METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 19, No. 2, pp. 187–210, June 2012
© 2012 International Press
005

A GEOMETRICAL VIEW OF THE NEHARI MANIFOLD*

JOSÉ MARIA GOMES†

Abstract. We study the Nehari manifold $\mathcal{N}$ associated to the boundary value problem

$$-\Delta u = f(u)\,, \quad u \in H_0^1(\Omega)\,,$$

where Ω is a bounded regular domain in $\mathbb{R}^n$. Using elementary tools from Differential Geometry, we provide a local description of $\mathcal{N}$ as an hypersurface of the Sobolev space $H_0^1(\Omega)$. We prove that, at any point $u \in \mathcal{N}$, there exists an exterior tangent sphere whose curvature is the limit of the increasing sequence of principal curvatures of $\mathcal{N}$. Also, the H^1-norm of $u \in \mathcal{N}$ depends on the number of principal negative curvatures. Finally, we study basic properties of an angle decreasing flow on the Nehari manifold associated to homogeneous non–linearities.

Key words. Nonlinear elliptic problems, Nehari manifold, principal curvature.

AMS subject classifications. 35J15, 35J25, 35J50, 53A07.

1. Introduction.

The variational method introduced by Nehari in [10]–[11] was a significant outcome of his research on the non–oscillating nature of solutions to certain classes of second order equations. For instance, concerning the linear problem

$$y'' + p(x)y = 0\,, \quad y(a) = y'(b) = 0\,,$$

where p is a continuous positive function, [Theorem 1, [9]] sets the equivalence between the existence of a positive solution in $[a, +\infty[$ and the fact that the lowest eigenvalue

$$\lambda := \min \frac{\int_a^b y'^2\,dx}{\int_a^b py^2\,dx}$$

satisfies $\lambda > 1$ for all $b > a$. In [8], a solution to the non–linear equation

$$y'' + p(x)y^{2n+1} = 0\,, \quad y(a) = y(b) = 0$$

with a prescribed number m of intermediate zeros $a < a_1 < ... < a_m < b$ is obtained by minimizing the functional

$$\tilde{J}(u; a_1, ..., a_m) := \sum_{\nu=1}^{m+1} [\tilde{J}_\nu]^{\frac{1}{n}}\,,$$

where $u \in C_0^{0,1}[a, b]$ satisfies $u(a_1) = ... = u(a_m) = 0$ and

$$\tilde{J}_\nu(w) = \frac{\left(\int_{a_\nu}^{a_{\nu+1}} w'^2\,dx\right)^{n+1}}{\int_{a_\nu}^{a_{\nu+1}} pw^{2n+2}\,dx}\,.$$

*Received March 6, 2012; accepted for publication November 8, 2012.
†Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia de Universidade Nova de Lisboa, 2829-516 Caparica, Portugal (jm.gomes@fct.unl.pt). This work is partially supported by FCT (Portuguese Foundation For Science and Technology) through PEst-OE/MAT/U10297/2011 (CMA).

is the Rayleigh coefficient on $C_0^{0,1}([a_\nu, a_{\nu+1}])$. Similar ideas were later exploited in [15] and [16]. In fact, as it was defined in [10], the "normalization condition" (known *a posteriori* as the Nehari constrain)

$$\int_a^b {y'}^2\, dx = \int_a^b y^2 F(y^2, x)\, dx \qquad (u \neq 0)\,,$$

was the basis of a more comprehensive method allowing the proof of the existence of solutions to a second order non-linear equation of type

$$y'' + yF(y^2, x) = 0\,,$$

where the non-homogeneous term prevented the method of minimizing a Rayleigh coefficient.

In the past few decades, the Nehari method has been extensively used on the study of existence of ground–state, nodal, multi-spike or multi-bump solutions, in what can be considered as a natural enlargement of Nehari's concerns about oscillatory aspects of second order non-linear differential equations (see for instance [2], [5],[6] and [13]). For the interested reader on an abstract treatment of the Nehari method (or on further references about the subject) we recommend the survey [14]. Our purpose to bring out a clearer picture of a variational framework known since 1960 was, in some sense, stimulated by the study of [3].

Along this article we consider the space $H_0^1(\Omega)$, where Ω is a bounded and regular domain of $\mathbb{R}^N$. We assume $H_0^1(\Omega)$ is endowed with the norm

$$\|u\|^2 = \langle u, u\rangle := \int_\Omega |\nabla u|^2(x)\, dx\,.$$

As usual, we denote $2^* = \frac{2N}{N-2}$ and $2^* = +\infty$ if $N = 2$, so that the embedding

$$H_0^1(\Omega) \subset L^q(\Omega)$$

is compact for $1 \leq q < 2^*$. Under well known assumptions on the non-linear term f (see, for instance [12]), solutions of the equation

$$-\Delta u = f(u) \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{1.1}$$

are critical points Euler-Lagrange functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2(x)\, dx - \int_\Omega F(u)(x)\, dx\,, \tag{1.2}$$

defined over $H_0^1(\Omega)$ where $F(u) = \int_0^u f(s)\, ds$. In our case, we require

(f1) $f \in C^2(\mathbb{R}, \mathbb{R})$.

(f2) $f(u)u \leq \beta f'(u)u^2$ where $0 < \beta < 1$.

(f3) There exist positive constants $\xi_1 \leq \xi_2$ such that

$$\xi_1 |u|^{p-2} \leq f'(u) \leq \xi_2 |u|^{p-2}\,,$$

where $2 < p < 2^*$.

Note that condition (f2) implies that $f(0) = 0$ as well as

$$\zeta F(u) \leq f(u)u\,, \tag{1.3}$$

for some $\zeta > 2$, which is the classical Ambrosetti-Rabinowitz condition. Condition (f3) implies

$$\frac{\xi_1}{p-1}|u|^p \leq f(u)u \quad \text{and} \quad \frac{\xi_1}{p(p-1)}|u|^p \leq F(u)\,. \tag{1.4}$$

Further, we will require

(f3') There exist positive constants $\xi_1 \leq \xi_2$ such that

$$\xi_1|u|^{p-2} \leq f''(u)u \leq \xi_2|u|^{p-2}\,.$$

Condition (f3') implies (f3) (adapting, if necessary, the constants ξ_1 and ξ_2).
The Nehari manifold is defined as

$$\mathcal{N} := \{u \in H_0^1(\Omega) : u \neq 0 \quad \text{and} \quad \langle \nabla J(u), u\rangle = 0\}. \tag{1.5}$$

Condition $\langle \nabla J(u), u\rangle = 0$ writes

$$\int_\Omega |\nabla u|^2(x)\,dx - \int_\Omega f(u)u(x)\,dx = 0\,. \tag{1.6}$$

Our paper is organized as follows. In section 2 we recall well known facts about the Nehari manifold. In particular, we prove that, given a sequence (u_j) of functions in $\mathcal{N}$ and a corresponding sequence of finite dimensional subspaces (V_j) such that $\lim \dim V_j = \infty$ and

$$D^2_{vv}J(u_j) \leq 0 \qquad \forall v \in V_j\,,$$

then

$$\lim_{j\to\infty} J(u_j) = +\infty\,. \tag{1.7}$$

In section 3, we use basic notions of Differential Geometry to describe the Nehari manifold as an hypersurface of $H_0^1(\Omega)$ (see for instance, [1] or [7]). More precisely, the tangent space $\mathfrak{T}_u$ of $\mathcal{N}$ at u is orthogonal to the normal vector

$$n(u) = \frac{N(u)}{\|N(u)\|}$$

with $N(u) = 2u + \Delta^{-1}(f'(u)u + f(u))$. As

$$L_u := Dn(u)[.] : \ \mathfrak{T}_u \mapsto \mathfrak{T}_u$$

is an operator of type $2(I + T_u)$, where I is the identity and T_u is compact, the Weingarten map L_u has a sequence of eigenvalues (k_n) that naturally extend the principal curvatures of an hypersurface in a finite-dimensional space. Our main result is the following:

THEOREM 1. *Suppose that conditions (f1), (f2) and (f3') are verified. Let* $u \in W_0^{1,\infty}(\Omega) \cap \mathcal{N}$ *and* (k_n) *be the corresponding increasing sequence of eigenvalues of the Weingarten map* L_u*. Then:*

1.

$$\lim_{n\to\infty} k_n = \frac{2}{\|N(u)\|}\,.$$

2. There exists $C > 0$ independent of u such that

$$-\frac{C(2+\|u\|^{2(p-2)/p})}{\|u\|} \le k_n(u) \le \frac{C}{\|u\|}\,, \qquad \forall n \in \mathbb{N}\,.$$

3. Suppose that

$$k_i(u) \le 0\,, \qquad i = 1, ..., n\,.$$

Then, there exist positive constants C_1 and C_2 independent of u such that

$$J(u) \ge \max\{C_1 \|e_n\|^{\frac{2p}{p-2}}\,,\, C_2\}\,,$$

where (e_n) is an orthogonal sequence of vectors in $H^1_0(\Omega)$ such that $\lim \|e_n\| = \infty$.

