

ON THE EXISTENCE OF SMOOTH SOLUTIONS FOR FULLY NONLINEAR ELLIPTIC EQUATIONS WITH MEASURABLE “COEFFICIENTS” WITHOUT CONVEXITY ASSUMPTIONS*

NICOLAI V. KRYLOV[†]

Abstract. We show that for any uniformly elliptic fully nonlinear second-order equation with bounded measurable “coefficients” and bounded “free” term one can find an approximating equation which has a unique continuous and having the second derivatives locally bounded solution in a given smooth domain with smooth boundary data. The approximating equation is constructed in such a way that it modifies the original one only for large values of the unknown function and its derivatives.

Key words. Fully nonlinear elliptic equations, Bellman’s equations, finite differences.

AMS subject classifications. 35J60,39A14.

1. Introduction and main result. In this article, we consider elliptic equations

$$(1.1) \quad H[v](x) := H(v(x), Dv(x), D^2v(x), x) = 0$$

in subdomains of \mathbb{R}^d , where

$$\mathbb{R}^d = \{x = (x_1, \dots, x_d) : x_1, \dots, x_d \in \mathbb{R} = (-\infty, \infty)\}.$$

Here

$$D^2u = (D_{ij}u), \quad Du = (D_iu), \quad D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = D_iD_j.$$

We introduce \mathcal{S} as the set of symmetric $d \times d$ matrices, fix a constant $\delta \in (0, 1]$, and set

$$\mathcal{S}_\delta = \{a \in \mathcal{S} : \delta|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d\},$$

where and everywhere in the article the summation convention is enforced unless specifically stated otherwise.

Recall that Lipschitz continuous functions are almost everywhere differentiable.

ASSUMPTION 1.1. (i) The function $H(u, x)$, $u = (u', u'')$,

$$u' = (u'_0, u'_1, \dots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathcal{S}, \quad x \in \mathbb{R}^d,$$

is measurable with respect to x for any u , and Lipschitz continuous in u for every $x \in \mathbb{R}^d$.

(ii) For any x , at all points of differentiability of $H(u, x)$ with respect to u , we have

$$(H_{u''_{ij}}) \in \mathcal{S}_\delta, \quad |H_{u'_k}| \leq \delta^{-1}, \quad k = 1, \dots, d, \quad 0 \leq -H_{u'_0} \leq \delta^{-1}.$$

*Received April 1, 2012; accepted for publication September 5, 2012.

[†]127 Vincent Hall, University of Minnesota, Minneapolis, MN 55455, USA (krylov@math.umn.edu). The author was partially supported by NSF Grant DMS-1160569.

(iii) Finally,

$$\bar{H} := \sup_{x \in \mathbb{R}^d} |H(0, x)| < \infty.$$

Let Ω be an open bounded subset of \mathbb{R}^d with C^2 boundary and take a function $g \in C^{1,1}(\bar{\Omega})$. Here is our main result, in which $K \geq 0$ is a fixed constant.

THEOREM 1.1. *There is a constant $\hat{\delta} \in (0, \delta]$ depending only on δ and d and there exists a function $P(u)$ (independent of x), satisfying Assumption 1.1 with $\hat{\delta}$ in place of δ , such that the equation*

$$(1.2) \quad \max(H[v], P[v] - K) = 0$$

in Ω (a.e.) with boundary condition $v = g$ on $\partial\Omega$ has a unique solution $v \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$. In addition, for all i, j , and $p \in (d, \infty)$,

$$(1.3) \quad |v|, |D_i v|, \rho |D_{ij} v| \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)}) \quad \text{in } \Omega \quad (\text{a.e.}),$$

$$(1.4) \quad \|v\|_{W_p^2(\Omega)} \leq N_p(\bar{H} + K + \|g\|_{W_p^2(\Omega)}),$$

$$(1.5) \quad \|v\|_{C^\alpha(\Omega)} \leq N(\|H[0]\|_{L_d(\Omega)} + \|g\|_{C^\alpha(\Omega)}),$$

where

$$\rho(x) = \text{dist}(x, \mathbb{R}^d \setminus \Omega),$$

$\alpha \in (0, 1)$ is a constant depending only on d and δ , N is a constant depending only on Ω and δ , whereas N_p only depends on the same objects and p .

Finally, $P(u)$ is constructed on the sole basis of δ and d , it is positive homogeneous of degree one and convex in u .

REMARK 1.1. If we drop (1.4) and replace $C^{1,1}$ in (1.3) with $C^{1,\alpha}$, $\alpha \in (0, 1]$, the assumptions of Theorem 1.1 about smoothness of Ω and g can be somewhat relaxed. It is sufficient to have the exterior ball condition on Ω and $g \in C^{1,\alpha}(\Omega)$. Furthermore, if we multiply the derivatives in (1.3) by one more ρ , then one can deal with Ω such that, for each boundary point x_0 and all $r > 0$ small enough with the smallness independent of x_0 , there is a ball of radius εr at the distance r from x_0 lying outside Ω . Here $\varepsilon > 0$ is a fixed constant. Of course, in that case the asserted regularity should be changed to $v \in C^{0,\beta}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$, where $\beta \in (0, 1]$ is determined by other parameters of the problem. All these and other possible extensions and generalizations are left to the interested reader.

To the best of the author's knowledge Theorem 1.1 is the first uniqueness and *existence* result for *general* fully nonlinear elliptic equations with *measurable* coefficients without *convexity* assumptions. In case H is Lipschitz continuous in x the theory of viscosity solutions provides the existence and uniqueness. Generally, one only knows that such solutions are in $C^{1+\alpha}$ (see Trudinger [15]). N. Nadirashvili and S. Vlăduț [14] found an example in which viscosity solutions even for H independent of x do not have bounded second-order derivatives.

It is also worth mentioning that M. G. Crandall, M. Kocan, and A. Świąch [4] developed a theory of L_p -viscosity solutions for equations with measurable coefficients (see also the references therein).

As far as a priori estimates in Sobolev spaces are concerned, L. Caffarelli was the first author who derived interior W_p^2 estimates under an assumption that certain estimates hold for equations with zero “free” term, which are known to hold only for H that are either convex or concave with respect to v, Dv, D^2v (see [1] and [2]). A particular case of $C^{2+\alpha}$ a priori estimates without this assumption is presented in [3]. Another case is found in [7].

The activity which started in [1] was continued by L. Wang in [16] who obtained similar interior a priori estimates for parabolic equations, by M. G. Crandall, M. Kocan, and A. Świąch [4] who established the *solvability* in local Sobolev spaces of the boundary-value problems for fully nonlinear parabolic equations, and by N. Winter [17] who established the solvability in the global W_p^2 -space of the associated boundary-value problem in the elliptic case. In the existence parts in [4] and [17] the function H is supposed to be convex with respect to D^2v and continuous in x (concerning the latter assumption see [17, Remark 2.3], [9], and [4, Example 8.3]). However, in the above references the authors consider equations like (1.1) with the right-hand side which is not zero but rather a function from an L_p -space. In our setting we can only treat bounded right-hand sides.

Recently a new method, very different from the methods in the above cited references, emerged in [5] for treating fully nonlinear elliptic and parabolic equations with VMO “coefficients”. Still the convexity of H with respect to D^2v is required in [5] while proving the existence result.

In our Theorem 1.1 we do not impose any convexity assumption on H and allow it to be just measurable in x . By the way, this theorem is obviously applicable to *linear* equations. Yet we approximate them with nonlinear ones.

The methods of the present article are quite elementary and, apart from what is related to (1.4) and (1.5) and uniqueness, are not using anything from any existing theory of partial differential equations in the main case where H depends only on pure second-order derivatives and is continuous in x . Our main tool is finite-difference approximations, best demonstrated in Sections 5 and 6, which the reader may like to read first.

REMARK 1.2. It is almost obvious that Assumption 1.1 (ii) is equivalent to the requirement that, for any $u \in \mathbb{R}^{d+1} \times \mathcal{S}$, $x, \xi \in \mathbb{R}^d$, $\eta \in \{\pm e_1, \dots, \pm e_d\}$, where e_1, \dots, e_d is the set of standard basis vectors in \mathbb{R}^d , and $r \geq 0$, we have

$$\delta|\xi|^2 \leq H(u', u'' + \xi\xi^*, x) - H(u', u'', x) \leq \delta^{-1}|\xi|^2,$$

$$|H(u' + r(0, \eta), u'', x) - H(u', u'', x)| \leq \delta^{-1}r,$$

$$H(u', u'', x) - \delta^{-1}r \leq H(u' + r(1, 0), u'', x) \leq H(u', u'', x),$$

where $(0, \eta) = (0, \eta_1, \dots, \eta_d)$ and $(1, 0) = (1, 0, \dots, 0)$.

REMARK 1.3. Estimate (1.5) follows from other assertions of Theorem 1.1 and the classical results about linear equations with measurable coefficients (see, for instance,

Section 9.9 of [6]). Indeed, as is easy to see for $v \in W_p^2(\Omega)$ satisfying (1.2) we have that

$$\begin{aligned} -\max(H[0], P[0] - K) &= \max(H[v], P[v] - K) - \max(H[0], P[0] - K) \\ &= a_{ij}D_{ij}v + b_iD_iv - cv \end{aligned}$$

with some functions $a = (a_{ij}) \in \mathcal{S}_{\hat{\delta}}$, $|b_i| \leq \hat{\delta}^{-1}$, $0 \leq c \leq \hat{\delta}^{-1}$ (cf. the proof of Lemma 2.2). Furthermore, $|\max(H[0], P[0] - K)| \leq |H[0]|$.

The assertion of Theorem 1.1 concerning uniqueness in our class of functions is also a classical result derived from the Alexandrov estimate.

Here is an almost trivial generalization of Theorem 1.1 which may be useful in some applications.

THEOREM 1.2. *Let $\phi \in C^2(\bar{\Omega})$ be a strictly positive function. Then there is a constant $\hat{\delta} \in (0, \delta]$ depending only on δ , ϕ , and d and there exists a function $P(u)$ (independent of x), satisfying Assumption 1.1 with $\hat{\delta}$ in place of δ , such that all assertions of Theorem 1.1 hold true if we replace $P[u]$ with $P[\phi u]$ and allow the constants to also depend on ϕ .*

This result is obtained from Theorem 1.1 just by replacing there u and g with ϕu and ϕg , respectively.

Here is a version of Theorem 1.1 which is obtained by just replacing $H(u, x)$ with $-H(-u, x)$.

THEOREM 1.3. *With P from Theorem 1.1 the equation*

$$\min(H[v], -P[-v] + K) = 0$$

in Ω (a.e.) with boundary condition $v = g$ on $\partial\Omega$ has a unique solution $v \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$. In addition, for all i, j , and $p \in (d, \infty)$,

$$|v|, |D_iv|, \rho|D_{ij}v| \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)}) \quad \text{in } \Omega \quad (\text{a.e.}),$$

$$\|v\|_{W_p^2(\Omega)} \leq N_p(\bar{H} + K + \|g\|_{W_p^2(\Omega)}),$$

$$\|v\|_{C^\alpha(\Omega)} \leq N(\|H[0]\|_{L_d(\Omega)} + \|g\|_{C^\alpha(\Omega)}),$$

where α , N , and N_p are the constants from Theorem 1.1.

