A QUASILINEAR DELAYED HYPERBOLIC NAVIER-STOKES SYSTEM: GLOBAL SOLUTION, ASYMPTOTICS AND RELAXATION LIMIT[*]

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Abstract. We consider a hyperbolic quasilinear fluid model, that arises from a delayed version for the constitutive law for the deformation tensor in the incompressible Navier-Stokes equation. We prove global existence of small solutions and asymptotic results in $\mathbb{R}^3$ and the half-space with slip boundary conditions. Futhermore we show that this relaxed system is close to the classical Navier-Stokes equation in the sense that for small times t the solutions converge in high Sobolev norms to the solution of the incompressible Navier-Stokes equation.

1. Introduction. Let $n \geq 2$ and $T, \tau, \mu > 0$. In this note the fluid model

$$
\begin{aligned}
(1) \quad \tau u_{tt} - \mu \Delta u &+ u_t + \nabla p + \tau \nabla p_t \\
&= -(u \cdot \nabla)u - (\tau u_t \cdot \nabla)u - (\tau u \cdot \nabla)u_t \quad \text{in } (0,T) \times \mathbb{R}^n, \\
\operatorname{div} u &= 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \\
u(0,\cdot) &= u_0, \quad u_t(0,\cdot) = u_1 \quad \text{in } \mathbb{R}^n
\end{aligned}
$$

for the velocity field $u = u(t,x) : (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ and the pressure $p = p(t,x) : (0,T) \times \mathbb{R}^n \to \mathbb{R}$, where u_0 and u_1 are given initial data, will be considered. This model arises from a delayed version for the constitutive law for the deformation tensor in the incompressible Navier-Stokes equation (compare below).

First of all there will be given some well-posedness and asymptotic results to the model above in $\mathbb{R}^3$, that improve and complete known results due to Racke and Saal from [11] and [12].

Furthermore, thanks to a reflection technique (cp. [15]), the equation in the half-space $\mathbb{R}^n_+$ with slip-boundary conditions can also be solved.
As another main result it will be shown, that the model above is closely related to the incompressible Navier-Stokes equation, in the sense that for small times t the solutions $(u^\tau)_\tau$ of (1) corresponding to $\tau > 0$ converge for $\tau \to 0$ in high Sobolev norms to the solution of the incompressible Navier-Stokes equation.

The model can be derived from the incompressible Navier-Stokes equation as follows.

The incompressible Navier-Stokes equation (in the following also referred to as classical Navier-Stokes equation) is given by

$$
\begin{aligned}
(2) \quad u_t + (u \cdot \nabla)u + \nabla p &= \operatorname{div} 2S \quad \text{in } (0,T) \times \Omega, \\
\operatorname{div} u &= 0 \quad \text{in } (0,T) \times \Omega, \\
u|_{t=0} &= u_0 \quad \text{in } \Omega,
\end{aligned}
$$

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where the deformation tensor S is determined by

$$(3) \qquad S = \frac{\mu}{2} \left(\nabla u + (\nabla u)' \right).$$

Like the change from Fourier type law to Cattaneo in heat conduction (cp. [4]), this law is replaced by a delayed version

$$(4) \qquad S + \tau S_t = \frac{\mu}{2} \left(\nabla u + (\nabla u)' \right),$$

for a small relaxation parameter $\tau > 0$, where the left hand side is a formal first order Taylor approximation of $S(t + \tau)$. By the second line in (2) one gets

$$\operatorname{div} 2(S + \tau S_t) = \mu \Delta u$$

and therefore (1) can be derived by differentiating the first line in (2) with respect to t and after a multiplication with τ adding the resulting line to the original one. The equation (1) will be refered to as hyperbolic Navier-Stokes equation.

This model was already derived in [2] and [3]. But in a mathematical rigorous way, concerning well-posedness in high Sobolev norms, it has been treated first in [11] an [12]. Further results have been established in the diploma thesis of the author [13], which in parts will be presented in the following.

Paicu and Raugel considered in [9] another hyperbolic fluid model by just adding a hyperbolic perturbation τu_{tt} to the classical Navier-Stokes equation, but this model rather differs to the one above, because their model remains semilinear, which makes it possible to carry over the proofs for the classical Navier-Stokes equation. For an introduction to the Navier-Stokes equation we refer to [14] and [7].

Furthermore in [1] the model from [9] was examined for $\tau \to 0$ and it was shown, that the solutions $(u^\tau)_\tau$ corresponding to $\tau > 0$ converge for $\tau \to 0$ to the solution of the incompressible Navier-Stokes equation. However this result on an approximation of the classical Navier-Stokes equation by relaxation cannot be compared to the Theorem 3.1 in this note, because different systems on different domains are considered. Formally, regarding the derivation of the hyperbolic Navier-Stokes equation, the result on the approximation seems to be not so astonishing but one has to keep in mind the works [5] and [6], where it was shown that delayed systems, that are formally close together, can behave differently. For example in [6] it was shown, that a equation, coming from a Cattaneo type law, might not be exponentially stable, although the same system with a Fourier type law is. In [5] it is even shown, that formal high taylor expansions of the delayed term, can lead to ill-posedness. Altogether the result on the approximation feeds hope to gain new ideas for the classical Navier-Stokes equation by examination of the hyperbolic Navier-Stokes equation. Futhermore it makes the conjecture of Racke and Saal in [12], that the hyperbolic Navier-Stokes equation has a blow-up for large data, even more interesting.

The paper is organized as follows. In section two well-posedness and asymptotic results to the model (1) will be proven, where the whole space is treated in subsection 2.1 and the half-space in subsection 2.2.

In section three we will prove the result on the convergence to the classical Navier-Stokes equation for $\tau \to 0$. In the first part of the section the boundedness in high Sobolev norms of $(u^\tau)_\tau$ for small times t in τ is proven, which makes it possible to show in the second part the convergence to the solution of the classical Navier-Stokes equation.

2. Well-posedness and asymptotic results. In this section there will be given some well-posedness and asymptotic results for (1), but first of all some remarks on the notation used in this note.

Let X be a general Banach space and $\Omega \subseteq \mathbb{R}^n$ a set, then $C^m(\Omega, X)$ denotes the space of m-times continuously differentiable functions with values in X. Analogously $L^p(\Omega, X)$ with $1 \leq p \leq \infty$ denotes the standard Lebesgue space of X-valued functions with norm $\| \cdot \|_p$. For the Hilbert space $L^2(\Omega, X)$ we write $\langle \cdot, \cdot \rangle$ for the scalar product.

As usual $W^{k,p}(\Omega, X)$ denotes the Sobolev space of k-th order, with the norm

$$(5) \qquad \|u\|_{W^{m,p}} := \|u\|_{m,p} := \begin{cases} \left(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha u\|_p^p \right)^{\frac{1}{p}}, & \text{for } \leq p < \infty, \\ \max_{0 \leq |\alpha| \leq m} \|\partial^\alpha u\|_\infty, & \text{for } p = \infty, \end{cases}$$

where the multi-index notation is used, that means for $\alpha \in \mathbb{N}_0^n$ we set $\nabla^\alpha := \partial^\alpha := \partial_1^{\alpha_1} \cdot ... \cdot \partial_n^{\alpha_n}$.

For a vector-valued function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n, x \mapsto u(x) = (u_1(x), ..., u_n(x))$ u_i denotes the i-th component.

The Helmholtz projection onto the space

$$(6) \qquad L_\sigma^q(\Omega) := \overline{\{u \in (C_0^\infty(\Omega))^n : \operatorname{div} u = 0\}}^{\|\cdot\|_q} \qquad \text{for } \Omega \in \{\mathbb{R}^n, \mathbb{R}_+^n\}$$

is denoted by P.

2.1. Whole space. In [11] the well-posedness in the whole space $\mathbb{R}^n$ has been proven under a smallness condition on the initial data, where the existence time depends on the highest norm of the initial data. One can improve this dependence to the result

THEOREM 2.1 (Local existence). *Let $n \geq 2$ and $s \geq m > \frac{n}{2}$. For all*

$$(u_0, u_1) \in \left(W^{s+2,2}(\mathbb{R}^n) \cap L_\sigma^2(\mathbb{R}^n) \right) \times \left(W^{s+1,2}(\mathbb{R}^n) \cap L_\sigma^2(\mathbb{R}^n) \right)$$

there exists a $T > 0$ and a unique solution (u, p) to (1) satisfying

$$u \in C^0([0,T], W^{s+2,2}(\mathbb{R}^n) \cap L_\sigma^2(\mathbb{R}^n)) \cap C^1([0,T], W^{s+1,2}(\mathbb{R}^n)) \cap C^2([0,T], W^{s,2}(\mathbb{R}^n)),$$

$$\nabla(p + \tau p_t) \in C^0([0,T], W^{s,2}(\mathbb{R}^n)).$$

The existence time T only depends on $\|u_0\|_{m+2,2}$ and $\|u_1\|_{m+1,2}$.

Hints on the proof. One has to check where in [11] the existence time is determined. The basic idea of the proof in [11] is to construct solutions to a linearized system and then use Majda's fixed point iteration to treat the full system. To this end the boundedness of the iteration sequence is needed, which can only been proven with a restriction to the existence time (compare [11, Lemma 5.2]). Therefore one has to find a better proof of [11, Lemma 5.2].

One approximates the initial data with smooth functions, then gets a smoother iteration sequence, which justifies to carry over the first part of the calculations in [11, Theorem 4.5 Step 2], and finally gets with the idea used in [10, Theorem 5.8] the desired energy estimate with better dependence of the existence time. The convergence of the iteration sequence to the solution can be proven like in [11]. □

In [12] the global existence in $\mathbb{R}^2$ and $\mathbb{R}^3$ was proven with a method by Klainerman and Ponce, like it is for example described in [10]. The proof uses convergence rates

coming from the damped wave equation, which in two dimensions are worse than in three dimensions and therefore Racke and Saal focused on the two dimensional case. Actually if one compares the equation with the damped wave equation, where quadratic nonlinearties in $\mathbb{R}^2$ touch the critical borderline, one would expect that the two dimensional case is more complicated (compare [12, 1. Introduction]). The remark in [12] that the result in $\mathbb{R}^3$ can be improved has now been carried out in detail and the following Theorem was proven.