Assertion 3 in the above Theorem may be re-phrased as in (1.7) considering, instead of the dimension of the subspaces V_j, the number n_j of non-positive principal curvatures at u_j.

In the last section, we propose an alternative flow on the Nehari manifold (assuming an homogeneous nonlinearity) whose stable stationnary points are, under appropriate conditions, solutions of the second order equation

$$-\Delta u = f(u)\,, \qquad u \in H^1_0(\Omega)\,.$$

This work is a personal tribute to Nehari's pioneering works [10]–[11] fifty years after their publication. I thank Luis Sanchez and Pedro Girao for their interest and support.

2. Preliminary results. We define a sequence (e_n) in $H^1_0(\Omega)$ in the following way. Let e_1 be such that

$$\|e_1\|^2 = \min\left\{\|u\|^2 \,:\, \int_\Omega F(u)(x)\,dx = 1\right\}\,,$$

and for $n > 1$

$$\|e_n\|^2 = \min\left\{\|u\|^2 \,:\, \int_\Omega F(u)(x)\,dx = 1\,, \quad u \in (\,\mathrm{span}\{e_1, ..., e_{n-1}\})^\perp\right\}\,. \tag{2.1}$$

We have the following fact whose proof we postpone to the appendix:

The sequence (e_n) is an orthogonal basis of $H^1_0(\Omega)$. Also $(\|e_n\|)$ is non-decreasing and

$$\lim_{n\to\infty} \|e_n\| = \infty.$$

Remark 1. *Each e_n satisfies the relation*

$$-\Delta e_n = \lambda_{nn} f(e_n) + \sum_{i=1}^{n-1} \lambda_{ni}(-\Delta e_i) \tag{2.2}$$

for some Lagrange multipliers λ_{ni}. In particular, $e_n \in C^{3,\alpha}(\Omega) \cap C_0(\overline{\Omega})$. Multiplying (2.2) by e_n, and integrating by parts we conclude

$$\lambda_{nn} = \frac{\|e_n\|^2}{\int_\Omega f(e_n) e_n(x)\, dx} > 0\,.$$

A similar argument yields, for all $m > n$,

$$0 = \int_\Omega \nabla e_n \nabla e_m(x)\, dx = \lambda_{nn} \int_\Omega f(e_n) e_m(x)\, dx. \tag{2.3}$$

Then (2.3) implies

$$\text{for all } m > n \qquad \langle \nabla J(e_n), e_m \rangle = 0\,.$$

In the next Proposition we obtain estimates on a function $u \in \mathcal{N}$ based on the dimension of a space where the second derivative of J at u is negative definite.

Proposition 2. *Assume $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies (f2)–(f3). Let $u \in \mathcal{N}$ and V_j be a j-dimensional subspace of $H_0^1(\Omega)$ such that*

$$D^2 J_{vv}(u) \le 0 \quad \text{for all } v \in V_j\,. \tag{2.4}$$

Then

$$J(u) \ge \max\{C_1 \|e_j\|^{\frac{2p}{p-2}}\,,\, C_2\},$$

where e_j was defined in (2.1) and C_1, C_2 are positive constants independent of u.

Proof. By (1.6), assumption (f2) and Sobolev's Embedding Theorem we have, for some constant c_p,

$$\|u\|^2 \le \beta\xi_2 \int_\Omega |u|^p(x)\, dx \le \beta\xi_2 c_p \|u\|^p. \tag{2.5}$$

Then, for $C = (\beta\xi_2 c_p)^{-\frac{1}{p-2}}$, we conclude

$$\|u\| \ge C\,. \tag{2.6}$$

By (1.2), (1.3) and (1.6),

$$J(u) \ge \left(\frac{1}{2} - \frac{1}{\zeta}\right) \|u\|^2\,. \tag{2.7}$$

The previous estimates prove that $J(u) \ge C_2$ with $C_2 = (1/2 - 1/\zeta)C^2$.

Let

$$S = \{v \in V_j : \|v\| = 1\}\,.$$

We have $\gamma(S) = j$ where γ is the the genus of a closed symmetric set (see [12]). Let

$$E_j = (\text{span}\{e_1, ..., e_{j-1}\})^\perp\,.$$

Since $\gamma(S) >$ codimension E_j, we conclude by [Proposition 7.8, [12]] that

$$S \cap E_j \neq \emptyset\,.$$

We may therefore choose $v \in V_j \cap E_j$ and, multiplying if necessary by an appropriate constant, assume $\int_\Omega F(v)(x)\,dx = 1$. We have

$$D^2 J_{vv}(u) = \int_\Omega |\nabla v|^2(x)\,dx - \int_\Omega f'(u)v^2(x)\,dx \leq 0\,. \tag{2.8}$$

By (2.8), Holder inequality and (1.4),

$$\begin{aligned} \int_\Omega |\nabla v|^2(x) &\leq \left(\int_\Omega |f'(u)|^{\frac{p}{p-2}}(x)\,dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |v|^p(x)\,dx\right)^{\frac{2}{p}} \leq \\ &K\left(\int_\Omega |u|^p(x)\,dx\right)^{\frac{p-2}{p}} \left(\int_\Omega F(v)(x)\,dx\right)^{\frac{2}{p}} = K\left(\int_\Omega |u|^p(x)\,dx\right)^{\frac{p-2}{p}} \end{aligned} \tag{2.9}$$

where $K = \xi_2^{\frac{p-2}{p}} \left(\frac{(p-1)p}{\xi_1}\right)^{\frac{2}{p}}$. By the definition of (e_n) and our assumptions on v we have,

$$\int_\Omega |\nabla v|^2(x)\,dx \geq \int_\Omega |\nabla e_j|^2(x)\,dx. \tag{2.10}$$

We conclude, by (2.7), (1.4), (2.9) and (2.10)

$$\begin{aligned} J(u) \geq \left(\frac{1}{2} - \frac{1}{\zeta}\right) \int_\Omega |\nabla u|^2(x)\,dx = \left(\frac{1}{2} - \frac{1}{\zeta}\right) \int_\Omega f(u)u(x)\,dx \geq \\ \left(\frac{1}{2} - \frac{1}{\zeta}\right) \frac{\xi_1}{p-1} \int_\Omega |u|^p(x)\,dx \geq C_1 \|e_j\|^{\frac{2p}{p-2}} \end{aligned} \tag{2.11}$$

where $C_1 = K^{-p/(p-2)} \left(\frac{1}{2} - \frac{1}{\zeta}\right) \frac{\xi_1}{p-1}$. □

Remark 2. *We conclude from Proposition 2 and Lemma 6 that, given a sequence* (u_j) *of functions in* $\mathcal{N}$ *and a corresponding sequence of finite dimensional subspaces* (V_j) *such that* $\lim dim\, V_j = \infty$ *and*

$$D^2_{vv} J(u_j) \leq 0 \qquad \forall v \in V_j\,,$$

then

$$\lim_{j\to\infty} J(u_j) = +\infty\,.$$

Given $u \in \mathcal{N}$ the tangent space $\mathfrak{T}_u$ of $\mathcal{N}$ at u consists on the functions $v \in H^1_0(\Omega)$ such that

$$2\int_\Omega \nabla u \nabla v\,(x)\,dx - \int_\Omega f'(u)uv(x)\,dx - \int_\Omega f(u)v(x)\,dx = 0\,. \tag{2.12}$$

The next proposition sets some well–known facts.

Proposition 3. *Assume f satisfies (f1)–(f3). There exists $C' > 0$ such that*

$$u \in \mathcal{N} \Rightarrow \|u\| \geq C' . \tag{2.13}$$

Moreover, $\mathcal{N}$ is locally diffeomorphic to

$$S := \{u \in H_0^1(\Omega), \|u\| = 1\} .$$

Given $u \in \mathcal{N}$,

$$\nabla J(u) = 0 \quad \Leftrightarrow \quad \Pi_u(\nabla J(u)) = 0 , \tag{2.14}$$

where Π_u is the orthogonal projection on $\mathfrak{T}_u$.

Proof. Condition (2.13) was already proved in Proposition 2. Given $u \in H_0^1(\Omega)\backslash\{0\}$, consider the function

$$g(t) := \langle \nabla J(tu), tu \rangle = t^2 \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega t f(tu) u \, (x) \, dx .$$

By (f2)–(f3), we have $g(t) > 0$ if $0 < t < \epsilon$ for ϵ sufficiently small. Also

$$\lim_{t \to +\infty} g(t) = -\infty .$$

Therefore there exists $t_0 > 0$ such that $g(t_0) = 0$. By (1.6) and (f2),

$$g'(t_0) = 2t_0 \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega f(t_0 u)u + f'(t_0 u)u^2 \, dx < 0 .$$

Consequently, $t_0 > 0$ is uniquely determined. Also, by the Implicit Function Theorem,

$$t_0(u) \in C^2(H_0^1(\Omega)\backslash\{0\}), \mathbb{R}\backslash\{0\}) .$$

Consider the C^2–application

$$P_{\mathcal{N}} : H_0^1(\Omega)\backslash\{0\} \mapsto \mathcal{N} \qquad u \to t_0(u)u .$$

Clearly, the restriction

$$P_{\mathcal{N}}|_S \mapsto \mathcal{N}$$

is a local diffeomorphism.