It is an interesting issue as to what is happening to $v = v_K$ as $K \rightarrow \infty$, where v_K is taken from Theorem 1.1. We have the following

Conjecture. Assume that $H(u, x)$ is Lipschitz continuous with respect to x with Lipschitz constant equal to a constant times $1 + |u|$. Let w be a unique viscosity solution of equation (1.1) in $\Omega \in C^3$ with boundary condition $g \in C^3$. Then $|w - v_K| \leq N/K$ where N is a constant.

To conclude our comments about Theorem 1.1 we show how P is constructed. By Theorems 3.1 of [10] there exists a set

$$\{l_1, \dots, l_m\} \subset \mathbb{Z}^d,$$

$m = m(\delta, d) \geq d$, chosen on the sole basis of knowing δ and d and there exists a constant

$$\hat{\delta} = \hat{\delta}(\delta, d) \in (0, \delta/4]$$

such that:

(i) We have

$$e_i, e_i \pm e_j \in \{l_1, \dots, l_m\} = \{-l_1, \dots, -l_m\}$$

for all $i, j = 1, \dots, d$ (recall that e_1, \dots, e_d is the standard orthonormal basis of \mathbb{R}^d);

(ii) There exist real-analytic functions $\lambda_1(a), \dots, \lambda_m(a)$ on $\mathcal{S}_{\delta/4}$ such that for any $a \in \mathcal{S}_{\delta/4}$

$$(1.6) \quad a \equiv \sum_{k=1}^m \lambda_k(a) l_k l_k^*, \quad \hat{\delta}^{-1} \geq \lambda_k(a) \geq \hat{\delta}, \quad \forall k.$$

Now introduce

$$\mathcal{P}(z) = \max_{\substack{\delta/2 \leq a_k \leq 2\hat{\delta}^{-1} \\ k=1, \dots, m}} \max_{\substack{|b_k| \leq 2\hat{\delta}^{-1} \\ k=1, \dots, d_1}} \max_{\delta/2 \leq c \leq 2\hat{\delta}^{-1}} \left[\sum_{k=1}^m a_k z_k'' + \sum_{k=1}^d b_k z_k' - cz_0' \right],$$

and for $u = (u', u'') \in \mathbb{R}^{d+1} \times \mathcal{S}$ define

$$P(u', u'') = \mathcal{P}(u', \langle u'' l_1, l_1 \rangle, \dots, \langle u'' l_m, l_m \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^d .

The rest of the article is organized as follows. In Section 2 we show that one may safely impose an additional assumption while proving Theorem 1.1. In Section 3 Theorem 1.1 is deduced from Theorem 3.2 in which even more additional assumptions are made. Then in Section 4 the function H is rewritten in terms of pure second-order derivatives along certain directions.

In a quite long Section 5 we consider finite-difference approximations for equations with “constant” coefficients and prove interior estimates for the second-order differences of solutions. In Section 6 we use the results of the previous section in order to prove an analog of Theorem 1.1 for H , that include only pure second-order derivatives. Here the reader will see the main underlying idea of the paper, which roughly speaking is that on the set, say Γ , where the second-order derivatives of v are large we have $P[v] = K$ and estimates similar to the ones from Section 5 show that the second order derivative on Γ are controlled by their values on the boundary of Γ , where they are under control by the definition of Γ . Of course, the implementation of this idea requires first proving that there are sufficiently regular solutions of (1.2). Since we do not know how to do that, we apply the above idea at the level of finite differences.

In the final short Section 7 we prove Theorem 3.2.

In the proofs of various results in this article we use the symbol N sometimes with indices to denote constants which may change from one occurrence to another and we do not always specify on which data these constants depend. In these cases the reader should remember that, if in the statement of a result there are constants called N which claimed to depend only on certain parameters, then in the proof of the result the constants N also depend only on the same parameters unless specifically stated otherwise.

2. Reducing Theorem 1.1 to a particular case where $-H_{u'_0} \geq \delta$. Suppose that Theorem 1.1 is true under the additional assumption that

$$(2.1) \quad -H_{u'_0} \geq \delta$$

at all points of differentiability of $H(u, x)$ with respect to u . Then we are going to prove it in the original form. Take an H satisfying only Assumption 1.1, take $n > 0$, and consider the mapping $T_n : w \rightarrow v$ defined for any $w \in C(\bar{\Omega})$ and mapping it into a unique solution of

$$(2.2) \quad \max(H[v] - v + n\chi(w/n), P[v] - K) = 0$$

in Ω (a.e.) with boundary condition $v = g$, where

$$\chi(t) = (-1) \vee t \wedge 1.$$

By assumption v is well defined and $v = T_n w \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$ and

$$|v|, |D_i v|, \rho |D_{ij} v| \leq N(\bar{H} + n + K + \|g\|_{C^{1,1}(\Omega)}) \quad \text{in } \Omega \quad (a.e.),$$

$$\|v\|_{W_p^2(\Omega)} \leq N_p(\bar{H} + n + K + \|g\|_{W_p^2(\Omega)})$$

if $p > d$. It follows that, for each n , T_n maps $C(\bar{\Omega})$ into its compact subset.

LEMMA 2.1. *For each n , the mapping T_n is continuous in $C(\bar{\Omega})$.*

Proof. Let $w, w_m \in C(\bar{\Omega})$, $m = 1, 2, \dots$, and assume that $\|w - w_m\|_{0,\Omega} \rightarrow 0$ as $m \rightarrow \infty$, where $\|\cdot\|_{0,\Omega}$ is the sup norm in $C(\bar{\Omega})$. In light of uniqueness of solutions of (2.2) with boundary condition $v = g$, to prove the lemma, it suffices to show that, at least along a subsequence, $\|T_n w - v_m\|_{0,\Omega} \rightarrow 0$, where $v_m = T_n w_m$. Since $T_n C(\bar{\Omega})$ is a compact set, there is a subsequence and a $v \in C(\bar{\Omega})$ such that $\|v - v_m\|_{0,\Omega} \rightarrow 0$ and $v = g$ on $\partial\Omega$. Without losing generality we may assume that the above convergence holds along the original sequence. Now we need only show that $v = T_n w$.

Observe that for $m \geq r$ we have

$$\max(H[v_m] - v_m + n \sup_{k \geq r} \chi(w_k/n), P[v_m] - K) \geq 0$$

in Ω (a.e.). Since the norms $\|v_m\|_{W_d^2(\Omega)}$ are bounded, by Theorems 3.5.9 and 3.5.15 of [8], whose conditions are easily checked on the basis of Remark 1.2, we have (a.e.)

$$\max(H[v] - v + n \sup_{k \geq r} \chi(w_k/n), P[v] - K) \geq 0.$$

By letting $r \rightarrow \infty$ we get (a.e.)

$$\max(H[v] - v + n\chi(w/n), P[v] - K) \geq 0.$$

One obtains the opposite inequality starting with

$$\max(H[v_m] - v_m + n \inf_{k \geq r} \chi(w_k/n), P[v_m] - K) \leq 0.$$

It follows that $v = T_n w$ indeed and the lemma is proved.

Now by Tikhonov's theorem we conclude that, for each n , there exists $v^n \in C(\bar{\Omega})$ such that $v^n = T_n v^n$. By assumption $v^n \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$ and

$$|D_i v^n|, \rho |D_{ij} v^n| \leq N(\bar{H} + \|v^n\|_{0,\Omega} + K + \|g\|_{C^{1,1}(\Omega)}) \quad \text{in } \Omega \quad (a.e.),$$

$$(2.3) \quad \|v^n\|_{W_p^2(\Omega)} \leq N_p(\bar{H} + \|v^n\|_{0,\Omega} + K + \|g\|_{W_p^2(\Omega)}).$$

LEMMA 2.2. *There is a constant N depending only on the diameter of Ω and δ such that*

$$\|v^n\|_{0,\Omega} \leq N(\bar{H} + K + \|g\|_{C(\Omega)}).$$

Proof. Introduce

$$H_K^n(u, x) = \max(H(u, x) - u'_0 + n\chi(u'_0/n), P(u) - K)$$

and observe that $H_{K u'_0}^n \leq 0$ and by Hadamard's formula

$$(2.4) \quad \begin{aligned} H_K^n(u', u'', x) - H_K^n(0, x) &= u''_{ij} \int_0^1 H_{K u''_{ij}}^n(tu', tu'', x) dt \\ &+ \sum_{i \geq 1} u'_i \int_0^1 H_{K u'_i}^n(tu', tu'', x) dt + u'_0 \int_0^1 H_{K u'_0}^n(tu', tu'', x) dt \end{aligned}$$

provided that $H^n(u, x)$ is differentiable with respect to u at (tu, x) for almost all $t \in [0, 1]$. Since this happens to be the case for almost all u , we see that, for each n , there exist \mathcal{S}_δ -valued function a and real-valued functions b_1, \dots, b_d, c , and f satisfying $|b_i| \leq \delta^{-1}$, $c \geq 0$, $|f| \leq \bar{H} + K$ such that in Ω (a.e.)

$$a_{ij} D_{ij} v^n + b_i D_i v^n - c v^n = f.$$

Now our result follows by the Alexandrov maximum principle (see, for instance, Section 3.3 of [8]). The lemma is proved.

Due to this lemma one can drop $\|v^n\|_{0,\Omega}$ in the right-hand sides of estimates (2.3). After that it only remains to observe that for $n \geq \|v^n\|_{0,\Omega}$, the function v^n satisfies (1.2) since $\chi(v^n/n) = v^n/n$ and Theorem 1.1 holds in its original form.

Hence, in the rest of the article we suppose that (2.1) holds at all points of differentiability of H with respect to u .

3. Further reductions of Theorem 1.1.

1. First, we show that we may additionally assume that for any $x, y \in \mathbb{R}^d$ and $u = (u', u'')$

$$(3.1) \quad |H(u, x) - H(u, y)| \leq N|x - y|(1 + |u|),$$

where N is independent of x, y, u .

Indeed, if Theorem 1.1 is true in this particular case, take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^d)$, which integrates to one, set $\zeta^n(x) = n^d \zeta(nx)$, and introduce $H^n(u, x)$ as the convolution of $H(u, x)$ and ζ^n performed with respect to x . Observe that H^n satisfies (2.1) and Assumption 1.1 with the same constant δ , whereas

$$|H^n(u, x) - H^n(u, y)| \leq n|x - y| \sup_z |H(u, z)| \sup |D\zeta|$$

and (3.1) is satisfied since $|H(u, z)| \leq |H(0, z)| + N(d)\delta^{-1}|u|$. Then assuming that the assertions of Theorem 1.1 are true under our additional assumption, we conclude that there exist solutions $v^n \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$ of

$$(3.2) \quad \max(H^n[v^n], P[v^n] - K) = 0$$

in Ω (a.e.) with boundary condition $v^n = g$, for which estimates (1.3) and (1.4) hold with v^n in place of v with the constants N and N_p from Theorem 1.1 and with

$$\overline{H^n} = \sup_{x \in \mathbb{R}^d} |H^n(0, x)| \quad (\leq \bar{H})$$

in place of \bar{H} . In particular,

$$(3.3) \quad \check{H}_K^n[v^m] \geq 0$$

in Ω (a.e.) for all $m \geq n$, where

$$\check{H}_K^n(u, x) := \sup_{k \geq n} \max(H^k(u, x), P(u) - K).$$

Furthermore, being uniformly bounded and uniformly continuous, the sequence $\{v^n\}$ has a subsequence uniformly converging to a function v , for which (1.3) and (1.4), of course, hold and $v \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$. In light of (3.3) and the fact that the norms $\|v^n\|_{W_p^2(\Omega)}$ are bounded, by Theorems 3.5.9 and 3.5.15 of [8] (the applicability of which is shown by an argument similar to the one in Remark 1.3) we have

$$(3.4) \quad \check{H}_K^n[v] \geq 0$$

in Ω (a.e.).

