assumption. Note that the constraint $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$ on $2 < \alpha < 3$ is equivalent to the constraint $\frac{3}{4}(17^{\frac{1}{2}} - 1) < \alpha < 3$. This indicates that the condition $u \in L^\infty(L^{\alpha,\infty})$ with such a $\alpha$ lying in $(\frac{3}{4}(17^{\frac{1}{2}} - 1), 3)$ is beyond the classical Prodi-Serrin-Ladyzhenskaya range and the $L^\infty(L^3)$ criterion of [7]. This means that this extra assumption, which is the first technical condition in the hypothesis of Theorem 1.2, is reasonable.

On the other hand, we *do not* require the Leray-Hopf solution $u$ which we consider in Theorem 1.2 to be a suitable weak solution. This is due to the fact that the function spaces (such as the $L^p$ spaces) on which we are working in the proof of Theorem 1.2 are in the $\mathbb{R}^3$ setting, and that there is nowhere in the proof at which the localized version of the energy inequality about the solution $u$ is employed. Indeed, the smoothness condition as imposed on the Leray-Hopf solution $u : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ priori to the possible blow up time $T$ is more than enough to justify the validity of relation (4.2) on $[0,T) \times \mathbb{R}^3$ in the pointwise sense, on which our mathematical argument leading to the conclusion of Theorem 1.2 is based.

Another related matter is the suitable assumption as imposed on the pressure term $P$ associated to the solution $u$ in Theorem 1.2. In order to ensure that $P$ will have good enough far range decay in $\mathbb{R}^3$ for us to carry out basic integration by part procedure, we require that the condition $P \in L^{\frac{5}{3}}([0,T) \times \mathbb{R}^3)$ has to be imposed on the pressure term $P$ associated to the solution $u$ in Theorem 1.2. This integrable condition as imposed on $P$ is a natural one, since conventional wisdom from fluid mechanics teaches that $P$ should behave like $|u|^2$, while $u \in L^{\frac{10}{3}}([0,T) \times \mathbb{R}^3)$ is a natural consequence which follows from the Leray-Hopf condition imposed on u, through an application of Sobolev embedding theorem and interpolation inequality. Under the condition $P \in L^{\frac{5}{3}}([0,T) \times \mathbb{R}^3)$, we can always identify $P$ with $\sum R_i R_j(u_i u_j)$ with $R_i$ to be the Riesz's transforms on $\mathbb{R}^3$.

Next, let us mention that there is nothing deep about the second condition in Theorem 1.2, which says that the large velocity region $\{x \in \mathbb{R}^3 : |u(t,x)| > M_0\}$ of the solution is restricted within the open ball $\{x \in \mathbb{R}^3 : |x| < r_0\}$ of some given radius

$r_0$. Even in the case when this second condition is lacking (i.e. not available) in the hypothesis of Theorem 1.2, it can be derived as a consequence from the notion of suitable weak solutions through an application of the partial regularity theorem in [3] , *provided the Leray-Hopf solution $u$ which we consider is further known to be a suitable weak solution.* So, from a technical stand-point, imposing the second condition in the hypothesis of Theorem 1.2 is the price we have to paid in order to get rid of the use of the notion of suitable weak solutions (and hence to avoid any localization procedure on the solution $u$ in the proof of Theorem 1.2 ). Indeed, if $u$ is a Leray-Hopf solution to the Navier-Stokes equation on the time-space region $(-2,0]\times\mathbb{R}^3$, and suppose further that $u$ is also simultaneously a suitable weak solution in that $u$ satisfies the following differential inequality on $(-2,0]\times\mathbb{R}^3$ in the distributional sense.

$$\partial_t(\frac{|u|^2}{2})+|\nabla u|^2-\triangle(\frac{|u|^2}{2})+\mathrm{div}(\frac{|u|^2}{2}u)+u\cdot\nabla P\leqslant 0. \tag{1.3}$$

Then, the condition $u\in L^{\frac{10}{3}}((-2,0)\times\mathbb{R}^3)$ immediately follows from the fact that u is a Leray-Hopf solution, through an application of the Sobolev embedding theorem and standard interpolation inequality. With the natural condition $P\in L^{\frac{5}{3}}((-2,0)\times\mathbb{R}^3)$ as imposed on the associated pressure $P$, one readily sees that $\int_{(-2,0)\times\{x:|x|>R\}}|u|^3+|P|^{\frac{3}{2}}$ trends to 0 as $R$ trends to $+\infty$. As a result, when $R$ is chosen to be large enough, then, $\int_{(-2,0)\times\{x:|x|>R\}}|u|^3+|P|^{\frac{3}{2}}$ will eventually be smaller than the absolute constant $\epsilon_0>0$, which is the one as required in the following key lemma which was due to Caffarelli, Kohn, and Nirenberg in [3] .

Theorem 1.3. *(Partial Regularity Theorem [3]) There exists an absolute constant $\epsilon_0>0$ such that the following assertion holds for any suitable weak solution $(u,P)$ to the Navier-Stokes equation on $(-1,0]\times B(1)$.*

- *If it happens that $\int_{(-1,0)\times B(1)}|u|^3+|P|^{\frac{3}{2}}<\epsilon_0$, then it follows that $|u|\leqslant 1$ holds on $[-\frac{1}{4},0]\times B(\frac{1}{2})$.*

As a result, for such a large enough $R>4$, one sees that, for any $(t,x)\in[-1,0]\times\{x:|x|>R+1\}$, the integral of $|u|^3+|P|^{\frac{3}{2}}$ over the parabolic cube $Q_{(t,x)}(1)=(t-1,t]\times B_x(1)$ will be smaller than the absolute constant $\epsilon_0>0$ of the above Lemma, and this in turns implies, through an application of the above key lemma that, $|u|$ is bounded by 1 on $Q_{(t,x)}(\frac{1}{2})=(t-\frac{1}{4},t]\times B_x(\frac{1}{2})$. This argument shows that $|u|$ is essentially bounded by 1 on $(-1,0]\times\{x\in\mathbb{R}^3:|x|>R+1\}$, for some sufficiently large $R>0$. The above argument, which can be founded in [7], is standard and well-known to PDE specialists working in the area of the regularity theory of Navier-Stokes equation. It shows that the same qualitative result which says that large velocity region of a solution $u$ to (1.1) has to be confined within a certain ball with some sufficiently large radius $R_0$ (depending on $u$) can be deduced by means of an application of the partial regularity theorem provided the solution $u$ is further known to be a suitable weak solution. So, to a certain extent, the second condition imposed on Theorem 1.2 is not very crucial and is imposed here for the sole purpose of avoiding to impose the notion of suitable weak solution as an extra condition upon the solution $u$ in the hypothesis of Theorem 1.2.

Before we finish the discussion about Theorem 1.2, we point out that the proof of Theorem 1.2 as presented in Section 4 of our paper closely follows the proof of the regularity criterion in [4]. However, we also point out that we have given completely *new* idea which allows us to use the extra weak $L^\alpha$ space condition with $\frac{3}{4}(17^{\frac{1}{2}}-1) < \alpha < 3$ to lower down the index $\beta$ in (1.2) from the typical value of $\frac{5}{3}$ or $\frac{4}{3}$ to become as close to 1 as possible. For those readers who are interested only in those new ideas contributed to the proof of Theorem 1.2, we have given, in Section 3 of our paper, an outline of those crucial and important ideas which make the old argument of [4] become strong enough to arrive at Theorem 1.2. But we also give, in Section 4 of our paper, the complete details of the proof of Theorem 1.2 by including those new ideas outlined in Section 3 in the technical argument.

Besides the main result in Theorem 1.2, we will also give some sufficient mathematical conditions, through the use of geometric languages such as those in Definitions 2.1 through 2.7, for a given divergence free velocity field $u$ within a stream-tube segment in an attempt to obtain a geometric visualization of the possible increasing twisting (i.e. increasing swirl) among the streamlines of $u$ towards the ending cross section of the stream-tube. The geometric conditions or characterizations, as given in Section 2, for a given velocity field $u$ demonstrates the possible way in which *excessive* twisting of streamlines towards the ending cross section of the stream-tube can result in the *blow up* of the quantities $\|u\|_{L^\alpha}$ (for some fixed choice of $\alpha \in (2,3)$) and $\|\operatorname{div}(\frac{u}{|u|})\|_{L^6}$ while at the same time preserving the finite energy property $u \in L^2(\mathbb{R}^3)$ of the fluid. We *do not* claim that such geometric characterizations being given in Section 2 would in anyway imply the *actual existence* of such a divergence free vector field having the increasing twisting property which we desire. The true purpose of such a demonstration is just to illustrate the possibility of having a finite energy velocity field with increasing swirl which is beyond the scope covered by the regularity criterion of Vasseur in [16] and the $L^\infty(L^3)$ criterion of [7]. In a certain sense, the possible excessive twisting of streamlines of the velocity field as considered in Section 2 within a stream-tube segment with *almost constant cross section everywhere* (see Definition 2.7) will cause the streamlines to become *densely packed* together towards the ending cross section of the stream-tube, and this potentially denser and denser packing of streamlines eventually leads to the blow up of the velocity field at a singular point lying at the center of the ending cross section of the stream-tube. According to the regularity criterion in Theorem 1.2, one can speculate that if the velocity field $u$ as described in Section 2 with increasing swirl towards the ending cross section of the stream-tube (provided it exists) can be *realized* as *an instantaneous profile* $v(T,\cdot)$ of a time-dependent solution $v : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ to (1.1) in which singularity occurs at the blow up time $T$, then, it must be that the rate of increase of $F = \operatorname{div}(\frac{u}{|u|})$ along those streamlines with increasing twisting must go *beyond* the exponential growth rate. Even though the geometric description in Section 2, with the main result to be summarized in the form of an assertion in proposition 2.14, is interesting, it is totally independent of the regularity criterion of Theorem 1.2, and the reader should treat this as a separate topic.

**2. Possible blow-up profile of a velocity field with large swirl.** In this section, we give geometric conditions, such as those in Definitions 2.10, 2.11, and 2.12, in order to characterize a possible divergence free velocity field $u$ which is specified in a stream-tube segment around a representative streamline (with an incoming cross section and an ending cross section), and whose streamlines will have unbounded increasing swirl (ie increasing twisting around the representative streamline) *towards the ending cross section of the stream-tube segment*(see Definition 2.3). The uncontrolled increasing swirl of those streamlines towards the ending

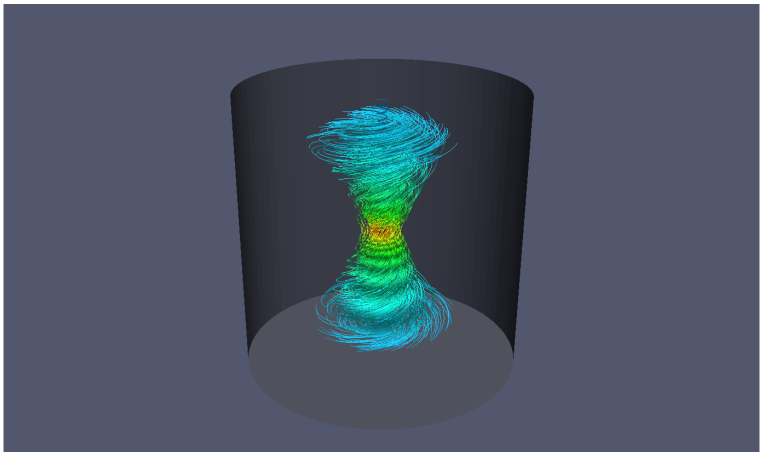

FIG. 1. *Increasing twisting streamlines.*

cross section of the stream-tube segment will lead to an isolated singularity of the velocity field which will be located at the point of intersection between the center representative streamline and the ending cross section of the stream-tube. Even though the conditions as given in Definitions 2.10, 2.11, and 2.12 look rather technical, the geometric intuitions hidden behind them are demonstrated by **figure 1** depicting a bundle of streamlines with increasing swirl (with regard to the source and credit of **figure 1**, please see the **Acknowledgments** on the last page of this article). After we state the required geometric setting and concepts in Definitions 2.1 through 2.7, we will give *sufficient* conditions, in Definitions 2.10, 2.11, and 2.12, for the divergence free velocity field $u$ under consideration to characterize the properties $u \in L^2$, $u \notin L^{\alpha}$ (for some given $2 < \alpha < 3$) and $\operatorname{div} \frac{u}{|u|} \notin L^6$ of such a velocity field with increasing swirl. However, we *do not* claim that we have showed the *actual existence* of such a divergence velocity field which satisfies the three characterizing conditions as given in Definitions 2.10, 2.11, and 2.12 respectively. Indeed, the purpose of imposing such geometric conditions upon the given divergence free velocity field is just to give a way to visualize how such a velocity profile might look like. Moreover, within this section, we will confine our discussion within the class of $L^p$-integrable velocity fields, and that no weak $L^p$ space will be involved. Before we start our discussion, let us mention that the main result of this section is summarized in the form of an assertion in proposition 2.14 at the end of Section 2 .

In order to describe such a velocity field $u$ with increasing swirl towards the ending cross section of the stream-tube segment, we first specify the center representative streamline $\gamma_\eta : [0, S) \to \mathbb{R}^3$ as follow.

DEFINITION 2.1. *(Representative stream line.) Let $\gamma_\eta : [0, S) \to \mathbb{R}^3$ be such that*

$$\partial_s \gamma_\eta(s) = \frac{u}{|u|}(\gamma_\eta(s)) \quad \textit{and} \quad \gamma_\eta(0) = \eta \in \mathbb{R}^3. \tag{2.1}$$

Note that the ending value $S$ is excluded from the definition of the representative streamline, since $\gamma_\eta(S)$ is supposed to be the isolated singularity point created by the unbounded increasing swirl of those streamlines close to the representative streamline.