THEOREM 2.2 (Global solution). *Let $m \geq 3$, then there exists a $\delta > 0$ such that for initial data $(u_0, u_1) \in \left(W^{m+3,2}(\mathbb{R}^3) \cap L_\sigma^2(\mathbb{R}^3)\right) \times \left(W^{m+2,2}(\mathbb{R}^3) \cap L_\sigma^2(\mathbb{R}^3)\right)$ satisfying*

$$(7) \qquad \|u_0\|_{m+3,2} + \|u_1\|_{m+2,2} + \|u_0\|_1 + \|u_1\|_1 < \delta,$$

there exists a unique global solution (u, p) to the hyperbolic Navier-Stokes equation (1) with

$$(8) \qquad \begin{aligned} u \in C^0([0,\infty), W^{m+3,2}(\mathbb{R}^3) \cap L_\sigma^2(\mathbb{R}^3)) \cap C^1([0,\infty), W^{m+2,2}(\mathbb{R}^3)) \\ \cap C^2([0,\infty), W^{m+1,2}(\mathbb{R}^3)), \end{aligned}$$

$$(9) \qquad \nabla(p + \tau p_t) \in C^0([0,\infty), W^{m+1,2}(\mathbb{R}^3)).$$

Furthermore

$$(10) \qquad \begin{aligned} \|u(t)\|_{m,2} = \mathcal{O}\left(t^{-\frac{3}{4}}\right), \qquad \|u_t(t)\|_{m,2} = \mathcal{O}\left(t^{-\frac{3}{2}}\right), \\ \|\nabla u(t)\|_{m,2} = \mathcal{O}\left(t^{-\frac{5}{4}}\right) \qquad \text{for } t \to \infty \end{aligned}$$

holds.

REMARK 2.3 (Improvements). *Obviously the decay rates are better than in the two dimensional case, but furthermore the conditions on the initial conditions were weakened and the proof was shortened and therefore the Theorem is a interesting result. Of course it is also possible to show $\|\cdot\|_{m_1,q}$- decay rates like in [12], but for the proof of the global existence in $\mathbb{R}^3$ this is not necessary.*

Proof of Theorem 2.2. One applies the Helmholtz projection P to (1) and gets for $u \in L_\sigma^2(\mathbb{R}^3)$ the equation

$$(11) \qquad \begin{aligned} \tau u_{tt} - \mu \Delta u + u_t = & -P((u \cdot \nabla)u) - P((\tau u_t \cdot \nabla)u) \\ & -P((\tau u \cdot \nabla)u_t) \equiv N_1 + N_2 + N_3 \qquad \text{in } (0,\infty) \times \mathbb{R}^3, \\ u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1 & \qquad \text{in } \mathbb{R}^3. \end{aligned}$$

According to Theorem 2.1 let u be the local solution to (11). Then as usual the proof consists of a high energy estimate and a weighted a priori estimate for u, which together guarantee that the local existence theorem can be applied successively to obtain a global solution.

The high energy estimate can be quoted from [12, Theorem 4.1].

THEOREM 2.4 (High energy estimate). *For*

$$(12) \qquad E_m(t) := E_m(u(t)) := \frac{1}{2} \sum_{|\alpha| \leq m+1} (\tau \|\nabla^\alpha u_t\|_2^2 + \mu \|\nabla^\alpha \nabla u\|_2^2 + \varepsilon_2 \|\nabla^\alpha u\|_2^2)(t)$$

with $\varepsilon_2 > 0$ appropriate, there exists a $c > 0$ independent from T and the initial data $(u_0, u_1) \in \left(W^{m+2,2} \cap L_\sigma^2\right) \times \left(W^{m+1,2} \cap L_\sigma^2\right)$ such that for $0 \le t \le T$ and $m > \frac{3}{2} + 1$

$$\text{(13)} \qquad E_m(t) \le cE_m(0)e^{c \int_0^t (\|u\|_\infty^2 + \|u_t\|_{1,\infty} + \|\nabla u\|_\infty)(r)\mathrm{d}r}$$

holds.

The improvement in the proof of the global existence in the three dimensional case to [12] lies in the weighted a priori estimate, that thanks to better decay rates, can be formulated and proven easier (compare the following Theorem to [12, Theorem 5.3]).

THEOREM 2.5 (Weighted a priori estimate). *Let $m > \frac{3}{2} + 1$, then there exists a $\delta_1 > 0$ such that for initial data $(u_0, u_1) \in \left(W^{m+3,2} \cap L_\sigma^2\right) \times \left(W^{m+2,2} \cap L_\sigma^2\right)$ with*

$$\text{(14)} \qquad \|u_0\|_{m+3,2} + \|u_1\|_{m+2,2} + \|u_0\|_1 + \|u_1\|_1 < \delta_1,$$

there exists a $M_0 > 0$ independent from T, such that for the solution u of (1)

$$\text{(15)} \qquad \begin{aligned} M(T) := \sup_{0 \le t \le T} & \left\{ (1+t)^{\frac{3}{4}} \|u(t)\|_{m,2} + (1+t)^{\frac{3}{2}} \|u_t(t)\|_{m,2} + (1+t)^{\frac{5}{4}} \|\nabla u(t)\|_{m,2} \right\} \\ & \le M_0 \end{aligned}$$

holds.

Proof. We quote the representation formula [12, Lemma 5.2] for the solution u

LEMMA 2.6 (Representation formula). *Let $w(t)g$ denote the solution to*

$$\begin{aligned} \tau v_{tt} - \mu \Delta v + v_t &= 0 && in \ (0, \infty) \times \mathbb{R}^3, \\ v(0, \cdot) &= 0, \quad v_t(0, \cdot) = g && in \ \mathbb{R}^3. \end{aligned}$$

Then u can be represented as

$$\text{(16)} \qquad u(t) = w(t)(u_1 + \frac{1}{\tau}u_0) + \partial_t w(t)u_0 + \frac{1}{\tau} \int_0^t w(t-r) \sum_{j=1}^3 N_j(r)\mathrm{d}r.$$

which makes it possible two obtain convergence rates for the solution u from the damped wave equation. In the case of $\mathbb{R}^3$ one only (compare to [12, Lemma 5.1]) needs the following convergence rates.

LEMMA 2.7. *Let v denote the solution to*

$$\begin{aligned} \tau v_{tt} - \mu \Delta v + v_t &= 0 && in \ (0, \infty) \times \mathbb{R}^3, \\ v(0, \cdot) &= v_0, \quad v_t(0, \cdot) = v_1 && in \ \mathbb{R}^3. \end{aligned}$$

Then for all $\alpha \in \mathbb{N}_0^3$ and $j \in \mathbb{N}_0$

$$\text{(17)} \qquad \|\nabla^\alpha \partial_t^j v(t, \cdot)\|_2 \le c(1+t)^{-\left(\frac{|\alpha|}{2} + j\right)} \|(v_0, v_1)\|_{X_2},$$

$$\text{(18)} \qquad \|\nabla^\alpha \partial_t^j v(t, \cdot)\|_2 \le c(1+t)^{-\left(\frac{1}{4} + \frac{|\alpha|}{2} + j\right)} \|(v_0, v_1)\|_{X_{\frac{3}{2}}},$$

$$\text{(19)} \qquad \|\nabla^\alpha \partial_t^j v(t, \cdot)\|_2 \le c(1+t)^{-\left(\frac{3}{4} + \frac{|\alpha|}{2} + j\right)} \|(v_0, v_1)\|_{X_1},$$

with

$$X_k := \begin{cases} L^2 \times L^2 \cap L^k \times L^k, & \text{for } |\alpha| + j = 0 \\ W^{|\alpha|+j,2} \times W^{|\alpha|+j-1,2} \cap L^k \times L^k, & \text{for } |\alpha| + j \geq 1. \end{cases}$$

holds.

Proof. Use [8, Lemma 1] with $n = 3$ and $m = 2$, $m = \frac{3}{2}$ resp. $m = 1$. $\square$

Now one successively deals with each term of $M(T)$. In contrast to the two dimensional case one always uses the following derivative structure of the nonlinearities. Because $\operatorname{div} u = 0$, one can write

$$N_1 = -P\left((u \cdot \nabla)u\right) = -P\left(\nabla \cdot (u \otimes u)\right) \equiv P(\nabla \cdot \widetilde{N}_1) = \nabla \cdot P\widetilde{N}_1,$$
$$N_2 = -P\left((\tau u_t \cdot \nabla)u\right) = -P\left(\nabla \cdot (\tau u_t \otimes u)\right) \equiv P(\nabla \cdot \widetilde{N}_2) = \nabla \cdot P\widetilde{N}_2,$$
$$N_3 = -P\left((\tau u \cdot \nabla)u_t\right) = -P\left(\nabla \cdot (\tau u \otimes u_t)\right) \equiv P(\nabla \cdot \widetilde{N}_3) = \nabla \cdot P\widetilde{N}_3.$$

I. Estimate for $\|u(t)\|_{m,2}$

Using the representation formula from Lemma 2.6

$$u(t) = w(t)(u_1 + \frac{1}{\tau}u_0) + \partial_t w(t)u_0 + \frac{1}{\tau}\int_0^t w(t-r)\sum_{j=1}^3 N_j(r)\mathrm{d}r \equiv v_1(t) + v_2(t) + v_3(t)$$

one gets with (19) the estimates

(20)
$$\|v_1(t)\|_{m,2} \leq c(1+t)^{-\frac{3}{4}}\left(\|u_0\|_{m-1,2} + \|u_1\|_{m-1,2} + \|u_0\|_1 + \|u_1\|_1\right) < c\delta_1(1+t)^{-\frac{3}{4}},$$
(21)
$$\|v_2(t)\|_{m,2} \leq c(1+t)^{-(\frac{3}{4}+1)}(\|u_0\|_{m+1-1,2} + \|u_0\|_1) < c\delta_1(1+t)^{-(\frac{3}{4}+1)}.$$