We now turn to (2.14). The first implication is trivial. Consider the constraint $\phi(u) := \langle \nabla J(u), u \rangle = 0$. By (f2), for any $u \in \mathcal{N}$,

$$\langle \nabla \phi(u), u \rangle = \int_\Omega |\nabla u|^2(x) \, dx - \int_\Omega f'(u)u^2(x) \, dx = \int_\Omega f(u)u - f'(u)u^2(x) \, dx < 0 ,$$

i.e., $u \notin \mathfrak{T}_u$. Then, $\Pi_u(\nabla J(u)) = 0$ and $\langle \nabla J(u), u \rangle = 0$ imply $\nabla J(u) = 0$. □

3. Local geometry of the Nehari manifold. In this section we prove Theorem 1. We assume that assumptions (f1), (f2) and (f3') are verified. We denote by Δ^{-1} the inverse of the Laplacian operator with Dirichlet boundary conditions. Using the Riesz representation of a linear functional in $H_0^1(\Omega)$ and (2.12), the tangent space can also be characterized as

$$\mathfrak{T}_u := \{v \in H_0^1(\Omega) : \quad \langle N(u), v\rangle = 0\},$$

with $N(u) = 2u + \Delta^{-1}(h(u))$ and

$$h(u) = f'(u)u + f(u)\,. \tag{3.1}$$

We recall that, by standard regularity theory (see for instance [4]), the operator $u \mapsto \Delta^{-1}(h(u))$ maps $H_0^1(\Omega)$ into itself and is compact. Prescribe

$$n(u) = \frac{N(u)}{\|N(u)\|}\,,$$

as unitary normal to $\mathfrak{T}_u$. By (f2),

$$\langle n(u), u\rangle < 0 \tag{3.2}$$

for all $u \in \mathcal{N}$. Our assumptions on f imply that the map $u \to n(u)$ is of class C^1 in $H_0^1(\Omega)\backslash\{0\}$. Given $u \in \mathcal{N}$, we formally define a Weingarten map

$$L_u : \mathfrak{T}_u \mapsto \mathfrak{T}_u \qquad L_u(v) = Dn(u)[v]\,.$$

In fact, given $u \in \mathcal{N}$, $v \in \mathfrak{T}_u$ and a regular path γ such that

$$\gamma : \quad]-1,1[\mapsto \mathcal{N}\,, \quad \gamma(0) = u\,, \quad \gamma'(0) = v,$$

we have

$$\langle n(\gamma(t)), n(\gamma(t))\rangle = 1 \qquad \forall t \in]-1,1[\,.$$

In particular

$$\langle Dn(\gamma(0))[\gamma'(0)], n(\gamma(0))\rangle = 0\,,$$

i.e.

$$Dn(u)[v] \in \mathfrak{T}_u$$

for all $v \in \mathfrak{T}_u$. We also recall the classical formula

$$Dn(u)[v] = -D\Pi_u(v, n(u))\,. \tag{3.3}$$

Computing,

$$Dn(u)[v] = \frac{1}{\|N(u)\|}\left(2v + \Delta^{-1}(h'(u)v) - n(u)\left\langle 2v + \Delta^{-1}(h'(u)v)\,,\, n(u)\right\rangle\right)\,. \tag{3.4}$$

If we assume $u \in W_0^{1,\infty}(\Omega) \subset H_0^1(\Omega)$ the operator

$$L_u(v) := Dn(u)[v] = \frac{1}{\|N(u)\|}(2I + T_u(v))$$

where

$$T_u(v) = \Delta^{-1}(h'(u)v) - n(u)\left\langle 2v + \Delta^{-1}(h'(u)v)\,,\, n(u)\right\rangle$$

is well-defined for all $v \in H_0^1(\Omega)$. Moreover the operator

$$T_u : \mathfrak{T}_u \mapsto \mathfrak{T}_u$$

is self-adjoint and compact (note that the term $\left\langle 2v + \Delta^{-1}(h'(u)v)\,,\, n(u)\right\rangle$ maps into $\mathbb{R}$). We may therefore provide an orthogonal basis for $\mathfrak{T}_u$ of eigenvectors of T_u. To an eigenvector v of T_u with associated eigenvalue λ corresponds the same eigenvector v of L_u with associated eigenvalue

$$k = \frac{2+\lambda}{\|N(u)\|}\,. \tag{3.5}$$

Remark 3. *The assumption that $u \in W_0^{1,\infty}(\Omega)$ may be weakened. Consider the case where Ω is a bounded regular subset of $\mathbb{R}^2$. As $H_0^1(\Omega) \subset L^q(\Omega)$ for any $q \in [1, +\infty[$ with compact embedding, the application of $H_0^1(\Omega)$ into $H_0^1(\Omega)^*$ defined by*

$$v \mapsto H_v\,, \quad H_v(w) \equiv \int_\Omega h'(u)vw\,dx \quad (w \in H_0^1(\Omega))$$

is compact. Consequently, identifying $H_0^1(\Omega)$ with its dual, we conclude that $\Delta^{-1}(h'(u)v)$ is self-adjoint and compact so that the principal curvatures are defined for all $u \in H_0^1(\Omega) \cap \mathcal{N}$. However, the class of functions in $W_0^{1,\infty}(\Omega)$ is of special interest regarding its invariance property for relevant energy decreasing flows associated to Euler-Lagrange functionals.

We have the following property of the non-zero eigenvalues of the compact operator T_u.

Lemma 1. *Given $u \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$, the distinct non-zero eigenvalues of T_u form an increasing sequence $(\lambda_n(u))$ converging to zero.*

Proof. As usual, we determine the sequence of the non-zero eigenvalues and corresponding eigenvectors of T_u by means of a recurrent sequence of minimization problems:

$$\lambda_n := \inf\left\{\langle T_u(v), v\rangle : v \in \mathfrak{T}_u\,, \|v\| = 1\,, v \in (\mathrm{span}\{v_1, ..., v_{n-1}\})^\perp\right\}$$

and a corresponding eigenvector v_n is a function where the infimum is attained. Necessarily, (λ_n) is an increasing sequence. In case $\lambda_{n+1} = \lambda_n$ the eigenvalue λ_n has multiplicity greater than 1. Since $\langle n(u), v\rangle = 0$, we have

$$\langle T_u(v), v\rangle = \langle \Delta^{-1}(h'(u)v), v\rangle = -\int_\Omega h'(u)v^2(x)\,dx\,,$$

and conclude $\lambda_n \leq 0$ for all $n \in \mathbb{N}$.

Assume, for some n, that $\lambda_n = 0$ and $\lambda_{n-1} < 0$. Then for any $k \geq n$, we have $\lambda_k = 0$ and a corresponding eigenfunction v_k satisfies

$$\int h'(u)v_k^2(x)\,dx = 0\,.$$

Then, by (f3),

$$v_k \equiv 0 \qquad \text{in} \quad \text{supp}(u) \qquad \forall k \geq n\,.$$

As any w such that

$$\text{support}(w) \subset \text{support}(u)$$

is orthogonal to v_k with $k \geq n$, w necessarily belongs to $\text{span}\{v_1, ..., v_{n-1}\}$. This would imply, for any bounded regular domain ω such that $\omega \subset \text{supp}(u)$,

$$\big(H_0^1(\omega) \cap \mathfrak{T}_u\big) \subset \text{span}\{v_1, ..., v_{n-1}\}$$

which is absurd since the first subspace is infinite dimensional. □

If

$$\int_\Omega h'(u) v^2(x)\, dx \,>0\,, \quad \forall v \in \mathfrak{T}_u \backslash \{0\}\,,$$

the sequence (v_i) of eigenvectors associated to the sequence of non-zero eigenvalues (λ_i) provides an Hilbert basis of $\mathfrak{T}_u$. This is the case if $u(x) \neq 0$ a.e. in Ω. In general, we may write

$$\mathfrak{T}_u = \text{Ker}(T_u) \oplus R(T_u)\,,$$

where $R(T_u)$ is the closure of the subspace generated by the family $\{v_i\}$.