Then we notice that by the Lebesgue differentiation theorem for any u

$$(3.5) \quad \lim_{n \rightarrow \infty} \check{H}_K^n(u, x) = \max(H(u, x), P(u) - K)$$

for almost all x . Since $\check{H}_K^n(u, x)$ are Lipschitz continuous in u with a constant independent of x and n , there exists a subset of Ω of full measure such that (3.5) holds on this subset for all u .

We conclude that in Ω (a.e.)

$$(3.6) \quad \max(H[v], P[v] - K) \geq 0.$$

The opposite inequality is obtained by considering

$$\inf_{k \geq n} \max(H^k(u, x), P(u) - K).$$

2. Next, we show that one may assume that H is boundedly inhomogeneous with respect to u . Introduce

$$P_0(u) = \max_{a \in \mathcal{S}_{\delta/2}} \max_{\substack{|b_i| \leq 2\delta^{-1} \\ i=1, \dots, d}} \max_{c \in [\delta/2, 2\delta^{-1}]} (a_{ij}u''_{ij} + b_i u'_i - cu'_0),$$

where the summations are performed before the maximum is taken. It is easy to see that $P_0[u]$ is a kind of Pucci's operator:

$$P_0(u) = -(\delta/2) \sum_{k=1}^d \lambda_k^-(u'') + 2\delta^{-1} \sum_{k=1}^d \lambda_k^+(u'') + 2\delta^{-1} \sum_{k=1}^d |u'_k| - (\delta/2)(u'_0)^+ + 2\delta^{-1}(u'_0)^-,$$

where $\lambda_1(u''), \dots, \lambda_d(u'')$ are the eigenvalues of u'' and $a^\pm = (1/2)(|a| \pm a)$.

Recall that the function P is introduced in the end of Section 1 and observe that

$$P(u) = \max_{\substack{\delta/2 \leq a_k \leq 2\hat{\delta}^{-1} \\ k=1, \dots, m}} \max_{\substack{|b_i| \leq 2\hat{\delta}^{-1} \\ i=1, \dots, d}} \max_{\delta/2 \leq c \leq 2\hat{\delta}^{-1}} \left[\sum_{i,j=1}^d \sum_{k=1}^m a_k l_{ki} l_{kj} u''_{ij} + \sum_{i=1}^d b_i u'_i - c u'_0 \right].$$

Moreover, owing to property (ii) in the end of Section 1, the collection of matrices

$$\sum_{k=1}^m a_k l_k l_k^*$$

such that $\hat{\delta} \leq a_k \leq \hat{\delta}^{-1}$, $k = 1, \dots, m$, covers $\mathcal{S}_{\delta/4}$. By combining this with the fact that $\hat{\delta} \leq \delta/2$ (actually, $\hat{\delta} \leq \delta/4$, which will be used much later) we see that

$$\begin{aligned} P(u) &\geq -(\delta/4) \sum_{k=1}^d \lambda_k^-(u'') + 4\delta^{-1} \sum_{k=1}^d \lambda_k^+(u'') \\ &\quad + 4\delta^{-1} \sum_{k=1}^d |u'_k| - (\delta/4)(u'_0)^+ + 4\delta^{-1}(u'_0)^- \\ (3.7) \quad &\geq P_0(u) + (\delta/4) \sum_{k=1}^d |\lambda_k(u'')| + (\delta/4) \sum_{k=0}^d |u'_k|. \end{aligned}$$

In particular, $P_0 \leq P$ and therefore,

$$\max(H, P - K) = \max(H_K, P - K),$$

where $H_K = \max(H, P_0 - K)$. It is easy to see that the function H_K satisfies Assumption 1.1 and (2.1) with $\delta/2$ in place of δ . It also satisfies (3.1) with the same constant N .

Furthermore, we have the following.

LEMMA 3.1. *There is a constant $\kappa > 0$ depending only on δ and d such that for all $x \in \Omega$ and $u = (u', u'')$*

$$(3.8) \quad H \leq P_0 - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H(0, x),$$

$$(3.9) \quad H_K \leq P - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H^+(0, x).$$

Furthermore,

$$H(u, x) \leq N \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H(0, x),$$

$$|H(u, x)| \leq N \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + |H(0, x)|,$$

where the constant N depends only on δ .

Proof. Observe that if a number $p \in (a, b)$, $a < b$, and $y \in \mathbb{R}$, then

$$yp \leq y^+b - y^-a.$$

Then from Hadamard's formula

$$\begin{aligned} H(u', u'', x) - H(0, 0, x) &= u''_{ij} \int_0^1 H_{u''_{ij}}(tu', tu'', x) dt \\ &+ \sum_{i \geq 1} u'_i \int_0^1 H_{u'_i}(tu', tu'', x) dt + u'_0 \int_0^1 H_{u'_0}(tu', tu'', x) dt \end{aligned}$$

we obtain (see our comments regarding (2.4))

$$\begin{aligned} H(u', u'', x) - H(0, 0, x) &\leq \delta^{-1} \sum_k \lambda_k^+(u'') - \delta \sum_k \lambda_k^-(u'') + \delta^{-1} \sum_{i \geq 1} |u'_i| \\ &\quad - \delta(u'_0)^+ + \delta^{-1}(u'_0)^- \\ &= P_0(u', u'') - \delta^{-1} \sum_k \lambda_k^+(u'') - (\delta/2) \sum_k \lambda_k^-(u'') \\ &\quad - \delta^{-1} \sum_{i \geq 1} |u'_i| - \delta^{-1}(u'_0)^- - (\delta/2)(u'_0)^+ \end{aligned}$$

and (3.8) follows since

$$\begin{aligned} \left[\sum_k (\lambda_k^+(u'') + \lambda_k^-(u'')) \right]^2 &= \left(\sum_k |\lambda_k(u'')| \right)^2 \\ &\geq \sum_k |\lambda_k(u'')|^2 = \sum_{i,j} |u''_{ij}|^2 \geq d^{-2} \left(\sum_{i,j} |u''_{ij}| \right)^2. \end{aligned}$$

Estimate (3.9) follows from (3.8) and (3.7). Finally, the second assertion of the lemma follows directly from the above Hadamard's formula. The lemma is proved.

In addition, H_K is boundedly inhomogeneous with respect to u in the sense that at all points of differentiability of $H_K(u, x)$ with respect to u

$$(3.10) \quad |H_K(u, x) - H_{Ku''_{ij}}(u, x)u''_{ij} - H_{Ku'_r}(u, x)u'_r| \leq N(|H_K(0, x)| + K),$$

where N depends only on δ and d .

Indeed, if

$$(3.11) \quad \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) \geq H^+(0, x) + K,$$

then by Lemma 3.1

$$H(u, x) \leq P_0(u) - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H^+(0, x) \leq P_0(u) - K,$$

so that $H_K(u, x) = P_0(u) - K$ and the left-hand side of (3.10) is just K owing to the fact that P_0 is positive homogeneous of degree one. On the other hand, if the opposite inequality holds in (3.11), then again in light of Lemma 3.1 the left-hand side of (3.10) is dominated by

$$N \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + |H_K(0, x)| \leq N(|H_K(0, x)| + H^+(0, x) + K),$$

where

$$H(0, x) \leq \max(H(0, x), -K) = H_K(0, x), \quad H^+(0, x) \leq |H_K(0, x)|.$$

Furthermore, as we have noticed above H_K satisfies Assumption 1.1 and (2.1) (with $\delta/2$ in place of δ) and as is easy to see $|H_K[0]| \leq |H[0]| + K$, which shows that in the rest of the article we may (and will) assume that not only Assumption 1.1 and (2.1) are satisfied with $\delta/2$ in place of δ and (3.1) holds with a constant N , but also at all points of differentiability of H with respect to u

$$(3.12) \quad |H(u, x) - H_{u''_{ij}}(u, x)u''_{ij} - H_{u'_r}(u, x)u'_r| \leq N_0,$$

where N_0 is a constant and

$$(3.13) \quad H \leq P - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + |H(0, \cdot)|,$$

where κ is the constant from Lemma 3.1. By the way we keep track of the value of δ in Assumption 1.1 and (2.1) because $P(u)$ is already fixed and defined by d and δ .

3. Finally, we show that we may assume that,

$$(3.14) \quad H(u, x) = \text{tr } u'' - u'_0 \text{ for all } u \text{ if } x \text{ is in a neighborhood of } \partial\Omega,$$

that is, for an $\varepsilon > 0$, we have $H(u, x) = \text{tr } u'' - u'_0$ if $\rho(x) \leq \varepsilon$. Indeed, take a continuous function $\zeta(t)$, $t \geq 0$ such that $\zeta(t) = 1$ for $t \in [0, 1]$, $\zeta(t) = 0$ for $t \geq 2$, and $0 \leq \zeta \leq 1$. Introduce

$$H^{1/\varepsilon}(u, x) = (1 - \zeta(\rho(x)/\varepsilon))H(u, x) + \zeta(\rho(x)/\varepsilon)(\text{tr } u'' - u'_0).$$

Notice that $H^{1/\varepsilon}$ satisfies Assumption 1.1 and (2.1) with $\delta/2$ in place of δ , satisfies (3.1) with a constant N depending on ε but independent of x, y, u , and satisfies (3.12) with the same constant N_0 . As long as (3.13) is concerned, observe that by Lemma 3.1 applied to $H = \text{tr } u'' - u'_0$ and by the inequality $P_0 \leq P$ we have

$$\text{tr } u'' - u'_0 \leq P - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right).$$

Then owing to (3.13)

$$\begin{aligned} H^{1/\varepsilon} &\leq P - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + |(1 - \zeta(\rho/\varepsilon))H(0, \cdot)| \\ &= P - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + |H^{1/\varepsilon}(0, \cdot)|. \end{aligned}$$

Therefore, if the assertions of Theorem 1.1 hold under the above additional assumptions, then we have a sequence of functions $v^n \in C^{0,1}(\bar{\Omega}) \cap C^{1,1}_{loc}(\Omega)$ satisfying (3.2) (with new $H^n = H^{1/\varepsilon}$ for $\varepsilon = 1/n$).

After that by repeating literally the argument in no. **1** we come to (3.4) and since, obviously, $H^{1/\varepsilon}(u, x) \rightarrow H(u, x)$ as $\varepsilon \downarrow 0$ for any $x \in \Omega$, we conclude that equation (3.6) holds (a.e.) and we finish the argument as in no. **1**.

As a result of the above arguments we see that to prove Theorem 1.1 it suffices to prove the following.

THEOREM 3.2. *Suppose that Assumption 1.1 is satisfied with $\delta/2$ in place of δ . Also assume that (3.13) and (3.14) hold. Finally, assume that estimate (3.1) holds for any $x, y \in \mathbb{R}^d$ and $u = (u', u'')$ with a constant N and (2.1) and (3.12) hold at all points of differentiability of $H(u, x)$ with respect to u .*

Then the assertions of Theorem 1.1 hold true with P introduced in the end of Section 1.