To estimate v_3 one uses the derivative structure of the nonlinearities and the continuity of the Helmholtz projection together with (18) and gets

(22) $$\|v_3(t)\|_{m,2} \leq c\int_0^t (1+t-r)^{-(\frac{1}{4}+\frac{1}{2})}\sum_{j=1}^3 (\|\widetilde{N}_j(r)\|_{m+1-1,2} + \|\widetilde{N}_j(r)\|_{\frac{3}{2}})\mathrm{d}r.$$

REMARK 2.8. *The concerning the convergence rate better estimate (19) cannot be used here, because the Helmholtz projection is not continuous on L^1. Otherwise the estimate (17) is not sufficient (this would avoid the $\|\cdot\|_{\frac{3}{2}}$ terms), since then the convergence rate of $\|u(t)\|_{m,2}$ would be too small (compare (32)).*

With Theorem 4.1 and the Sobolev embedding theorem ($m > \frac{3}{2}$) it follows

$$\|\widetilde{N}_1(r)\|_{m,2} \leq c\|u(r)\|_\infty \|u(r)\|_{m,2}$$
$$\leq c\|u(r)\|_{m,2}\|u(r)\|_{m,2}$$
$$= c(1+r)^{-\frac{3}{4}}\left((1+r)^{\frac{3}{4}}\|u(r)\|_{m,2}\right)(1+r)^{-\frac{3}{4}}\left((1+r)^{\frac{3}{4}}\|u(r)\|_{m,2}\right)$$
$$\leq c(1+r)^{-\frac{3}{2}}M(T)^2$$

and

$$\|\widetilde{N}_2(r)\|_{m,2} + \|\widetilde{N}_3(r)\|_{m,2} \le c(1+r)^{-\frac{3}{4}-\frac{3}{2}} M(T)^2.$$

Because of $\frac{1}{\frac{3}{2}} = \frac{1}{2} + \frac{1}{6}$, one can use Hölder, which together with the Sobolev embedding theorem implies

$$\|\widetilde{N}_1(r)\|_{\frac{3}{2}} \le c\|u(r)\|_2\|u(r)\|_6 \le c\|u(r)\|_{m,2}\|u(r)\|_{m,2} \le c(1+r)^{-\frac{3}{2}} M(T)^2$$

and

$$\|\widetilde{N}_2(r)\|_{\frac{3}{2}} + \|\widetilde{N}_3(r)\|_{\frac{3}{2}} \le c(1+r)^{-\frac{3}{4}-\frac{3}{2}} M(T)^2.$$

Plugging in all estimates for the $\widetilde{N}_j(r)$ in (22), it follows

$$(23) \qquad \|v_3(t)\|_{m,2} \le cM(T)^2(1+t)^{-\frac{3}{4}} \int_0^t (1+t-r)^{-(\frac{1}{4}+\frac{1}{2})}(1+r)^{-\frac{3}{2}}(1+t)^{\frac{3}{4}}\mathrm{d}r.$$

Using Lemma 4.2, one finally gets

$$(24) \qquad \|v_3(t)\|_{m,2} \le cM(T)^2(1+t)^{-\frac{3}{4}}.$$

Summing up (20), (21) and (24)

$$(25) \qquad (1+t)^{\frac{3}{4}}\|u(t)\|_{m,2} < c\delta_1 + cM(T)^2$$

is shown.

II. Now to the estimate for $\|u_t(t)\|_{m,2}$

Differentiating the representation formula from Lemma 2.6 with respect to t yields

$$u_t(t) = \partial_t w(t)(u_1 + \frac{1}{\tau}u_0) + \partial_t^2 w(t)u_0 + \partial_t \frac{1}{\tau} \int_0^t w(t-r) \sum_{j=1}^3 N_j(r)\mathrm{d}r$$

$$= \partial_t w(t)(u_1 + \frac{1}{\tau}u_0) + \partial_t^2 w(t)u_0 + \underbrace{\frac{1}{\tau}w(0)\sum_{j=1}^3 N_j(t)}_{=0} + \frac{1}{\tau}\int_0^t \partial_t w(t-r)\sum_{j=1}^3 N_j(r)\mathrm{d}r$$

$$\equiv v_1(t) + v_2(t) + v_3(t).$$

Analogous to I. it follows

$$\|v_1(t)\|_{m,2} \le c(1+t)^{-(\frac{3}{4}+1)} \left(\|u_0\|_{m+1-1,2} + \|u_1\|_{m+1-1,2} + \|u_0\|_1 + \|u_1\|_1\right)$$

$$(26) \qquad\qquad\qquad < c\delta_1(1+t)^{-(\frac{3}{4}+1)},$$

$$(27) \qquad \|v_2(t)\|_{m,2} \le c(1+t)^{-(\frac{3}{4}+2)}(\|u_0\|_{m+2-1,2} + \|u_0\|_1) < c\delta_1(1+t)^{-(\frac{3}{4}+2)}.$$

Like above, but with (17) one gets

$$(28) \qquad \|v_3(t)\|_{m,2} \le c \int_0^t (1+t-r)^{-(\frac{1}{2}+1)} \sum_{j=1}^3 \|\widetilde{N}_j(r)\|_{m+1+1-1,2}\mathrm{d}r.$$

Theorem 4.1 and the Sobolev embedding Theorem imply

$$\|\widetilde{N}_1(r)\|_{m+1,2} \leq c\|u(r)\|_{m,2}\|u(r)\|_{m+1,2}$$

$$(29) \qquad\qquad \leq c\|u(r)\|_{m,2}(\|u(r)\|_{m,2} + \|\nabla^{m+1}u(r)\|_2),$$

$$\|\widetilde{N}_2(r)\|_{m+1,2} + \|\widetilde{N}_3(r)\|_{m+1,2} \leq c(\|u(r)\|_{m,2}\|u_t(r)\|_{m+1,2} + \|u_t(r)\|_{m,2}\|u(r)\|_{m+1,2})$$

$$\leq c\big(\|u(r)\|_{m,2}(\|u_t(r)\|_{m,2} + \|\nabla^{m+1}u_t(r)\|_2)$$

$$(30) \qquad\qquad\qquad + \|u_t(r)\|_{m,2}(\|u(r)\|_{m,2} + \|\nabla^{m+1}u(r)\|_2)\big).$$

The highest derivative can be estimated with the representation formula from Lemma 2.6 and (17) to

$$\|\nabla^{m+1}u(r)\|_2 \leq c(1+r)^{-\frac{m+1}{2}}\|u_1 + \frac{1}{\tau}u_0\|_{m+1-1,2} + c(1+r)^{-(\frac{m+1}{2}+1)}\|u_0\|_{m+1+1-1,2}$$

$$+ c\int_0^r (1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2})}\sum_{j=1}^3 \|P\widetilde{N}_j(\lambda)\|_{m+1+1-1,2}d\lambda,$$

$$\|\nabla^{m+1}u_t(r)\|_2 \leq c(1+r)^{-(\frac{m+1}{2}+1)}\|u_1 + \frac{1}{\tau}u_0\|_{m+1+1-1,2} + c(1+r)^{-(\frac{m+1}{2}+2)}\|u_0\|_{m+1+2-1,2}$$

$$+ c\int_0^r (1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2}+1)}\sum_{j=1}^3 \|P\widetilde{N}_j(\lambda)\|_{m+1+1+1-1,2}d\lambda.$$

The high norm $\|P\widetilde{N}_j(\lambda)\|_{m+2,2}$ can be estimated by contiuity of the Helmholtz projection, Theorem 4.1 and the high energy estimate (Theorem 2.4) for $\overline{m} = m+1$ to

$$\|P\widetilde{N}_j(\lambda)\|_{m+2,2} \leq c(\|u(\lambda)\|_\infty + \|u_t(\lambda)\|_\infty)(\|u(\lambda)\|_{m+2,2} + \|u_t(\lambda)\|_{m+2,2})$$

$$\leq c(\|u(\lambda)\|_\infty + \|u_t(\lambda)\|_\infty)\sqrt{E_{m+1}(\lambda)}$$

$$\leq c(\|u(\lambda)\|_\infty + \|u_t(\lambda)\|_\infty)c\sqrt{E_{m+1}(0)}e^{c\int_0^\lambda(\|u\|_\infty^2 + \|u_t\|_{1,\infty} + \|\nabla u\|_\infty)(\varrho)d\varrho}.$$

Using Sobolev $(m > \frac{3}{2}+1)$ and $\sqrt{E_{m+1}(0)} \leq c\|(u_0,u_1)\|_{W^{m+3,2}\times W^{m+2,2}} < c\delta_1$ it follows

$$(31) \quad \|P\widetilde{N}_j(\lambda)\|_{m+2,2} \leq c\delta_1(\|u(\lambda)\|_{m,2} + \|u_t(\lambda)\|_{m,2})e^{c\int_0^\lambda(\|u\|_{m,2}^2 + \|u_t\|_{m,2} + \|\nabla u\|_{m,2})(\varrho)d\varrho}$$

and finally

$$(1+r)^{\frac{3}{4}}\big(\|\nabla^{m+1}u(r)\|_2 + \|\nabla^{m+1}u_t(r)\|_2\big)$$

$$\leq c(1+r)^{-\frac{m+1}{2}+\frac{3}{4}}\|u_1 + \frac{1}{\tau}u_0\|_{m+1,2} + c(1+r)^{-(\frac{m+1}{2}+1)+\frac{3}{4}}\|u_0\|_{m+2,2}$$

$$+ c\delta_1\int_0^r \bigg((1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2})}(1+\lambda)^{-\frac{3}{4}}(1+r)^{\frac{3}{4}}\Big[(1+\lambda)^{\frac{3}{4}}\|u(\lambda)\|_{m,2}\Big]$$

$$+ (1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2})}(1+\lambda)^{-\frac{3}{2}}(1+r)^{\frac{3}{4}}\Big[(1+\lambda)^{\frac{3}{2}}\|u_t(\lambda)\|_{m,2}\Big]\bigg)$$

$$\cdot e^{c\int_0^\lambda(1+\varrho)^{-2\cdot\frac{3}{4}}\left[(1+\varrho)^{\frac{3}{4}}\|u(\varrho)\|_{m,2}\right]^2 + (1+\varrho)^{-\frac{5}{4}}\left[(1+\varrho)^{\frac{5}{4}}(\|u_t(\varrho)\|_{m,2} + \|\nabla u(\varrho)\|_{m,2})\right]d\varrho}d\lambda.$$