Before we can create the stream-tube segment around the representative streamline $\gamma_\eta$, we need to specify the initial streamplane $A$ with parameter r as follow.

Definition 2.2. *(Initial stream plane $A$ with parameter $r$.) Let $\{\bar{A}_0(r)\}_{r\in(0,1]}$ be a smooth family of smoothly bounded open set in $\mathbb{R}^2$ s.t. $\bar{A}_0(r) \subset \bar{A}_0(r')$ $(r < r')$, $\bar{A}_0(r) \to \{0\}$ $(r \to 0)$. Let*

$$A_0(r) = \{x \in \mathbb{R}^3 : R(x-\eta) \in \bar{A}_0(r)\}, \tag{2.2}$$

*where $R$ is a rotation matrix s.t. $R(\frac{u}{|u|}(\eta,t)) = (0,0,1)$.*

The initial streamplane $A_0(1)$ is exactly the incoming cross section of the stream-tube segment which will be specified. To construct the stream-tube segment with $A_0(1)$ as its incoming cross section, we just need to specify, for each $s \in (0,S)$, the associated stream-plane $A(r,s)$ intersecting $\gamma_\eta$ at the point $\gamma_\eta(s)$ as follow.

Definition 2.3. *(Stream-planes.) Let*

$$\begin{aligned} A(r,s) := \bigcup_{\eta'\in A_0(r)} \{\gamma_{\eta'}(s') : s' &\text{ is the minimum among all possible } \tau > 0 \text{ for which} \\ \gamma_{\eta'}(\tau) &\text{ belongs the plane which passes through the point } \gamma_\eta(s) \\ &\qquad\qquad \text{and is perpendicular to } \partial_s\gamma_\eta(s)\}. \end{aligned}$$

For simplicity, we just set $A(s) := A(1,s)$. Then, we can define the stream-tube to be

$$T^A_{[0,S)} = \bigcup_{0\leqslant s<S} A(s) \tag{2.3}$$

We remark that, for any $x \in A(s)$, there is $r$ s.t. $x \in \partial A(r,s)$. This is due to the fact that for each $s \in [0,S)$, $A(r,s)$ is strictly shrinking towards the representative streamline as $r \to 0^+$. Based on this observation, we introduce an orthonormal coordinate frame system within the stream-tube $T^A_{[0,S)}$ in the following definition.

Definition 2.4. *For $x,y \in \partial A(r,s)$, let $e_\theta(x) := \lim_{y\to x} \frac{x-y}{|x-y|}$, $e_z(x) := \frac{u}{|u|}(\gamma_\eta(s))$ and let $e_r(x)$ be s.t.*

$$\langle e_z(x), e_r(x)\rangle = \langle e_\theta(x), e_r(x)\rangle = 0 \quad \text{and} \quad |e_r(x)| = 1. \tag{2.4}$$

We emphasize that the notations $e_\theta$, $e_z$ and $e_r$ are borrowed from the notations of the standard cylindrical coordinate frame $\partial_r$, $\frac{1}{r}\partial_\theta$ and $\partial_z$ for axi-symmetric velocity field about the $z$-axis. This is a good choice of notation, since one can imagine that the representative streamline $\gamma_\eta$ plays a similar role as the axi-symmetric axis provided $\gamma_\eta$ is *relatively straight*. Next, in order to describe the increasing swirl of $u$ towards the ending cross section $A(S) = A(1,S)$ of the stream-tube segment $T^A_{[0,S)}$, we will now decompose $\frac{u}{|u|}$ into its radial component, $z$-component, and swirl component as in the following definition.

Definition 2.5. *(Decomposition of normalized streamline.) Let $\omega_\theta$, $\omega_r$ and $\omega_z$ be s.t.*

$$\frac{u}{|u|}(x) = \omega_\theta(x)e_\theta(x) + \omega_r(x)e_r(x) + \omega_z(x)e_z(x). \tag{2.5}$$

Remark 2.6. We see that $\omega_\theta^2+\omega_r^2+\omega_z^2=1$ and $\omega_z(x)\to 1$ $(x\to\gamma_\eta(s))$ if $u$ is smooth.

At a glance, $\omega_\theta$, $\omega_r$ and $\omega_z$ do not give us any information about the size of $|u|$. Nevertheless, the balance of $\omega_\theta$, $\omega_r$ and $\omega_z$ will tell us something about the size of $|u|$. In particular, $|u|$ is proportional to $1/|\omega_z|$. Roughly speaking, the successive increase in the twisting of the streamlines will eventually lead to $|\omega_\theta|\to 1$. It means that $|\omega_z|$ tends to zero and thus $|u|$ becomes bigger and bigger (see Remark 2.9).

In order to give a model of possible blow-up situation, we need to define "uniform bundle" as follows:

Definition 2.7. *We call that "the stream-tube segment $T^A_{[0,S)}$ has a uniform bundle" if the following two properties hold*

- *For any $B(0)\subset A(0)$ and any $s\in[0,S]$, we have $C^{-1}\leqslant\frac{|B(s)|}{|B(0)|}\leqslant C$, for some universal constant $C>0$. Here, $B(s)$ is defined in the same way as $A(r,s)$ through replacing $A_0(r)$ by $B(0)$ in Definition 2.3.*
- *For the same universal constant $C>0$, we have $\sup_{y\in A(0)}u\cdot e_z(y)\leq C\inf_{y\in A(0)}u\cdot e_z(y)$.*

Remark 2.8. Since $\int_{B(0)}u\cdot e_z(y)d\sigma_y=\int_{B(s)}u\cdot e_z(y)d\sigma_y$ by divergence free, we see $u\cdot e_z(x)=\lim_{B(s)\ni x}\frac{1}{|B(s)|}\int_{B(s)}u\cdot e_z(y)d\sigma_y\approx\lim_{B(0)\ni x'}\frac{1}{|B(0)|}\int_{B(0)}u\cdot e_z(y)d\sigma_y=u\cdot e_z(x')$ for any two points $x\in A(s)$ and $x'\in A(0)$ connected by a streamline passing through $A(0)$ and $A(s)$ , if $A(s)$ has a uniformly bundle.

Remark 2.9. If $A(s)$ has a uniformly bundle, we can see from divergence free condition

$$\int_{B(0)}u\cdot e_z(y)d\sigma_y=\int_{B(s)}u\cdot e_z(y)d\sigma_y\approx|B(s)||u\cdot e_z|=|B(s)||u|\omega_z \tag{2.6}$$

and then

$$\frac{\int_{B(0)}u\cdot e_z(y)d\sigma_y}{|B(s)||\omega_z(x)|}\approx|u(x)|\quad\text{for}\quad x\in B(s)\subset A(s). \tag{2.7}$$

Now, we want to characterize the properties $u\in L^2$, $u\notin L^\alpha$ (for some given $2<\alpha<3$) and $\operatorname{div}\frac{u}{|u|}\notin L^6$ in terms of some conditions specifying *how fast the streamlines are increasing their swirl towards the ending cross section $A(s)$ of the stream-tube segment $T^A_{[0,S)}$*.

To specify the increasing swirl of streamlines towards the ending cross section $A(S)$ of the stream-tube segment $T^A_{[0,S)}$, we need to decompose each stream-plane $A(s)$ into the disjoint union of a countable list of ring-shaped regions $A_j(s)$ as follow. We first select a decreasing sequence of positive numbers $\{r_j\}_{j=1}^\infty$ dropping down to $0(r_j\searrow 0)$ as $j\to\infty$. We then set $A_j(s):=A(r_j,s)\setminus A(r_{j+1},s)$. Notice that $A_j(s)$ is shrinking towards the representative streamline $\gamma_\eta$ as $j$ becomes large. We also set

$$\omega_z^{A_j}(s):=\int_{A_j(s)}\omega_z(y)d\sigma_y/|A_j(s)|. \tag{2.8}$$

That is, $\omega_z^{A_j}(s)$ is the average of $\omega_z$ over the ring-shaped region $A_j(s)$ in the stream-plane $A(s)$. We can assume, according to Definition 2.7, that

$$|A_j(s)| \approx |A_j(0)| \quad \text{for} \quad s \in (0, S]. \tag{2.9}$$

Since we require that $u$ blows up at the isolated singular point $\gamma_\eta(S)$ lying in $A(S)$, in light of condition (2.9) and (2.7), we would require that, as $s$ becomes close to $S$, $\omega_z^{A_j}(s)$ should become small as $j$ becomes large, which indicates that the average swirl (or twisting) of those streamlines passing through $A_j(s)$ should become large as $s \to S$ and $j \to \infty$.

Now, in order to ensure that $u \in L^2$, we impose $(S-s)^{\frac{1}{2}-\epsilon}$ as the lower bound for $\omega_z$ as follow.

Definition 2.10. *( The condition to ensure $u \in L^2$ ) For any $0 < s < S$, we have $(S-s)^{\frac{1}{2}-\epsilon} \leqslant \omega_z(\gamma_{\eta'}(s)) < 1$ for any $\eta' \in A(0)$.*

The purpose of the above condition is to prevent the swirl of streamlines passing through $A_j(s)$ to become *too large* as $s \to S$ and $j \to \infty$, because we want to have the finite energy property for $u$. Under the condition $(S-s)^{\frac{1}{2}-\epsilon} < \omega_z(\gamma_{\eta'}(s)) < 1$, a direct calculation yields the finite $L^2$ property of $u$ as follow.

$$\begin{aligned} \|u\|_{L^2}^2 \approx \|u\|_{L^2(T_{[0,S)}^A)}^2 \approx \int_{T_{[0,S)}^A} \left|\frac{\omega_z^A(0)}{\omega_z(x)}\right|^2 dx \approx \int_0^S \int_A \left|\frac{\omega_z^A(0)}{\omega_z(\gamma_\eta(s))}\right|^2 d\eta ds \\ \leqslant \int_0^S \frac{C}{(S-s)^{1-2\epsilon}} ds < \infty, \end{aligned} \tag{2.10}$$

where

$$\omega_z^A(s) := \int_{A(s)} \omega_z(y) d\sigma_y / |A(s)|$$

as defined in (2.8). The second approximation in (2.10) follows from (2.7), which gives

$$|u(x)| \approx \frac{|\omega_z^A(s)|}{|B(s)||\omega_z(x)|}$$

together with $B(s) \approx B(0)$ and $\omega_z^A(s) \approx \omega_z^A(0)$ (see Definition 2.7 and Remark 2.9).

In order to ensure that $u \notin L^\alpha$, we impose $(S-s)^{\frac{1}{\alpha}}$ as the upper bound for $\omega_z$ as follow.

Definition 2.11. *(The condition to ensure $u \notin L^\alpha$) Let $\{S_j\}_j \subset [0, S)$ be s.t. $S_j \to S$ $(j \to \infty)$ and*

$$|A_j(0)| \int_0^{S_j} (S-s)^{-1} ds \geqslant C.$$

*For any $0 < s < S_j$, we have*

$$|\omega_z(\gamma_{\eta'}(s))| \leq (S-s)^{1/\alpha} \quad \text{for} \quad \eta' \in A_j(0).$$

We show $u \notin L^\alpha$. By Remark 2.6, we see $\omega^{A_j}(0) \approx 1$. Thus

$$\|u\|_{L^\alpha}^\alpha \approx \|u\|_{L^\alpha(T_{[0,S)}^A)}^\alpha \approx \int_{T_{[0,S)}^A} \left|\frac{\omega_z^A(0)}{\omega_z(x)}\right|^\alpha dx \tag{2.11}$$

$$\geqslant \sum_j \int_0^{S_j} \int_{A_j} \left|\frac{\omega_z^{A_j}(0)}{\omega_z(\gamma_\eta(s))}\right|^\alpha d\eta ds \geqslant \sum_j \int_0^{S_j} \frac{|A_j(0)|}{(S-s)} ds = \infty. \tag{2.12}$$

In order to show $\left\|\operatorname{div}\frac{u}{|u|}\right\|_{L^6}=\infty$, we impose $(S-s)^{-1}$ as the upper bound for $|\partial_s\omega_z^{A_j}(s)|^6$ as follow.

DEFINITION 2.12. (*The condition to ensure div$(u/|u|)\notin L^6$*) *Let* $\{\tilde{S}_j\}_j\subset[0,S)$ *be s.t.* $S_j<\tilde{S}_j<S$ *and*

$$|A_j(0)|\int_{S_j}^{\tilde{S}_j}(S-s)^{-1}ds\geqslant C,$$

*where* $C$ *is a universal constant. For any* $S_j<s<\tilde{S}_j$*, we have*

$$|\partial_s\omega_z^{A_j}(s)|^6>(S-s)^{-1}.$$

REMARK 2.13. There exists $\omega_z$ which satisfies the above three conditions. In fact, we can choose $\omega_z$ in order to satisfy $\omega(\gamma_{\eta'}(s))=(S-s)^{1/2-\epsilon}$ and $\omega_z^{A_j}(s)=(S-s)^{1/2-\epsilon}$ for $\eta'\in A_j(0)$, $s\in[0,\tilde{S}_j)$ and $j=1,2\cdots$.