REMARK 3.1. One may wonder why we need (3.12) with a constant which does not enter the assertions of Theorem 3.2 in any way. The only reason to reduce general H to boundedly inhomogeneous ones is that for those we can rewrite $H[u]$ in such a way that only pure second-order derivatives of $u(t, x)$ with respect to x enter. Then the whole operator $\max(H[u], P[u] - K)$ also has this form.

Another possible question is: Why don't we start with $\max(H, P - K)$, which is already boundedly inhomogeneous by the above? The point is that our way to transform boundedly inhomogeneous operators does not preserve the particular structure of $\max(H, P - K)$.

4. Writing H in Theorem 3.2 in a special form. Here we suppose that the assumptions of Theorem 3.2 are satisfied and take the objects introduced in the end of Section 1. Owing to the the assumptions of Theorem 3.2 by Theorem 7.1 of [10] (see the beginning of its proof in [10]) there exists a function $\mathcal{H}(z, x)$ defined for

$$z = (z', z''), \quad z' = (z'_0, \dots, z'_d) \in \mathbb{R}^{d+1}, \quad z'' \in \mathbb{R}^m, \quad x \in \mathbb{R}^d$$

such that:

(i) The function \mathcal{H} is Lipschitz continuous in z with Lipschitz constant $\hat{\delta}^{-1}$ and there exists a constant N' such that

$$|\mathcal{H}(z, x) - \mathcal{H}(z, y)| \leq N'|x - y|(1 + |z|)$$

for all $x, y \in \mathbb{R}^d$ and z .

(ii) We have $\mathcal{H}(z, x) = H(u, x)$ if $z' = u'$ and for all $j = 1, \dots, m$

$$z''_j = \langle u'' l_j, l_j \rangle.$$

In particular, $\mathcal{H}(0, x) = H(0, x)$ and if $v(x)$ is a real-valued function which is twice differentiable at a point $x \in \mathbb{R}^d$, at this point we have

$$H[v](x) = \mathcal{H}[v](x)$$

where

$$\mathcal{H}[v](x) = \mathcal{H}(v(x), Dv(x), D_{l_1}^2 v(x), \dots, D_{l_m}^2 v(x), x), \quad D_l^2 v = v_{x_i x_j} l_i l_j.$$

(iii) At all points (z, x) at which $\mathcal{H}(z, x)$ is differentiable with respect to z we have

$$(4.1) \quad |\mathcal{H}_{z'_i}(z, x)| \leq 4\delta^{-1}, \quad i = 1, \dots, d,$$

$$(4.2) \quad \delta/4 \leq -\mathcal{H}_{z''_0}(z, x) \leq 4\delta^{-1}, \quad \hat{\delta}^{-1} \geq \mathcal{H}_{z''_j}(z, x) \geq \hat{\delta}, \quad j = 1, \dots, m.$$

The proofs in [10] use the fact that (3.12) holds and yield the function \mathcal{H} such that, in addition, at all points (z, x) at which $\mathcal{H}(z, x)$ is differentiable with respect to z we also have

$$|\mathcal{H}(z, x) - \langle z, D_z \mathcal{H}(z, x) \rangle| \leq 2N_0.$$

However, the latter property of \mathcal{H} will not be used in the future, so that we only used assumption (3.12) to be sure that \mathcal{H} with the properties (i)-(iii) exists.

5. An auxiliary equation. Some notation in this section are different from the previous ones. Fix an $h \in (0, 1]$ and for $\xi \in \mathbb{R}^d$ and any function ϕ on \mathbb{R}^d introduce

$$T_\xi \phi(x) = \phi(x + h\xi), \quad \delta_\xi = h^{-1}(T_\xi - 1), \quad \Delta_\xi = h^{-2}(T_\xi - 2 + T_{-\xi}).$$

Notice that h enters the definition of T_ξ and δ_ξ and Δ_ξ are usual approximations for the first and second-order derivatives along ξ .

Let $m \geq 1$ be an integer and let $\ell_{-m}, \dots, \ell_{-1}, \ell_1, \dots, \ell_m$ be some fixed vectors in \mathbb{R}^d such that

$$\ell_{-k} = -\ell_k.$$

Next denote $\Lambda = \{\ell_k : k = \pm 1, \dots, \pm m\}$,

$$\Lambda_1 = \Lambda, \quad \Lambda_{n+1} = \Lambda_n + \Lambda, \quad n \geq 1, \quad \Lambda_\infty = \bigcup_n \Lambda_n.$$

Let $m' \geq 0$ be an integer $\leq m$ and let $A = \{\alpha = (a, b, c)\}$ be a closed bounded set in $\mathbb{R}^{2m} \times \mathbb{R}^{m'} \times \mathbb{R}$, so that

$$a = (a_{-m}, a_{-m+1}, \dots, a_{-1}, a_1, \dots, a_m) \in \mathbb{R}^{2m},$$

$$b = (b_1, \dots, b_{m'}) \in \mathbb{R}^{m'},$$

and $c \in \mathbb{R}$. Also let $f(\alpha, x)$ be a real-valued function defined for $\alpha \in A$, $x \in \mathbb{R}^d$.

Fix an $r \in \{1, \dots, m\}$ and for $k = \pm 1, \dots, \pm m$ set

$$\delta_{h,k} = \delta_k = \delta_{\ell_k}, \quad \Delta_{h,k} = \Delta_k = \Delta_{\ell_k}.$$

ASSUMPTION 5.1. There are constants $\delta > 0$ and $K_1, K_2 \in [0, \infty)$ such that

(i) For any $(a, b, c) \in A$ and all k we have

$$a_k = a_{-k}, \quad \delta \leq a_k \leq \delta^{-1}, \quad |b_k| \leq \delta^{-1}, \quad hb_k^- \leq a_k, \quad c \geq 0;$$

(ii) The function f is continuous in α for any x and $|\delta_r f| \leq K_1$, $\Delta_r f \geq -K_2$ on \mathbb{R}^d .

For $u = (u', u'')$ with

$$u' = (u'_0, u'_1, \dots, u'_{m'}), \quad u'' = (u''_{-m}, \dots, u''_{-1}, u''_1, \dots, u''_m),$$

introduce

$$\mathcal{P}(u, x) = \max_{\alpha=(a,b,c) \in A} \left(\sum_{|k|=1}^m a_k u''_k + \sum_{k=1}^{m'} b_k u'_k - cu'_0 + f(\alpha, x) \right).$$

For any function u on \mathbb{R}^d define

$$\mathcal{P}[u](x) = \mathcal{P}(u(x), \delta u(x), \delta^2 u(x), x),$$

where

$$\delta u = (\delta_1 u, \dots, \delta_{m'} u),$$

$$\delta^2 u = (\Delta_{-m} u, \dots, \Delta_{-1} u, \Delta_1 u, \dots, \Delta_m u).$$

In connection with this notation a natural question arises as to why use ℓ_k along with $\ell_{-k} = -\ell_k$ since $\Delta_k = \Delta_{-k}$ and

$$a_k \Delta_k = 2 \sum_{k \geq 1} a_k \Delta_k$$

owing to the assumption that $a_k = a_{-k}$. This is done for the sake of convenience of computations. For instance,

$$\Delta_k(uv) = u \Delta_k v + v \Delta_k u + (\delta_k u)(\delta_k v) + (\delta_{-k} u)(\delta_{-k} v)$$

(no summation in k). At the same time

$$a_k \Delta_k(uv) = u a_k \Delta_k v + v a_k \Delta_k u + 2 a_k (\delta_k u)(\delta_k v)$$

as if we were dealing with usual partial derivatives.

Fix a nonempty finite set $Q^\circ \subset h\Lambda_\infty$ and let

$$Q := Q^\circ \cup \{x + h\Lambda : x \in Q^\circ\}.$$

Next take a function $\eta \in C^\infty(\mathbb{R}^d)$ with bounded derivatives, such that $|\eta| \leq 1$ and set $\zeta = \eta^2$,

$$|\eta'(x)| = |\eta'(x)|_h = \sup_k |\delta_k \eta(x)|, \quad |\eta''(x)| = |\eta''(x)|_h = \sup_k |\Delta_k \eta(x)|,$$

$$\|\eta'\| = \|\eta'\|_h = \sup_{h\Lambda_\infty} |\eta'|_h, \quad \|\eta''\| = \|\eta''\|_h = \sup_{h\Lambda_\infty} |\eta''|_h.$$

Finally, let u be a function on \mathbb{R}^d such that in Q°

$$(5.1) \quad \mathcal{P}[u] = 0$$

and $\mathcal{P}[u] \leq 0$ on $Q \setminus Q^\circ$.

THEOREM 5.1. *There exist constants $N = N(m, \delta) \geq 1$ and $N^* = N^*(m, \delta)$ such that for any constant ν satisfying*

$$\nu \geq N^* \|\eta'\| + N(\|\eta''\| + \|\eta'\|^2),$$

we have in Q that (recall that $a^\pm = (1/2)(|a| \pm a)$)

$$(5.2) \quad \zeta^2 [(\Delta_r u)^-]^2 \leq \max_{Q \setminus Q^\circ} \zeta^2 [(\Delta_r u)^-]^2 + (N\nu + N^*) \bar{W}_r + N\nu^{-2} K_2^2 + \nu^{-1} K_1^2,$$

where

$$\bar{W}_r = \max_Q (|\delta_r u|^2 + |\delta_{-r} u|^2).$$

Furthermore, $N^* = 0$ if $b \equiv 0$.

REMARK 5.1. Theorem 5.1 looks very much like Theorem 1.1 of [11]. However, in the latter the boundary of Q° is “twice fatter” and all mixed second-order differences are present under the maximum sign in the corresponding counterpart of (5.2). Our idea is to apply Theorem 5.1 to regions where at least one of pure second-order differences is large. Then outside the region all of them will be under control. Yet this does not provide any control of *mixed* differences on the boundary of the region and makes it impossible to apply Theorem 1.1 of [11], where the driving goal was to obtain estimates for equations with variable coefficients and estimating all mixed second-order finite differences was necessary.

In the following arguments no summation with respect to r is done. The number r is fixed in the beginning of the section. For simplicity of notation set

$$u_{rr} = \Delta_r u, \quad u_r = \delta_r u, \quad u_{kr} = -\delta_{-k} \delta_r u.$$

Notice that in the above line the last notation when $k = r$ is consistent with the first one.

In the following two lemmas the fact that u is a solution of (5.1) is not used and

$$u_{rr}^- = (u_{rr})^-.$$

LEMMA 5.2. *There exists $N = N(m, \delta)$ and $N^* = N^*(m, \delta)$ such that, if $N^* h \leq 1$, on Q° for any $\alpha = (a, b, c) \in A$ we have*

$$(5.3) \quad \begin{aligned} -2\zeta^2 u_{rr}^- [a_k \Delta_k + b_k \delta_k] u_{rr}^- &\geq -[a_k \Delta_k + b_k \delta_k] (\zeta^2 (u_{rr}^-)^2) \\ &\quad - N^* |\eta'| \zeta a_k u_{kr}^2 - N (|\eta''| + |\eta'|^2) \zeta (u_{rr}^-)^2 - (N^* |\eta'|^2 + N |\eta'|^4) \bar{W}_r, \end{aligned}$$

$$(5.4) \quad \begin{aligned} -2\zeta u_r [a_k \Delta_k + b_k \delta_k] u_r &\geq -[a_k \Delta_k + b_k \delta_k] (\zeta u_r^2) \\ &\quad + \zeta a_k u_{kr}^2 - N (|\eta''| + |\eta'|^2) \bar{W}_r - N^* |\eta'| \bar{W}_r. \end{aligned}$$

Furthermore, $N^* = 0$ if $b \equiv 0$.