In view of (3.5), we will refer an eigenvalue k_i of L_u as a (signed) principal curvature of $\mathcal{N}$ at u if the corresponding eigenvalue λ_i of T_u satisfies $\lambda_i < 0$. The sequence (k_i) is increasing and converges to $2/\|N(u)\|$. We denote by $\mathfrak{K}_u$ the set of all eigenvalues of L_u. We have

$$\mathfrak{K}_u \subseteq \{k_i\}_{i \in \mathbb{N}} \cup \{2/\|N(u)\|\}\,, \tag{3.6}$$

with equality of sets in the degenerate case $\text{Ker}(T_u) \neq \{0\}$. In particular, at any point $u \in \mathcal{N}$, the principal curvatures are positive, except at most for a finite number.

Remark 4. *Let P be a plane containing the inward normal $n(u)$ and a direction $v(u)$ associated to a positive curvature. Using the reference frame of center u and vectors $v(u)$ and $n(u)$, if $w \in P \cap \mathcal{N} \backslash \{u\}$ is sufficiently close to u, then*

$$w = x\, v(u) + y\, n(u) \quad \text{with} \quad (x, y) \in \mathbb{R}^2\,, \quad y < 0\,.$$

In view of (3.6), we may locally describe the Nehari manifold saying that, at any point $u \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$, there exists an "exterior" tangent sphere to the Nehari manifold, with center

$$C(u) = u - \frac{\|N(u)\|}{2} \cdot n(u) = -\frac{1}{2}\Delta^{-1}(h(u))\,,$$

and radius $\|N(u)\|/2$, whose curvature is the limit of the sequence of principal curvatures (k_i).

We have the following estimates on the curvatures of the Nehari manifold.

LEMMA 2. *There exists $C > 0$ such that, for every $u \in W_0^{1,\infty}(\Omega) \cap \mathcal{N}$ and $i \in \mathbb{N}$*

$$-\frac{C(2 + \|u\|^{2(p-2)/p})}{\|u\|} \le k_i(u) \le \frac{C}{\|u\|}. \tag{3.7}$$

Proof. As

$$\langle N(u), u\rangle = 2\|u\|^2 - \int f(u)u\,dx - \int f'(u)u^2\,dx$$

by (1.6) and (f2)

$$|\langle N(u), u\rangle| \ge \frac{1-\beta}{\beta} \cdot \|u\|^2\,,$$

and, by Schwarz inequality,

$$\|N(u)\| \ge \frac{1-\beta}{\beta}\,\|u\|\,. \tag{3.8}$$

In view of (3.5), we conclude from Lemma 1 and (3.8) the right hand-side of (3.7). In order to prove the complete estimate it suffices to set the inequality to k_1. Assume $\|v\| = 1$. Necessarily

$$\lambda_1 \ge \lambda := \min_{\|v\|=1} -\int_\Omega h'(u)v^2(x)\,dx\,.$$

By (f3') and (3.1),

$$h'(u) \le C_1\,|u|^{p-2}$$

for $C_1 = \xi_2/(p-1)$. Then, by Holder inequality, (1.6) and Sobolev Imbedding Theorem, for some constant $C_2 > 0$

$$\begin{aligned}\int_\Omega h'(u)v^2(x)\,dx &\le C_1\left(\int_\Omega |u|^p(x)\,dx\right)^{\frac{p-2}{p}}\left(\int_\Omega |v|^p(x)\,dx\right)^{\frac{2}{p}}\\ &\le C_2\left(\int_\Omega f(u)u(x)\,dx\right)^{(p-2)/p} = C_2\|u\|^{2(p-2)/p}\,,\end{aligned} \tag{3.9}$$

thereby proving inequality (3.7). ◻

REMARK 5. *Note that, if $p \le 4$, the curvatures are uniformly bounded below on the Nehari manifold by a negative constant. In particular, there exists $\overline{K} > 0$ such that, for all $u \in \mathcal{N}$,*

$$|k_i(u)| \le \overline{K} \qquad \forall i \in \mathbb{N}\,.$$

Analogously to Proposition 2, we obtain lower bounds on the the energy of $u \in \mathcal{N}$ based on the number of negative principal curvatures of the Weingarten map L_u.

LEMMA 3. *Let $u \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$ be such that*

$$k_i(u) \le 0\,, \qquad i = 1, ..., j\,.$$

Then, there exist positive constants C_1 and C_2 independent of u such that

$$J(u) \geq \max\{C_1 \|e_j\|^{\frac{2p}{p-2}}, C_2\},$$

where e_j was defined in (2.1).

Proof. The proof is similar to the proof of Proposition 2 so we omit the details. Consider the subspaces

$$V_j := \text{span}\{v_1, ...v_{\bar{j}}\} \quad \text{and} \quad E_j = (\text{span}\{e_1, ..., e_{j-1}\})^{\perp}$$

where the v_i's are eigenvectors associated to $k_1, ..., k_j$ (necessarily, $\bar{j} \geq j$). For any $v \in V_j$,

$$\langle Dn_u(v), v\rangle = \frac{1}{\|N(u)\|}\langle 2v + T_u(v), v\rangle = \frac{1}{\|N(u)\|}\left(2\|v\|^2 - \int_\Omega h'(u)v^2(x)\,dx\right) \leq 0\,, \tag{3.10}$$

or

$$\|v\|^2 - \frac{1}{2}\int_\Omega h'(u)v^2(x)\,dx \leq 0\,.$$

As in Lemma 2, we may choose $v \in V_j \cap E_j$ such that $\int_\Omega H(v)(x)\,dx = 1$ for $H(v) = \int_0^v h(s)\,ds$. Recalling that, by (3.1), $h'(u) = 2f'(u) + f''(u)u$, we may estimate as in (2.9)–(2.11) and conclude the proof. ◻

Remark 6. *We may assert the existence of points on the Nehari manifold with an arbitrarily large number of negative principal curvatures. In fact, let us consider a multi-bump function*

$$u := \sum_{k=1}^{n} v_k$$

where, for $i \neq j$,

$$\textit{support}\,(v_i) \cap \textit{support}\,(v_j) = \emptyset$$

and

$$v_k \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$$

for all $k = 1, ..., n$. As

$$2\|v_i\|^2 - \int_\Omega h'(u)v_i^2(x)\,dx = 2\|v_i\|^2 - \int_\Omega h'(v_i)v_i^2(x)\,dx < 0$$

and the set of functions $\{v_i\}_{i=1,...,n}$ is orthogonal, we may conclude the existence of $n-1$ orthogonal vectors $\tilde{v}_i$ in $\mathfrak{T}_u$ such that $\langle L_u(\tilde{v}_i), \tilde{v}_i\rangle < 0$. In particular

$$k_1 < ... < k_{n-1} < 0\,,$$

where k_i is the sequence of eigenvalues of L_u.

4. An angle-decreasing flow. In the next section, we assume

$$f(u) = \begin{cases} c_1|u|^{p-2}u & \text{if } u \le 0 \\ c_2|u|^{p-2}u\,, & \text{if } u > 0\,, \end{cases} \tag{4.1}$$

where $c_1, c_2 > 0$. In case where the non-linearity f is as in (4.1), then

$$J(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 \quad \forall u \in \mathcal{N}\,.$$

In particular, critical points of the distance functional $u \mapsto \|u\|$ constrained to $\mathcal{N}$ are solutions of (1.1).

We introduce an auxiliary functional on the Nehari manifold:

$$\theta_u \equiv \theta(u) =: \left\langle n(u), \frac{u}{\|u\|} \right\rangle .$$

The functional θ is the restriction to $\mathcal{N}$ of a functional of class $C^1(H_0^1(\Omega)\backslash\{0\}, \mathbb{R})$ that we will also denote by θ. Note that, by (3.2) and Schwarz inequality

$$\theta(\mathcal{N}) \subset [-1, 0[\,.$$

Also, $\arccos(\theta_u)$ corresponds to the angle between the vectors u and $n(u)$.

Assuming $u \in W_0^{1,\infty}(\Omega)$, we use our previous decomposition of the tangent space $\mathfrak{T}_u$ to calculate

$$\Pi_u(\nabla\theta_u)\,.$$

For any $v \in \mathfrak{T}_u$,

$$\langle \nabla\theta_u, v\rangle = D\theta_u(v) = \left\langle Dn(u)[v], \frac{u}{\|u\|} \right\rangle - \langle n(u), u\rangle \frac{\langle u, v\rangle}{\|u\|^3}. \tag{4.2}$$

Choosing v an eigenvector with corresponding eigenvalue k, as $\langle n, v\rangle = 0$ we obtain by (3.4),

$$\langle \nabla\theta_u, v\rangle = \left(k - \frac{\theta_u}{\|u\|}\right)\left\langle v, \frac{u}{\|u\|}\right\rangle . \tag{4.3}$$

We may write, in the non-degenerate case $\mathrm{Ker}(T_u) = \{0\}$,

$$\Pi_u(\nabla\theta_u) = \frac{1}{\|u\|}\sum_{i=1}^{\infty}\left(k_i - \frac{\theta_u}{\|u\|}\right)\langle v_i, u\rangle \cdot v_i\,. \tag{4.4}$$

More generally, denoting by Π_u^0 the projection on $\mathrm{Ker}(T_u) \subset \mathfrak{T}_u$,

$$\Pi_u(\nabla\theta_u) = \frac{1}{\|u\|}\sum_{i=1}^{\infty}\left(k_i - \frac{\theta_u}{\|u\|}\right)\langle v_i, u\rangle \cdot v_i + \frac{1}{\|u\|}\left(\frac{2}{\|N(u)\|} - \frac{\theta_u}{\|u\|}\right)\cdot \Pi_u^0(u). \tag{4.5}$$

Remark 7. *Using (4.4)–(4.5) and Lemma 2, a simple estimate shows that, for some $C > 0$,*

$$\|\Pi_u(\nabla\theta_u)\| \le C\|u\|^{(p-4)/p} \le C\|u\|\,, \qquad \forall u \in \mathcal{N}\,.$$

In case $\nabla J(u) = 0$ then $\nabla\theta_u = 0$ but the inverse is not true. However, in case $\theta_u/\|u\| \notin \mathfrak{K}_u$,

$$\nabla J(u) = 0 \quad \Leftrightarrow \quad \nabla\theta_u = 0\,.$$

In the next proposition we establish the existence of an "angle–decreasing" flow.

