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THE TRUNCATED MATRIX HAUSDORFF MOMENT PROBLEM*

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Abstract. In this paper we obtain a description of all solutions of the truncated matrix Hausdorff moment problem in a general case (no conditions besides solvability are assumed). We use the basic results of Krein and Ovcharenko about generalized sc-resolvents of Hermitian contractions. Necessary and sufficient conditions for the determinateness of the moment problem are obtained, as well. Several numerical examples are provided.

Key words. Moment problem, generalized resolvent, contraction.

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1. Introduction. In this paper we analyze the following problem: to find a non-decreasing matrix-valued function $M(x) = (m_{k,j}(x))_{k,j=0}^{N-1}$ on $[a,b]$, which is left-continuous in (a,b), $M(a) = 0$, and such that

$$\int_a^b x^n dM(x) = S_n, \qquad n = 0, 1, ..., \ell, \tag{1}$$

where $\{S_n\}_{n=0}^{\ell}$ is a prescribed sequence of Hermitian $(N \times N)$ complex matrices, $N \in \mathbb{N}$, $\ell \in \mathbb{Z}_+$. Here $a, b \in \mathbb{R}$: $a < b$. This problem is said to be the **truncated matrix Hausdorff moment problem**. If this problem has a unique solution, it is said to be **determinate**. In the opposite case it is said to be **indeterminate**.
In the scalar case this problem was solved by Krein, see [1] and references therein. The operator moment problem on $[-1,1]$ with an odd number of prescribed moments $\{S_n\}_{n=0}^{2d}$ was considered by Krein and Krasnoselskiy in [2]. Among other results, conditions for the solvability of the moment problem were obtained there. The operator moment problem on $[0,1]$ with an arbitrary number of given moments $\{S_n\}_{n=0}^{\ell}$ was considered by Ando in [3]. In particular, conditions for the solvability of the moment problem were derived.

Recently, a detailed investigation of the matrix moment problem (1) by matrix methods was done by Choque Rivero, Dyukarev, Fritzsche and Kirstein, see [4], [5]. These authors used the Potapov method for interpolating problems which was enriched by the Sachnovich method of operator identities.
Set

$$\Gamma_k = (S_{i+j})_{i,j=0}^k = \begin{pmatrix} S_0 & S_1 & \dots & S_k \\ S_1 & S_2 & \dots & S_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_k & S_{k+1} & \dots & S_{2k} \end{pmatrix}, \qquad k \in \mathbb{Z}_+ : \ 2k \le \ell; \tag{2}$$

$$\widetilde{\Gamma}_k = (-abS_{i+j} + (a+b)S_{i+j+1} - S_{i+j+2})_{i,j=0}^{k-1}, \qquad k \in \mathbb{N} : \ 2k \le \ell. \tag{3}$$

If we choose an arbitrary element $f = (f_0, f_1, \dots, f_{N-1})$, where all f_k are some polynomials and calculate $\int_a^b f dM f^*$, one can easily deduce that

$$\Gamma_k \ge 0, \qquad k \in \mathbb{Z}_+ : \ 2k \le \ell. \tag{4}$$

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In the case of an odd number of prescribed moments $\ell = 2d$, the result of Choque Rivero, Dyukarev, Fritzsche and Kirstein states that conditions

$$\Gamma_d \geq 0, \ \widetilde{\Gamma}_d \geq 0, \tag{5}$$

are necessary and sufficient for the solvability of the matrix moment problem (1), see [5, Theorem 1.3, p. 106]. In the case $\Gamma_d > 0, \widetilde{\Gamma}_d > 0$, they parameterized all solutions of the moment problem by a linear fractional transformation where the set of parameters consisted of some distinguished pairs of meromorphic matrix-valued functions. In the case $[a,b] = [-1,1]$, conditions (5) were obtained by Krein and Krasnoselskiy in [2]. In the case $[a,b] = [0,1]$, conditions (5) were established by Ando in [3].
Set

$$H_k = (-aS_{i+j} + S_{i+j+1})_{i,j=0}^k, \ \widetilde{H}_k = (bS_{i+j} - S_{i+j+1})_{i,j=0}^k, \quad k \in \mathbb{Z}_+ : \ 2k+1 \leq \ell. \tag{6}$$