Proof. Here is the result of simple computations, which can be found, for instance, in the proof of Lemma 5.1 of [11]. For any $\alpha \in A$ we have

$$(5.5) \quad \begin{aligned} a_k \Delta_k (\zeta^2 (u_{rr}^-)^2) &= 2\zeta^2 u_{rr}^- a_k \Delta_k u_{rr}^- + 2a_k [\delta_k (\zeta u_{rr}^-)]^2 + 4a_k u_{rr}^- (\delta_k \zeta) \delta_k (\zeta u_{rr}^-) \\ &\quad + 2a_k (u_{rr}^-)^2 [\zeta \Delta_k \zeta - 2(\delta_k \zeta)^2] - 4h u_{rr}^- a_k (\delta_k \zeta)^2 \delta_k u_{rr}^-. \end{aligned}$$

We also know from Lemma 4.3 of [11] that

$$|\Delta_k \zeta| \leq 2(|\eta''| + |\eta'|^2), \quad (\delta_k \zeta)^2 \leq N |\eta'|^2 \zeta + N h^2 |\eta'|^4 \leq N |\eta'|^2.$$

It follows that, for any $\varepsilon > 0$,

$$|4a_k u_{rr}^- (\delta_k \zeta) \delta_k (\zeta u_{rr}^-)| \leq \varepsilon a_k [\delta_k (\zeta u_{rr}^-)]^2 + N \varepsilon^{-1} (u_{rr}^-)^2 (|\eta'|^2 \zeta + h^2 |\eta'|^4),$$

where

$$(5.6) \quad (u_{rr}^-)^2 h^2 \leq |h u_{rr}|^2 = |(T_{\ell_r} - 1) u_{-r}|^2,$$

so that $|\eta'|^4 (u_{rr}^-)^2 h^2 \leq 4|\eta'|^4 \bar{W}_r$ in Q^o . Therefore, in Q^o

$$|4a_k u_{rr}^- (\delta_k \zeta) \delta_k (\zeta u_{rr}^-)| \leq \varepsilon a_k [\delta_k (\zeta u_{rr}^-)]^2 + N \varepsilon^{-1} |\eta'|^2 \zeta (u_{rr}^-)^2 + N \varepsilon^{-1} \bar{W}_r |\eta'|^4.$$

Similarly

$$|2a_k (u_{rr}^-)^2 [\zeta \Delta_k \zeta - 2(\delta_k \zeta)^2]| \leq N (|\eta''| + |\eta'|^2) \zeta (u_{rr}^-)^2 + N \bar{W}_r |\eta'|^4.$$

By Lemma 4.3 of [11] for any $\varepsilon \in (0, 1]$

$$h(\delta_k \zeta)^2 |u_{rr}^- \delta_i u_{rr}^-| \leq \varepsilon |\delta_i (\zeta u_{rr}^-)|^2 + \varepsilon |\eta'|^2 \zeta (u_{rr}^-)^2 + N \varepsilon^{-1} |\eta'|^4 [(h u_{rr}^-)^2 + (h^2 \delta_i u_{rr}^-)^2].$$

Estimate (5.6) leads to

$$h(\delta_k \zeta)^2 |u_{rr}^- \delta_i u_{rr}^-| \leq \varepsilon |\delta_i (\zeta u_{rr}^-)|^2 + |\eta'|^2 \zeta (u_{rr}^-)^2 + N \varepsilon^{-1} \bar{W}_r |\eta'|^4 + N \varepsilon^{-1} |\eta'|^4 (h^2 \delta_i u_{rr}^-)^2$$

on Q^o , where the last term is estimated by using the fact that $|\delta_i \psi^-| \leq |\delta_i \psi|$ for any function ψ implying that

$$|\eta'|^4 (h^2 \delta_i u_{rr}^-)^2 \leq |\eta'|^4 |h \delta_i (u_r + u_{-r})|^2 = |\eta'|^4 |(T_{\ell_i} - 1)(u_r + u_{-r})|^2 \leq N |\eta'|^4 \bar{W}_r.$$

Hence on Q^o

$$h(\delta_k \zeta)^2 |u_{rr}^- \delta_i u_{rr}^-| \leq \varepsilon |\delta_i (\zeta u_{rr}^-)|^2 + |\eta'|^2 \zeta (u_{rr}^-)^2 + N \varepsilon^{-1} \bar{W}_r |\eta'|^4.$$

Upon combining these estimates, choosing ε appropriately, and coming back to (5.5), we find on Q^o that

$$(5.7) \quad \begin{aligned} -2\zeta^2 u_{rr}^- a_k \Delta_k u_{rr}^- &\geq -a_k \Delta_k (\zeta^2 (u_{rr}^-)^2) + a_k [\delta_k (\zeta u_{rr}^-)]^2 \\ &\quad - N (|\eta''| + |\eta'|^2) \zeta (u_{rr}^-)^2 - N \bar{W}_r |\eta'|^4. \end{aligned}$$

Next,

$$\begin{aligned} b_k \delta_k (\zeta^2 (u_{rr}^-)^2) &= 2\zeta u_{rr}^- b_k \delta_k (\zeta u_{rr}^-) + b_k h [\delta_k (\zeta u_{rr}^-)]^2 \\ &= 2\zeta^2 u_{rr}^- b_k \delta_k u_{rr}^- + 2\zeta (u_{rr}^-)^2 b_k \delta_k \zeta \\ &\quad + 2h b_k u_{rr}^- \zeta (\delta_k \zeta) \delta_k u_{rr}^- + b_k h [\delta_k (\zeta u_{rr}^-)]^2. \end{aligned}$$

Here $|\delta_k \zeta| \leq 2|\eta'|$, since $|\eta| \leq 1$. Also $a_k \geq \delta$, so that

$$|2\zeta (u_{rr}^-)^2 b_k \delta_k \zeta| \leq N^* |\eta'| \zeta a_k u_{kr}^2.$$

Furthermore, $hT_{\ell_k}u_{rr}^- = T_{\ell_k}(u_r + u_{-r})^-$, implying that on Q^o

$$\begin{aligned} |2hb_k u_{rr}^- \zeta(\delta_k \zeta) \delta_k u_{rr}^-| &= |2hb_k u_{rr}^- (\delta_k \zeta) [\delta_k (\zeta u_{rr}^-) - (\delta_k \zeta) T_{\ell_k} u_{rr}^-]| \\ &\leq N^* |\eta'| \bar{W}_r^{1/2} a_k^{1/2} |\delta_k (\zeta u_{rr}^-)| + N^* u_{rr}^- \sum_k (\delta_k \zeta)^2 \bar{W}_r^{1/2}, \end{aligned}$$

where, owing to the inequality $h|\eta'| \leq 2$, the last term is dominated by

$$\begin{aligned} N^* \bar{W}_r^{1/2} u_{rr}^- (\zeta |\eta'|^2 + h^2 |\eta'|^4) &\leq (N^* \zeta^{1/2} |\eta'|^{3/2} \bar{W}_r^{1/2}) (|\eta'|^{1/2} \zeta^{1/2} u_{rr}^-) \\ &\quad + N^* |\eta'|^3 \bar{W}_r \leq N^* |\eta'| \zeta a_k u_{kr}^2 + N^* |\eta'|^3 \bar{W}_r. \end{aligned}$$

Hence, in Q^o

$$(5.8) \quad \begin{aligned} -2\zeta^2 u_{rr}^- b_k \delta_k u_{rr}^- &\geq -b_k \delta_k (\zeta^2 (u_{rr}^-)^2) - N^* |\eta'| \zeta a_k u_{kr}^2 \\ &\quad - (N^* h + 1/2) a_k [\delta_k (\zeta u_{rr}^-)]^2 - N^* (|\eta'|^2 + |\eta'|^3) \bar{W}_r. \end{aligned}$$

For $N^* h \leq 1/2$ estimates (5.7) and (5.8) and the fact that $|\eta'|^3 \leq |\eta'|^2 + |\eta'|^4$ lead to (5.3).

To prove (5.4) observe that (recall that $\delta_k u_r = -u_{-kr}$ and $a_{-k} = a_k$)

$$a_k \Delta_k (\zeta u_r^2) = a_k \zeta [2u_r \Delta_k u_r + 2u_{kr}^2] + a_k u_r^2 \Delta_k \zeta + 2a_k (\delta_k \zeta) (hu_{-kr}^2 - 2u_r u_{-kr}),$$

where

$$u_r^2 |\Delta_k \zeta| \leq N (|\eta''| + |\eta'|^2) \bar{W}_r,$$

and owing to (5.6)

$$\begin{aligned} |2a_k (\delta_k \zeta) (hu_{-kr}^2 - 2u_r u_{-kr})| &\leq N a_k (|\eta'| \zeta^{1/2} + h |\eta'|^2) (|u_r u_{kr}| + hu_{kr}^2) \\ &\leq N a_k |u_{kr}| (|\eta'| \zeta^{1/2} + h |\eta'|^2) \bar{W}_r^{1/2} \\ &\leq N a_k |u_{kr}| |\eta'| \zeta^{1/2} \bar{W}_r^{1/2} + N |\eta'|^2 \bar{W}_r \\ &\leq (1/2) \zeta a_k u_{kr}^2 + N |\eta'|^2 \bar{W}_r. \end{aligned}$$

It follows that in Q^o

$$(5.9) \quad -2\zeta a_k u_r \Delta_k u_r \geq -a_k \Delta_k (\zeta u_r^2) + (3/2) \zeta a_k u_{kr}^2 - N (|\eta''| + |\eta'|^2) \bar{W}_r.$$

Also

$$b_k \delta_k (\zeta u_r^2) = 2\zeta u_r b_k \delta_k u_r + \zeta b_k h u_{-kr}^2 + u_r^2 b_k \delta_k \zeta + h b_k (\delta_k \zeta) [h u_{-kr}^2 - 2u_r u_{-kr}].$$

Here in Q^o

$$|u_r^2 b_k \delta_k \zeta| \leq N^* |\eta'| W_r, \quad |h b_k (\delta_k \zeta) [h u_{-kr}^2 - 2u_r u_{-kr}]| \leq N^* |\eta'| \bar{W}_r,$$

where the last estimate follows from an equality similar to (5.6). Furthermore,

$$|\zeta b_k h u_{-kr}^2| \leq (1/2)\zeta a_k u_{kr}^2$$

if $N^*h \leq 1$ and N^* is chosen appropriately.

Upon combining this estimates with (5.9) we come to (5.4) and the lemma is proved.

For a constant $\nu \geq 0$ introduce an operator (recall that r is fixed)

$$L_\nu \phi = \zeta^2 u_{rr}^- \Delta_r \phi - \nu \zeta u_r \delta_r \phi.$$

Observe that

$$(5.10) \quad L_\nu u = -\zeta^2 (u_{rr}^-)^2 - \nu \zeta u_r^2 =: -V_\nu.$$

LEMMA 5.3. *There exists $N = N(m, \delta) \geq 1$ and $N^* = N^*(m, \delta)$ such that if*

$$(5.11) \quad \nu \geq N^* \|\eta'\| + N(\|\eta''\| + \|\eta'\|^2)$$

and $N^*h \leq 1$, then on Q^o for any $\alpha = (a, b, c) \in A$ we have

$$(5.12) \quad 2L_\nu[a_k \Delta_k + b_k \delta_k]u \geq -[a_k \Delta_k + b_k \delta_k]V_\nu - (N\nu^2 + N^*\nu)\bar{W}_r + (\nu/2)\zeta a_k u_{kr}^2.$$

Furthermore, $N^* = 0$ if $b \equiv 0$.