PROPOSITION 4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and regular domain and*

$$\Phi : W_0^{1,\infty}(\Omega) \cap \mathcal{N} \mapsto W_0^{1,\infty}(\Omega)\,, \quad \Phi(u) = \Pi_u(\nabla\theta_u)\,.$$

Given $u_0 \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$, the initial value problem

$$\eta(0, u_0) = u_0\,, \qquad \frac{d\eta}{dt}(t) = -\Phi(\eta(t, u_0))\,. \tag{4.6}$$

has a unique solution

$$\eta : \mathcal{N} \cap W_0^{1,\infty}(\Omega) \times [0, \tau_0[\mapsto \mathcal{N} \cap W_0^{1,\infty}(\Omega)\,,$$

for some $\tau_0 > 0$. In case Ω is a bounded regular domain of $\mathbb{R}^2$, then the same conclusion holds with $\tau_0 = +\infty$.

The proof of Proposition 4 will follow from a sequence of lemmas.

LEMMA 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Define*

$$\Psi : W_0^{1,\infty}(\Omega) \mapsto W_0^{1,\infty}(\Omega)\,, \quad u \to \Delta^{-1}(f(u))\,.$$

Then Ψ is locally Lipschitz continuous.

Proof. Trivially, $W_0^{1,\infty}(\Omega) \subset C_0(\Omega)$ with continuous injection. By standard regularity theory (see [4], theorems 8.33-8.34) we have $\Delta^{-1}(f(u)) \in C_0^{1,\alpha}\left(\overline{\Omega}\right)$ so Ψ is well-defined.

Let $B_\epsilon(u)$ be the ball of radius ϵ and center u in $C_0(\Omega)$. By our assumptions on f, for any $v \in B_\epsilon(u)$, we have

$$|f(u) - f(v)| \le K_\epsilon |u - v|\,,$$

for some $K_\epsilon > 0$. We conclude that the functional

$$\psi : W_0^{1,\infty}(\Omega) \mapsto C\left(\overline{\Omega}\right)\,, \qquad u \to f(u)$$

is locally Lipschitz continuous. Since $\Delta^{-1} : C\left(\overline{\Omega}\right) \mapsto C_0^{1,\alpha}(\Omega)$ is Lipschitz continuous, we conclude that $\Psi = \Delta^{-1} \circ \psi$ is locally Lipschitz continuous. The proof is complete. □

REMARK 8. *With similar arguments, we may prove that, for locally Lipschitz functions $f, g : \mathbb{R} \mapsto \mathbb{R}$,*

$$u \mapsto \Delta^{-1}[\Delta^{-1}(f(u))g(u)]$$

is locally Lipschitz continuous in $W_0^{1,\infty}(\Omega)$.

LEMMA 5. *Let*

$$\Phi : W_0^{1,\infty}(\Omega) \cap \mathcal{N} \mapsto W_0^{1,\infty}(\Omega)\,, \qquad \Phi(u) = \Pi_u(\nabla\theta_u)\,.$$

For any $u_1 \in W_0^{1,\infty}(\Omega) \cap \mathcal{N}$*, there exists a* $W^{1,\infty}$*-ball* B_1 *centered at* u_1 *and a Lipschitz continuous function*

$$F : B_1 \cap W_0^{1,\infty}(\Omega) \mapsto W_0^{1,\infty}(\Omega)$$

such that

$$F(u) = \Phi(u)\,, \qquad \forall u \in B_1 \cap W_0^{1,\infty}(\Omega) \cap \mathcal{N}\,.$$

Proof. Let B_1 be a $W^{1,\infty}$-ball centered at u_1 such that $\|u\|$ and $\|N(u)\|$ are uniformly bounded below by a positive constant in B_1. We consider the following extensions $N, n : B_1 \mapsto W_0^{1,\infty}(\Omega)$ and $\theta : B_1 \mapsto \mathbb{R}$,

$$N(u) = 2u + \Delta^{-1}(h(u))\,, n(u) = \frac{N(u)}{\|N(u)\|}\,, \theta(u) = \left\langle n(u), \frac{u}{\|u\|} \right\rangle\,.$$

In the homogeneous case, $h(u) = f(u) + f'(u)u = pf(u)$. Then,

$$\theta(u) = \frac{1}{\|N(u)\| \cdot \|u\|} \left(2\|u\|^2 - p \int_\Omega f(u)u\,dx \right)\,.$$

Define

$$J_1(u) = \|N(u)\|^2 = 4\|u\|^2 + 4p\langle u, \Delta^{-1}(f(u))\rangle + p^2\|\Delta^{-1}(f(u))\|^2\,,$$

and

$$J_2(u) = \left(2\|u\| - \frac{p}{\|u\|} \int_\Omega f(u)u\,dx \right)\,,$$

so that

$$\theta(u) = \frac{J_2(u)}{\sqrt{J_1(u)}}\,. \tag{4.7}$$

By (3.8) and Proposition 3, we have, for all $u \in \mathcal{N}$,

$$J_1(u) \geq \frac{C'(1-\beta)}{\beta} > 0\,.$$

As the square root function is Lipschitz in any interval $[\delta, +\infty[$ with $\delta > 0$, in order to prove that $\nabla\theta$ is locally Lipschitz it will suffice to prove that $\nabla\theta_1$ and $\nabla\theta_2$ are locally Lipschitz. By Lemma 4 $J_1 : W_0^{1,\infty}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Moreover

$$\begin{aligned} &\langle \nabla J_1(u), v\rangle = \\ &8\langle u, v\rangle + 4p\langle v, \Delta^{-1}(f(u))\rangle + \\ &4p\langle u, \Delta^{-1}(f'(u)v)\rangle + 2p^2\langle \Delta^{-1}(f'(u)v), \Delta^{-1}(f(u))\rangle\,. \end{aligned} \tag{4.8}$$

As

$$\langle u, \Delta^{-1}(f'(u)v)\rangle = -\int_\Omega f'(u)uv\,dx = (p-1)\langle v, \Delta^{-1}(f(u))\rangle\,,$$

and

$$\langle \Delta^{-1}(f'(u)v), \Delta^{-1}(f(u))\rangle = (p-1)\langle v, \Delta^{-1}[\Delta^{-1}(f(u))f'(u)]\rangle\,,$$

we conclude that

$$\nabla J_1(u) = 8u + 4p^2\Delta^{-1}(f(u)) + 2p^2(p-1)\Delta^{-1}[\Delta^{-1}(f(u))f'(u)]\,. \tag{4.9}$$

Then, by Remark 8, we conclude that $\nabla J_1 : W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ is locally Lipschitz continuous. Similarly, we may prove that $J_2 : W_0^{1,\infty}(\Omega) \to \mathbb{R}$ and $\nabla J_2 : W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ are locally Lipschitz continuous.