We need to get a *rough* expression of $\frac{1}{A_j(s)}\int_{A_j(s)}\operatorname{div}(\frac{u}{|u|})dy$ as follow. Let $s>0$ be fixed. Then, for any $s_1>s$ to be sufficiently close to $s$, we consider the following stream-tube $T^{A_j}_{[s,s_1]}$ connecting the stream-plane $A_j(s)$ to $A_j(s_1)$.

$$T^{A_j}_{[s,s_1]}=\bigcup_{s\leqslant\tau\leqslant s_1}A_j(s). \tag{2.13}$$

From Definition 2.3, we can view the stream-tube $T^{A_j}_{[s,s_1]}$ as being formed by the union of those streamlines which first pass into the stream-tube through the cross section $A_j(s)$ and eventually leave the same stream-tube through the cross section $A_j(s_1)$. Since $s_1$ is chosen to be close to $s$, the stream-tube $T^{A_j}_{[s,s_1]}$ is roughly the same as the product $A_j(s)\times[s,s_1]$, which, together with condition (2.9), makes the following deduction justifiable.

$$\begin{aligned}
\frac{1}{A_j(s)}\int_{A_j(s)}\operatorname{div}(\frac{u}{|u|})dy &= \lim_{s_1\to s}\frac{1}{(s_1-s)}\int_s^{s_1}\frac{1}{|A_j(\tau)|}\int_{A_j(\tau)}\operatorname{div}(\frac{u}{|u|})dyd\tau\\
&\approx \lim_{s_1\to s}\frac{1}{(s_1-s)|A_j(s)|}\int_{T^{A_j}_{[s,s_1]}}\operatorname{div}(\frac{u}{|u|})dy\\
&= \lim_{s_1\to s}\frac{1}{|A_j(s)|(s_1-s)}\{\int_{A_j(s_1)}\frac{u}{|u|}\cdot e_z d\sigma-\int_{A_j(s)}\frac{u}{|u|}\cdot e_z d\sigma\}\\
&= \lim_{s_1\to s}\frac{1}{|A_j(s)|(s_1-s)}\{\int_{A_j(s_1)}\omega_z d\sigma-\int_{A_j(s)}\omega_z d\sigma\}\\
&= \lim_{s_1\to s}\frac{1}{|A_j(s)|(s_1-s)}\left(\omega_z^{A_j}(s_1)|A_j(s_1)|-\omega_z^{A_j}(s)|A_j(s)|\right)\\
&= \frac{1}{|A_j(s)|}\partial_s\{\omega_z^{A_j}(s)|A_j(s)|\}=\left(\partial_s\omega_z^{A_j}(s)+\frac{\partial_s|A_j(s)|}{|A_j(s)|}\omega_z^{A_j}(s)\right).
\end{aligned}$$

Hence, it follows from the above calculation and an application of Holder inequality that

$$\begin{aligned}\int_{A_j(s)} |\operatorname{div}(\frac{u}{|u|})|^6 &\geqslant \frac{1}{|A_j(s)|^5}\left|\int_{A_j(s)} \operatorname{div}(\frac{u}{|u|})\right|^6 \\ &\approx |A_j(s)|\left(\partial_s\omega_z^{A_j}(s) + \frac{\partial_s|A_j(s)|}{|A_j(s)|}\omega_z^{A_j}(s)\right)^6 \\ &\approx |A_j(0)|\left(\partial_s\omega_z^{A_j}(s) + \frac{\partial_s|A_j(s)|}{|A_j(s)|}\omega_z^{A_j}(s)\right)^6 \\ &\geq |A_j(0)||\partial_s\omega_z^{A_j}(s)|^6.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\operatorname{div}\frac{u}{|u|}\right\|^6_{L^6(T^A_{[0,S)})} \approx \sum_j \int_{[0,S)}\int_{A_j(s)} |\operatorname{div}(\frac{u}{|u|})|^6 &\geq \sum_j \int_{[S_j,\tilde{S}_j)} |A_j(0)||\partial_s\omega_z^{A_j}(s)|^6 ds \\ &\geqslant \sum_j C = \infty.\end{aligned}$$

Before we leave this section, let us summarize what we have done in the form of the following statement.

PROPOSITION 2.14. *Consider a locally defined smooth velocity field $u$ as given in a stream-tube around a representative streamline in the sense of Definitions 2.1, 2.2, and 2.3. Moreover, suppose further that such a locally defined velocity field $u$ satisfies the three conditions as stated in Definitions 2.10, 2.11, and 2.12. Then, it follows that within such a stream-tube region, the following properties of $u$ hold: $u \in L^2$, $u \notin L^\alpha$ for the same $\alpha \in (2,3)$ which appears in Definition 2.11, and div$\frac{u}{|u|} \notin L^6$.*

**3. Outline of the proof of Theorem 1.2.** The proof of Theorem 1.2 is quite similar to the one in [4]. The purpose of this section is just to outline those *crucial and important* changes which have to be made to the structure of the proof as presented in [4], so that the modified proof will be strong enough to give the result of Theorem 1.2. In other words, we will only state the essential changes to the main argument in [4] which are the new ideas contributed in this paper.

Just in the same way as [4], we will follow the parabolic De Giorgi's method developed by Vasseur in [17]. So, let us fix our notation as follow. We remark that, without the lost of generality, we will assume that the possible blow up time $T$ is just 1.

- for each $k \geqslant 0$, let $Q_k = [T_k, 1] \times \mathbb{R}^3$, in which $T_k = \frac{3}{4} - \frac{1}{4^{k+1}}$.
- for each $k \geqslant 0$, let $v_k = \{|u| - R(1 - \frac{1}{2^k})\}_+$.
- for each $k \geqslant 0$, let $w_k = \{|u| - R^\beta(1 - \frac{1}{2^k})\}_+$ , with $\beta > 1$ to be selected later.
- for each $k \geqslant 0$, let $d_k^2 = \frac{R(1-\frac{1}{2^k})}{|u|}\chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2$.
- for each $k \geqslant 0$, let $D_k^2 = \frac{R^\beta(1-\frac{1}{2^k})}{|u|}\chi_{\{|u|>R^\beta(1-\frac{1}{2^k})\}}|\nabla|u||^2 + \frac{w_k}{|u|}|\nabla u|^2$.
- for each $k \geqslant 0$, let $U_k = \frac{1}{2}\|v_k\|^2_{L^\infty(T_k,1;L^2(\mathbb{R}^3))} + \int_{T_k}^1\int_{\mathbb{R}^3} d_k^2 dx\, dt$.

With the above setting, the first author proved the following proposition (see [4]).

Proposition 3.1. *Let $u$ be a suitable weak solution for the Navier-Stokes equation on $[0,1]\times\mathbb{R}^3$ which satisfies the condition that $|\frac{u\cdot\nabla F}{|u|^\gamma}|\leqslant A|F|$, where $A$ is some finite-positive constant, and $\gamma$ is some positive number satisfying $0<\gamma<\frac{1}{3}$. Then, there exists some constant $C_{p,\beta}$, depending only on $1<p<\frac{5}{4}$, and $\beta>\frac{6-3p}{10-8p}$,and also some constants $0<\alpha, K<\infty$, which do depend on our suitable weak solution $u$, such that the following inequality holds*

$$\begin{aligned} U_k \leqslant & C_{p,\beta}2^{\frac{10k}{3}}\{\frac{1}{R^{\beta\frac{10-8p}{3p}-\frac{2-p}{p}}}\|u\|^{2(1-\frac{1}{p})}_{L^\infty(0,1;L^2(\mathbb{R}^3))}U_{k-1}^{\frac{5-p}{3p}}+ \\ & (1+A)(1+\frac{1}{\alpha})(1+K^{1-\frac{1}{p}})(1+\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))})\times \\ & [(\frac{1}{R^{\frac{10}{3}-2p\beta+1-\gamma-p}})^{\frac{1}{p}}U_{k-1}^{\frac{5}{3p}}+\frac{1}{R^{\frac{10}{3}-2\beta-\gamma}}U_{k-1}^{\frac{5}{3}}]\}, \end{aligned} \tag{3.1}$$

*for every sufficiently large $R>1$.*

The nonlinear recurrence relation as given in (3.1) was indeed the main cornerstone leading to the regularity criterion in [4]. More precisely, the structure of (3.1) directly gives the smallness of $U_1$ as long as $R$ is sufficiently large. The smallness of $U_1$, together with the nonlinear recurrence structure of relation (3.1), then allowed us to deduce in [4] the decay of $U_k$ to 0 (as $k\to\infty$) by means of the following useful lemma as appeared in [17].

Lemma 3.2. *For any given constants $B$, $b>1$, there exists some constant $C_0^*$ such that for any sequence $\{a_k\}_{k\geqslant 1}$ satisfying $0\leqslant a_1\leqslant C_0^*$ and $0\leqslant a_k\leqslant B^k a_{k-1}^b$, for any $k\geqslant 1$, we have $lim_{k\to\infty}a_k=0$.*

The resulting decay of $U_k$ to 0 as $k\to\infty$ allowed the first author to draw the conclusion that $u$ is essentially bounded by some sufficiently large constant $R>1$ over $[\frac{3}{4},1)\times\mathbb{R}^3$, and this lead to the following theorem in [4].

Theorem 3.3. *Let $u:[0,T)\times\mathbb{R}^3\to\mathbb{R}^3$ be a Leray-Hopf solution to* (1.1) *which is smooth on $[0,T)\times\mathbb{R}^3$ (with $T$ to be the possible blow up time) and which satisfies the condition that $|\frac{u\cdot\nabla F}{|u|^\gamma}|\leqslant A|F|$, in which $A$ is some positive constant, and $\gamma$ is some positive constant for which $0<\gamma<\frac{1}{3}$. Then, it follows that the $u$ is $L^\infty$-bounded on $[\frac{3}{4},1)\times\mathbb{R}^3$ and hence the smoothness of $u$ can be extended beyond $T$.*

In this paper we will refine the $\gamma$ in Theorem 3.3 to be 1. As indicated in the introduction, the problem we face here is that those powers of $U_{k-1}$ such as $\frac{5-p}{3p}$, $\frac{5}{3p}$ and $\frac{5}{3}$ (appearing in Proposition 3.1), are *too far* from 1. However, the use of Lemma 3.2 only requires that $\beta>1$, so the extra condition $u\in L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))$, with $\alpha\in(2,3)$ satisfying $1+2(\frac{\alpha}{3}-\frac{3}{\alpha})>0$ can help us to bring the powers of $U_{k-1}$ to become very close to 1, and this in turn allows us to replace the old condition $|\frac{u\cdot\nabla F}{|u|^\gamma}|\leq A|F|$ with $\gamma\in(0,\frac{1}{3})$ by the new one $|\frac{u\cdot\nabla F}{|u|}|\leq A|F|$.

Technically speaking, the key idea which allows us to use the condition $u\in L^\infty(L^{\alpha,\infty})$ (with $\alpha\in(2,3)$ satisfying $1+2(\frac{\alpha}{3}-\frac{3}{\alpha})>0$) to lower down the powers of $U_{k-1}$ to become close to 1 is the following lemma. We can establish the following lemma for any truncations $w_{k-1}=(|u|-R^\beta(1-\frac{1}{2^{k-1}}))_+$ (with $k\geq 2$) of a Leray-Hopf solution $u\in L^\infty(0,1;L^2(\mathbb{R}^3))\cap L^2(0,1;\dot{H}^1(\mathbb{R}^3))$ satisfying the condition $u\in L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))$ for some given $\alpha\in(2,3)$.

Lemma 3.4. *Consider a Leray-Hopf weak solution $u\in L^\infty(0,1;L^2(\mathbb{R}^3))\cap L^2(0,1;\dot{H}^1(\mathbb{R}^3))$ which satisfies the condition $u\in L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))$ for some given*

$\alpha \in (2,3)$. *Then, the truncation* $w_{k-1} = (|u| - R^\beta(1 - \frac{1}{2^{k-1}}))_+$ *of* $|u|$ *satisfies the following inequality for each* $k \geqslant 2$ *and each* $\delta$ *with* $0 < \delta < \frac{4}{3}$.

$$\int_{Q_{k-1}} w_{k-1}^{\frac{10}{3}} \leqslant C_0\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))}\}^{\frac{2}{3}-\delta}\frac{U_{k-1}^{1+\delta}}{R^{\beta(\alpha-2)(\frac{2}{3}-\delta)}}, \tag{3.2}$$

*in which* $C_0$ *is a universal constant essentially arising from the Sobolev embedding theorem. In the same way, the truncation* $v_k = (|u| - R(1 - \frac{1}{2^k}))_+$ *also satisfies the following inequality for each* $k \geqslant 2$ *and each* $\delta$ *with* $0 < \delta < \frac{4}{3}$.

$$\int_{Q_{k-1}} v_{k-1}^{\frac{10}{3}} \leqslant C_0\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))}\}^{\frac{2}{3}-\delta}\frac{U_{k-1}^{1+\delta}}{R^{(\alpha-2)(\frac{2}{3}-\delta)}}. \tag{3.3}$$

*Proof.* To begin, let $u \in L^\infty(0,1;L^2(\mathbb{R}^3)) \cap L^2(0,1;\dot{H}^1(\mathbb{R}^3))$ to be a Leray-Hopf solution which satisfies the condition $u \in L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))$ for some given $\alpha$ with $2 < \alpha < 3$. Recall that the truncation $w_{k-1} = (|u| - R^\beta(1 - \frac{1}{2^{k-1}}))_+$ satisfies the property that $|\nabla w_{k-1}| \leqslant D_{k-1} \leqslant 5^{\frac{1}{2}} d_{k-1}$, for $k \geqslant 2$ (The relation $|\nabla w_{k-1}| \leqslant D_{k-1}$ can be verified easily, while the relation $D_{k-1} \leqslant 5^{\frac{1}{2}} d_{k-1}$ was justified in Lemma 4.1 of [4]). So, it follows from standard interpolation inequality that

$$\begin{aligned}
\int_{Q_{k-1}} w_{k-1}^{\frac{10}{3}} &\leqslant C_0\|w_{k-1}\|^{\frac{4}{3}}_{L^\infty(T_{k-1},1;L^2(\mathbb{R}^3))}\|\nabla w_{k-1}\|^2_{L^2(Q_{k-1})} \\
&\leqslant C_0\{\sup_{t\in[T_{k-1},1]}\int_{\mathbb{R}^3} w_{k-1}^2(t,x)dx\}^{\frac{2}{3}}U_{k-1} \\
&\leqslant C_0 U_{k-1}^{1+\delta}\{\sup_{t\in[T_{k-1},1]}\int_{\mathbb{R}^3} w_{k-1}^2(t,x)dx\}^{\frac{2}{3}-\delta}.
\end{aligned} \tag{3.4}$$