Proof. Since, for each k , the operators $a_k \Delta_k + b_k \delta_k$ respect the maximum principle, it follows by Lemma 4.2 of [11] that

$$u_{rr}^-(a_k \Delta_k + b_k \delta_k)u_{rr} \geq -u_{rr}^-[a_k \Delta_k + b_k \delta_k]u_{rr}^-.$$

Hence,

$$I := \zeta^2 u_{rr}^- \Delta_r [a_k \Delta_k + b_k \delta_k]u = \zeta^2 u_{rr}^- [a_k \Delta_k + b_k \delta_k]u_{rr} \geq -\zeta^2 u_{rr}^- [a_k \Delta_k + b_k \delta_k]u_{rr}^-,$$

which by Lemma 5.2 and the observation that

$$\zeta (u_{rr}^-)^2 \leq N \zeta a_k u_{kr}^2,$$

for $N^*h \leq 1$ yields

$$(5.13) \quad \begin{aligned} 2I &\geq -[a_k \Delta_k + b_k \delta_k](\zeta^2 (u_{rr}^-)^2) - N(N^*|\eta'| + |\eta''| + |\eta'|^2)\zeta a_k u_{kr}^2 \\ &\quad - (N^*|\eta'|^2 + N|\eta'|^4)\bar{W}_r. \end{aligned}$$

Furthermore, by Lemma 5.2

$$\begin{aligned} -2\nu \zeta u_r \delta_r [a_k \Delta_k + b_k \delta_k]u &= -2\nu \zeta u_r [a_k \Delta_k + b_k \delta_k]u_r \\ &\geq -[a_k \Delta_k + b_k \delta_k](\nu \zeta u_r^2) + \nu \zeta a_k u_{kr}^2 \\ &\quad - N\nu(|\eta''| + |\eta'|^2 + N^*|\eta'|)\bar{W}_r. \end{aligned}$$

By combining this with (5.13) and recalling (5.10) we find

$$(5.14) \quad \begin{aligned} 2L_\nu[a_k\Delta_k + b_k\delta_k]u &\geq -[a_k\Delta_k + b_k\delta_k]V_\nu \\ &\quad +(\nu - N_1^*|\eta'| - N_1|\eta''| - N_1|\eta'|^2)\zeta a_k u_{kr}^2 \\ &\quad -[N^*(|\eta'|^2 + \nu|\eta'|) + N(|\eta'|^4 + \nu|\eta''| + \nu|\eta'|^2)]\bar{W}_r. \end{aligned}$$

We may assume that $N_1 \geq 1$ and then, if

$$\nu \geq 2N_1^*|\eta'| + 2N_1(|\eta''| + |\eta'|^2),$$

we have that $|\eta'|^2 \leq \nu$, $|\eta''| \leq \nu$, and

$$|\eta'|^4 + \nu|\eta''| + \nu|\eta'|^2 \leq 3\nu^2.$$

Also

$$N^*(|\eta'|^2 + \nu|\eta'|) \leq N^*(\nu + \nu^{3/2}) \leq N^*(\nu + \nu^2) \leq N^*\nu + N\nu^2.$$

After that (5.14) clearly yields (5.12) and the lemma is proved.

Proof of Theorem 5.1. Denote by N_0 and N_0^* the constants N and N^* in Lemma 5.3 and take and fix a ν satisfying (5.11) (with N_0 and N_0^* in place of N and N^*).

Observe that (5.2) obviously holds on $Q \setminus Q^\circ$ and we only need to prove it on Q° . Also notice that

$$|u_{rr}| = h^{-1}|u_r + u_{-r}| \leq 2h^{-1}\bar{W}_r^{1/2},$$

which shows that (5.2) holds if $h \geq \nu^{-1/2}$ or if $N_0^*h \geq 1$. Therefore below we assume that

$$(5.15) \quad h \leq \nu^{-1/2}, \quad N_0^*h \leq 1.$$

Introduce

$$D^\circ = \{x \in Q^\circ : \zeta(x)u_{rr}^-(x) \geq \nu h u_r(x)\}.$$

If $x \in Q^\circ \setminus D^\circ$, then $\zeta(x)u_{rr}^-(x) \leq \nu h u_r(x)$ and (5.2) holds at x in light of (5.15). Therefore, in the remaining part of the proof we concentrate on establishing (5.2) for $x \in D^\circ$, assuming, of course, that $D^\circ \neq \emptyset$.

Denote

$$D = D^\circ \cup \{x + h\Lambda : x \in D^\circ\} \quad (\subset Q).$$

If V_ν in D° is less than its maximum over $D \setminus D^\circ$, then in D°

$$\zeta^2[u_{rr}^-]^2 \leq V_\nu \leq \max[\max_{Q \setminus Q^\circ} \zeta^2[u_{rr}^-]^2, \max_{Q^\circ \setminus D^\circ} \zeta^2[u_{rr}^-]^2] + \nu\bar{W}_r,$$

where the maximums are less than the right-hand side of (5.2) by the above. Hence, in the rest of the proof we consider the case that the maximum over D of V_ν is attained at a point $x_0 \in D^\circ$.

Notice that if a function $\phi(x)$ is such that $\phi(x) \leq \phi(x_0)$ for $x \in x_0 + h\Lambda$, then at x_0

$$\begin{aligned} h^2 L_\nu \phi(x_0) &= \zeta[\phi(x_0 + h\ell_r)(\zeta u_{rr}^- - \nu h u_r) + \phi(x_0 - h\ell_r)\zeta u_{rr}^-] - \zeta[2\zeta u_{rr}^- - \nu h u_r]\phi \\ &\leq \zeta[(\zeta u_{rr}^- - \nu h u_r)\phi + \zeta u_{rr}^- \phi] - \zeta[2\zeta u_{rr}^- - \nu h u_r]\phi, \end{aligned}$$

where the last expression is zero. Thus

$$L_\nu \phi(x_0) \leq 0,$$

which in the terminology from [11] means that L_ν respects the maximum principle.

Next, we can find an $\bar{\alpha} = (\bar{a}, \bar{b}, \bar{c}) \in A$ such that

$$\bar{a}_k \Delta_k u(x_0) + \bar{b}_k \delta_k u(x_0) - \bar{c} u(x_0) + f(\bar{\alpha}, x_0) = \mathcal{P}[u](x_0) = 0.$$

Since $\mathcal{P}[u] \leq 0$ in Q , we have that

$$\phi(x) := \bar{a}_k \Delta_k u(x) + \bar{b}_k \delta_k u(x) - \bar{c} u(x) + f(\bar{\alpha}, x) \leq 0$$

for $x \in x_0 + h\Lambda$. Hence,

$$0 \geq 2L_\nu \phi(x_0) = \bar{a}_k 2L_\nu \Delta_k u(x_0) + \bar{b}_k 2L_\nu \delta_k u(x_0) - \bar{c} 2L_\nu u(x_0) + 2L_\nu f(\bar{\alpha}, \cdot)(x_0),$$

which owing to (5.10) and (5.12) yields

$$0 \leq [\bar{a}_k \Delta_k + \bar{b}_k \delta_k - 2\bar{c}]V_\nu(x_0) - (\nu/2)\zeta \bar{a}_k u_{kr}^2(x_0) + (N\nu^2 + N^*\nu)\bar{W}_r - 2L_\nu f(\bar{\alpha}, \cdot)(x_0).$$

Here the last term is dominated by

$$K_2 \zeta^2 u_{rr}^-(x_0) + \nu |u_r(x_0)| K_1 \leq N\nu^{-1} K_2^2 + (\nu/4)\zeta \bar{a}_k u_{kr}^2(x_0) + K_1^2 + \nu^2 \bar{W}_r.$$

Furthermore, by the maximum principle

$$[\bar{a}_k \Delta_k + \bar{b}_k \delta_k - 2\bar{c}]V_\nu(x_0) \leq 0,$$

since $V_\nu \geq 0$ attains its maximum at x_0 .

We now conclude that

$$(\nu/4)\zeta \bar{a}_k u_{kr}^2(x_0) \leq (N\nu^2 + N^*\nu)\bar{W}_r + N\nu^{-1} K_2^2 + K_1^2,$$

which implies that in D°

$$\zeta^2 (u_{rr}^-)^2 \leq V_\nu(x_0) \leq N\zeta \bar{a}_k u_{kr}^2(x_0) + \nu \bar{W}_r \leq (N\nu + N^*)\bar{W}_r + N\nu^{-2} K_2^2 + \nu^{-1} K_1^2.$$

Thus, estimate (5.2) holds on D° as well and this proves the theorem.

6. A model cut-off equation. We will work in the setting of Section 5. However now $h > 0$ is not fixed. Take a function $\mathcal{H}(u, x)$, where $x \in \mathbb{R}^d$, $u = (u', u'') \in \mathbb{R}^{1+m'+2m}$.

ASSUMPTION 6.1. (i) The function \mathcal{H} is Lipschitz continuous in u for every x , and at all points of differentiability of \mathcal{H} with respect to u we have

$$\delta \leq \mathcal{H}_{u_k''} \leq \delta^{-1}, \quad k = \pm 1, \dots, \pm m, \quad \delta \leq -\mathcal{H}_{u_0'} \leq \delta^{-1},$$

$$|\mathcal{H}_{u'_k}| \leq \delta^{-1}, \quad k = 1, \dots, m';$$

(ii) The number $\bar{\mathcal{H}} = \sup_x |\mathcal{H}(0, 0, x)|$ is finite;

(iii) The function \mathcal{H} is locally Lipschitz continuous in x for every u and there exists a constant N' such that at all points of differentiability of \mathcal{H} with respect to x we have

$$|\mathcal{H}_{x_i}(u, x)| \leq N'(1 + |u|), \quad \forall i;$$

(iv) We have $\text{Span}(\ell_1, \dots, \ell_m) = \mathbb{R}^d$.

Define

$$\begin{aligned} \mathcal{P}(u', u'', x) &= \mathcal{P}(u', u'') \\ &= 2\delta^{-1} \sum_k (u''_k)^+ - (\delta/2) \sum_k (u''_k)^- \\ &\quad + 2\delta^{-1} \sum_{k \geq 1} |u'_k| - (\delta/2)(u'_0)^+ + 2\delta^{-1}(u'_0)^- \\ (6.1) \quad &= \max_{\substack{\delta/2 \leq a_k \leq 2/\delta \\ |k|=1, \dots, m}} \max_{\substack{|b_k| \leq 2/\delta \\ |k|=1, \dots, m'}} \max_{\delta/2 \leq c \leq 2/\delta} \left[\sum_{|i|=1}^m a_i u''_i + \sum_{i=1}^{m'} b_i u'_i - c u'_0 \right]. \end{aligned}$$

For functions $v(x)$ introduce

$$H[v](x) = \mathcal{H}(v(x), \partial v(x), \partial^2 v(x), x)$$

whenever and wherever it makes sense, where

$$\partial v = (v_{(\ell_1)}, \dots, v_{(\ell_{m'})}),$$

$$\partial^2 v = (v_{(\ell_{-m})(\ell_{-m})}, \dots, v_{(\ell_{-1})(\ell_{-1})}, v_{(\ell_1)(\ell_1)}, \dots, v_{(\ell_m)(\ell_m)}),$$

and $v_{(\ell)} = \ell_i v_{x_i}$, $v_{(\ell)(\ell)} = \ell_i \ell_j v_{x_i x_j}$. Similarly,

$$P[v](x) = \mathcal{P}(v(x), \partial v(x), \partial^2 v(x)).$$

Let Ω be a bounded C^2 domain in \mathbb{R}^d , g be a bounded function on \mathbb{R}^d such that $g \in C^{1,1}(\bar{\Omega})$, and let $K \geq 0$ be a finite number.