Finally, writing

$$F(u) := \nabla\theta(u) - \langle \nabla\theta(u), n(u)\rangle n(u)$$

we conclude from (4.7) that

$$F : W_0^{1,\infty}(\Omega) \cap B_1 \mapsto W_0^{1,\infty}(\Omega)\,,$$

is locally Lipschitz continuous and

$$F(u) = \Pi_u(\nabla\theta_u)\,, \quad \forall u \in \mathcal{N}\,. \qquad \square$$

Proof of Proposition 4. Assuming $[0,\tau_0[$ is the maximal domain of $\eta(t,u_0)$ in $W_0^{1,\infty}(\Omega)$, one easily verifies that $\eta(t,u_0) \in \mathcal{N}$ for all $t \in [0,\tau_0[$. Consider the case where Ω is a bounded regular domain of $\mathbb{R}^2$. Suppose, in view of a contradiction, that $\tau_0 < \infty$. Then, by Remark 7 and classical Gronwall estimates, as $t \to \tau_0$ necessarily $\eta(t,u_0) \to w \in \mathcal{N}$ in H^1-norm. Consider the H^1-ball $B_R(w)$ centered at w and radius $R = \|w\|/2$. Noting that $B_R(w)$ is bounded in $L^q(\Omega)$ for arbitrarily large q, by standard regularity theory (see section 8.11–[4]), we have, for all $u \in B_R(w)$,

$$\|\Delta^{-1}(f(u))\|_{W^{1,\infty}(\Omega)} \le \|\Delta^{-1}(f(u))\|_{C^{1,\alpha}(\Omega)} \le C$$

and

$$\|\Delta^{-1}[\Delta^{-1}(f(u))f'(u)]\|_{W^{1,\infty}(\Omega)} \le \|\Delta^{-1}[\Delta^{-1}(f(u))f'(u)]\|_{C^{1,\alpha}(\Omega)} \le C$$

for some $C > 0$. Also, $\|u\|$ and $\|N(u)\|$ are uniformly bounded below in $B_R(w)$ by a positive constant. Adapting the arguments in Lemma 5 we may consider $F : B_R(w) \cap W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ such that

$$\Pi_\eta(\nabla\theta_\eta) = F(\eta)\,, \quad \forall \eta \in B_R(w) \cap W_0^{1,\infty}(\Omega) \cap \mathcal{N}$$

and, for some $K_B > 0$,

$$|F(w_1) - F(w_2)|_{W^{1,\infty}} \le K_B|w_1 - w_2|_{W^{1,\infty}}\,, \;\; \forall\, w_1, w_2 \in B_R(w) \cap W_0^{1,\infty}(\Omega)\,.$$

Then there exists a constant ϵ such that, for any $w' \in B_{R/2}(w) \cap W_0^{1,\infty}(\Omega)$, the maximal domain of definition in $W_0^{1,\infty}(\Omega)$ of $\eta(w',t)$ contains $[0,\epsilon[$. This implies that the maximal domain of $\eta(t,u_0)$ contains $[0,\tau_0+\epsilon[$, contradicting our assumption on τ_0. $\square$

Remark 9. *In case Ω is a bounded regular domain in $\mathbb{R}^2$, similar estimates to the ones in Lemmas 4–5 (with $W_0^{1,\infty}(\Omega)$ replaced by $H_0^1(\Omega)$) allow us to prove the existence and uniqueness of the solution to (4.6) when $u_0 \in \mathcal{N}$.*

In the next proposition we prove some essential facts allowing us a clearer understanding of the angle decreasing flow.

Proposition 5. *Let $\eta(t,u_0)$ be the flow defined in (4.6).*
(i) For $0<t_1<t_2<\tau_0$

$$-1 \le \theta(\eta(t_2,u_0)) \le \theta(\eta(t_1,u_0)) < 0\,. \tag{4.10}$$

(ii) Let $t_0\in[0,\tau_0[$ and denote $\eta=\eta(t_0,u_0)$. Then
(4.11)
$$\frac{d}{dt}\left(\frac12\|\Pi_\eta\eta\|^2\right)(t_0)=\sum_{i=1}^{\infty}\left(k_i(\eta)-\frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i(\eta)-\frac{1}{\|\eta\|}\right)\langle\eta,v_i(\eta)\rangle^2+K_1\|\Pi^0_\eta(\eta)\|^2\,.$$

where $K_1=(2/\|N(\eta)\|-\theta_\eta\|\eta\|^{-1})(2\theta_\eta/\|N(\eta)\|-\|\eta\|^{-1})$ and

$$\frac{d}{dt}\left(\frac12\langle\eta(t),n(\eta(t))\rangle^2\right)(t_0)=\sum_{i=1}^{\infty}\theta_\eta k_i(\eta)\left(-k_i(\eta)+\frac{\theta_\eta}{\|\eta\|}\right)\langle\eta,v_i(\eta)\rangle^2+K_2\|\Pi^0_\eta(\eta)\|^2\,. \tag{4.12}$$

where $K_2=2\theta_\eta/\|N(\eta)\|\,(-2/\|N(\eta)\|+\theta_\eta/\|\eta\|)$.

(iii) Denote $\eta^\top=\Pi_\eta\eta$ and $\eta^\perp=\eta-\eta^\top$. Then

$$\mathfrak{K}_u\cap\,]\,(\theta_u\|u\|)^{-1},\theta_u\|u\|^{-1}\,[\;=\;\emptyset\quad\Rightarrow\quad\frac{d}{dt}\|\eta^\top\|_{\eta=u}\le 0\,,$$

and

$$\mathfrak{K}_u\cap]\theta_u/\|u\|,0[\;=\;\emptyset\quad\Rightarrow\quad\frac{d}{dt}\|\eta^\perp\|_{\eta=u}\ge 0\,.$$

Proof. Assertion (i) follows trivially from (4.6) and the formula

$$\frac{d}{dt}\theta(\eta(t,u_0))=\langle\nabla\theta(\eta(t)),\eta'(t)\rangle\,.$$

As $\eta([0,\tau_0[)\subset\mathcal{N}\cap W^{1,\infty}_0(\Omega)$, for any $u\in\eta([0,\tau_0[)$ we may provide an orthonormal basis of $\mathfrak{T}_u$ consisting of eigenvectors of L_u. Let us study how the norm of the projection $\Pi_\eta(\eta)$ and of the normal component $\langle\eta,n\rangle\cdot n$ evolve along the flow defined in (4.6). For simplicity of notation, we assume $\mathrm{Ker}(T_u)=\{0\}$ although minor changes provide the more general case.

$$\frac{d}{dt}\left(\frac12\|\Pi_{\eta(t)}\eta(t)\|^2\right)=$$
$$\big\langle D\Pi_{\eta(t)}(\eta'(t),\eta(t)),\Pi_{\eta(t)}(\eta(t))\big\rangle+\big\langle\Pi_{\eta(t)}(\eta'(t)),\Pi_{\eta(t)}(\eta(t))\big\rangle \tag{4.13}$$

Denoting $\eta(t)=u$ and $n(u)=n$, we have, by (4.4),

$$\big\langle\Pi_{\eta(t)}(\eta'(t)),\Pi_{\eta(t)}(\eta(t))\big\rangle=\langle-\Pi_u(\nabla\theta_u),u\rangle=\frac{1}{\|u\|}\sum_{i=1}^{\infty}\left(-k_i+\frac{\theta_u}{\|u\|}\right)\langle v_i,u\rangle^2\,. \tag{4.14}$$

Also

$$\big\langle D\Pi_{\eta(t)}(\eta'(t),\eta(t)),\Pi_{\eta(t)}(\eta(t))\big\rangle=\langle D\Pi_u(-\Pi_u(\nabla\theta_u),u),\Pi_u(u)\rangle\,. \tag{4.15}$$

We decompose

$$D\Pi_u(-\Pi_u(\nabla\theta_u), u) = D\Pi_u(-\Pi_u(\nabla\theta_u), \Pi_u(u) + \langle u, n\rangle n) =$$
$$D\Pi_u(-\Pi_u(\nabla\theta_u), \Pi_u(u)) + \langle u, n\rangle D\Pi_u(-\Pi_u(\nabla\theta_u), n)$$

and observe that

$$D\Pi_u(-\Pi_u(\nabla\theta_u), \Pi_u(u)) \in \mathfrak{T}_u^{\perp}$$

(since it is the second fundamental form of $\mathcal{N}$ at u). Then, by (3.3), we may re-write (4.15)

$$\langle D\Pi_u(-\Pi_u(\nabla\theta_u), u), \Pi_u(u)\rangle = \langle u, n\rangle\langle Dn(u)[\Pi_u(\nabla\theta_u)], u\rangle =$$
$$\sum_{i=1}^{\infty} \theta_u k_i \left(k_i - \frac{\theta_u}{\|u\|}\right) \langle u, v_i\rangle^2 . \tag{4.16}$$

Combining (4.13), (4.14) and (4.16) we obtain, for $u = \eta(t)$,

$$\frac{d}{dt}\left(\frac{1}{2}\|\Pi_\eta \eta\|^2\right) = \sum_{i=1}^{\infty}\left(k_i(\eta) - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i(\eta) - \frac{1}{\|\eta\|}\right)\langle \eta, v_i(\eta)\rangle^2 . \tag{4.17}$$

Let us turn to the study of the normal component $\langle \eta, n\rangle n$. Differentiating in t, assuming $\eta(t) = u$, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\langle \eta(t), n(\eta(t))\rangle^2\right) = \langle u, n\rangle \left(\langle u, Dn_u(-\Pi_u(\nabla\theta_u))\rangle + \langle -\Pi_u(\nabla\theta_u), n\rangle\right) .$$