But according to the assumption that $u \in L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))$, we can control $\int_{\mathbb{R}^3} w_{k-1}^2(t,x)dx$ (for each $k \geqslant 2$) uniformlly over $t \in [0,1]$ as follow.

$$\begin{aligned}
\int_{\mathbb{R}^3} w_{k-1}^2(t,x)dx &= 2\int_0^\infty r|\{x\in\mathbb{R}^3 : w_{k-1}(t,x) > r\}|dr \\
&\leqslant 2\int_0^\infty r|\{x\in\mathbb{R}^3 : |u(t,x)| > r + R^\beta(1-\frac{1}{2^{k-1}})\}|dr \\
&\leqslant 2\int_0^\infty (r+\frac{R^\beta}{2})|\{x\in\mathbb{R}^3 : |u(t,x)| > r + \frac{R^\beta}{2}\}|dr \\
&= 2\int_{\frac{R^\beta}{2}}^\infty r|\{x\in\mathbb{R}^3 : |u(t,x)| > r\}|dr \\
&\leq 2\|u\|_{L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))}\int_{\frac{R^\beta}{2}}^\infty r^{1-\alpha}dr \\
&= \frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))}\frac{1}{R^{\beta(\alpha-2)}}.
\end{aligned} \tag{3.5}$$

Hence, inequality (3.2) follows from the above two inequality estimations. By the same way, we can also derive inequality (3.3) by replacing $w_k$ by $v_k = (|u| - R(1-\frac{1}{2^k}))$ and $R^\beta$ by $R$. □

As a corollary of Lemma 3.4, we have the following result which allows us to raise up the index for the terms $\|\chi_{\{w_k>0\}}\|_{L^q(Q_{k-1})}$ and $\|\chi_{\{v_k>0\}}\|_{L^q(Q_{k-1})}$.

LEMMA 3.5. *Suppose that the given suitable weak solution* $u : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ *satisfies the condition* $u \in L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))$ *for some given* $\alpha\in(2,3)$*. Then, for any* $1<q<\infty$*, and any* $1<\delta<\frac{4}{3}$*, we have*

$$\|\chi_{\{w_k>0\}}\|_{L^q(Q_{k-1})} \leqslant C_{(\alpha,\delta,q)} \frac{2^{\frac{10k}{3q}}}{R^{\frac{1}{q}[\frac{10\beta}{3}+\beta(\alpha-2)(\frac{2}{3}-\delta)]}} \cdot \|u\|^{(\frac{2}{3}-\delta)\frac{1}{q}}_{L^\infty(L^{\alpha,\infty})} U^{(1+\delta)\frac{1}{q}}_{k-1}, \qquad (3.6)$$

*in which the constant* $C_{(\alpha,\delta,q)}$ *is given by* $C_{(\alpha,\delta,q)} = C_0^{\frac{1}{q}}[\frac{2^{\alpha-1}}{(\alpha-2)}]^{(\frac{2}{3}-\delta)\frac{1}{q}}$*, with* $C_0$ *to be a universal constant arising from the Sobolev embedding theorem and standard interpolation.*

*In the same way, we have the following estimate for* $\|\chi_{\{v_k>0\}}\|_{L^q(Q_{k-1})}$*, with* $1<q<\infty$ *and* $1<\delta<\frac{4}{3}$*.*

$$\|\chi_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \leqslant C_{(\alpha,\delta,q)} \frac{2^{\frac{10k}{3q}}}{R^{\frac{1}{q}[\frac{10}{3}+(\alpha-2)(\frac{2}{3}-\delta)]}} \cdot \|u\|^{(\frac{2}{3}-\delta)\frac{1}{q}}_{L^\infty(L^{\alpha,\infty})} U^{(1+\delta)\frac{1}{q}}_{k-1}. \qquad (3.7)$$

REMARK. Notice that the constant $C_{(\alpha,\delta,q)}$ as appears in inequality (3.6) blows up to $\infty$ as the choice of $\alpha$ approaches to 2, which means that inequality (3.6) applies only in the case of $\alpha>2$. We also point out that replacing the old Lemma 3.2 and Lemma 3.3 in [4] by the above lemma (i.e. Lemma 3.5) is the *crucial* decision leading to the final success of our new proof of Theorem 1.2 (see the next section, in which we will give all the details of the new proof of Theorem 1.2).

*Proof.* We recall that the sequence of truncations $w_k$ is defined to be $w_k = (|u| - R^\beta(1-\frac{1}{2^k}))_+$. So, it is easy to see that $\{w_k>0\}\subset\{w_{k-1}>\frac{R^\beta}{2^k}\}$. Hence, it follows from inequality (3.2) that

$$\begin{aligned}\int_{Q_{k-1}} \chi_{\{w_k>0\}} &\leqslant \int_{Q_{k-1}} \chi_{\{w_{k-1}>\frac{R^\beta}{2^k}\}} \\ &\leqslant \frac{2^{\frac{10k}{3}}}{R^{\frac{10\beta}{3}}} \int_{Q_{k-1}} w_{k-1}^{\frac{10}{3}} \\ &\leqslant \frac{2^{\frac{10k}{3}}}{R^{\frac{10\beta}{3}}} \cdot C_0\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^\infty(0,1;L^{\alpha,\infty}(\mathbb{R}^3))}\}^{\frac{2}{3}-\delta} \frac{U_{k-1}^{1+\delta}}{R^{\beta(\alpha-2)(\frac{2}{3}-\delta)}}.\end{aligned} \qquad (3.8)$$

Hence, inequality (3.6) follows from taking the power $\frac{1}{q}$ on both sides of the above inequality. The deduction of inequality (3.7) follows in the same way. □

In order to adopt to the new hypothesis $|u\cdot\nabla F| \leqslant A|u|\cdot|F|$ on $\{(t,x)\in[0,1)\times B(r_0) : |F(t,x)|\geqslant L\}$ (for some given constant $L>0$) , the second refinement is on the function $\psi$ appearing in Step five of the proof in [4]. We *redefine* the function $\psi:\mathbb{R}\to\mathbb{R}$ as the one which satisfies the following conditions

- $\psi(t)=1$, for all $t\geqslant L+1$.
- $0<\psi(t)<1$, for all $t$ with $L<t<L+1$.
- $\psi(t)=0$, for all $-L\leqslant t\leqslant L$ .
- $-1<\psi(t)<0$, for all $t$ with $-L-1<t<-L$.

- $\psi(t) = -1$, for all $t \leqslant -L-1$.
- $0 \leqslant \frac{d}{dt}\psi \leqslant 2$, for all $t \in \mathbb{R}$.

We further remark that the smooth function $\psi : \mathbb{R} \to \mathbb{R}$ characterized by the above properties must also satisfy the property that $\frac{d\psi}{dt}|_{(t)} = 0$, on $t \in (-\infty, -L-1) \cup (-L, L) \cup (L+1, \infty)$.

Up to this point, we have already spelled out *all* the important changes that have to be made to the old argument in [4]. In the next section, we will redo the old argument in [4] by including all those important changes given here, and see the way in which the modified new argument will lead to the result of Theorem 1.2.

**4. Appendix: Technical steps of the proof of Theorem 1.2.** The purpose of this section is to convince the readers of the correctness of the outline in the previous section through giving all the technical details of the proof of Theorem 1.2. Except those *crucial and important* changes as given in the outline of the previous section, the structure of the proof of theorem 1.2 is in many aspects the same as the one in [4]. It is also not surprising that some of the technical aspects of the proof of Theorem 1.2 as given below are directly transported (or copied) from that of [4] (This is justified for those parts to which no change is necessary). So, in a certain sense, all the new ideas of the proof of Theorem 1.2 has already been given in the outline of the previous section, and we spell out all the details of the proof of Theorem 1.2 here only for the sake of completeness. Moreover, we remark that, within this section, the definitions of $T_k$, $Q_k$, $v_k$, $w_k$ $d_k$ etc were given in the beginning of Section 3. Moreover, the possible finite blow up time for the solution $u; [0,1) \times \mathbb{R}^3 \to \mathbb{R}^3$ under consideration is assumed to be 1.
We will begin our discussion by stating and proving the following technical lemma, from which we can easily derive the conclusion of Theorem 1.2 in the final part of this section.

LEMMA 4.1. *Consider $u : [0,1) \times \mathbb{R}^3 \to \mathbb{R}^3$ to be a Leray-Hopf solution which is smooth up to the possible blow up time $T = 1$, and which also satisfies all the hypothesis of Theorem 1.2. Let $\{U_k\}_{k=0}^{\infty}$ to be the sequence of truncated energies associated to $u$, which is defined precisely as in the beginning of section 3. Then, for any fixed choice of positive constant $\beta \in (\frac{3}{\alpha}, \frac{1}{2} + \frac{\alpha}{3})$ (whose existence is ensured by the condition $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$ in the hypothesis of Theorem 1.2), the following nonlinear recurrence relation of $\{U_k\}_{k=0}^{\infty}$ holds for any positive parameter $p > 1$ which is sufficiently close to 1, and any positive parameter $\delta > 0$ which is sufficiently close to 0.*

$$
\begin{aligned}
U_k \leqslant\ & \frac{2^{\frac{10k}{3}}}{R^{\frac{4}{3}}} C_0 U_{k-1}^{\frac{5}{3}} + C(\beta, A, L, p, \delta, \|u\|_{L^\infty L^2}, \|u\|_{L^\infty L^{\alpha,\infty}}) 2^{\frac{10k}{3}} \\
& \times \{ \frac{U_{k-1}^{\frac{1}{p} + \delta(\frac{2-p}{2p})}}{R^{\beta[\frac{10-8p}{3p} + (\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)] - (\frac{2-p}{p})}} \\
& + (\frac{U_{k-1}^{(1+\delta)}}{R^{\frac{10}{3} - 2p\beta + (\alpha-2)(\frac{2}{3}-\delta) - p}})^{\frac{1}{p}} + \frac{U_{k-1}^{1+\delta}}{R^{\frac{10}{3} - 2\beta + (\alpha-2)(\frac{2}{3}-\delta) - 1}} \},
\end{aligned} \tag{4.1}
$$

*where the constant $C_0$ is basically the one arising from the standard Sobolev embedding theorem from $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, while the constant $C(\beta, A, L, p, \delta, \|u\|_{L^\infty L^2}, \|u\|_{L^\infty L^{\alpha,\infty}})$ is the one which depends only on $\beta$ ,$A$ ,$L$*

*, p, δ , $\|u\|_{L^\infty L^2}, \|u\|_{L^\infty L^{\alpha,\infty}}$ (recall that A, and L are the two constants as given in the hypothesis of Theorem 1.2).*

In that which follows, we will give a detailed proof of Lemma 4.1, which by itself will be splitted into six different steps (from Step1 to Step6).

*Proof.*

**Step one.** To begin the argument, we notice that the truncations $v_k = \{|u| - R(1 - \frac{1}{2^k})\}$ of the solution $u : [0,1) \times \mathbb{R}^3 \to \mathbb{R}^3$ as considered in Theorem 1.2 satisfy the following equality for every point $(t,x) \in [0,1) \times \mathbb{R}^3$.

$$\partial_t(\frac{v_k^2}{2}) + d_k^2 - \triangle(\frac{v_k^2}{2}) + \mathrm{div}(\frac{v_k^2}{2}u) + \frac{v_k}{|u|}u\nabla P \leqslant 0. \tag{4.2}$$

We note that there is no difficulty in justifying the validity of the above equality for every point $(t,x) \in [0,1) \times \mathbb{R}^3$. this is simply because the Leray-Hopf solution $u : [0,1) \times \mathbb{R}^3 \to \mathbb{R}^3$ as considered in Theorem 1.2 is assumed to be smooth on $[0,1) \times \mathbb{R}^3$, even though $t = 1$ is the possible blow up time.