THEOREM 6.1. *In addition to Assumption 6.1 suppose that $\pm e_i, \pm(e_i + e_j), e_i - e_j \in \Lambda$, $i, j = 1, \dots, d$, where e_1, \dots, e_d is the standard orthonormal basis in \mathbb{R}^d and assume that all vectors in Λ have rational coordinates. Then there exists a unique $v \in C^{0,1}(\bar{\Omega}) \cap C^{1,1}_{loc}(\Omega)$ such that $v = g$ on $\partial\Omega$ and*

$$(6.2) \quad H_K[v] = 0$$

(a.e.) in Ω , where

$$H_K[v] = \max(H[v], P[v] - K).$$

Furthermore,

$$(6.3) \quad |v|, |D_i v|, \rho |D_{ij} v| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})$$

in Ω (a.e.) for all i, j , where N is a constant depending only on Ω , $\{\ell_1, \dots, \ell_m\}$, d , and δ (but not on N').

To prove the theorem, we are going to use finite-difference approximations of the operators $H[v]$ and $P[v]$. For $h > 0$ introduce

$$P_h[v](x) = \mathcal{P}(v(x), \delta_h v(x), \delta_h^2 v(x)),$$

where naturally

$$\delta_h u = (\delta_{h,1} u, \dots, \delta_{h,m'} u),$$

$$\delta_h^2 u = (\Delta_{h,-m} u, \dots, \Delta_{h,-1} u, \Delta_{h,1} u, \dots, \Delta_{h,m} u).$$

Similarly we introduce H_h and $H_{K,h}$.

LEMMA 6.2. Under Assumptions 6.1 (i), (ii)

$$(6.4) \quad \mathcal{H} \leq \mathcal{P} - (\delta/2) \left(\sum_k |u_k''| + \sum_k |u_k'| \right) + \bar{\mathcal{H}}.$$

Proof. Basically, we just repeat part of the proof of Lemma 3.1. From Hadamard's formula

$$\begin{aligned} \mathcal{H}(u', u'', x) - H(0, 0, x) &= u_k'' \int_0^1 \mathcal{H}_{u_k''}(tu', tu'', x) dt \\ &\quad + \sum_{k \geq 1} u_k' \int_0^1 \mathcal{H}_{u_k'}(tu', tu'', x) dt + u_0' \int_0^1 \mathcal{H}_{u_0'}(tu', tu'', x) dt \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{H}(u', u'', x) - \mathcal{H}(0, 0, x) &\leq \delta^{-1} \sum_k (u_k'')^+ - \delta \sum_k (u_k'')^- + \delta^{-1} \sum_{k \geq 1} |u_k'| \\ &\quad - \delta (u_0')^+ + \delta^{-1} (u_0')^- \\ &= \mathcal{P}(u', u'') - \delta^{-1} \sum_k (u_k'')^+ - (\delta/2) \sum_k (u_k'')^- - \delta^{-1} \sum_{k \geq 1} |u_k'| \\ &\quad - (\delta/2) (u_0')^+ - \delta^{-1} (u_0')^- \end{aligned}$$

and (6.4) follows. The lemma is proved.

Introduce B as the smallest closed ball containing Λ and set

$$\Omega_h = \{x \in \Omega : x + hB \subset \Omega\} = \{x : \rho(x) > \lambda h\},$$

where λ is the radius of B .

Consider the equation

$$(6.5) \quad H_{K,h}[v] = 0 \quad \text{in } \Omega_h$$

with boundary condition

$$(6.6) \quad v = g \quad \text{on } \mathbb{R}^d \setminus \Omega_h.$$

It is a rather simple fact that for sufficiently small $h > 0$ there exists a unique bounded solution $v = v_h$ of (6.5)–(6.6) (see, for instance, [13] or Theorem 8.2 in [10] or else Theorem 2.2 in [12]). By the way, we do not include K in the notation v_h since K is a fixed number.

Below by h_0 and N with occasional indices we denote various (finite) constants depending only on Ω , $\{\ell_1, \dots, \ell_m\}$, d , and δ .

In the following lemma the additional assumption of Theorem 6.1 concerning the e_i 's and the ℓ_k 's is not used.

LEMMA 6.3. *Under Assumptions 6.1 (i), (ii), (iv) there are constants $h_0 > 0$ and N such that for all $h \in (0, h_0]$ and $|r| = 1, \dots, m$*

$$(6.7) \quad |v_h - g| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})\rho,$$

$$(6.8) \quad |\delta_{h,r}v_h| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})$$

on Ω_h .

Proof. Introduce

$$\mathcal{H}_K = \max(\mathcal{H}, \mathcal{P} - K).$$

Clearly, \mathcal{H}_K satisfies Assumption 6.1 with $\delta/2$ in place of δ . Therefore, by Hadamard's formula there exist functions a_k, b_k , $k = \pm 1, \dots, \pm m$, and c such that

$$(6.9) \quad \delta/2 \leq a_k \leq 2\delta^{-1}, \quad |b_k| \leq 2\delta^{-1}, \quad \delta/2 \leq c \leq 2\delta^{-1}$$

and in Ω_h we have

$$\begin{aligned} -H_K[0] &= H_{K,h}[v_h] - H_K[0] \\ &= a_k \Delta_{h,k} v_h + b_k \delta_{h,k} v_h - c v_h \\ &= a_k \Delta_{h,k} (v_h - g) + b_k \delta_{h,k} (v_h - g) - c(v_h - g) + f, \end{aligned}$$

where

$$f = a_k \Delta_{h,k} g + b_k \delta_{h,k} g - c g.$$

After that (6.7) is proved by using the barrier function Φ from Lemma 2.4 of [12] (cf. Lemma 2.5 in [12]). It implies that

$$(6.10) \quad |v_h - g| \leq N_1(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})h \quad \text{on} \quad \Omega \setminus \Omega_{3h}.$$

To prove (6.8), fix an r and define

$$Q^o = \{x \in \Omega_{2h} : (\delta/2)|\delta_{h,r}v_h| \geq \bar{\mathcal{H}} + K\}.$$

If $Q^o = \emptyset$, then $(\delta/2)|\delta_{h,r}v_h| \leq \bar{\mathcal{H}} + K$ in Ω_{2h} , and by virtue of (6.10),

$$|\delta_{h,r}(v_h - g)| \leq 2N_1(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})$$

in $\Omega_h \setminus \Omega_{2h}$. In that case (6.8) obviously holds.

Therefore, we assume that $Q^o \neq \emptyset$ and owing to Lemma 6.2 conclude that

$$(6.11) \quad P_h[v_h] = K$$

in Q° . Furthermore, (6.5) implies that

$$(6.12) \quad P_h[v_h] \leq K$$

in Ω_h .

Now use again the mean value theorem to conclude that

$$\delta_{h,r}P_h[v_h] = a_k\Delta_{h,k}(\delta_{h,r}v_h) + b_k\delta_{h,k}(\delta_{h,r}v_h) - c(\delta_{h,r}v_h)$$

for some functions $a_k(x), b_k(x), c(x)$ satisfying (6.9). In addition, $\delta_{h,r}P_h[v_h] \leq 0$ in Q° owing to (6.11) and (6.12), that is in Q°

$$a_k\Delta_{h,k}(\delta_{h,r}v_h) + b_k\delta_{h,k}(\delta_{h,r}v_h) - c(\delta_{h,r}v_h) \leq 0.$$

For small enough h_0 the operator $a_k\Delta_{h,k} + b_k\delta_{h,k} - c$ with $h \in (0, h_0]$ respects the maximum principle and therefore in Q° (see Theorem 2.2 in [12])

$$(6.13) \quad (\delta_{h,r}v_h)^- \leq \sup_{\Omega_h \setminus Q^\circ} (\delta_{h,r}v_h)^-.$$

While estimating the right-hand side of (6.13), notice that if $x \in \Omega_h \setminus Q^\circ$, then either $x \notin \Omega_{2h}$, in which case (6.8) holds by the above, or else $x \in \Omega_{2h}$ but $(\delta/2)|\delta_{h,r}v_h| \leq \bar{\mathcal{H}} + K$. It follows that the right-hand side of (6.13) is dominated by the right-hand side of (6.8), if $h \in (0, h_0]$ and $h_0 > 0$ is sufficiently small.

Thus, in all cases

$$(\delta_{h,r}v_h)^- \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})$$

on Ω_h . Upon replacing here r with $-r$, we get

$$T_{h,-\ell_r}(\delta_{h,r}v_h)^+ \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})$$

in Ω_h , which after being combined with the previous estimate proves (6.8) in Ω_{2h} . In $\Omega_h \setminus \Omega_{2h}$ estimate (6.8) has been established above. The lemma is proved.

LEMMA 6.4. *Suppose that Assumptions 6.1 (i), (ii), (iv) are satisfied. Assume also that all vectors in Λ have rational coordinates. Then there are constants $h_0 > 0$ and N such that for all $h \in (0, h_0]$ and $|r| = 1, \dots, m$*

$$(6.14) \quad (\rho - 6\lambda h)|\Delta_{h,r}v_h| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})$$

on \mathbb{R}^d (we remind the reader that λ is the radius of B).

Proof. Clearly, the assertion of the lemma would follow if we can prove that (6.14) holds on $y + h\Lambda_\infty$ for any $y \in \mathbb{R}^d$ with a constant N independent of h and y . Without losing generality we concentrate on $y = 0$. Then for a fixed r define

$$Q^\circ := \{x \in (h\Lambda_\infty) \cap \Omega_{3h} : (\delta/2)|\Delta_{h,r}v_h(x)| \geq \bar{\mathcal{H}} + K\}.$$

If $x \in h\Lambda_\infty$ is such that $x \notin Q^\circ$, then either $x \notin \Omega_{3h}$, so that $\rho(x) \leq 3\lambda h$ and (6.14) holds, or else $x \in \Omega_{3h}$ but $(\delta/2)|\Delta_{h,r}v_h(x)| \leq \bar{\mathcal{H}} + K$, in which case (6.14) holds again.

Thus we need only prove (6.14) on Q° assuming, of course, that $Q^\circ \neq \emptyset$. Then define

$$Q = \{x + h\Lambda : x \in Q^\circ\}.$$

Observe that Q is a finite set since ℓ_k have rational coordinates and there is a number M such that the coordinates of all points in $M\Lambda_{1,\infty}$ are integers and the number of points with integral coordinates lying in a bounded domain is finite.

Next by Lemma 6.2 we have that (6.11) holds in Q° and (6.12) holds in $Q \setminus Q^\circ$.