In the case $\ell = 2d+1$, the result of Choque Rivero, Dyukarev, Fritzsche and Kirstein states that conditions

$$H_d \geq 0, \ \widetilde{H}_d \geq 0, \tag{7}$$

are necessary and sufficient for the solvability of the matrix moment problem (1), see [4, Theorem 1.3, p. 127]. In the case $H_d > 0, \widetilde{H}_d > 0$, they parameterized all solutions of the moment problem by a linear fractional transformation. The set of parameters consisted of some distinguished pairs of meromorphic matrix-valued functions. In the case $[a,b] = [0,1]$, conditions (7) were obtained by Ando in [3]. The scalar truncated Hausdorff moment problem was also studied in [6], [7], [8].

In this work we shall study the truncated matrix Hausdorff moment problem (1) by virtue of the operator approach. The operator approach to the moment problem originates from the papers of Neumark [9], [10] and Krein, Krasnoselskiy [2] (see the remarkable books [11], [12] for more references). Different versions of the operator approach appeared afterwards. Our approach in this paper is close to the "pure operator" approach of Szökefalvi-Nagy and Koranyi to the Nevanlinna-Pick interpolation problem [13], [14]. As usual, the moment problem (1) generates a positive definite kernel constructed by the prescribed moments. Then the well-known construction [14, p.177] provides a Hilbert space and a sequence of elements in it such that the kernel is generated by the scalar products of these elements (see more precise statements below). This construction goes back to the paper of Gelfand, Naimark [15]. A similar construction, from the point of view of linear functionals on $*$-algebras, is called the Gelfand-Naimark-Segal construction (GNS-construction) [16]. Also a similar construction can be found in the book of Berezansky and Kondratiev [17, pp. 401, 418, 432]. We also refer to a survey of Fuglede [18, Section 5] (see also [19]).

After the construction of a Hilbert space, we consider the shift operator in it. The generalized resolvents of this operator are in a bijective correspondence with the solutions of the moment problem. This situation is similar to the situation in the case of the scalar Hamburger moment problem studied in the above-mentioned works. However all that works used orthogonal polynomials and the Jacobi matrix related to the Hamburger moment problem. Therefore it was not possible to study the degenerate case of the moment problem in this framework. Lately, we showed that the Hamburger moment problem can be studied using generalized resolvents both in the nondegenerate and degenerate cases, see [19]. In particular, the Nevanlinna formula for solutions was derived.

Our goal here is to describe all solutions of the matrix moment problem (1) in a general case. This means that no conditions besides the solvability of the moment problem will be assumed. Firstly, we study the case of an odd number of prescribed moments $\ell = 2d$. Then we shall reduce the case of an even number of moments $\ell = 2d+1$ to the previous case ($d \in \mathbb{Z}_+$). In our investigation we shall use the basic results of Krein and Ovcharenko on generalized sc-resolvents of Hermitian contractions, as well as Krein's theory of self-adjoint extensions of semi-bounded symmetric operators, see [20], [21], [22]. The necessary and sufficient conditions for the determinacy of the moment problem (1) (in the both cases $\ell = 2d$ and $\ell = 2d+1$) are obtained, as well. Several numerical examples are provided.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. The space of n-dimensional complex vectors $a = (a_0, a_1, \dots, a_{n-1})$, will be denoted by $\mathbb{C}^n$, $n \in \mathbb{N}$. If $a \in \mathbb{C}^n$ then a^* means the complex conjugate vector. For a complex matrix M, by M^T we mean its transposed matrix. By $\mathbb{P}$ we denote the set of all complex polynomials and by $\mathbb{P}_d$ we mean all complex polynomials with degrees less or equal to d, $d \in \mathbb{Z}_+$, (including the zero polynomial). Let $M(x)$ be a left-continuous non-decreasing matrix function $M(x) = (m_{k,\ell}(x))_{k,\ell=0}^{N-1}$ on $\mathbb{R}$, $M(-\infty) = 0$, and $\tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x)$; $\Psi(x) = (dm_{k,\ell}/d\tau_M)_{k,\ell=0}^{N-1}$. We denote by $L^2(M)$ a set (of classes of equivalence) of vector functions $f : \mathbb{R} \to \mathbb{C}^N$, $f(x) = (f_0(x), f_1(x), \dots, f_{N-1}(x))$, such that (see, e.g., [23])