Next, let us consider the variables $\sigma$ , $t$ verifying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. Then, we have

- $\int_\sigma^t \int_{\mathbb{R}^3} \partial_t(\frac{v_k^2}{2})dx\,ds = \int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx - \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2}dx.$
- $\int_\sigma^t \int_{\mathbb{R}^3} \triangle(\frac{v_k^2}{2})dx\,ds = 0.$
- $\int_\sigma^t \int_{\mathbb{R}^3} div(\frac{v_k^2}{2}u)dx\,ds = 0.$

So, it is straightforward to see that

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx + \int_\sigma^t \int_{\mathbb{R}^3} d_k^2 dx\,ds \leqslant \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2}dx + \int_\sigma^t |\int_{\mathbb{R}^3} \frac{v_k}{|u|}u\nabla P dx|ds,$$

for any $\sigma$, $t$ satisfying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. By taking the average over the variable $\sigma$, we yield

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx + \int_{T_k}^t \int_{\mathbb{R}^3} d_k^2 dx\,ds \leqslant \frac{4^{k+1}}{6}\int_{T_{k-1}}^{T_k}\int_{\mathbb{R}^3} v_k^2(s,x)dx\,ds + \int_{T_{k-1}}^t |\int_{\mathbb{R}^3} \frac{v_k}{|u|}u\nabla P dx|ds.$$

By taking the sup over $t \in [T_k, 1]$. the above inequality will give the following

$$U_k \leqslant \frac{4^{k+1}}{6}\int_{Q_{k-1}} v_k^2 + \int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|}u\nabla P dx|ds.$$

But, by using the interpolation inequality $\|f\|_{L^{\frac{10}{3}}(Q_k)} \leqslant \|f\|_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}^{\frac{2}{5}} \|\nabla f\|_{L^2(Q_k)}^{\frac{3}{5}}$ (see Lemma 3.1 of [4] or [17]) and the inequality $\|\chi_{v_k>0}\|_{L^q(Q_{k-1})} \leqslant (\frac{2^k}{R})^{\frac{10}{3q}} C^{\frac{1}{q}} U_{k-1}^{\frac{5}{3q}}$ (see Lemma 3.2 of [4] or [17]), we can carry out the following estimate.

$$\begin{aligned}
\int_{Q_{k-1}} v_k^2 &= \int_{Q_{k-1}} v_k^2 \chi_{\{v_k>0\}} \\
&\leqslant (\int_{Q_{k-1}} v_k^{\frac{10}{3}})^{\frac{3}{5}} \|\chi_{\{v_k>0\}}\|_{L^{\frac{5}{2}}(Q_{k-1})} \\
&\leqslant \|v_k\|^2_{L^{\frac{10}{3}}(Q_{k-1})} \frac{2^{\frac{4k}{3}}}{R^{\frac{4}{3}}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant \|v_{k-1}\|^2_{L^{\frac{10}{3}}(Q_{k-1})} \frac{2^{\frac{4k}{3}}}{R^{\frac{4}{3}}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant C U_{k-1}^{\frac{5}{3}} \frac{2^{\frac{4k}{3}}}{R^{\frac{4}{3}}}.
\end{aligned}$$

As a result, we have the following conclusion

$$U_k \leqslant \frac{2^{\frac{10k}{3}}}{R^{\frac{4}{3}}} C U_{k-1}^{\frac{5}{3}} + \int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla p dx| ds. \tag{4.3}$$

**Step two.** Now, in order to estimate the term $\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx| ds$, we would like to carry out the following computation

$$\begin{aligned}
-\triangle P &= \sum \partial_i \partial_j (u_i u_j) \\
&= \sum \partial_i \partial_j \{(1 - \frac{w_k}{|u|}) u_i (1 - \frac{w_k}{|u|}) u_j\} + 2 \sum \partial_i \partial_j \{(1 - \frac{w_k}{|u|}) u_i \frac{w_k}{|u|} u_j\} \\
&+ \sum \partial_i \partial_j \{\frac{w_k}{|u|} u_i \frac{w_k}{|u|} u_j\},
\end{aligned}$$

in which $w_k$ is given by $w_k = \{|u| - R^\beta (1 - \frac{1}{2^k})\}_+$, and $\beta > 1$ is some arbitrary index which will be determined later. This motivates us to decompose $P$ as $P = P_{k1} + P_{k2} + P_{k3}$, in which

$$-\triangle P_{k1} = \sum \partial_i \partial_j \{(1 - \frac{w_k}{|u|}) u_i (1 - \frac{w_k}{|u|}) u_j\}, \tag{4.4}$$

$$-\triangle P_{k2} = \sum \partial_i \partial_j \{2(1 - \frac{w_k}{|u|}) u_i \frac{w_k}{|u|} u_j\} \tag{4.5}$$

$$-\triangle P_{k3} = \sum \partial_i \partial_j \{\frac{w_k}{|u|} u_i \frac{w_k}{|u|} u_j\}. \tag{4.6}$$

Here, we have to remind ourself that the cutting functions which are used in the decomposition of the pressure are indeed $w_k = \{|u| - R^\beta (1 - \frac{1}{2^k})\}_+$, for all $k \geqslant 0$ , in which $\beta$ is some suitable index strictly greater than 1. With respect to the cutting functions $w_k$, we need to define the respective $D_k$ as follow:

$$D_k^2 = \frac{R^\beta (1 - \frac{1}{2^k})}{|u|} \chi_{\{w_k>0\}} |\nabla |u||^2 + \frac{w_k}{|u|} |\nabla u|^2.$$

Then, just like what happens to the cutting functions $v_k$, we have the following assertions about the cutting functions $w_k$, which are easily verified (see [17]).

- $|\nabla w_k| \leqslant D_k$, for all $k \geqslant 0$.
- $|\nabla(\frac{w_k}{|u|}u_i)| \leqslant 3D_k$, for all $k \geqslant 0$, and $1 \leqslant i \leqslant 3$.
- $|\nabla(\frac{w_k}{|u|})u_i| \leqslant 2D_k$, for any $k \geqslant 0$, and $1 \leqslant i \leqslant 3$.
- $D_k \leq 5^{\frac{1}{2}} d_k$ as long as $R$ is larger than some fixed constant $R_0$ (see Lemma 4.1 of [4] for a proof of this).

Now, let us recall that we have already used the cutting functions $w_k$ to obtain the decomposition $P = P_{k1} + P_{k2} + P_{k3}$, in which $P_{k1}$, $P_{k2}$, and$P_{k3}$ are described in equations (4.4), (4.5), and (4.6) respectively.
Due to the incompressible condition $\operatorname{div}(u) = 0$, we have the following two identities

- $\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P_{k2} dx = \int_{\mathbb{R}^3} (\frac{v_k}{|u|} - 1) u\nabla P_{k2} dx$.
- $\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P_{k3} dx = \int_{\mathbb{R}^3} (\frac{v_k}{|u|} - 1) u\nabla P_{k3} dx$.

Hence, it follows that

$$\int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P dx| dt \leqslant \int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| dt + \int_{Q_{k-1}} (1 - \frac{v_k}{|u|})|u||\nabla P_{k2}| + \int_{Q_{k-1}} (1 - \frac{v_k}{|u|})|u||\nabla P_{k3}|. \tag{4.7}$$

**Step three.** We are now ready to deal with the term $\int_{Q_{k-1}} (1 - \frac{v_k}{|u|})|u||\nabla P_{k2}|$. For this purpose, let $p$ be such that $1 < p < \frac{5}{4}$, and let $q = \frac{p}{p-1}$, so that $2 < q < \infty$. We remark that the purpose of the condition $1 < p < \frac{5}{4}$ is to ensure that the quantity $\frac{2p}{2-p}$ will satisfy the condition $2 < \frac{2p}{2-p} < \frac{10}{3}$, which is required in the forthcoming inequality estimation (4.10). Next, by applying Holder's inequality, we find that

$$\begin{aligned} \|(1 - \frac{v_k}{|u|})u\|_{L^q(\mathbb{R}^3)} &\leqslant \|(1 - \frac{v_k}{|u|})u\|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \|(1 - \frac{v_k}{|u|})u\|_{L^\infty(\mathbb{R}^3)}^{1-\frac{2}{q}} \\ &\leqslant R^{1-\frac{2}{q}} \|(1 - \frac{v_k}{|u|})u\|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \\ &\leqslant R^{\frac{2}{p}-1} \|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})}. \end{aligned}$$

Hence, it follows from Holder's inequality that

$$\int_{\mathbb{R}^3} (1 - \frac{v_k}{|u|})|u||\nabla P_{k2}| dx \leqslant R^{\frac{2}{p}-1} \|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})} \{\int_{\mathbb{R}^3} |\nabla P_{k2}|^p dx\}^{\frac{1}{p}}.$$

Hence, we have

$$\int_{Q_{k-1}} (1 - \frac{v_k}{|u|})|u||\nabla P_{k2}| \leqslant R^{\frac{2}{p}-1} \|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})} \|\nabla P_{k2}\|_{L^p(Q_{k-1})}. \tag{4.8}$$

But, we recognize that

$$\nabla P_{k2} = \sum R_i R_j \{2(1 - \frac{w_k}{|u|}) u_i \nabla[\frac{w_k}{|u|} u_j] + 2(1 - \frac{w_k}{|u|}) u_j [\frac{w_k}{|u|} \nabla u_i] - 2\nabla[\frac{w_k}{|u|}] u_i \frac{w_k}{|u|} u_j\}.$$

Moreover, it is straightforward to see that for any $1 \leqslant i, j \leqslant 3$, we have

- $|2(1 - \frac{w_k}{|u|}) u_i \nabla[\frac{w_k}{|u|} u_j] + 2(1 - \frac{w_k}{|u|}) u_j [\frac{w_k}{|u|} \nabla u_i]| \leqslant 8R^\beta D_k$.

- $|2\nabla[\frac{w_k}{|u|}]u_i\frac{w_k}{|u|}u_j| \leqslant 8w_kD_k$.

So, we can decompose $\nabla P_{k2}$ as $\nabla P_{k2} = G_{k21} + G_{k22}$, where $G_{k21}$ and $G_{k22}$ are given by

- $G_{k21} = \sum R_iR_j\{2(1-\frac{w_k}{|u|})u_i\nabla[\frac{w_k}{|u|}u_j] + 2(1-\frac{w_k}{|u|})u_j[\frac{w_k}{|u|}\nabla u_i]\}$.
- $G_{k22} = -\sum R_iR_j\{2\nabla[\frac{w_k}{|u|}]u_i\frac{w_k}{|u|}u_j\}$.

In order to use inequality (4.8), we need to estimate $\|G_{k21}\|_{L^p(Q_{k-1})}$ and $\|G_{k22}\|_{L^p(Q_{k-1})}$ respectively, for $p$ with $1 < p < \frac{5}{4}$. Indeed, by applying the Zygmund-Calderon Theorem, we can deduce that

- $\|G_{k21}\|_{L^p(Q_{k-1})} \leqslant C_pR^\beta\|D_k\|_{L^p(Q_{k-1})}$,
- $\|G_{k22}\|_{L^p(Q_{k-1})} \leqslant C_p\|w_kD_k\|_{L^p(Q_{k-1})}$,

where $C_p$ is some constant depending only on $p$. But it turns out that

$$\begin{aligned}
\|D_k\|^p_{L^p(Q_{k-1})} &= \int_{Q_{k-1}} D_k^p\chi_{\{w_k>0\}} \\
&\leqslant \{\int_{Q_{k-1}} D_k^2\}^{\frac{p}{2}}\|\chi_{\{w_k>0\}}\|_{L^{\frac{2}{2-p}}(Q_{k-1})} \\
&\leqslant 5^{\frac{p}{2}}\|d_k\|^p_{L^2(Q_{k-1})}C_{\alpha,p}\frac{2^{\frac{5(2-p)k}{3}}}{R^{\beta(\frac{2-p}{2})[\frac{10}{3}+(\alpha-2)(\frac{2}{3}-\delta)]}} \cdot \|u\|^{(\frac{2}{3}-\delta)(\frac{2-p}{2})}_{L^\infty(L^{\alpha,\infty})}U_{k-1}^{(1+\delta)(\frac{2-p}{2})} \\
&\leqslant C_{\alpha,p,\delta}\frac{2^{\frac{5(2-p)k}{3}}}{R^{\beta(\frac{2-p}{2})[\frac{10}{3}+(\alpha-2)(\frac{2}{3}-\delta)]}} \cdot \|u\|^{(\frac{2}{3}-\delta)(\frac{2-p}{2})}_{L^\infty(L^{\alpha,\infty})} \cdot U_{k-1}^{1+\delta(\frac{2-p}{2})}.
\end{aligned}$$

That is , we have

$$\|D_k\|_{L^p(Q_{k-1})} \leqslant C_{\alpha,p,\delta}\frac{2^{\frac{5(2-p)k}{3p}}}{R^{\beta(\frac{2-p}{2p})[\frac{10}{3}+(\alpha-2)(\frac{2}{3}-\delta)]}} \cdot \|u\|^{(\frac{2}{3}-\delta)(\frac{2-p}{2p})}_{L^\infty(L^{\alpha,\infty})} \cdot U_{k-1}^{\frac{1}{p}+\delta(\frac{2-p}{2p})}.$$

Hence, it follows that

$$\|G_{k21}\|_{L^p(Q_{k-1})} \leqslant C_{\alpha,p,\delta}\frac{2^{\frac{5(2-p)k}{3p}}}{R^{\beta[\frac{10-8p}{3p}+(\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)]}} \cdot \|u\|^{(\frac{2}{3}-\delta)(\frac{2-p}{2p})}_{L^\infty(L^{\alpha,\infty})} \cdot U_{k-1}^{\frac{1}{p}+\delta(\frac{2-p}{2p})}. \quad (4.9)$$

On the other hand, we have

$$\begin{aligned}
\|w_kD_k\|^p_{L^p(Q_{k-1})} &= \int_{Q_{k-1}} w_k^pD_k^p \\
&\leqslant \{\int_{Q_{k-1}} w_k^{\frac{2p}{2-p}}\}^{\frac{2-p}{2}}\{\int_{Q_{k-1}} D_k^2\}^{\frac{p}{2}} \\
&\leqslant C_p\{\int_{Q_{k-1}} w_k^{\frac{2p}{2-p}}\}^{\frac{2-p}{2}}U_{k-1}^{\frac{p}{2}}.
\end{aligned}$$

Now, let us recall that $1 < p < \frac{5}{4}$, and put $r = \frac{2p}{2-p}$. we then recognize that $2 < r = \frac{2p}{2-p} < \frac{10}{3}$, if $1 < p < \frac{5}{4}$. So, we can have the following estimation

$$
\begin{aligned}
\int_{Q_{k-1}} w_k^{\frac{2p}{2-p}} &= \int_{Q_{k-1}} w_k^r \chi_{\{w_k>0\}} \\
&\leqslant \int_{Q_{k-1}} w_k^r \chi_{\{w_{k-1}>\frac{R^\beta}{2^k}\}} \\
&\leqslant \frac{1}{R^{\beta(\frac{10}{3}-r)}} 2^{k(\frac{10}{3}-r)} \int_{Q_{k-1}} w_k^{\frac{10}{3}} \\
&\leqslant \frac{C_{\alpha,\delta}\|u\|_{L^\infty(L^{\alpha,\infty})}^{\frac{2}{3}-\delta}}{R^{\beta[\frac{20-16p}{3(2-p)}+(\alpha-2)(\frac{2}{3}-\delta)]}} 2^{\frac{k(20-16p)}{3(2-p)}} U_{k-1}^{1+\delta}.
\end{aligned}
\tag{4.10}
$$