Lemma 4.2 yields

$$\sup_{r\geq 0}\int_0^r (1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2})}(1+\lambda)^{-\frac{3}{4}}(1+r)^{\frac{3}{4}}d\lambda < \infty,$$

$$\sup_{r\geq 0}\int_0^r (1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2})}(1+\lambda)^{-\frac{3}{2}}(1+r)^{\frac{3}{4}}d\lambda < \infty.$$

Obviously it holds

$$(32) \qquad \sup_{\lambda \geq 0} \int_0^\lambda (1+\varrho)^{-\frac{3}{2}} + (1+\varrho)^{-\frac{5}{4}} \mathrm{d}\varrho < \infty.$$

REMARK 2.9 (Comparison to the two dimensional case). *This estimate is the essential difference between the two and the three dimensional case. In $\mathbb{R}^2$ it is not possible to estimate the norms in the exponent of e against $W^{m,2}$-norms, because the integral (32) would be divergent, since the convergence rates in $\mathbb{R}^2$ are not good enough. Therefore in [12] $W^{m_1,q}$-estimates were used.*

With these estimates and by definition of δ_1 (14) and $M(T)$ (15) it follows

$$(1+r)^{\frac{3}{4}} \|\nabla^{m+1} u(r)\|_2 + \|\nabla^{m+1} u_t(r)\|_2 \leq c\delta_1 \left(1 + M(T)e^{c\left(M(T)^2 + M(T)\right)} \right).$$

Plugging in in (29) resp. (30) yields

$$(1+r)^{\frac{3}{2}} \sum_{j=1}^3 \|\widetilde{N}_j(r)\|_{m+1,2}$$

$$\leq c(1+r)^{\frac{3}{4}} \|u(r)\|_{m,2} \left((1+r)^{\frac{3}{4}} \|u(r)\|_{m,2} + (1+r)^{\frac{3}{4}} \|\nabla^{m+1} u(r)\|_2 \right)$$

$$+ c\Bigg((1+r)^{\frac{3}{4}} \|u(r)\|_{m,2} \left((1+r)^{\frac{3}{4}} \|u_t(r)\|_{m,2} + (1+r)^{\frac{3}{4}} \|\nabla^{m+1} u_t(r)\|_{m,2} \right)$$

$$+ (1+r)^{\frac{3}{4}} \|u_t(r)\|_{m,2} \left((1+r)^{\frac{3}{4}} \|u(r)\|_{m,2} + (1+r)^{\frac{3}{4}} \|\nabla^{m+1} u(r)\|_{m,2} \right) \Bigg)$$

$$\leq cM(T) \left(M(T) + c\delta_1 \left(1 + M(T)e^{c\left(M(T)^2 + M(T)\right)} \right) \right).$$

If one puts this in the estimate (28) for $\|v_3(t)\|$, it follows

$$(33) \qquad \begin{aligned} \|v_3(t)\|_{m,2} \leq\, & cM(T) \left(M(T) + c\delta_1 \left(1 + M(T)e^{c\left(M(T)^2 + M(T)\right)} \right) \right) \\ & \cdot (1+t)^{-\frac{3}{2}} \int_0^t (1+t-r)^{-\left(\frac{1}{2}+1\right)} (1+r)^{-\frac{3}{2}} (1+t)^{\frac{3}{2}} \mathrm{d}r. \end{aligned}$$

Again by an application of Lemma 4.2 one finally gets

$$(34) \quad (1+t)^{\frac{3}{2}} \|u_t(t)\|_{m,2} < c\delta_1 + cM(T) \left(M(T) + c\delta_1 \left(1 + M(T)e^{c\left(M(T)^2 + M(T)\right)} \right) \right).$$

III. Estimate of $\|\nabla u(t)\|_{m,2}$

Differentiating the representation formula from Lemma 2.6 with respect to x yields

$$\nabla u(t) = \nabla w(t)\left(u_1 + \frac{1}{\tau} u_0\right) + \nabla \partial_t w(t) u_0 + \frac{1}{\tau} \int_0^t \nabla w(t-r) \sum_{j=1}^3 N_j(r) \mathrm{d}r$$

$$\equiv v_1(t) + v_2(t) + v_3(t).$$

Like above it follows

$$\|v_1(t)\|_{m,2} \leq c(1+t)^{-\left(\frac{3}{4}+\frac{1}{2}\right)} \left(\|u_0\|_{m+1-1,2} + \|u_1\|_{m+1-1,2} + \|u_0\|_1 + \|u_1\|_1\right)$$

$$(35) \qquad < c\delta_1(1+t)^{-\frac{5}{4}},$$

$$(36) \quad \|v_2(t)\|_{m,2} \leq c(1+t)^{-\left(\frac{3}{4}+1+\frac{1}{2}\right)}(\|u_0\|_{m+1+1-1,2} + \|u_0\|_1) < c\delta_1(1+t)^{-\left(\frac{5}{4}+1\right)}.$$

Like in I. one gets with (18) the estimate

$$(37) \qquad \|v_3(t)\|_{m,2} \leq c\int_0^t (1+t-r)^{-\left(\frac{1}{4}+\frac{2}{2}\right)} \sum_{j=1}^3 (\|\widetilde{N}_j(r)\|_{m+2-1,2} + \|\widetilde{N}_j(r)\|_{\frac{3}{2}})\mathrm{d}r.$$

Now $\sum_{j=1}^3 \|\widetilde{N}_j(r)\|_{m+1,2}$ can be estimated like in II., while $\sum_{j=1}^3 (\|\widetilde{N}_j(r)\|_{\frac{3}{2}}$ can be estimated like in I. such that

$$(1+r)^{\frac{3}{2}} \sum_{j=1}^3 (\|\widetilde{N}_j(r)\|_{m+1,2} + \|\widetilde{N}_j(r)\|_{\frac{3}{2}})$$

$$(38) \qquad \leq cM(T)\left(M(T) + c\delta_1\left(1 + M(T)e^{c\left(M(T)^2+M(T)\right)}\right)\right)$$

follows.
Plugging in in (37), Lemma 4.2 and (35) and (36) yield

$$(39) \ (1+t)^{\frac{5}{4}}\|\nabla u(t)\|_{m,2} < c\delta_1 + cM(T)\left(M(T) + c\delta_1\left(1 + M(T)e^{c\left(M(T)^2+M(T)\right)}\right)\right).$$

IV. Summary of I.-III. and proof of the Theorem

Putting together (25), (34) and (39) it is shown

$$(1+t)^{\frac{3}{4}}\|u(t)\|_{m,2} + (1+t)^{\frac{3}{2}}\|u_t(t)\|_{m,2} + (1+t)^{\frac{5}{4}}\|\nabla u(t)\|_{m,2}$$

$$< c\delta_1 + cM(T)^2 + c\delta_1 + cM(T)\left(M(T) + c\delta_1\left(1 + M(T)e^{c\left(M(T)^2+M(T)\right)}\right)\right)$$

$$\leq c\delta_1 + cM(T)^2 + c\delta_1 M(T)\left(1 + M(T)e^{c\left(M(T)^2+M(T)\right)}\right).$$

By definition of $M(T)$ (15) one gets

$$(40) \qquad M(T) \leq c\delta_1 + cM(T)^2 + c\delta_1 M(T)\left(1 + M(T)e^{c\left(M(T)^2+M(T)\right)}\right)$$

and therefore the claim follows by standard arguments (compare [12]). □

Theorem 2.2 can now be followed as usual by applying the local existence Theorem 2.1 successively. □

2.2. Half-space with slip boundary conditions. In this subsection the solvability of the hyperbolic Navier-Stokes equation in the half-space with slip boundary conditions is proven with a reflection technique. For simplicity we only deal with the three dimensional case. To be precise, the equation

$$(41) \quad \begin{aligned} \tau u_{tt} - \mu\Delta u + u_t + \nabla p + \tau\nabla p_t & \\ = -(u\cdot\nabla)u - (\tau u_t \cdot\nabla)u - (\tau u\cdot\nabla)u_t \quad & \text{in } (0,\infty)\times\mathbb{R}_+^3, \\ \mathrm{div}\, u = 0 \quad & \text{in } (0,\infty)\times\mathbb{R}_+^3, \\ \frac{\partial u_1}{\partial\vec{n}} = 0, \quad \frac{\partial u_2}{\partial\vec{n}} = 0, \quad u_3 = 0 \quad & \text{in } \partial\mathbb{R}_+^3, \\ u(0,\cdot) = v, \quad u_t(0,\cdot) = w \quad & \text{in } \mathbb{R}_+^3. \end{aligned}$$

for $u = u(t,x) : (0,\infty) \times \mathbb{R}_+^3 \to \mathbb{R}^3$ and $p = p(t,x) : (0,\infty) \times \mathbb{R}_+^3 \to \mathbb{R}$ with $\mu, \tau > 0$ and given initial conditions v and w is treated.