$$\|f\|^2_{L^2(M)} := \int_{\mathbb{R}} f(x)\Psi(x)f^*(x)d\tau_M(x) < \infty.$$

The space $L^2(M)$ is a Hilbert space with the scalar product

$$(f,g)_{L^2(M)} := \int_{\mathbb{R}} f(x)\Psi(x)g^*(x)d\tau_M(x), \qquad f,g \in L^2(M).$$

For a separable Hilbert space H we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ the scalar product and the norm in H, respectively. The indices may be omitted in obvious cases.

For a linear operator A in H we denote by $D(A)$ its domain, by $R(A)$ its range, and by A^* we denote its adjoint if it exists. If A is bounded, then $\|A\|$ stands for its operator norm. For a set of elements $\{x_n\}_{n\in B}$ in H, we denote by $\mathrm{Lin}\{x_n\}_{n\in B}$ and $\mathrm{span}\{x_n\}_{n\in B}$ the linear span and the closed linear span (in the norm of H), respectively. Here B is an arbitrary set of indices. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. By E_H we denote the identity operator in H, i.e. $E_H x = x$, $x \in H$. If H_1 is a subspace of H, by $P_{H_1} = P^H_{H_1}$ we denote the operator of the orthogonal projection on H_1 in H. A set of all linear bounded operators which map H into H is denoted by $[H]$.

2. The case of an odd number of prescribed moments: a description of all solutions and the determinacy. We shall use the following important fact (e.g. [14, p.177]):

Theorem 2.1. *Let $K = (K_{n,m})_{n,m=0}^r \geq 0$ be a positive semi-definite complex $((r+1)\times(r+1))$ matrix, $r \in \mathbb{Z}_+$. Then there exist a finite-dimensional Hilbert space H with a scalar product $(\cdot,\cdot)$ and a sequence $\{x_n\}_{n=0}^r$ in H, such that*

$$K_{n,m} = (x_n, x_m), \qquad n,m = 0,1,...,r, \tag{8}$$

and $\mathrm{span}\{x_n\}_{n=0}^r = H$.

Proof. (We place the proof for the convenience of the reader). Let $\{x_n\}_{n=0}^r$ be an arbitrary orthonormal basis in $\mathbb{C}^n$. Introduce the following functional:

$$[x,y] = \sum_{n,m=0}^{r} K_{n,m} a_n \overline{b_m}, \tag{9}$$

for $x, y \in \mathbb{C}^n$,

$$x = \sum_{n=0}^{r} a_n x_n, \quad y = \sum_{m=0}^{r} b_m x_m, \quad a_n, b_m \in \mathbb{C}.$$

The space $\mathbb{C}^n$ equipped with $[\cdot,\cdot]$ will be a quasi-Hilbert space. Factorizing and making the completion we obtain the required space H (e.g. [12, p. 10-11]). $\Box$