Hence, it follows that

$$
\begin{aligned}
\|G_{k22}\|_{L^p(Q_{k-1})} &\leqslant C_p\|w_k D_k\|_{L^p(Q_{k-1})} \\
&\leqslant C_{\alpha,p,\delta} \frac{2^{\frac{(10-8p)k}{3p}}}{R^{\beta[\frac{10-8p}{3p}+(\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)]}} \|u\|_{L^\infty(L^{\alpha,\infty})}^{(\frac{2}{3}-\delta)(\frac{2-p}{2p})} U_{k-1}^{\frac{1}{p}+\delta(\frac{2-p}{2p})}.
\end{aligned}
\tag{4.11}
$$

By combining inequalities (4.8), (4.9), (4.11), we deduce that

$$
\int_{Q_{k-1}} (1-\frac{v_k}{|u|})|u||\nabla P_{k2}| \leqslant \frac{2^{\frac{(10-8p)k}{3p}} C(\alpha,p,\delta;u)}{R^{\beta[\frac{10-8p}{3p}+(\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)]-(\frac{2-p}{p})}} U_{k-1}^{\frac{1}{p}+\delta(\frac{2-p}{2p})}, \tag{4.12}
$$

in which the constant $C(\alpha,p,\delta;u)$ is in the form of

$$
C(\alpha,p,\delta;u) = C_{\alpha,p,\delta}\|u\|_{L^\infty(L^2)}^{2(1-\frac{1}{p})}\|u\|_{L^\infty(L^{\alpha,\infty})}^{(\frac{2}{3}-\delta)(\frac{2-p}{2p})}. \tag{4.13}
$$

As for the term $\int_{Q_{k-1}}(1-\frac{v_k}{|u|})|u||\nabla P_{k3}|$. We first notice that

$$
P_{k3} = \sum R_i R_j \{\frac{w_k}{|u|}u_i \frac{w_k}{|u|}u_j\}.
$$

So, we know that

$$
\nabla P_{k3} = \sum R_i R_j \{\nabla[\frac{w_k}{|u|}u_i]\frac{w_k}{|u|}u_j + \frac{w_k}{|u|}u_i \nabla[\frac{w_k}{|u|}u_j]\},
$$

with

$$
|\nabla[\frac{w_k}{|u|}u_i]\frac{w_k}{|u|}u_j + \frac{w_k}{|u|}u_i\nabla[\frac{w_k}{|u|}u_j]| \leqslant 6 w_k D_k.
$$

Again, by the Riesz's theorem, we have $\|\nabla P_{k3}\|_{L^p(\mathbb{R}^3)} \leqslant C_p\|w_k D_k\|_{L^p(\mathbb{R}^3)}$, in which $C_p$ is some constant depending only on $p$. So, we can repeat the same type of estimation, just as what we have done to the term $\int_{Q_{k-1}}(1-\frac{v_k}{|u|})|u||\nabla P_{k2}|$, to conclude that

$$
\begin{aligned}
\int_{Q_{k-1}} (1-\frac{v_k}{|u|})|u||\nabla P_{k3}| &\leqslant R^{\frac{2}{p}-1}\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})}\|\nabla P_{k3}\|_{L^p(Q_{k-1})} \\
&\leqslant \frac{2^{\frac{(10-8p)k}{3p}} C(\alpha,p,\delta;u)}{R^{\beta[\frac{10-8p}{3p}+(\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)]-(\frac{2-p}{p})}} U_{k-1}^{\frac{1}{p}+\delta(\frac{2-p}{2p})},
\end{aligned}
\tag{4.14}
$$

in which the constant $C(\alpha, p, \delta; u)$ is again in the form of (4.13).

We have to ensure that the quantity $\beta[\frac{10-8p}{3p} + (\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)] - (\frac{2-p}{p})$ is *strictly greater than* 0. To this end, recall that $p > 1$ can be as close to 1 as possible, and $\delta > 0$ can also be as close to 0 as possible. So, by passing to the limit as $p \to 1^+$, and $\delta \to 0^+$, we have

$$\lim_{p\to 1^+, \delta\to 0^+} \beta[\frac{10-8p}{3p} + (\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)] - (\frac{2-p}{p}) = \beta(\frac{\alpha}{3}) - 1. \tag{4.15}$$

Now, we insist that the choice of $\beta$ has to satisfy the condition $\beta > \frac{3}{\alpha}$, under which we must have the limiting value $\beta(\frac{\alpha}{3}) - 1$ to be strictly positive. Hence, for such a choice of $\beta$, it follows from (4.15) that the following relation holds for all $p > 1$ to be sufficiently close to 1, and all $\delta > 0$ to be sufficiently close to 0.

$$\beta[\frac{10-8p}{3p} + (\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)] - (\frac{2-p}{p}) > 0. \tag{4.16}$$

**Step four.**

We now have to raise up the index for the term $\int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|})uP_{k1}dx|ds$.

Recall that, in the hypothesis of Theorem 1.2, there is some constant $M_0 > 0$ for which $|u| \leqslant M_0$ is valid on the outer region $[0,1) \times \{x \in \mathbb{R}^3 : |x| \geqslant r_0\}$ for some given radius $r_0 > 0$. As a result, we will now choose $R > 2M_0$ so that, for each $k \geq 1$ and $t \in [0,1)$ we have $\{|u(t,\cdot)| > R(1-\frac{1}{2^k})\} \subset B(r_0)$, which means that both $v_k(t,\cdot)$ and $d_k(t,\cdot)$ are compactly supported in $B(r_0)$. Hence, for such a choice of $R > 2M_0$, we always can express $U_k$ as

$$U_k = \frac{1}{2}\sup_{t\in[T_k,1)} \int_{B(r_0)} v_k^2(t,\cdot)dx + \int_{T_k}^1 \int_{B(r_0)} d_k^2 dxdt.$$

Since $\nabla(\frac{v_k}{|u|})u = -R(1-\frac{1}{2^k})F\chi_{\{v_k>0\}}$, we have *for any* $R > 2M_0$ that

$$\begin{aligned} |\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|})uP_{k1}dx| &= |\int_{B(r_0)} R(1-\frac{1}{2^k})F\chi_{\{v_k>0\}}P_{k1}dx| \\ &\leqslant R\int_{B(r_0)} |F|\chi_{\{v_k>0\}}|P_{k1} - (P_{k1})_{B(r_0)}|dx \\ &+ R\int_{B(r_0)} |F|\chi_{\{v_k>0\}}|(P_{k1})_{B(r_0)}|dx, \end{aligned}$$

for all $k \geqslant 1$, and all $\frac{1}{2} < t < 1$ (here, the symbol $(P_{k1})_B$ stands for the average value of $P_{k1}$ over the ball $B$ ). From now on, we will *always assume, within this section, that our choice of R has to satisfy* $R > 2M_0$ . Now, since $P_{k1} = \sum R_iR_j\{(1-\frac{w_k}{|u|})u_i(1-\frac{w_k}{|u|})u_j\}$, it follows from the Riesz's Theorem in the theory of singular integral that $\|P_{k1}(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leqslant C_2R^\beta\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)}$, for all $t \in [0,1]$, in which $C_2$ is some constant depending only on 2. So, we can use the Holder's inequality to carry out the following estimation

$$
\begin{aligned}
|(P_{k1})_{B(r_0)}(t)| &\leqslant \frac{1}{|B(r_0)|}\int_{B(r_0)}|P_{k1}(t,x)|dx\\
&\leqslant \frac{1}{|B(r_0)|^{\frac{1}{2}}}\|P_{k1}(t,\cdot)\|_{L^2(B(r_0))}\\
&\leqslant \frac{1}{|B(r_0)|^{\frac{1}{2}}}C_2R^{\beta}\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)}\\
&\leqslant C(r_0)R^{\beta}\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))},
\end{aligned}
$$

in which the constant $C(r_0)=\frac{1}{|B(r_0)|^{\frac{1}{2}}}C_2$ depends on $r_0$. As a result, it follows that

$$
\begin{aligned}
|\int_{\mathbb{R}^3}\nabla(\frac{v_k}{|u|})uP_{k1}dx| \leqslant{}& R\int_{B(r_0)}|F|\chi_{\{v_k>0\}}|P_{k1}-(P_{k1})_{B(r_0)}|dx\\
&+C(r_0)R\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}\int_{B(r_0)}R^{\beta}|F|\chi_{\{v_k>0\}}.
\end{aligned}
\tag{4.17}
$$

Indeed, the operator $R_iR_j$ is indeed a Zygmund- Calderon operator, and so $R_iR_j$ must be a bounded operator from $L^\infty(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$. Hence we can deduce that

$$
\begin{aligned}
\|P_{k1}(t,\cdot)-(P_{k1})_{B(r_0)}(t)\|_{BMO} &= \|P_{k1}(t,\cdot)\|_{BMO}\\
&\leqslant C_0\|(1-\frac{w_k}{|u|})u_i(1-\frac{w_k}{|u|})u_j\|_{L^\infty(\mathbb{R}^3)}\\
&\leqslant C_0R^{2\beta},
\end{aligned}
$$

for all $t\in(0,1)$, in which $C_0$ is some constant depending only on $\mathbb{R}^3$.

Just as the proof of the main result in [4], at this stage, we need the assistant of the following Lemma, which is a straightforward corollary of the famous $BMO$ result [8] of John and Nirenberg. For a proof of this lemma, we refer to Lemma 4.3 of [4].

Lemma 4.2. *(see [4])Let $B$ be a ball with finite radius sitting in $\mathbb{R}^3$. There exists some finite positive constants $\nu$ and $K$,depending only on $B$, such that for every measurable function $\mu\geqslant 0$, and every $f\in BMO(\mathbb{R}^3)$ with $\int_B fdx=0$, and $p$ with $1<p<\infty$, we have $\int_B\mu|f|\leqslant\frac{2p}{\nu(p-1)}\{1+K^{1-\frac{1}{p}}\}\|f\|_{BMO}\{(\int_B\mu)^{\frac{1}{p}}+\int_B\mu log^+\mu\}$.*

So, we now apply Lemma 4.2 with $\mu=|F|\chi_{\{v_k>0\}}$, and $f=P_{k1}-(P_{k1})_{B(r_0)}$ to deduce that

$$
\begin{aligned}
\int_{B(r_0)}|F|\chi_{\{v_k>0\}}|P_{k1}-(P_{k1})_{B(r_0)}|dx \leqslant{}& \frac{2pC_0}{\nu(p-1)}\{1+K^{1-\frac{1}{p}}\}\times\\
&\{(\int_{B(r_0)}R^{2p\beta}|F|\chi_{\{v_k>0\}})^{\frac{1}{p}}+\int_{B(r_0)}R^{2\beta}|F|log^+|F|\cdot\chi_{\{v_k>0\}}\},
\end{aligned}
$$

in which the symbol $(P_{k1})_{B(r_0)}$ stands for the mean value of $P_{k1}$ over the open ball $B(r_0)$. Since we know that $\{v_k>0\}$ is a subset of $\{|u|>\frac{R}{2}\}$, for all $k\geqslant 1$, so it follows from the above inequality that

$$\begin{aligned}\int_{B(r_0)}|F|\chi_{\{v_k>0\}}|P_{k1}-(P_{k1})_{B(r_0)}|dx \leqslant \frac{2C_0}{\nu}\frac{p}{p-1}4^{p\beta}\{1+K^{1-\frac{1}{p}}\}\times \\ \{(\int_{B(r_0)}|u|^{2p\beta}|F|\chi_{\{v_k>0\}})^{\frac{1}{p}} \\ +\int_{B(r_0)}|u|^{2\beta}|F|\log^+|F|\cdot\chi_{\{v_k>0\}}\}.\end{aligned}$$

So, we can conclude from inequality (4.17), and the above inequality that

$$\begin{aligned}\int_{T_{k-1}}^1|\int_{\mathbb{R}^3}\nabla(\frac{v_k}{|u|})uP_{k1}dx|dt \leqslant R\frac{2C_0}{\nu}\frac{p}{p-1}4^{p\beta}(1+K^{1-\frac{1}{p}})\times \\ \{(\int_{T_{k-1}}^1\int_{B(r_0)}|u|^{2p\beta}|F|\chi_{\{v_k>0\}})^{\frac{1}{p}} \\ +\int_{T_{k-1}}^1\int_{B(r_0)}|u|^{2\beta}|F|\log(1+|F|)\chi_{\{v_k>0\}}\} \\ +C(r_0)2^\beta R\|u\|_{L^\infty(L^2)}\int_{T_{k-1}}^1\int_{B(r_0)}|u|^\beta|F|\chi_{\{v_k>0\}}.\end{aligned} \tag{4.18}$$

In order to use the given hypothesis that $|u\cdot\nabla F|(t,x)\leqslant A|u(t,x)||F(t,x)|$, for any $(t,x)\in[0,1)\times B(r_0)$ satisfying $|F(t,x)|\geqslant L$ (with $L>0$ to be the given constant in Theorem 1.2), we carry out the following estimate.

$$\begin{aligned}&\int_{T_{k-1}}^1\int_{B(r_0)}|u|^{2\beta}|F|\log(1+|F|)\chi_{\{v_k>0\}} \\ &\leqslant \int_{T_{k-1}}^1\int_{B(r_0)}|u|^{2\beta}|F|\log(1+|F|)\chi_{\{|F|\leqslant L+1\}}\chi_{\{v_k>0\}} \\ &+\int_{T_{k-1}}^1\int_{B(r_0)}|u|^{2\beta}|F|\log(1+|F|)\chi_{\{|F|>L+1\}}\chi_{\{v_k>0\}} \\ &\leqslant (L+1)\log(L+2)\int_{T_{k-1}}^1\int_{B(r_0)}|u|^{2\beta}\chi_{\{v_k>0\}} \\ &+\int_{T_{k-1}}^1\int_{B(r_0)}|u|^{2\beta}|F|\log(1+|F|)\chi_{\{|F|>L+1\}}\chi_{\{v_k>0\}}.\end{aligned} \tag{4.19}$$

**Step five.** To deal with the second term in the last line of inequality (4.19), we consider the sequence $\{\phi_k\}_{k=1}^\infty$ of nonnegative continuous functions on $[0,\infty)$, which are defined by

- $\phi_k(t)=0$, for all $t\in[0,C_k]$.
- $\phi_k(t)=t-C_k$, for all $t\in(C_k,C_k+1)$.
- $\phi_k(t)=1$, for all $t\in[C_k+1,+\infty)$.

where the symbol $C_k$ stands for $C_k=R(1-\frac{1}{2^k})$, for every $k\geqslant 1$. Here, we remark that, for the purpose of taking spatial derivative, the composite function $\phi_k(|u|)$ is a good substitute for $\chi_{\{v_k>0\}}=\chi_{\{|u|>R(1-\frac{1}{2^k})\}}$, since $\phi_k$ is Lipschitz. Moreover, we also need a smooth function $\psi:\mathbb{R}\to\mathbb{R}$ satisfying the following conditions that:

- $\psi(t) = 1$, for all $t \geqslant L+1$.
- $0 < \psi(t) < 1$, for all $t$ with $L < t < L+1$.
- $\psi(t) = 0$, for all $-L \leqslant t \leqslant L$.
- $-1 < \psi(t) < 0$, for all $t$ with $-L-1 < t < -L$.
- $\psi(t) = -1$, for all $t \leqslant -L-1$.
- $0 \leqslant \frac{d}{dt}\psi \leqslant 2$, for all $t \in \mathbb{R}$.