Assuming for the initial conditions v and w for $m \in \mathbb{N}$ and $\delta > 0$

(V1($\boldsymbol{\delta, m}$)) $(v,w) \in \left(W^{m+3,2}(\mathbb{R}_+^3) \cap L_\sigma^2(\mathbb{R}_+^3)\right) \times \left(W^{m+2,2}(\mathbb{R}_+^3) \cap L_\sigma^2(\mathbb{R}_+^3)\right)$ with $\|v\|_{m+3,2} + \|w\|_{m+2,2} + \|v\|_1 + \|w\|_1 < \delta$

(V2($\boldsymbol{m}$))   For all odd integers $k \in [3, m+2]$ the partial derivatives satisfy $\partial_3^k v_1, \partial_3^k v_2 \in H_0^1(\mathbb{R}_+^n)$. Analogously for all odd integers $k \in [3, m+1]$, it holds $\partial_3^k w_1, \partial_3^k w_2 \in H_0^1(\mathbb{R}_+^n)$ (v_j denotes the j-th component of the vector v).
Furthermore v and w satisfy the boundary condition.

the following Theorem can be proven.

THEOREM 2.10 (Global solution in the half-space). *Let $m \geq 3$, then there exists a $\delta > 0$ such that if the initial conditions v and w satisfy ($V1(\delta, m)$) and ($V2(m)$) then there exists a unique global solution (u,p) to the hyperbolic Navier-Stokes equation* (41) *with*

$$(42) \qquad u \in C^0([0,\infty), W^{m+3,2}(\mathbb{R}_+^3) \cap L_\sigma^2(\mathbb{R}_+^3)) \cap C^1([0,\infty), W^{m+2,2}(\mathbb{R}_+^3)) \\ \cap C^2([0,\infty), W^{m+1,2}(\mathbb{R}_+^3)),$$

$$(43) \qquad \nabla(p + \tau p_t) \in C^0([0,\infty), W^{m+1,2}(\mathbb{R}_+^3))$$

Furthermore

$$(44) \qquad \|u(t)\|_{m,2} = \mathcal{O}\left(t^{-\frac{3}{4}}\right), \qquad \|u_t(t)\|_{m,2} = \mathcal{O}\left(t^{-\frac{3}{2}}\right), \\ \|\nabla u(t)\|_{m,2} = \mathcal{O}\left(t^{-\frac{5}{4}}\right) \qquad \text{for } t \to \infty$$

holds.

Hints on the proof. The basic idea is to reflect the initial data to functions in the whole space, then apply the Theorem 2.2 for the whole space and finally check that the restriction of this solution to the half-space solves (41). In [15] this idea was described in a general context and applied for the classical Navier-Stokes equation. Therefore one actually applies Theorems from [15].

The boundary conditions dictate how to reflect the initial conditions, which means even extension of the first and second component of the vector and odd extension of the last component.

$$(45) \qquad \widetilde{f}(x_1, x_2, x_3) := \begin{cases} \begin{pmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{pmatrix}, & \text{for } x_3 \geq 0, \\ \begin{pmatrix} f_1(x_1,x_2,-x_3) \\ f_2(x_1,x_2,-x_3) \\ -f_3(x_1,x_2,-x_3) \end{pmatrix}, & \text{for } x_3 < 0 \end{cases} \qquad \text{for } f \in \{v, w\}.$$

The assumption ($V2(m)$) guarantees thanks to [15, Theorem 5.1] that this extension is smooth, which means

$$(46) \qquad (\widetilde{v}, \widetilde{w}) \in \left(W^{m+3,2}(\mathbb{R}^3) \cap L_\sigma^2(\mathbb{R}^3)\right) \times \left(W^{m+2,2}(\mathbb{R}^3) \cap L_\sigma^2(\mathbb{R}^3)\right).$$

Choosing $\delta < \frac{\bar{\delta}}{2}$ with $\bar{\delta}$ from Theorem 2.2 the assumption (V1(δ, m)) ensures that Theorem 2.2 can be applied to the initial data $(\widetilde{v}, \widetilde{w})$ which yields a solution $(\widetilde{u}, \widetilde{p})$ to the whole space problem.

Now one only has to check that

$$(u, p) := (\widetilde{u}|_{\mathbb{R}^3_+}, \widetilde{p}|_{\mathbb{R}^3_+}) \tag{47}$$

has the right boundary conditions. But this is clear because the equation preserves the symmetry of the initial conditions $(\widetilde{v}, \widetilde{w})$ (compare [15, Proposition 4.1]).

The claimed convergence rates follow by construction of the solution directly from Theorem 2.2. The uniqueness of the solution can be proven with the multiplier method (compare Section 3.2). $\square$

3. Relaxation parameter $\tau \to 0$. Looking at the derivation of the hyperbolic Navier-Stokes equation one would expect that for vanishing relaxation parameter τ the corresponding solutions u^τ converge to the solution v of the classical Navier-Stokes equation, but as mentioned in the introduction this is not clear.

It will be shown that in the whole space $\mathbb{R}^n$ with $n = 2$ or $n = 3$ for sufficiently smooth initial data, $T_1 > 0$ appropiate and $m > \frac{n}{2}$

$$\sup_{0 \le t \le T_1} \|u^\tau(t) - v(t)\|_{m+2,2} = \mathcal{O}(\tau) \quad \text{and} \quad \sup_{0 \le t \le T_1} \|u_t^\tau(t) - v_t(t)\|_{m+1,2} = \mathcal{O}(\sqrt{\tau}) \tag{48}$$

for $\tau \to 0$ holds.

This will be proven with the multiplier method applied to an equation for the difference $w := u^\tau - v$. Therefore one first needs an appropriate equation for w.

Differentiating the Helmholtz-projected Navier-Stokes equation for $v \in L^2_\sigma(\mathbb{R}^n)$ with respect to t yields

$$\begin{aligned} v_t - \mu \Delta v + P\left((v \cdot \nabla)v\right) &= 0 && \text{in } (0, T) \times \mathbb{R}^n, \\ v(0, \cdot) &= v_0 && \text{in } \mathbb{R}^n. \end{aligned} \tag{49}$$

This equation is now multiplied with τ and then added to the original equation and hence

$$\begin{aligned} \tau v_{tt} - \mu \Delta v &- \tau \mu \Delta v_t + v_t \\ &= -P\left((v \cdot \nabla)v\right) - P\left((\tau v_t \cdot \nabla)v\right) \\ &\quad - P\left((\tau v \cdot \nabla)v_t\right) && \text{in } (0, T) \times \mathbb{R}^n, \\ v(0, \cdot) = v_0, \quad v_t(0, \cdot) &= \mu \Delta v_0 - P\left((v_0 \cdot \nabla)v_0\right) =: v_1 && \text{in } \mathbb{R}^n. \end{aligned} \tag{50}$$

The initial condition for $v_t(0, \cdot)$ is a natural compability condition from (49). Subtraction of this equation from the Helmholtz-projected hyperbolic Navier-Stokes equation for $u \in L^2_\sigma(\mathbb{R}^n)$

$$\begin{aligned} \tau u_{tt} - \mu \Delta u + u_t &= -P\left((u \cdot \nabla)u\right) - P\left((\tau u_t \cdot \nabla)u\right) \\ &\quad - P\left((\tau u \cdot \nabla)u_t\right) \equiv N_1 + N_2 + N_3 && \text{in } (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = v_0, \quad u_t(0, \cdot) &= v_1 && \text{in } \mathbb{R}^n, \end{aligned} \tag{51}$$

gives the desired equation for $L^2_\sigma(\mathbb{R}^n) \ni w := u - v$

$$\begin{aligned} \tau w_{tt} - \mu \Delta w + w_t + \tau \mu \Delta v_t &= \sum_{j=1}^{6} M_j && \text{in } (0, T) \times \mathbb{R}^n, \\ w(0, \cdot) = 0, \quad w_t(0, \cdot) &= 0 && \text{in } \mathbb{R}^n. \end{aligned} \tag{52}$$

Where the M_j are determined by

$$- P(u \cdot \nabla)u + P(v \cdot \nabla)v = -P(u \cdot \nabla)(u - v) - P(u \cdot \nabla)v + P(v \cdot \nabla)v$$
$$= -P(u \cdot \nabla)(u - v) - P((u - v) \cdot \nabla)v = -P(u \cdot \nabla)w - P(w \cdot \nabla)v \equiv M_1 + M_2,$$

resp.

$$- \tau P(u_t \cdot \nabla)u - \tau P(u \cdot \nabla)u_t + \tau P(v_t \cdot \nabla)v + \tau P(v \cdot \nabla)v_t$$
$$= -\tau P(u_t \cdot \nabla)w - \tau P(w_t \cdot \nabla)v - \tau P(u \cdot \nabla)w_t - \tau P(w \cdot \nabla)v_t$$
$$\equiv M_3 + M_4 + M_5 + M_6.$$

The following energy estimate is similar to the methods used to proof the high energy estimate [12, Theorem 4.1], but first we have to specify the regularity of the solutions u and v. Actually the solutions should be just as good as that the following calculation works, which can surely be achieved if v_0 is smooth enough and $m > \frac{n}{2}$. To be precise Theorem 2.1 gives a solution

$$(53) \qquad \begin{aligned} u^\tau \in C^0([0, T^\tau], W^{m+3,2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)) \cap C^1([0, T^\tau], W^{m+2,2}(\mathbb{R}^n)) \\ \cap C^2([0, T^\tau], W^{m+1,2}(\mathbb{R}^n)) \end{aligned}$$

to the hyperbolic Navier-Stokes equation and for example with [14] one can derive a solution

$$(54) \qquad v \in C^1([0, T_v], W^{m+3,2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)) \cap C^2([0, T_v], W^{m+1,2}(\mathbb{R}^n))$$

to the classical Navier-Stokes equation.