Noting that $\langle -\Pi_u(\nabla\theta_u), n\rangle = 0$, we may write, for $u = \eta(t)$,

$$\frac{d}{dt}\left(\frac{1}{2}\langle \eta(t), n(\eta(t))\rangle^2\right) = \sum_{i=1}^{\infty} \theta_\eta k_i(\eta)\left(-k_i(\eta) + \frac{\theta_\eta}{\|\eta\|}\right)\langle \eta, v_i(\eta)\rangle^2 . \tag{4.18}$$

In case $\mathrm{Ker}(T_u) \neq \{0\}$, formulas (4.11)–(4.12) are trivially obtained from formulas (4.17)–(4.18) by adding to their right hand-side a term corresponding to the projection on $\mathrm{Ker}(T_u)$: K_1 and K_2 are obtained by plugging $k_0 = 2/\|N(u)\|$ on

$$\left(k_i(\eta) - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i(\eta) - \frac{1}{\|\eta\|}\right)$$

and

$$k_i(\eta)\left(-k_i(\eta) + \frac{\theta_\eta}{\|\eta\|}\right) .$$

This proves (ii). Finally, assertion (iii) follows by a simple evaluation of the above quadratic terms on the variable $k_i(\eta)$, recalling that $\theta_\eta < 0$. □

We will now study the convergence of the angle decreasing flow $\eta(t, u_0)$ to a critical point of the distance functional on $\mathcal{N}$ –i.e. a solution of (1.1)– when Ω is a bounded regular domain in $\mathbb{R}^2$. Note that, by Remark 3, the principal curvatures of the Nehari manifold are defined for all $u \in \mathcal{N}$ and therefore we may provide a basis of the tangent space $\mathfrak{T}_u$ composed by eigenvectors of the Weingarten Map L_u.

Let $\rho > 0$, $u \in \mathcal{N}$. We define

$$\mathfrak{K}_{\rho,u} = \left\{ k_i \in \mathfrak{K}_u : k_i \in \left] \frac{1}{\theta(u)\|u\|} - \rho, \frac{\theta(u)}{\|u\|} + \rho \right[\right\} ,$$

$\mathfrak{K}'_{\rho,u} := \mathfrak{K}_u \backslash \mathfrak{K}_{\rho,u}$ and the subspace

$$E_{\rho,u} = \{ v \in \mathfrak{T}_u : v \in \operatorname{span}\{v_k : k \in \mathfrak{K}_{\rho,u}\} \} .$$

Note that, in case $\rho < (2/\|N(u)\| - \theta(u)/\|u\|)$, $E_{\rho,u}$ is finite dimensional. We shall denote by $\Pi_{\rho,u}$ the projection on $E_{\rho,u}$ and by $\Pi'_{\rho,u} := \Pi_u - \Pi_{\rho,u}$.

Proposition 6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded regular domain and f defined by (4.1). Let $u_0 \in \mathcal{N}$, $\theta_0 = \theta(u_0)$. Assume there exist $0 < R \leq R_0 < \|u_0\|$, $\delta > 0$, $\rho > 0$ and $C_0 \geq 0$, such that, for all $u \in \mathcal{N} \cap B(u_0, R)$ we have*

(i)

$$\|\Pi_{\rho,u}(u)\|^2 \leq C_0 \|\Pi'_{\rho,u}(u)\|^2 \tag{4.19}$$

where

$$C_0 \leq \frac{\delta + |\theta_0|\rho^2}{(\theta_0/2 - (2\theta_0)^{-1})^2} \tag{4.20}$$

(ii)

$$2a\overline{K} \frac{\|u_0^\top\|}{\|u_0\| - R} \exp(b\|u_0^\top\|^2) < R \tag{4.21}$$

where $|k_i(u)| \leq \overline{K}$ for all $u \in \mathcal{N} \cap B(u_0, R_0)$, $a = \|u_0\|^2(1 + C_0)/(2\delta)$ and $b = \overline{K}^2(1 + C_0)/(2\delta)$.

Then $\eta(t, u_0)$ defined by (4.6) converges in H_1-norm to a critical point $u^* \in B(u_0, R) \cap \mathcal{N}$ of the distance functional –i.e a solution to (1.1). Moreover, $\|\eta^\top(t, u_0)\|$ is decreasing in t.

Remark 10. *Note that the existence of $\overline{K}$ verifying*

$$|k_i(u)| \leq \overline{K} \quad \textit{for all} \quad u \in \mathcal{N} \cap B(u_0, R_0)$$

with $R_0 < \|u_0\|$ is a simple consequence of Lemma 2.

Proof. By Proposition 4, in view of Remark 9, $\eta(t, u_0)$ is defined for all $u_0 \in \mathcal{N}$ and $t \in [0, +\infty[$. For simplicity, we denote $\eta(t) := \eta(t, u_0)$ and assume $\operatorname{Ker}(T_\eta) = \{0\}$ since the calculations remain essentially unchanged.

Step 1: Decreasing of $\|\eta^\top(t)\|$ as $\eta \in B(u_0, R)$.

We write

$$\frac{d}{dt}\frac{1}{2}\|\eta^\top(t, u_1)\|^2 =$$
$$\sum_{i \in \mathfrak{K}'_{\rho,\eta}} \left(k_i - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i - \frac{1}{\|\eta\|}\right)\langle \eta, v_i\rangle^2 + \sum_{i \in \mathfrak{K}_{\rho,\eta}} \left(k_i - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i - \frac{1}{\|\eta\|}\right)\langle \eta, v_i\rangle^2 .$$

By (i), estimating the quadratic term in k_i, recalling that $\theta_\eta \leq \theta_0 < 0$,

$$\sum_{i\in\mathfrak{K}'_{\rho,\eta}} \left(k_i - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i - \frac{1}{\|\eta\|}\right)\langle\eta, v_i\rangle^2 \leq -\frac{|\theta_0|\rho^2}{\|\eta\|^2}\|\Pi'_{\rho,\eta}(\eta)\|^2 \,. \tag{4.22}$$

Similarly,

$$\sum_{i\in\mathfrak{K}_{\rho,\eta}} \left(k_i - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i - \frac{1}{\|\eta\|}\right)\langle\eta, v_i\rangle^2 \leq \frac{1}{\|\eta\|^2}\left(\frac{\theta_0}{2} - \frac{1}{2\theta_0}\right)^2\|\Pi_{\rho,\eta}(\eta)\|^2 \,. \tag{4.23}$$

Therefore, by (i), (4.22)–(4.23),

$$\frac{d}{dt}\frac{1}{2}\|\eta^\top(t)\|^2 \leq -\frac{\delta}{\|\eta(t)\|^2}\|\Pi'_{\rho,\eta}(\eta)\|^2 \,. \tag{4.24}$$

Step 2: Convergence of $\eta(t)$ to a global minimum of θ

By the previous step, we have

$$\frac{d}{dt}\|\eta\|^2 = \frac{d}{dt}\|\eta^\perp\|^2 + \frac{d}{dt}\|\eta^\top\|^2 \leq \frac{d}{dt}\|\eta^\perp\|^2 \,. \tag{4.25}$$

Also, by a simple estimate

$$\frac{d}{dt}\|\eta^\perp\|^2 \leq \overline{K}^2\|\eta^\top\|^2 \,. \tag{4.26}$$

We conclude from (4.25)–(4.26)

$$\frac{d}{dt}\|\eta\|^2 \leq \overline{K}^2\|\eta^\top\|^2 \,,$$

or

$$\|\eta\|^2(t) \leq \|\eta\|^2(0) + \overline{K}^2\int_0^t \|\eta^\top\|^2(s)\,ds \,.$$

Then, by (4.19) and (4.24), as $\|\eta^\top\|^2 = \|\Pi_{\rho,\eta}\eta\|^2 + \|\Pi'_{\rho,\eta}\eta\|^2$,

$$\frac{d}{dt}\|\eta^\top(t)\|^2 \leq -\frac{2\delta(1+C_0)^{-1}\|\eta^\top(t)\|^2}{\|\eta(0)\|^2 + \overline{K}^2\int_0^t \|\eta^\top\|^2(s)\,ds} \,.$$

By Lemma 7 (Appendix), we conclude that, for any T such that $\eta([0,T]) \subset B(u_0, R)$

$$\int_0^T \|\eta^\top\|(t)\,dt \leq 2a\,\exp(b\|u_0^\top\|^2)\,\|u_0^\top\| \,. \tag{4.27}$$

where $a = \|u_0\|^2(1+C_0)/(2\delta)$ and $b = \overline{K}^2(1+C_0)/(2\delta)$. As

$$\|\eta(t)\| \geq \|u_0\| - R > 0 \,,$$

we conclude, by (4.4),

$$\left\| \frac{d\eta}{dt} \right\| = \|\Pi_\eta(\nabla\theta_\eta)\| \le \frac{\overline{K}}{\|\eta\|}\|\eta^\top\| \le \frac{\overline{K}}{\|u(0)\| - R}\|\eta^\top\| \,. \tag{4.28}$$