We further remark that the smooth function $\psi : \mathbb{R} \to \mathbb{R}$ characterized by the above properties must also satisfy the property that $\psi'(t) = \frac{d\psi}{dt}|_{(t)} = 0$, on $t \in (-\infty, -L-1) \cup (-L, L) \cup (L+1, \infty)$, which will be employed in forthcoming inequality estimations 4.21 and 4.24 without explicit mention. With the above preparation, let $\beta$ be such that $\frac{3}{\alpha} < \beta < \frac{10}{3\alpha}$, with $\alpha$ to be the given index as *specified in* Theorem 1.2.

We now consider the function $F = div(\frac{u}{|u|})$, and recall that our solution $u$ satisfies $|u \cdot \nabla F| \leqslant A|F| \cdot |u|$ on $\{(t,x) \in [0,1) \times B(r_0) : |F(t,x)| \geqslant L\}$. for some given constant $L > 0$.

it follows that

- $|u \cdot \nabla F|(t,x) \leqslant A(L+1)|u(t,x)|$, if $(t,x) \in [0,1) \times B(r_0)$ satisfies $L \leqslant |F(t,x)| \leqslant L+1$.
- $|\frac{u \cdot \nabla |F|}{1+|F|}| \leqslant \frac{|u \cdot \nabla |F||}{|F|} = \frac{|u \cdot \nabla F|}{|F|} \leqslant A|u|$ is valid on $[0,1) \times B(r_0) \cap \{|F(s)| \geq L\}$.

Then, we carry out the following calculation on $[0,1) \times B(r_0)$, for each $k \geqslant 1$.

$$
\begin{aligned}
div\{|u|^{2\beta-1} u \psi(F) \log(1+|F|)\phi_k(|u|)\} = &-(2\beta-1)|u|^{2\beta} F \psi(F) \log(1+|F|)\phi_k(|u|) \\
&- |u|^{2\beta+1} F \psi(F) \log(1+|F|)\chi_{\{C_k < |u| < C_k+1\}} \\
&+ |u|^{2\beta-1} \frac{d\psi}{dt}(F)(u \cdot \nabla F) \log(1+|F|)\phi_k(|u|) \\
&+ |u|^{2\beta-1} \psi(F) \frac{u \cdot \nabla |F|}{1+|F|} \phi_k(|u|),
\end{aligned}
\tag{4.20}
$$

Since $R > 2M_0$ ensures that, for each $t \in [0,1)$, $\phi_k(|u|)(t,\cdot)$ is *compactly supported* in $B(r_0)$, we have the following equality for each $t \in [0,1)$.

$$
\int_{B(r_0)} div\{|u|^{2\beta-1} u \psi(F) \log(1+|F|)\phi_k(|u|)\} = 0.
$$

So, it follows from inequality (4.20) that

$$
\begin{aligned}
\Lambda_1 + \Lambda_2 \leqslant &\int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta-1} |\frac{d\psi}{dt}(F)| \cdot |u \cdot \nabla F| \log(1+|F|)\phi_k(|u|) \\
&+ \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta-1} |\psi(F)| \cdot |\frac{u \cdot \nabla |F|}{1+|F|}| \phi_k(|u|) \\
\leqslant &\int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta-1} (2)(A(L+1)|u|) \log(L+2)\phi_k(|u|) \\
&+ \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta-1} \cdot A \cdot |u| \phi_k(|u|) \cdot \chi_{\{|F| \geqslant L\}} \\
\leqslant &A[2(L+1)\log(L+2)+1] \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta} \phi_k(|u|) \\
\leqslant &A[2(L+1)\log(L+2)+1] \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta} \chi_{\{v_k > 0\}},
\end{aligned}
\tag{4.21}
$$

in which the terms $\Lambda_1$, and $\Lambda_2$ are given by

- $\Lambda_1 = (2\beta - 1)\int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta} F\psi(F) \cdot log(1+|F|)\phi_k(|u|)$.
- $\Lambda_2 = \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta+1} (F\psi(F)) \cdot log(1+|F|)\chi_{\{C_k<|u|<C_k+1\}}$ .

We then notice that

- Since $\beta > \frac{3}{\alpha} > 1$, we have $\Lambda_1 \geqslant \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta}(F\psi(F))\log(1+|F|)\chi_{\{|u|\geqslant C_k+1\}}$.
- $\Lambda_2 \geqslant \frac{R}{2}\int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta} F\psi(F)\log(1+|F|)\chi_{\{C_k<|u|<C_k+1\}}$, for every $k \geqslant 1$. Notice that this is true because $C_k = R(1-\frac{1}{2^k})$, and that $(1-\frac{1}{2^k}) \geqslant \frac{1}{2}$, for every $k \geqslant 1$.

Since $|F|\chi_{\{|F|>L+1\}} \leqslant |F||\psi(F)| = F\psi(F)$, it follows from inequality (4.21) that

$$\begin{aligned}
&\int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta}|F|\log(1+|F|)\chi_{\{|F|>L+1\}}\chi_{\{v_k>0\}} \\
&\leqslant \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta} F\psi(F)\log(1+|F|)\chi_{\{v_k>0\}} \\
&\leqslant \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta} F\psi(F)\log(1+|F|)\chi_{\{C_k<|u|<C_k+1\}} \\
&+ \int_{T_{k-1}}^1 \int_{B(r_0)} |u|^{2\beta} F\psi(F) log(1+|F|)\chi_{\{|u|\geqslant C_k+1\}} \\
&\leqslant \frac{2}{R}\Lambda_2 + \Lambda_1 \\
&\leqslant 2A[2(L+1)\log(L+2)+1]\int_{Q_{k-1}} |u|^{2\beta}\chi_{\{v_k>0\}}.
\end{aligned} \tag{4.22}$$

By using inequality (3.3) in Lemma 3.4, we raise up the index for the term $\int_{Q_{k-1}} |u|^\theta \chi_{\{v_k>0\}}$, for any $\theta$ with $0<\theta<\frac{10}{3}$, in the following way

$$\begin{aligned}
\int_{Q_{k-1}} |u|^\theta \chi_{\{v_k>0\}} &= \int_{Q_{k-1}} \{R(1-\frac{1}{2^k}) + v_k\}^\theta \chi_{\{v_k>0\}} \\
&\leqslant C_\theta\{R^\theta \int_{Q_{k-1}} \chi_{\{v_k>0\}} + \int_{Q_{k-1}} v_k^\theta \chi_{\{v_k>0\}}\} \\
&\leqslant \frac{C_\theta}{R^{\frac{10}{3}-\theta}}\{2^{\frac{10k}{3}} + 2^{(\frac{10}{3}-\theta)k}\}\int_{Q_{k-1}} v_{k-1}^{\frac{10}{3}} \\
&\leqslant \frac{C_\theta}{R^{\frac{10}{3}-\theta+(\alpha-2)(\frac{2}{3}-\delta)}} 2^{\frac{10k}{3}}\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^\infty(L^{\alpha,\infty})}\}^{\frac{2}{3}-\delta} U_{k-1}^{1+\delta},
\end{aligned}$$

for every $\theta$ with $0<\theta<\frac{10}{3}$, where $C_\theta$ is some positive constant depending only on $\theta$.

Hence it follows from inequalities(4.19), (4.22), and our last inequality that

$$\begin{aligned}
&\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{2\beta}|F|\cdot log(1+|F|)\chi_{\{v_k>0\}}\\
&\leqslant (L+1)\log(L+2)\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{2\beta}\chi_{\{v_k>0\}}\\
&+\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{2\beta}|F|log(1+|F|)\chi_{\{|F|>L+1\}}\chi_{\{v_k>0\}}\\
&\leqslant \frac{(L+1)\log(L+2)C_{2\beta}2^{\frac{10k}{3}}}{R^{\frac{10}{3}-2\beta+(\alpha-2)(\frac{2}{3}-\delta)}}\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^\infty(L^{\alpha,*})}\}^{\frac{2}{3}-\delta}U_{k-1}^{1+\delta} \qquad (4.23)\\
&+C_{(A,L)}\int_{Q_{k-1}}|u|^{2\beta}\chi_{\{v_k>0\}}\\
&\leqslant C_{(\beta,A,L)}\cdot 2^{\frac{10k}{3}}\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^\infty(L^{\alpha,\infty})}\}^{\frac{2}{3}-\delta}U_{k-1}^{1+\delta}\\
&\times\{\frac{1}{R^{\frac{10}{3}-2\beta+(\alpha-2)(\frac{2}{3}-\delta)}}\},
\end{aligned}$$

in which $\beta > \frac{3}{\alpha}$, and that $\beta$ is sufficiently close to $\frac{3}{\alpha}$, and $C_{\beta,A,L}$ is some constant depending only on $\beta$, $A$, and $L$. Next, we also need to deal with $(\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{2p\beta}|F|\chi_{\{v_k\geqslant 0\}})^{\frac{1}{p}}$, and $\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\beta}|F|\chi_{\{v_k\geqslant 0\}}$, which appear in inequality (4.18). For this purpose, we will consider $\lambda$ which satisfies $\frac{3}{\alpha}<\lambda<\frac{10}{3}$ (we will take $\lambda$ to be $2p\beta$ and $\beta$ respectively in forthcoming inequality estimates 4.25 and 4.26 ), and let us carry out the following computation, in which $\psi$ and $\phi_k$ etc are just the same as before.

$$\begin{aligned}
div\{|u|^{\lambda-1}u\psi(F)\phi_k(|u|)\} &= -(\lambda-1)|u|^{\lambda}F\psi(F)\phi_k(|u|)\\
&+|u|^{\lambda-1}\frac{d\psi}{dt}(F)(u\cdot\nabla F)\phi_k(|u|)\\
&-|u|^{\lambda+1}F\psi(F)\chi_{\{C_k<|u|<C_k+1\}}.
\end{aligned}$$

Since $R>2M_0$ ensures that $\phi_k(|u|)$ is compactly supported in $B(r_0)$, we have, for each $t\in[0,1)$, that

$$\int_{B(r_0)}div\{|u|^{\lambda-1}u\psi(F)\phi_k(|u|)\}=0.$$

Hence, it follows from $|\frac{d\psi}{dt}(F)|\leqslant 2\chi_{\{L<|F|<L+1\}}$ and the above equality that

$$\begin{aligned}
(\lambda-1)\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda}F\psi(F)\phi_k(|u|)+\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda+1}F\psi(F)\chi_{\{C_k<|u|<C_k+1\}}&\\
\leqslant\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda-1}|\frac{d\psi}{dt}(F)|\cdot|u\cdot\nabla F|\phi_k(|u|)&\\
\leqslant\int_{Q_{k-1}}|u|^{\lambda-1}(2)(A(L+1)|u|)\chi_{\{v_k>0\}}&\\
\leqslant 2A(L+1)\int_{Q_{k-1}}|u|^{\lambda}\chi_{\{v_k>0\}}.&
\end{aligned}\qquad (4.24)$$