It will be shown that $T_1 > 0$ can be chosen independent from τ such that all solutions exist on a common interval $[0, T_1]$. Therefore the following definition and the subsequent calculation makes sense for $t \in [0, T_1]$.

One defines the energies by
$$(55)$$
$$E_m(\varphi(t)) := \frac{1}{2} \sum_{|\alpha| \le m+1} (\tau \|\nabla^\alpha \varphi_t\|_2^2 + \mu \|\nabla^\alpha \nabla \varphi\|_2^2 + \|\nabla^\alpha \varphi\|_2^2)(t) \qquad \text{for } \varphi \in \{u, v, w\}$$

and corresponding Lyapunov functionals by

$$(56) \qquad \widetilde{E}_m(\varphi(t)) := E_m(\varphi(t)) + \tau \sum_{|\alpha| \le m+1} \langle \nabla^\alpha \varphi_t, \nabla^\alpha \varphi \rangle(t) \qquad \text{for } \varphi \in \{u, w\}.$$

For $\tau \le \frac{1}{2}$ it holds

$$(57) \qquad \frac{1}{4} E_m(\varphi(t)) \le \widetilde{E}_m(\varphi(t)) \le \frac{7}{4} E_m(\varphi(t)).$$

In subsection 3.2 it will be shown that for $R > 0$ appropriate and τ small enough the estimate
$$(58)$$
$$E_m(w(t))$$

$$\le \int_0^t c\tau^2 R \,\mathrm{d}r \cdot \exp\left(c \int_0^t \underbrace{(E_m(u(r)) + E_{m+1}(v(r)) + E_m(u(r))^{\frac{1}{2}} + E_{m+1}(v(r))^{\frac{1}{2}} + 1)}_{=: E(r)} \,\mathrm{d}r \right)$$

holds, where c is a constant independent from τ (this shall hold for the whole section). Since in subsection 3.1 for $T_1 > 0$ appropriate the boundedness of $E_m(u(t))$ in τ for $t \in [0, T_1]$ is shown, $E(t)$ stays bounded in τ for $t \in [0, T_1]$ and the following Theorem is proven.

THEOREM 3.1. *For $m > \frac{n}{2}$ and v_0 sufficiently smooth there exists a $T_1 > 0$ such that*

$$\tag{59} \sup_{0 \le t \le T_1} E_m(w(t)) = \mathcal{O}(\tau^2) \qquad \text{for } \tau \to 0.$$

Especially

$$\tag{60} \sup_{0 \le t \le T_1} \|u^\tau(t) - v(t)\|_{m+2,2} = \mathcal{O}(\tau) \quad \text{and} \quad \sup_{0 \le t \le T_1} \|u_t^\tau(t) - v_t(t)\|_{m+1,2} = \mathcal{O}(\sqrt{\tau}),$$

holds. This means that the solutions u^τ of the hyperbolic Navier-Stokes equation converge for $\tau \to 0$ to the solution v of the classical Navier-Stokes equation.

3.1. Boundedness of u^τ in τ.

THEOREM 3.2. *There exists a $T_1 > 0$ such that for $m > \frac{n}{2}$ and $v_0 \in W^{m+4,2}(\mathbb{R}^n)$*

$$\tag{61} \sup_{0 \le t \le T_1} E_m(u(t)) = \mathcal{O}(1) \qquad \text{for } \tau \to 0.$$

Proof. Under the assumption that the solutions $(u^\tau)_\tau$ exist on a common interval $[0, T]$ with T independent from τ, for small τ the a priori estimate

$$\tag{62} \widetilde{E}_m(u(t)) \le \widetilde{E}_m(u(0)) + c \int_0^t (\widetilde{E}_m(u(r)) + 1)^2 \mathrm{d}r,$$

will be shown. Then one can apply a nonlinear version of the Lemma from Gronwall 4.3 and gets the claim.

For reasons of clarity the argument (t) is dropped.

Let $|\alpha| \le m + 1$. Applying ∇^α to (51) and multiplication with $\nabla^\alpha u_t$ in $L^2(\mathbb{R}^n)$ yields

$$\tag{63} \frac{\tau}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^\alpha u_t\|_2^2 + \frac{\mu}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^\alpha \nabla u\|_2^2 + \|\nabla^\alpha u_t\|_2^2 = \sum_{j=1}^3 \langle \nabla^\alpha N_j, \nabla^\alpha u_t \rangle.$$

Now one has to estimate the nonlinear terms. In doing so, one has to observe that terms like $\|\nabla^\alpha u_t\|_2^2$ can only be estimated τ-independent against $E_m(u)$ if they have a prefactor τ. If τ appears in front of other terms like $\tau \|\nabla^\alpha u\|_2^2$, one can estimate them τ-independent, if one assumes without restriction $\tau < 1$.

With the continuity of the Helmholtz projection and Theorem 4.1 one gets

$$\tag{64} |\langle \nabla^\alpha N_1, \nabla^\alpha u_t \rangle| \le c \left(\|u\|_\infty \|\nabla^{m+1} \nabla u\|_2 + \|\nabla u\|_\infty \|\nabla^{m+1} u\|_2 \right) \|\nabla^\alpha u_t\|_2.$$

Using Young's inequality and Sobolev yields

$$\begin{aligned} |\langle \nabla^\alpha N_1, \nabla^\alpha u_t \rangle| &\le c \left(\|u\|_{m,2}^2 \|\nabla^{m+1} \nabla u\|_2^2 + \|\nabla u\|_{m,2}^2 \|\nabla^{m+1} u\|_2^2 \right) + \tfrac{1}{2} \|\nabla^\alpha u_t\|_2^2 \\ \tag{65} &\le c E_m(u)^2 + \tfrac{1}{2} \|\nabla^\alpha u_t\|_2^2. \end{aligned}$$

Analogously

$$|\langle \nabla^\alpha N_2, \nabla^\alpha u_t\rangle| \leq c\tau \left(\|u_t\|_{m,2}^2\|\nabla^{m+1}\nabla u\|_2^2 + \|\nabla u\|_{m,2}^2\|\nabla^{m+1}u_t\|_2^2\right) + c\tau\|\nabla^\alpha u_t\|_2^2$$

$$(66) \qquad\qquad \leq cE_m(u)^2 + c(E_m(u)+1)^2.$$

The last term can be rearranged to

$$\langle \nabla^\alpha N_3, \nabla^\alpha u_t\rangle = -\tau\langle(u\cdot\nabla\nabla^\alpha)u_t, \nabla^\alpha u_t\rangle - \tau\langle\nabla^\alpha((u\cdot\nabla)u_t) - (u\cdot\nabla^\alpha\nabla)u_t, \nabla^\alpha u_t\rangle$$

$$\equiv R_1 + R_2.$$

With integration by parts one gets

$$R_1 = -\tau\langle u_j\partial_j\partial^\alpha\partial_t u_r, \partial^\alpha\partial_t u_r\rangle \overset{\mathrm{div}\,u=0}{=} -\tau\langle\partial_j u_j\partial^\alpha\partial_t u_r, \partial^\alpha\partial_t u_r\rangle$$

$$= \tau\langle u_j\partial^\alpha\partial_t u_r, \partial_j\partial^\alpha\partial_t u_r\rangle = \tau\langle\partial^\alpha\partial_t u_r, u_j\partial_j\partial^\alpha\partial_t u_r\rangle = -R_1,$$

and therefore $R_1 = 0$.
The term R_2 can be estimated with Theorem 4.1(ii) to

$$|R_2| \leq c\tau\|\nabla^\alpha((u\cdot\nabla)u_t) - (u\cdot\nabla^\alpha\nabla)u_t\|_2\|\nabla^\alpha u_t\|_2$$

$$\leq c\tau\left(\|\nabla u\|_\infty\|\nabla^m\nabla u_t\|_2 + \|\nabla u_t\|_\infty\|\nabla^{m+1}u\|_2\right)\|\nabla^\alpha u_t\|_2$$

$$\leq c\tau\left(\|\nabla u\|_{m,2}^2\|\nabla^m\nabla u_t\|_2^2 + \|\nabla u_t\|_{m,2}^2\|\nabla^{m+1}u\|_2^2\right) + c\tau\|\nabla^\alpha u_t\|_2^2$$

$$(67) \qquad\qquad \leq cE_m(u)^2 + c(E_m(u)+1)^2.$$

Now one again applies for all $|\alpha| \leq m+1$ the operator ∇^α to (51) but this time multiplies with $\nabla^\alpha u$ in $L^2(\mathbb{R}^n)$ to the end

$$(68)\quad \tau\frac{\mathrm{d}}{\mathrm{d}t}\langle\nabla^\alpha u_t, \nabla^\alpha u\rangle - \tau\|\nabla^\alpha u_t\|_2^2 + \mu\|\nabla^\alpha\nabla u\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla^\alpha u\|_2^2 = \sum_{j=1}^3\langle\nabla^\alpha N_j, \nabla^\alpha u\rangle.$$

As above one gets

$$(69) \qquad |\langle\nabla^\alpha N_1, \nabla^\alpha u\rangle| + |\langle\nabla^\alpha N_2, \nabla^\alpha u\rangle| \leq cE_m(u)^2 + c(E_m(u)+1)^2.$$