To proceed further observe a standard fact that there are constants $\mu_0 > 0$ and $N \in [0, \infty)$ depending only on Ω such that for any $\mu \in (0, \mu_0]$ there exists an $\eta_\mu \in C_0^\infty(\Omega)$ satisfying

$$\begin{aligned} \eta_\mu &= 1 \quad \text{on} \quad \Omega_{2\mu}, \quad \eta_\mu = 0 \quad \text{outside} \quad \Omega_\mu, \\ (6.15) \quad |\eta_\mu| &\leq 1, \quad |D\eta_\mu| \leq N/\mu, \quad |D^2\eta_\mu| \leq N/\mu^2. \end{aligned}$$

By Theorem 5.1 and Lemma 6.3 there are constants N and $h_0 > 0$ such that, for any number ν satisfying

$$\nu \geq N(\|\eta'_\mu\|_h + \|\eta'_\mu\|_h^2 + \|\eta''_\mu\|_h),$$

we have in Q° that

$$\eta_\mu^4 [(\Delta_r v_h)^-]^2 \leq \max_{Q \setminus Q^\circ} \eta_\mu^4 [(\Delta_r v_h)^-]^2 + N(\nu + 1)(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})^2$$

if $h \in (0, h_0]$. In light of (6.15) one can take $\nu = N\mu^{-2}$ for an appropriate N and then

$$\eta_\mu^4 [(\Delta_r v_h(x))^-]^2 \leq \max_{y \in Q \setminus Q^\circ} \eta_\mu^4 [(\Delta_r v_h(y))^-]^2 + N\mu^{-2}(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})^2$$

for $x \in Q^\circ$. We will only concentrate on $\mu \geq 3h$, when $\eta_\mu = 0$ outside Ω_{3h} . In that case, for any $y \in Q \setminus Q^\circ$, either $y \notin \Omega_{3h}$ implying that

$$\eta_\mu^4 [(\Delta_r v_h)^-]^2(y) = 0$$

or else $y \in \Omega_{3h} \cap (h\Lambda_\infty)$ but

$$(\delta/2)|\Delta_{h,r}v_h(y)| \leq \bar{\mathcal{H}} + K.$$

It follows that as long as $h \in (0, h_0]$, $x \in Q^\circ$, and $\mu \geq 3h$ we have

$$(6.16) \quad \eta_\mu^4 [(\Delta_r v_h)^-(x)]^2 \leq N\mu^{-2}(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)})^2.$$

If x is such that $\rho(x) \geq 6\lambda h$, take $\mu = \mu_0 \wedge (\rho(x)/(2\lambda))$, which is bigger than $3h$ provided that $h \leq \mu_0/3$. In that case also

$$\rho(x) = 2\lambda[\rho(x)/(2\lambda)] \geq 2\lambda\mu,$$

so that $\eta_\mu(x) = 1$ and we conclude from (6.16) that

$$\begin{aligned} \rho(x)(\Delta_r v_h)^-(x) &\leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)}), \\ (6.17) \quad (\rho(x) - 6\lambda h)(\Delta_r v_h)^-(x) &\leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,1}(\Omega)}) \end{aligned}$$

for $x \in Q^\circ$ such that $\rho(x) \geq 6\lambda h$. However, the second relation in (6.17) is obvious for $\rho(x) \leq 6\lambda h$.

As a result of all the above arguments we see that (6.17) holds in $h\Lambda_\infty$ for any r whenever $h \in (0, h_0]$.

Finally, since $P_h[v_h] \leq K$ in Ω_h we have that

$$\begin{aligned} 2\delta^{-1} \sum_r (\Delta_r v_h)_+ &\leq (\delta/2) \sum_r (\Delta_r v_h)_- \\ &\quad - 2\delta^{-1} \sum_{r \geq 1} |\delta_r v_h| + (\delta/2)(v_h)_+ - 2\delta^{-1}(v_h)_- + K, \end{aligned}$$

which after being multiplied by $\rho - 6h$ along with (6.17) and Lemma 6.3 leads to (6.14) on $h\Lambda_\infty$. As is explained in the beginning of the proof, this finishes proving the lemma.

Proof of Theorem 6.1. Owing to Assumption 6.1 (iii), by Corollary 2.7 of [12], which is applicable in light of Lemmas 6.3 and 6.4, there exists a constant M such that for all sufficiently small h and $x, y \in \mathbb{R}^d$ we have

$$|v_h(x) - v_h(y)| \leq M(|x - y| + h).$$

Here Assumptions 6.1(iii) plays a crucial role.

After that our theorem is proved in exactly the same way as Theorem 8.7 of [10] on the basis of Lemmas 6.3 and 6.4 and the fact that the derivatives of v are weak limits of finite differences of v_h as $h \downarrow 0$ (see the proof of Theorem 8.7 of [10]). One also uses the fact that there are sufficiently many pure second order derivatives in directions of the l_i 's to conclude from their boundedness that the Hessian of v is bounded. The theorem is proved.

7. Proof of Theorem 3.2. The functions \mathcal{H} from Section 4 and \mathcal{P} from Section 1 are instances of \mathcal{H} and \mathcal{P} from Section 6. To see this, of course, one has to change the constant δ in Section 6 and renumber the l_i 's in Section 1. We also take into account that $\hat{\delta} \leq \delta/4$ which allows us to match (4.1) and (4.2) with the requirements of Assumption 6.1 (i). Furthermore, $\bar{\mathcal{H}} = \bar{\mathcal{H}}$. Therefore, Theorem 6.1 is applicable and yields a unique $v \in C^{0,1}(\bar{\Omega}) \cap C_{loc}^{1,1}(\Omega)$ such that $v = g$ on $\partial\Omega$, estimates (6.3), that is (1.3), hold true, and

$$\max[\mathcal{H}(v, Dv, v_{(l_1)(l_1)}, \dots, v_{(l_m)(l_m)}, x), \mathcal{P}(v, Dv, v_{(l_1)(l_1)}, \dots, v_{(l_m)(l_m)}) - K] = 0$$

in Ω (a.s.). In light of the construction of \mathcal{H} and \mathcal{F} in Section 4 this equation coincides with (1.2), so that the only remaining assertions of Theorem 3.2 to prove are that for $p > d$

$$(7.1) \quad \|v\|_{W_p^2(\Omega)} \leq N_p(\bar{H} + K + \|g\|_{W_p^2(\Omega)})$$

and estimate (1.5) holds. The latter follows from other assertions of Theorem 3.2 by Remark 1.3, so that we may concentrate on (7.1).

Observe that

$$\max(H(u, x), P(u) - K) = P(u) + G(u, x),$$

where $G(u, x) = (H(u, x) - P(u) + K)_+ - K$ and, owing to condition (3.13), $G(u, x) = -K$ if

$$\kappa \left(\sum_{i,j} |u_{ij}| + \sum_i |u_i| \right) \geq \bar{H} + K.$$

If the opposite inequality holds, then

$$(7.2) \quad |G(u, x)| \leq |H(u, x) - H(0, x)| + |P(u)| + \bar{H} + K \leq N(\bar{H} + K),$$

where N depends only on δ and d . It follows that the inequality between the extreme terms in (7.2) holds for all u and x . This allows us to apply Theorem 1.2 of [5] and shows that (7.1) holds if $v \in W_p^2(\Omega)$ or if $w := v - g \in W_p^2(\Omega)$. In light of (1.3) it suffices to show that $w \in W_p^2(D \cap \Omega)$, where D is a neighborhood of $\partial\Omega$.

To prove the latter we use the fact that, according to (3.14), in a neighborhood D of $\partial\Omega$ intersected with Ω the function w satisfies the equation

$$(7.3) \quad \max(\Delta u - u + \Delta g - g, P[u + g] - K) = 0,$$

in which the left-hand side is given by a convex function of u and its derivatives. We may certainly assume that $D \in C^2$ and then, by relying on (1.3), find a $\zeta \in C_0^\infty(\Omega)$ such that $\zeta w = w$ on $\partial(D \cap \Omega)$ and $\zeta w \in W_p^2(D \cap \Omega)$. Then due to Theorem 1.2 of [5] equation (7.3) with boundary condition $u - \zeta w \in \overset{\circ}{W}_p^2(D \cap \Omega)$ has a (unique) solution $u \in W_p^2(D \cap \Omega)$.

By uniqueness of $W_{d,loc}^2(D \cap \Omega) \cap C(\overline{D \cap \Omega})$ -solutions we obtain $w = u \in W_p^2(D \cap \Omega)$ and the theorem is proved.

Acknowledgment. The author is sincerely grateful to Hongjie Dong for pointing out a few glitches in the first version of the paper and to the referees for careful reading resulting in finding quite a bit of misprints.

REFERENCES

- [1] L. A. CAFFARELLI, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. Math., 130 (1989), pp. 189–213.
- [2] L. A. CAFFARELLI AND X. CABRÉ, *Fully nonlinear elliptic equations*, American Mathematical Society, Providence, 1995.
- [3] X. CABRÉ AND L. A. CAFFARELLI, *Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations*, J. Math. Pures Appl. (9), 82:5 (2003), pp. 573–612.
- [4] M. G. CRANDALL, M. KOCAN, AND A. ŚWIĘCH, *L^p -theory for fully nonlinear uniformly parabolic equations*, Comm. Partial Differential Equations, 25:11-12 (2000), pp. 1997–2053.
- [5] H. DONG, N. V. KRYLOV, AND X. LI, *On fully nonlinear elliptic and parabolic equations in domains with VMO coefficients*, Algebra i Analiz, 24:1 (2012), pp. 54–95.
- [6] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Series: Classics in Mathematics, Springer, Berlin, Heidelberg, New York, 2001.
- [7] J. KOVATS, *Differentiability properties of solutions of nondegenerate Isaacs equations*, Nonlinear Anal., 71:12 (2009), pp. e2418–e2426.
- [8] N. V. KRYLOV, *Nonlinear elliptic and parabolic equations of second order*, Nauka, Moscow, 1985 in Russian; English translation Reidel, Dordrecht, 1987.
- [9] N. V. KRYLOV, *On Bellman's equations with VMO coefficients*, Methods and Applications of Analysis, 17:1 (2010), pp. 105–122.
- [10] N. V. KRYLOV, *On a representation of fully nonlinear elliptic operators in terms of pure second order derivatives and its applications*, Problems of Mathematical Analysis in Russian; English translation: Journal of Mathematical Sciences, New York, 177:1 (2011), pp. 1–26.
- [11] N. V. KRYLOV, *Interior estimates for second differences of solutions of finite-difference elliptic Bellman's equations*, to appear in Math. Comp.
- [12] N. V. KRYLOV, *Rate of convergence of difference approximations for uniformly nondegenerate elliptic Bellman's equations*, preprint.
- [13] H.-J. KUO AND N. S. TRUDINGER, *Discrete methods for fully nonlinear elliptic equations*. SIAM Journal on Numerical Analysis, 29:1 (1992), pp. 123–135.
- [14] N. NADIRASHVILI AND S. VLĀDUT, *Singular viscosity solutions to fully nonlinear elliptic equations*, Journal de Mathématiques Pures et Appliquées, 89:2 (2008), pp. 107–113.

- [15] N. S. TRUDINGER, *On regularity and existence of viscosity solutions of nonlinear second order, elliptic equations*, in Partial differential equations and the calculus of variations, Vol. II, pp. 939–957, Progr. Nonlinear Differential Equations Appl., 2, Birkhäuser Boston, Boston, MA, 1989.
- [16] L. WANG, *On the regularity theory of fully nonlinear parabolic equations: I*, Comm. Pure Appl. Math., 45 (1992), pp. 27–76.
- [17] N. WINTER, *$W^{2,p}$ and $W^{1,p}$ -estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, Z. Anal. Anwend., 28:2 (2009), pp. 129–164.