Consider the matrix moment problem (1) with $\ell = 2d$, $d \in \mathbb{N}$. Suppose that $\Gamma_d \geq 0$ (this condition is necessary for the solvability of the moment problem). Let $\Gamma_d = (\gamma_{d;n,m})_{n,m=0}^{(d+1)N-1}$, $\gamma_{d;n,m} \in \mathbb{C}$. By Theorem 2.1 there exist a finite-dimensional Hilbert space H and a sequence $\{x_n\}_{n=0}^{(d+1)N-1}$ in H, such that

$$(x_n, x_m) = \gamma_{d;n,m}, \qquad n,m = 0,1,...,(d+1)N-1, \tag{10}$$

and $\mathrm{span}\{x_n\}_{n=0}^{(d+1)N-1} = \mathrm{Lin}\{x_n\}_{n=0}^{(d+1)N-1} = H$. Notice that

$$\gamma_{d;rN+j,tN+n} = s_{r+t}^{j,n}, \qquad 0 \leq j,n \leq N-1; \quad 0 \leq r,t \leq d, \tag{11}$$

where

$$S_n = (s_n^{k,\ell})_{k,\ell=0}^{N-1}, \qquad n \in \mathbb{Z}_+,$$

are the given moments. From (11) it follows that

$$\gamma_{d;a+N,b} = \gamma_{d;a,b+N}, \qquad a = rN+j,\ b = tN+n,\ 0 \leq j,n \leq N-1; \quad 0 \leq r,t \leq d-1. \tag{12}$$

In fact, we can write

$$\gamma_{d;a+N,b} = \gamma_{d;(r+1)N+j,tN+n} = s_{r+t+1}^{j,n} = \gamma_{d;rN+j,(t+1)N+n} = \gamma_{d;a,b+N}.$$

Set $H_a = \mathrm{Lin}\{x_n\}_{n=0}^{dN-1}$. We introduce the following operator:

$$Ax = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \qquad x \in H_a,\ x = \sum_{k=0}^{dN-1} \alpha_k x_k,\ \alpha_k \in \mathbb{C}. \tag{13}$$

The following proposition provides conditions for the operator A to be correctly defined.

PROPOSITION 2.1. *Let the matrix moment problem (1) with $\ell = 2d$, $d \in \mathbb{N}$, be given and conditions (5) hold. Then the operator A in (13) is correctly defined and the following operator:*

$$Bx = \frac{2}{b-a}A - \frac{a+b}{b-a}E_H, \qquad x \in H_a, \tag{14}$$

is a contraction in H (i.e. $\|B\| \leq 1$). Moreover, the operators A and B are Hermitian.

Proof. Let the matrix moment problem (1) be given and conditions (5) be satisfied. Then the moment problem has a solution $M(x) = (m_{k,\ell}(x))_{k,\ell=0}^{N-1}$. Consider the space $L^2(M)$ and let Q be the operator of multiplication by an independent variable in $L^2(M)$. The operator Q is self-adjoint and its resolution of unity is (see [23])

$$E_b - E_a = E([a,b)) : h(x) \to \chi_{[a,b)}(x)h(x), \tag{15}$$

where $\chi_{[a,b)}(x)$ is the characteristic function of an interval $[a,b)$, $-\infty \leq a < b \leq +\infty$. Set $\vec{e}_k = (\delta_{k,0}, \delta_{k,1}, \ldots, \delta_{k,N-1})$, for $k = 0, 1, \ldots N-1$ (here $\delta_{k,j}$ is Kronecker's delta). A set of (classes of equivalence of) functions $f \in L^2(M)$ such that (the corresponding class includes) $f = (f_0, f_1, \ldots, f_{N-1})$, $f_j \in \mathbb{P}_d$, $0 \leq j \leq N-1$, is denoted by $\mathbb{P}_d^2(M)$. It is called a set of vector polynomials of order d in $L^2(M)$. Set $L_{d,0}^2(M) = \overline{\mathbb{P}_d^2(M)}$. Since $\mathbb{P}_d^2(M)$ is finite-dimensional, we have $L_{d,0}^2(M) = \mathbb{P}_d^2(M)$.