For the last term one distinguishes between $|\alpha| = 0$ and $|\alpha| \neq 0$. For $|\alpha| = 0$

$$|\langle N_3, u\rangle| \leq c\tau\|(u\cdot\nabla)u_t\|_2\|u\|_2 \leq c\tau\|\nabla u_t\|_\infty\|u\|_2^2$$

$$(70) \qquad\qquad \leq c\tau\|\nabla u_t\|_{m,2}^2 + c\tau\|u\|_2^4 \leq c(E_m(u)+1)^2 + cE_m(u)^2$$

holds, while for $|\alpha| \neq 0$ there exists a $1 \leq k \leq n$ with $\alpha_k \neq 0$ and therefore

$$|\langle\nabla^\alpha N_3, \nabla^\alpha u\rangle| \leq c\tau\left(\|u\|_\infty\|\nabla^m\nabla u_t\|_2 + \|\nabla u_t\|_\infty\|\nabla^m u\|_2\right)\|\nabla^{\alpha+e_k}u\|_2$$

$$\leq c\tau\left(\|u\|_{m,2}^2\|\nabla^m\nabla u_t\|_2^2 + \|\nabla u_t\|_{m,2}^2\|\nabla^m u\|_2^2\right) + c\tau\|\nabla^{\alpha+e_k}u\|_2^2$$

$$(71) \qquad\qquad \leq cE_m(u)^2 + c(E_m(u)+1)^2.$$

Adding (63) and (68), summing up over $|\alpha| \leq m+1$ and using the estimates for the nonlinearities (65), (66), (67) and (69), (70) resp. (71) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_m(u) + \sum_{|\alpha|\leq m+1}\left(\|\nabla^\alpha u_t\|_2^2 - \tau\|\nabla^\alpha u_t\|_2^2 + \mu\|\nabla^\alpha\nabla u\|_2^2\right)$$

$$\leq cE_m(u)^2 + c(E_m(u)+1)^2 + \sum_{|\alpha|\leq m+1}\frac{1}{2}\|\nabla^\alpha u_t\|_2^2.$$

Without loss of generality one sets $\tau \leq \frac{1}{2}$ and obtains

(72) $$\frac{d}{dt}\widetilde{E}_m(u) \leq cE_m(u)^2 + c(E_m(u) + 1)^2 \leq c(E_m(u) + 1)^2.$$

If one integrates in time from 0 to t and uses (57), the estimate (62) is proven. To apply the nonlinear version of the Lemma from Gronwall 4.3 one has to look at the differential equation

$$\dot{h}(t) = c(h(t) + 1)^2$$
$$h(0) = \frac{7}{4}E_m(u(0)) =: h_0,$$

which clearly has the solution

(73) $$h(t) = \frac{h_0 + 1}{1 - tc(h_0 + 1)} - 1 \qquad \text{for } t \in [0, \frac{1}{c(h_0+1)}).$$

The existence interval of h gets larger for $\tau \to 0$. Setting δ equal to the value of $\frac{7}{4}E_m(u(0))$ for $\tau = \frac{1}{2}$ and T_1 as $0 < T_1 < \frac{1}{c(\delta+1)}$, then the continuous function $h = h(t, \tau) : [0, T_1] \times [0, \frac{1}{2}] \to \mathbb{R}$ has a maximum M and therefore

(74) $$\sup_{0 \leq t \leq T_1} E_m(u(t)) \leq \sup_{0 \leq t \leq T_1} h(t) \leq M \qquad \text{for } \tau \in [0, \frac{1}{2}].$$

It remains to show, that $(u^\tau)_\tau$ for $\tau \in [0, \frac{1}{2}]$ arbitrary exist on the common interval $[0, T_1]$.

Let $\tau \in [0, \frac{1}{2}]$. Then u^τ exists on $[0, T^\tau]$, where $T^\tau = T^\tau(\|u_0\|_{m+2}, \|u_1\|_{m+1}, \tau)$. If $T^\tau \geq T_1$ nothing is to prove and therefore let $T^\tau < T_1$. From the estimate above one gets $E_m(u^\tau(T^\tau)) \leq M$. Now the local existence theorem can be applied to the initial data $u^\tau(T^\tau)$ and $u_t^\tau(T^\tau)$ and therefore one gets a solution on $[T^\tau, \widetilde{T}^\tau]$ with $\widetilde{T}^\tau = \widetilde{T}^\tau(M, \frac{M}{\tau}, \tau)$. If $\widetilde{T}^\tau < T_1$ again $E_m(u^\tau(\widetilde{T}^\tau)) \leq M$ holds and hence one can argument successively. $\square$

REMARK 3.3 (Dependences on T_1 and M). *T_1 is defined by the existence time of h (cp. (73)) and therefore depends on the norm of the initial data and on the constant c that comes from the estimate above. To be more precise c depends on the constant coming from the Moser inequalities, on μ and so on but most important c and T_1 do not depend on τ. M is by definition the maximum of h and so depends on the same quantities as c and T_1.*

3.2. Convergence for $\tau \to 0$. Now we want to proof Theorem 3.1.

The estimate is very similiar to the one of the previous subsection, especially one again has to observe that terms with time derivative must have a prefactor τ to estimate them against the energy. Of course for terms like $\|v_t\|_{m,2}$ this is not necessary, but for clarity and uniformity we do so. Without restriction let $T_1 \leq T_v$.

For a suitable $R > 0$ and τ small enough we prove with the multiplier method the estimate

(75)
$$\frac{d}{dt}\widetilde{E}_m(w(t))$$
$$\leq c\tau^2 R + c\left(E_m(u(t)) + E_{m+1}(v(t)) + E_m(u(t))^{\frac{1}{2}} + E_{m+1}(v(t))^{\frac{1}{2}} + 1\right)\widetilde{E}_m(w(t)),$$

from which one easily obtains (58) and the claim.

As above one gets

(76)
$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^\alpha w_t\|_2^2 + \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^\alpha\nabla w\|_2^2 + \|\nabla^\alpha w_t\|_2^2 = -\langle\mu\tau\nabla^\alpha\Delta v_t, \nabla^\alpha w_t\rangle + \sum_{j=1}^6\langle\nabla^\alpha M_j, \nabla^\alpha w_t\rangle.$$

Without restriction let again $\tau < 1$ hold. Analogously to the previous section one gets

$$
\begin{aligned}
|\langle\nabla^\alpha M_1, \nabla^\alpha w_t\rangle| &\le c\left(\|u\|_\infty\|\nabla^{m+1}\nabla w\|_2 + \|\nabla w\|_\infty\|\nabla^{m+1}u\|_2\right)\|\nabla^\alpha w_t\|_2 \\
&\le c(\|u\|_{m,2}^2 + \|\nabla^{m+1}u\|_2^2)(\|\nabla^{m+1}\nabla w\|_2^2 + \|\nabla w\|_{m,2}^2) + \tfrac{1}{6}\|\nabla^\alpha w_t\|_2^2 \\
&\le cE_m(u)E_m(w) + \tfrac{1}{6}\|\nabla^\alpha w_t\|_2^2
\end{aligned}
$$
(77)

and

$$
\begin{aligned}
|\langle\nabla^\alpha M_2, \nabla^\alpha w_t\rangle| &\le c\left(\|w\|_\infty\|\nabla^{m+1}\nabla v\|_2 + \|\nabla v\|_\infty\|\nabla^{m+1}w\|_2\right)\|\nabla^\alpha w_t\|_2 \\
&\le cE_m(v)E_m(w) + \tfrac{1}{6}\|\nabla^\alpha w_t\|_2^2.
\end{aligned}
$$
(78)

Furthermore it holds

$$
\begin{aligned}
|\langle\nabla^\alpha M_3, \nabla^\alpha w_t\rangle| &\le c\tau\left(\|u_t\|_\infty\|\nabla^{m+1}\nabla w\|_2 + \|\nabla w\|_\infty\|\nabla^{m+1}u_t\|_2\right)\|\nabla^\alpha w_t\|_2 \\
&\le c\tau^2(\|u_t\|_{m,2}^2 + \|\nabla^{m+1}u_t\|_2^2)(\|\nabla^{m+1}\nabla w\|_2^2 + \|\nabla w\|_{m,2}^2) + \tfrac{1}{6}\|\nabla^\alpha w_t\|_2^2 \\
&\le c\tau E_m(u)E_m(w) + \tfrac{1}{6}\|\nabla^\alpha w_t\|_2^2.
\end{aligned}
$$
(79)

and

$$
\begin{aligned}
|\langle\nabla^\alpha M_4, \nabla^\alpha w_t\rangle| &\le c\tau\left(\|w_t\|_\infty\|\nabla^{m+1}\nabla v\|_2 + \|\nabla v\|_\infty\|\nabla^{m+1}w_t\|_2\right)\|\nabla^\alpha w_t\|_2 \\
&\le c\tau(\|\nabla^{m+1}\nabla v\|_2 + \|\nabla v\|_\infty)(\|w_t\|_\infty^2 + \|\nabla^{m+1}w_t\|_2^2 + \|\nabla^\alpha w_t\|_2^2) \\
&\le cE_m(v)^{\frac{1}{2}}E_m(w).
\end{aligned}
$$
(80)

The term $|\langle\nabla^\alpha M_5, \nabla^\alpha w_t\rangle|$ is rearranged to

$$
\begin{aligned}
\langle\nabla^\alpha M_5, \nabla^\alpha w_t\rangle &= -\tau\langle(u\cdot\nabla\nabla^\alpha)\,w_t, \nabla^\alpha w_t\rangle - \tau\langle\nabla^\alpha\left((u\cdot\nabla)w_t\right) - (u\cdot\nabla^\alpha\nabla)w_t, \nabla^\alpha w_t\rangle \\
&\equiv R_1 + R_2
\end{aligned}
$$

and like above $R_1 = 0$ follows.