Then (4.27) implies

$$\int_0^T \left\| \frac{d\eta}{dt}(t) \right\| \, dt \le 2a\overline{K}\frac{\|u_0^\top\|}{\|u_0\| - R} \, \exp(b\|u_0^\top\|^2) \,,$$

for all $T > 0$ such that $\eta([0,T]) \subset B(u_0, R)$. By (4.21), the flow $\eta(t)$ necessarily converges in H^1-norm to $u^* \in B(u_0, R) \cap \mathcal{N}$. By (i), using a simple approximation argument, one concludes that $\Pi_{u^*}(u^*) = 0$. Then $\theta(u^*) = -1$, u^* is a critical point of the distance functional on the Nehari Manifold and a solution to (1.1). □

Remark 11. *In the conditions of Proposition 6, if u^* is a nontrivial solution of (1.1) such that $-\frac{1}{\|u^*\|} \notin \mathfrak{K}_{u^*}$, there exists a H^1-ball $B(u^*, R^*)$ such that, for all $u_0 \in B(u^*, R^*) \cap \mathcal{N}$, $\eta(t, u_0)$ converges to a solution $\tilde{u}$ of (1.1) (in H^1-norm) as t tends to infinity. In fact, by a simple continuity argument, we may choose $\rho > 0$ and $R^* > 0$ such that, for all $u \in B(u^*, R^*) \cap \mathcal{N}$,*

$$\mathfrak{K}_u \cap \left] \frac{1}{\theta_u \|u\|} - \rho, \frac{\theta_u}{\|u\|} + \rho \right[= \emptyset \,.$$

Moreover, fixing $\delta = 1$ and $C_0 = 0$ (considering an eventually smaller value for R^*) conditions (i)–(ii) of Proposition 6 are verified for all $u_0 \in B(u^*, R^*) \cap \mathcal{N}$. Of course, in case u^* is an isolated critical point of the distance functional, we may insure $\eta(t, u_0)$ will converge to u^* provided $\|u_0 - u^*\|$ is sufficiently small.

5. Appendix.

5.1. A suitable basis of $H_0^1(\Omega)$. Let $F \in C(\mathbb{R}, \mathbb{R})$ be such that $F(0) = 0$, $F(u) > 0$ if $u \neq 0$. Moreover, assume

$$\lim_{u \to \pm\infty} F(u) = +\infty \,, \tag{5.1}$$

and

$$\lim_{u \to \pm\infty} \frac{F(u)}{|u|^q} = 0 \,, \tag{5.2}$$

for some $1 \le q < 2^*$.

We define by recurrence a family of orthogonal vectors. Consider the following minimization problem:

$$\min \left\{ \int_\Omega |\nabla u|^2(x) \, dx \,:\, u \in H_0^1(\Omega), \int_\Omega F(u)(x) \, dx = 1 \right\}. \tag{5.3}$$

By (5.1)–(5.2), a minimizer exists, that we shall denote by e_1. More generally, we define e_n to be a minimizer of the Dirichlet integral $\int_\Omega |\nabla u|^2(x) \, dx$ over the weakly closed set

$$\left\{ u \in H_0^1(\Omega) : \int_\Omega F(u)(x) \, dx = 1 \text{ and } u \in \langle e_1, ..., e_{n-1} \rangle^\perp \right\} .$$

LEMMA 6. *The sequence* (e_n) *is an orthogonal basis of* $H_0^1(\Omega)$. *Also* $(\|e_n\|)$ *is non-decreasing and*

$$\lim_{n\to\infty} \|e_n\| = \infty.$$

Proof. Trivially, the sequence $(\|e_n\|)$ is non-decreasing. We assert that

$$\lim_{n\to\infty} \|e_n\| = \infty\,.$$

Suppose, in view of a contradiction, the existence of $C > 0$ such that $\|e_n\| \le C$ for all $n \in \mathbb{N}$. Passing to a weakly convergent subsequence, denoted by (e_{n_j}), we have

$$e_{n_j} \rightharpoonup v \quad \text{and} \quad \int_\Omega F(v)(x)\,dx = 1\,. \tag{5.4}$$

Let $n_j \in \mathbb{N}$ be fixed. We have

$$\langle v, e_{n_j}\rangle = \lim_{k\to\infty} \langle e_{n_k}, e_{n_j}\rangle = 0\,.$$

Now letting $n_j \to \infty$ we conclude $\|v\| = 0$ and contradict (5.4). The assertion is proved.

Let $w \in H_0^1(\Omega)$ be such that

$$\langle w, e_i\rangle = 0 \quad \text{for all } i \in \mathbb{N}\,. \tag{5.5}$$

If $w \neq 0$ assume (without loss of generality)

$$\int_\Omega F(w)(x)\,dx = 1\,.$$

The previous assertion, together with (5.5), imply that there exists $n \in \mathbb{N}$ such that $\|e_{n-1}\| \le \|w\| < \|e_n\|$. This, contradicts the definition of the function e_n. Then $w = 0$ and the proof is complete. □

5.2. A Gronwall type estimate.

LEMMA 7. *Let* $f \in C^1([0,+\infty[,\mathbb{R}^+)$ *be such that*

$$f'(t) \le -\frac{f(t)}{a + b\int_0^t f(u)\,du} \tag{5.6}$$

for some $a, b > 0$. *Then*

$$\int_0^\infty \sqrt{f}(u)\,du \le 2a\,e^{bf(0)}\sqrt{f(0)}\,. \tag{5.7}$$

Proof. Integrating equation (5.6),

$$f(t) - f(0) \le -\frac{1}{b}\left[\ln\left(a + b\int_0^s f(u)du\right)\right]_0^t,$$

or

$$f(t) + \frac{1}{b}\ln\left(a + b\int_0^t f(u)\,du\right) \leq f(0) + \frac{\ln(a)}{b}$$

and, as $f(t) \geq 0$, we conclude, by passing to the limit in t,

$$\ln\left(a + b\int_0^{+\infty} f(u)\,du\right) \leq bf(0) + \ln(a)$$

or

$$\int_0^{+\infty} f(u)\,du \leq C \tag{5.8}$$

where $C = (ae^{bf(0)} - a)/b$. Writing $f(t) = h^2(t)$ with $h(t) > 0$, inequality (5.6) becomes

$$2h(t)h'(t) \leq -\frac{h^2(t)}{a + b\int_0^t f(u)\,du}\,.$$

By (5.6)–(5.8), we conclude

$$h'(t) \leq -\frac{h(t)}{2(a+bC)} = -\frac{1}{2a}e^{-bf(0)}h(t)\,,$$

or

$$0 \leq h(t) \leq \sqrt{f(0)}e^{-C_2 t}\,,$$

where $C_2 = \frac{1}{2a}e^{-bf(0)}$. A simple integral comparison proves the lemma. □

REFERENCES

[1] A. Machado, *Geometria Diferencial: uma introdução fundamental*, Textos de Matemática vol. 9, Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, (1997).

[2] T. Bartsch and M. Willem, *Infinitely many radial solutions of a semilinear elliptic problem on* $\mathbb{R}^n$, Arch. Rational Mech. Anal., 124 (1993), pp. 261–276.

[3] P. Bates, K. Lu, and C. Zeng, *Existence and persistence of invariant manifolds for semiflows in Banach space*, Mem. of the Amer. Math. Soc., No. 645, Vol. 135, (1998).

[4] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer (2001).

[5] A. Castro, J. Cossio, and J. M. Neuberger, *A sign–changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math., 27 (1997), pp. 1041–1053.

[6] P. Girão and J. M. Gomes, *Multibump nodal solutions for an indefinite superlinear elliptic problem*, J. Differential Equations, 247 (2009), pp. 1001–1012.

[7] W. Kühnel, *Differential Geometry: Curves-Surfaces-Manifolds*, Student Mathematical Library vol 16, AMS, (2005).

[8] R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations*, Trans. Amer. Math. Soc., 93 (1959), pp. 30–52.

[9] Z. Nehari, *Oscillation criteria for second–order linear differential equations*, Trans. Amer. Math. Soc., 85 (1957), pp. 428–445.

[10] Z. Nehari, *On a class of non-linear second-order differential equations*, Trans. Amer. Math. Soc., 95 (1960), pp. 101–123.

[11] Z. Nehari, *Characteristic values associated with a class of non–linear second order differential equations*, Acta Math., 105 (1961), pp. 101–123.

[12] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, C.B.M.S.–A.M.S. 65 (1986).
[13] M. Ramos and H. Tavares, *Solutions with multiple spike patterns for an elliptic system*, Calc. Var. PDE, 31 (2008), pp. 1–25.
[14] A. Szulkin and T. Weth, *The method of the Nehari manifold*, Handbook of nonconvex analysis and applications, International Press, (2010).
[15] S. Terracini and G. Verzini, *Solutions of a prescribed number of zeroes to a class of superlinear ODE's systems*, NoDEA, 8 (2001), pp. 323–341.
[16] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.