By the same calculation as in inequality (4.21), we can see that

$$
\begin{aligned}
&\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda}F\psi(F)\chi_{\{v_k>0\}}\\
&\leqslant \int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda}F\psi(F)\chi_{\{C_k<|u|<C_k+1\}}\\
&+\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda}F\psi(F)\chi_{\{|u|\geqslant C_k+1\}}\\
&\leqslant \frac{2}{R}\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda+1}F\psi(F)\chi_{\{C_k<|u|<C_k+1\}}\\
&+\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda}F\psi(F)\phi_k(|u|)\\
&\leqslant (2+\frac{1}{\lambda-1})\{\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda+1}F\psi(F)\chi_{\{C_k<|u|<C_k+1\}}\\
&+(\lambda-1)\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\lambda}F\psi(F)\phi_k(|u|)\}\\
&\leqslant 2A(L+1)(2+\frac{1}{\lambda-1})\int_{Q_{k-1}}|u|^{\lambda}\chi_{\{v_k>0\}},
\end{aligned}
$$

in which $\lambda$ satisfies $\frac{3}{\alpha}<\lambda<\frac{10}{3}$. Now, put $\lambda=2p\beta$, with $\beta>\frac{3}{\alpha}$ to be sufficiently close to $\frac{3}{\alpha}$, and $p>1$ to be sufficiently close to 1. Since $|F|\chi_{\{|F|>L+1\}}\leqslant|F||\psi(F)|=F\psi(F)$, it follows from our last inequality that

$$
\begin{aligned}
&\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{2p\beta}|F|\chi_{\{v_k>0\}}\\
&=\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{2p\beta}|F|\chi_{\{|F|\leqslant L+1\}}\chi_{\{v_k>0\}}\\
&+\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{2p\beta}\chi_{\{|F|>L+1\}}\chi_{\{v_k>0\}}|F|\\
&\leqslant (L+1)\int_{Q_{k-1}}|u|^{2p\beta}\chi_{\{v_k>0\}}\\
&+2A(L+1)(2+\frac{1}{2p\beta-1})\int_{Q_{k-1}}|u|^{2p\beta}\chi_{\{v_k>0\}}\\
&\leqslant \frac{C_{(\beta,A,L,p)}}{R^{\frac{10}{3}-2p\beta+(\alpha-2)(\frac{2}{3}-\delta)}}\cdot 2^{\frac{10k}{3}}\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^{\infty}(L^{\alpha,\infty})}\}^{\frac{2}{3}-\delta}U_{k-1}^{1+\delta}.
\end{aligned}
\tag{4.25}
$$

In exactly the same way, by setting $\lambda$ to be $\beta$, with $\beta>\frac{3}{\alpha}$ to be sufficiently close to

$\frac{3}{\alpha}$, it also follows that

$$\begin{aligned}
&\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\beta}|F|\chi_{\{v_k>0\}}\\
&=\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\beta}|F|\chi_{\{|F|\leqslant L+1\}}\chi_{\{v_k>0\}}\\
&+\int_{T_{k-1}}^{1}\int_{B(r_0)}|u|^{\beta}|F|\chi_{\{|F|>L+1\}}\chi_{\{v_k>0\}}\\
&\leqslant (L+1)\int_{Q_{k-1}}|u|^{\beta}\chi_{\{v_k>0\}}+2A(L+1)(2+\frac{1}{\beta-1})\int_{Q_{k-1}}|u|^{\beta}\chi_{\{v_k>0\}}\\
&\leqslant \frac{C_{(\beta,A,L)}}{R^{\frac{10}{3}-\beta+(\alpha-2)(\frac{2}{3}-\delta)}}\cdot 2^{\frac{10k}{3}}\{\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^{\infty}(L^{\alpha,\infty})}\}^{\frac{2}{3}-\delta}U_{k-1}^{1+\delta}.
\end{aligned}\tag{4.26}$$

By combining inequalities (4.18), (4.23), and (4.25),and (4.26) we now conclude that

$$\begin{aligned}
&\int_{Q_{k-1}}|\int_{Q_{k-1}}\nabla(\frac{v_k}{|u|})uP_{k1}dx|ds\\
&\leqslant (1+\frac{1}{\alpha})C(\beta,A,L,p)(1+K^{1-\frac{1}{p}})(1+\|u\|_{L^{\infty}(L^2))})\\
&\{[\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^{\infty}(L^{\alpha,\infty})}]^{\frac{2}{3}-\delta}+[\frac{2^{\alpha-1}}{\alpha-2}\|u\|_{L^{\infty}(L^{\alpha,\infty})}]^{(\frac{2}{3}-\delta)\frac{1}{p}}\}\\
&\{(\frac{1}{R^{\frac{10}{3}-2p\beta+(\alpha-2)(\frac{2}{3}-\delta)-p}})^{\frac{1}{p}}2^{\frac{10k}{3p}}U_{k-1}^{\frac{1}{p}(1+\delta)}\\
&+\frac{1}{R^{\frac{10}{3}-2\beta+(\alpha-2)(\frac{2}{3}-\delta)-1}}2^{\frac{10k}{3}}U_{k-1}^{1+\delta}\}.
\end{aligned}\tag{4.27}$$

Before we proceed to the last step and complete the proof of Theorem 1.2, let us briefly explain why the condition $1+2(\frac{\alpha}{3}-\frac{3}{\alpha})>0$ imposed on $2<\alpha<3$ is necessary. Notice that if $p\to 1^{+}$, and $\beta\to \frac{3}{\alpha}^{+}$, and $\delta\to 0^{+}$, we have $(\frac{10}{3}-2p\beta+(\alpha-2)(\frac{2}{3}-\delta)-p)\to 1+2(\frac{\alpha}{3}-\frac{3}{\alpha})$, and that $(\frac{10}{3}-2\beta+(\alpha-2)(\frac{2}{3}-\delta)-1)\to 1+2(\frac{\alpha}{3}-\frac{3}{\alpha})$. This explains that the condition $1+2(\frac{\alpha}{3}-\frac{3}{\alpha})>0$ on $\alpha\in(2,3)$ is necessary if we insist that both $(\frac{10}{3}-2p\beta+(\alpha-2)(\frac{2}{3}-\delta)-p)$ and $(\frac{10}{3}-2\beta+(\alpha-2)(\frac{2}{3}-\delta)-1)$ have to be positive.

**Step six: Final step of the proof of Lemma 4.1**.

By combining inequalities (4.3), (4.7), (4.12), (4.14), and (4.27), we conclude that the following estimate is valid, and hence we arrive at the conclusion of Lemma 4.1.

$$\begin{aligned}
U_k &\leqslant \frac{2^{\frac{10k}{3}}}{R^{\frac{4}{3}}}C_0U_{k-1}^{\frac{5}{3}}+C(\beta,A,L,p,\delta,\|u\|_{L^{\infty}L^2},\|u\|_{L^{\infty}L^{\alpha,\infty}})2^{\frac{10k}{3}}\\
&\times\{\frac{U_{k-1}^{\frac{1}{p}+\delta(\frac{2-p}{2p})}}{R^{\beta[\frac{10-8p}{3p}+(\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)]-(\frac{2-p}{p})}}\\
&+(\frac{U_{k-1}^{(1+\delta)}}{R^{\frac{10}{3}-2p\beta+(\alpha-2)(\frac{2}{3}-\delta)-p}})^{\frac{1}{p}}+\frac{U_{k-1}^{1+\delta}}{R^{\frac{10}{3}-2\beta+(\alpha-2)(\frac{2}{3}-\delta)-1}}\}
\end{aligned}\tag{4.28}$$

□

Now, with the nonlinear recurrence relation of $\{U_k\}_{k=1}^{\infty}$ as given in Lemma 4.1 to be available, we will see in that which follows that Theorem 1.2 is nothing other than an easy consequence of Lemma 4.1.

**Proof of Theorem 1.2: A direct consequence of Lemma 4.1.** Again, let $u : [0,1) \times \mathbb{R}^3 \to \mathbb{R}^3$ to be the Leray-Hopf solution which is smooth up to the possible blow up time $T = 1$, and which also satisfies all the hypothesis of Theorem 1.2. Here, in order to derive the conclusion that $|u|$ is $L^{\infty}$-bounded on $[\frac{3}{4}, 1) \times \mathbb{R}^3$ by using inequality (4.28), we have to be very careful in the selection of the constants $\beta$, $p$, $\delta$ etc. This is due to the following observations.

On the one hand, we require all the powers of $U_{k-1}$ such as $\frac{1}{p} + \delta(\frac{2-p}{2p})$, $\frac{1}{p}(1+\delta)$, and $1+\delta$ to be strictly greater than 1. In order to select some suitable $p > 1$ and $\delta > 0$ for which the quantities $\frac{1}{p} + \delta(\frac{2-p}{2p})$, $\frac{1}{p}(1+\delta)$ will all be strictly greater than 1, we observe that the requirement $\frac{1}{p} + \delta(\frac{2-p}{2p}) > 1$ is equivalent to the condition $\frac{1+\delta}{1+\frac{\delta}{2}} > p$, which indicates that the positive parameter $p > 1$ has to be selected in the open interval $(1, \frac{1+\delta}{1+\frac{\delta}{2}})$ (for some $\delta > 0$ to be given). For such a choice of $p \in (1, \frac{1+\delta}{1+\frac{\delta}{2}})$, the second requirement $\frac{1}{p}(1+\delta) > 1$ is nothing but a consequence which follows from the fact that $(1+\delta) > p(1+\frac{\delta}{2}) > p$. Of course, the third condition $1+\delta > 1$ is always true as long as $\delta$ is strictly greater than 0. In summary, we see that we will have all the three conditions $\frac{1}{p} + \delta(\frac{2-p}{2p}) > 1$, $\frac{1}{p}(1+\delta) > 1$ and $1+\delta > 1$ to be valid simultaneously, provided we choose $\delta > 0$ and $p \in (1, \frac{1+\delta}{1+\frac{\delta}{2}})$.

On the other hand, the constant $C(\beta, A, L, p, \delta, \|u\|_{L^{\infty}L^2}, \|u\|_{L^{\infty}L^{\alpha,\infty}})$ will blow up to $\infty$ if $p \to 1^+$. So, to clarify the situation, we have to fix the choice of $\beta$ *first* by using the condition $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$ on $\alpha \in (2,3)$. Once the choice of $\beta$ is fixed, we will *fix* the parameter $\delta > 0$, and subsequently the parameter $p \in (1, \frac{1+\delta}{1+\frac{\delta}{2}})$.

Observe that the condition $1 + 2(\frac{\alpha}{3} - \frac{3}{\alpha}) > 0$ on $\alpha \in (2,3)$ is equivalent to $\frac{1}{2} + \frac{\alpha}{3} > \frac{3}{\alpha}$, and this allows us to select some $\beta$ to be in the interval $(\frac{3}{\alpha}, \frac{1}{2} + \frac{\alpha}{3})$. Now, let $\beta$ to be a *fixed* choice of positive number which satisfies $\frac{3}{\alpha} < \beta < \frac{1}{2} + \frac{\alpha}{3}$. Next, recall that we have the following limiting relations.

- $\lim_{p\to 1^+, \delta \to 0^+} \beta[\frac{10-8p}{3p} + (\frac{2-p}{2p})(\alpha - 2)(\frac{2}{3} - \delta)] - (\frac{2-p}{p}) = \beta(\frac{\alpha}{3}) - 1$.
- $\lim_{p\to 1^+, \delta \to 0^+} \{\frac{10}{3} - 2p\beta + (\alpha-2)(\frac{2}{3} - \delta) - p\} = 2\{\frac{1}{2} + \frac{\alpha}{3} - \beta\}$.
- $\lim_{\delta \to 0^+} \frac{10}{3} - 2\beta + (\alpha - 2)(\frac{2}{3} - \delta) - 1 = 2\{\frac{1}{2} + \frac{\alpha}{3} - \beta\}$ .

Notice that the fixed choice of $\beta$ with $\frac{3}{\alpha} < \beta < \frac{1}{2} + \frac{\alpha}{3}$ ensures that the limiting constants $\beta(\frac{\alpha}{3}) - 1$ and $2\{\frac{1}{2} + \frac{\alpha}{3} - \beta\}$ are both positive simultaneously. The above three limiting relations motivate us to first choose some *fixed* choice of $\delta > 0$ which is sufficiently close to 0, in a manner dependent on the choice of $\beta \in (\frac{3}{\alpha}, \frac{1}{2} + \frac{\alpha}{3})$. Due to the fact that the length of the open interval $(1, \frac{1+\delta}{1+\frac{\delta}{2}})$ is shrinking down to 0 as $\delta > 0$ becomes small, any choice of $p \in (1, \frac{1+\delta}{1+\frac{\delta}{2}})$ for sure will be sufficiently close to 1, as long as $\delta > 0$ is sufficiently small (that is, $p \to 1^+$, as long as $\delta \to 0^+$). So, according to the above three limiting relations, it follows that the following three constants will become *strictly positive*, as long as a sufficiently small parameter $\delta > 0$ is chosen in a way dependent on the choice of $\beta \in (\frac{3}{\alpha}, \frac{1}{2} + \frac{\alpha}{3})$, with the parameter $p \in (1, \frac{1+\delta}{1+\frac{\delta}{2}})$ to be selected subsequently.

- $\beta[\frac{10-8p}{3p} + (\frac{2-p}{2p})(\alpha-2)(\frac{2}{3}-\delta)] - (\frac{2-p}{p}) > 0.$
- $\{\frac{10}{3} - 2p\beta + (\alpha-2)(\frac{2}{3}-\delta) - p\} > 0.$
- $\frac{10}{3} - 2\beta + (\alpha-2)(\frac{2}{3}-\delta) - 1 > 0.$

This observation allows us to use nonlinear recurrence relation (4.28) (with $\beta \in (\frac{3}{\alpha}, \frac{1}{2}+\frac{\alpha}{3})$, $\delta > 0$ to be small, and $p \in (1, \frac{1+\delta}{1+\frac{\delta}{2}})$) to deduce that as long as $R > M_0 + 1$ is chosen to be sufficiently large, $U_1$ will become smaller than the universal constant $C_0^*$ as required by Lemma 3.2. According to Lemma 3.2, this smallness of $U_1$ will lead to the decay of $U_k$ to 0 as $k \to \infty$, and this in turn will lead to the conclusion that $|u| \leqslant R$ is valid over $[\frac{3}{4}, 1) \times \mathbb{R}^3$, for some sufficiently large constant $R$. Hence, it follows that the smoothness of $u$ can be extended beyond the possible blow up time 1.

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