Moreover one obtains

$$
\begin{aligned}
|R_2| &\le c\tau\|\nabla^\alpha\left((u\cdot\nabla)w_t\right) - (u\cdot\nabla^\alpha\nabla)w_t\|_2\|\nabla^\alpha w_t\|_2 \\
&\le c\tau\left(\|\nabla u\|_\infty\|\nabla^m\nabla w_t\|_2 + \|\nabla w_t\|_\infty\|\nabla^{m+1}u\|_2\right)\|\nabla^\alpha w_t\|_2 \\
&\le c\tau(\|\nabla u\|_\infty + \|\nabla^{m+1}u\|_2)(\|\nabla^m\nabla w_t\|_2^2 + \|\nabla w_t\|_\infty^2 + \|\nabla^\alpha w_t\|_2^2) \\
&\le cE_m(u)^{\frac{1}{2}}E_m(w).
\end{aligned}
$$
(81)

The last term $|\langle\nabla^\alpha M_6, \nabla^\alpha w_t\rangle|$ can be estimated by

$$
\begin{aligned}
|\langle\nabla^\alpha M_6, \nabla^\alpha w_t\rangle| &\le c\tau\left(\|w\|_\infty\|\nabla^{m+1}\nabla v_t\|_2 + \|\nabla v_t\|_\infty\|\nabla^{m+1}w\|_2\right)\|\nabla^\alpha w_t\|_2 \\
&\le c\tau^2(\|\nabla^{m+1}\nabla v_t\|_2^2 + \|\nabla v_t\|_{m,2}^2)(\|w\|_{m,2}^2 + \|\nabla^{m+1}w\|_2^2) + \tfrac{1}{6}\|\nabla^\alpha w_t\|_2^2 \\
&\le c\tau E_{m+1}(v)E_m(w) + \tfrac{1}{6}\|\nabla^\alpha w_t\|_2^2.
\end{aligned}
$$
(82)

Now one applies for all $|\alpha| \le m+1$ the operator ∇^α to (52), but this time multiplies with $\nabla^\alpha w$ in $L^2(\mathbb{R}^n)$ to the end

$$\tau \frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla^\alpha w_t, \nabla^\alpha w \rangle - \tau \|\nabla^\alpha w_t\|_2^2 + \mu \|\nabla^\alpha \nabla w\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|\nabla^\alpha w\|_2^2$$

(83)
$$= -\langle \mu \tau \nabla^\alpha \Delta v_t, \nabla^\alpha w \rangle + \sum_{j=1}^{6} \langle \nabla^\alpha M_j, \nabla^\alpha w \rangle.$$

With the same estimates as above one obtains

(84)
$$|\langle \nabla^\alpha M_1, \nabla^\alpha w \rangle| \le c E_m(u)^{\frac{1}{2}} E_m(w),$$

(85)
$$|\langle \nabla^\alpha M_2, \nabla^\alpha w \rangle| \le c E_m(v)^{\frac{1}{2}} E_m(w),$$

(86)
$$|\langle \nabla^\alpha M_3, \nabla^\alpha w \rangle| \le c \sqrt{\tau} E_m(u)^{\frac{1}{2}} E_m(w),$$

(87)
$$|\langle \nabla^\alpha M_4, \nabla^\alpha w \rangle| \le c E_m(v)^{\frac{1}{2}} E_m(w).$$

For the term $|\langle \nabla^\alpha M_5, \nabla^\alpha w \rangle|$ one distinguishes between $\alpha = 0$ and $\alpha \ne 0$.
For $|\alpha| = 0$ it holds

(88) $$|\langle M_5, w \rangle| \le c\tau \|(u \cdot \nabla) w_t\|_2 \|w\|_2 \le c\tau \|u\|_\infty \|\nabla w_t\|_2 \|w\|_2 \le c E_m(u)^{\frac{1}{2}} E_m(w),$$

while for $|\alpha| \ne 0$ one obtains as above

$$|\langle \nabla^\alpha M_5, \nabla^\alpha w \rangle| \le c\tau \left(\|u\|_\infty \|\nabla^m \nabla w_t\|_2 + \|\nabla w_t\|_\infty \|\nabla^m u\|_2 \right) \|\nabla^{\alpha + e_k} w\|_2$$

(89)
$$\le c E_m(u)^{\frac{1}{2}} E_m(w).$$

The last term can be estimated to

$$|\langle \nabla^\alpha M_6, \nabla^\alpha w \rangle| \le c\tau \left(\|w\|_\infty \|\nabla^{m+1} \nabla v_t\|_2 + \|\nabla v_t\|_\infty \|\nabla^{m+1} w\|_2 \right) \|\nabla^\alpha w\|_2$$

(90)
$$\le c \sqrt{\tau} E_{m+1}(v)^{\frac{1}{2}} E_m(w).$$

Now one adds (76) and (83), sums up over $|\alpha| \le m+1$ and uses the estimates for the nonlinearities (77), (78), (79), (80), (81), (82) and (84), (85), (86), (87), (88) resp. (89), (90) and ends up with

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{E}_m(w) + \sum_{|\alpha| \le m+1} \left(\|\nabla^\alpha w_t\|_2^2 - \tau \|\nabla^\alpha w_t\|_2^2 + \mu \|\nabla^\alpha \nabla w\|_2^2 \right)$$

$$\le c \left(E_m(u) + E_{m+1}(v) + E_m(u)^{\frac{1}{2}} + E_{m+1}(v)^{\frac{1}{2}} \right) E_m(w)$$

$$+ \sum_{|\alpha| \le m+1} \left(\frac{4}{6} \|\nabla^\alpha w_t\|_2^2 + |\langle \mu \tau \nabla^\alpha \Delta v_t, \nabla^\alpha w_t \rangle| + |\langle \mu \tau \nabla^\alpha \Delta v_t, \nabla^\alpha w \rangle| \right).$$

The scalarproducts are estimated by

$$|\langle \mu \tau \nabla^\alpha \Delta v_t, \nabla^\alpha w_t \rangle| + |\langle \mu \tau \nabla^\alpha \Delta v_t, \nabla^\alpha w \rangle|$$

$$\le \mu \left(\frac{3\mu}{2} \tau^2 \|\nabla^\alpha \Delta v_t\|_2^2 + \frac{1}{6\mu} \|\nabla^\alpha w_t\|_2^2 \right) + \mu \left(\frac{\tau^2}{2} \|\nabla^\alpha \Delta v_t\|_2^2 + \frac{1}{2} \|\nabla^\alpha w\|_2^2 \right)$$

(91)
$$\le c\tau^2 \|\nabla^\alpha \Delta v_t\|_2^2 + \frac{1}{6} \|\nabla^\alpha w_t\|_2^2 + c E_m(w).$$

Plugging this in and settig without restriction $\tau \leq \frac{1}{6}$ one finally obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_m(w) \leq c(E_m(u) + E_{m+1}(v) + E_m(u)^{\frac{1}{2}} + E_{m+1}(v)^{\frac{1}{2}} + 1)E_m(w)$$

$$(92) \qquad\qquad + c\tau^2 \sum_{|\alpha| \leq m+1} \|\nabla^\alpha \Delta v_t\|_2^2.$$

Now one defines $R := \sum_{|\alpha| \leq m+1} \|\nabla^\alpha \Delta v_t\|_2^2$ and uses (57) what yields (75). $\qquad \square$

4. Appendix.

THEOREM 4.1 (Moser-Inequalities). *Let $m \in \mathbb{N}$. Then there is a constant $c = c(m,n) > 0$ such that for all $f, g \in W^{m,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n, |\alpha| \leq m$, the following inequalities hold:*

$$(93) \qquad\qquad \|\nabla^\alpha(fg)\|_2 \leq c(\|f\|_\infty \|\nabla^m g\|_2 + \|\nabla^m f\|_2 \|g\|_\infty),$$

$$(94) \qquad \|\nabla^\alpha(fg) - f\nabla^\alpha g\|_2 \leq c(\|\nabla f\|_\infty \|\nabla^{m-1} g\|_2 + \|\nabla^m f\|_2 \|g\|_\infty)$$

Proof. see [10, Lemma 4.9]. $\square$

LEMMA 4.2. *Let $\alpha, \beta, \gamma \geq 0$. Then*

$$\sup_{t \geq 0} \int_0^t (1 + t - r)^{-\alpha}(1 + r)^{-\beta}(1 + t)^\gamma \mathrm{d}r < \infty$$

if and only if
  (i) $\alpha + \beta - \gamma \geq 1$,
  (ii) $\alpha \geq \gamma$ and $\beta \geq \gamma$,
  (iii) (if $\beta = 1$ then $\alpha > \gamma$) and (if $\alpha = 1$ then $\beta > \gamma$),
are satisfied.

Proof. see [10, Lemma 7.4]. $\square$

LEMMA 4.3 (Nonlinear version of the Lemma from Gronwall). *Let $v, w, g \in C^0([0,T], \mathbb{R})$*
  (i) Let $k \in C^0([0,T] \times [0,T] \times \mathbb{R}, \mathbb{R})$, $k(t, s, \cdot)$ be monotone increasing for all $s, t \in [0,T]$ and let

$$(95) \qquad \begin{aligned} v(t) &\leq g(t) + \int_0^t k(t, s, v(s))\mathrm{d}s, \\ w(t) &\geq g(t) + \int_0^t k(t, s, w(s))\mathrm{d}s \quad \text{for all } t \in [0,T], \end{aligned}$$

  hold, then

$$(96) \qquad\qquad v(t) \leq w(t) \qquad \text{for all } t \in [0,T].$$

  holds.
  (ii) Let $g \in C^1([0,T], \mathbb{R})$, $k \in C^1([0,T] \times \mathbb{R}, \mathbb{R})$, $k(t, \cdot)$ be monotone increasing for all $t \in [0,T]$ and let h be the local solution of

$$(97) \qquad \begin{aligned} \dot{h}(t) &= \dot{g}(t) + k(t, h(t)), \\ h(0) &= g(0). \end{aligned}$$

Then for all $v \in C^0([0,T],\mathbb{R})$ with

$$(98) \qquad\qquad v(t) \leq g(t) + \int_0^t k(s,v(s))\mathrm{d}s,$$

the estimate

$$(99) \qquad\qquad v(t) \leq h(t)$$

holds within the existence time of h.

Proof. (i) see [16, Theorem I.1.II].
(ii) cp. [16, I.5.IX]. It is well known that h is well defined and $h(t) = g(t) + \int_0^t k(s,h(s))\mathrm{d}s$ holds. Therefore part (i) with $w := h$ applies. $\square$

The standard Lemma from Gronwall is included in this lemma (cp. [16, I.1.III]).

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