For an arbitrary polynomial (in a class) from $\mathbb{P}_d^2(M)$ there exists a unique representation of the following form:

$$f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{d} \alpha_{k,j} x^j \vec{e}_k, \quad \alpha_{k,j} \in \mathbb{C}. \tag{16}$$

Let a polynomial $g \in \mathbb{P}_d^2(M)$ have a representation

$$g(x) = \sum_{\ell=0}^{N-1} \sum_{r=0}^{d} \beta_{\ell,r} x^r \vec{e}_\ell, \quad \beta_{\ell,r} \in \mathbb{C}. \tag{17}$$

We can write

$$(f,g)_{L^2(M)} = \sum_{k,\ell=0}^{N-1} \sum_{j,r=0}^{d} \alpha_{k,j} \overline{\beta_{\ell,r}} \int_{\mathbb{R}} x^{j+r} \vec{e}_k dM(x) \vec{e}_\ell^{\,*} = \sum_{k,\ell=0}^{N-1} \sum_{j,r=0}^{d} \alpha_{k,j} \overline{\beta_{\ell,r}} *$$

$$* \int_{\mathbb{R}} x^{j+r} dm_{k,\ell}(x) = \sum_{k,\ell=0}^{N-1} \sum_{j,r=0}^{d} \alpha_{k,j} \overline{\beta_{\ell,r}} s_{j+r}^{k,\ell}. \tag{18}$$

On the other hand, we can write

$$\left(\sum_{j=0}^{d} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=0}^{d} \sum_{\ell=0}^{N-1} \beta_{\ell,r} x_{rN+\ell} \right)_H = \sum_{k,\ell=0}^{N-1} \sum_{j,r=0}^{d} \alpha_{k,j} \overline{\beta_{\ell,r}} (x_{jN+k}, x_{rN+\ell})_H =$$

$$= \sum_{k,\ell=0}^{N-1} \sum_{j,r=0}^{d} \alpha_{k,j} \overline{\beta_{\ell,r}} \gamma_{d;jN+k,rN+\ell} = \sum_{k,\ell=0}^{N-1} \sum_{j,r=0}^{d} \alpha_{k,j} \overline{\beta_{\ell,r}} s_{j+r}^{k,\ell}, \tag{19}$$

where the space H and the elements $\{x_k\}$ were constructed before the statement of the Proposition. From relations (18),(19) it follows that

$$(f,g)_{L^2(M)} = \left(\sum_{j=0}^{d} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=0}^{d} \sum_{\ell=0}^{N-1} \beta_{\ell,r} x_{rN+\ell} \right)_H. \tag{20}$$

Set

$$Vf = \sum_{j=0}^{d} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \tag{21}$$

for $f(x) \in \mathbb{P}^2_d(M)$, $f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{d} \alpha_{k,j} x^j \vec{e}_k$, $\alpha_{k,j} \in \mathbb{C}$.
If f, g have representations (16),(17), and $\|f-g\|_{L^2(M)} = 0$, then from (20) it follows that

$$\|Vf - Vg\|^2_H = (V(f-g), V(f-g))_H = (f-g, f-g)_{L^2(M)} = \|f-g\|^2_{L^2(M)} = 0.$$

Thus, V is a correctly defined operator from $\mathbb{P}^2_d(M)$ to H.
Relation (20) shows that V is an isometric transformation from $\mathbb{P}^2_d(M)$ onto $\mathrm{Lin}\{x_n\}_{n=0}^{d(N+1)-1}$. Thus, V is an isometric transformation from $L^2_{d,0}(M)$ onto H. In particular, we note that

$$Vx^j \vec{e}_k = x_{jN+k}, \qquad 0 \le j \le d; \quad 0 \le k \le N-1. \tag{22}$$

Set $L^2_{d,1}(M) := L^2(M) \ominus L^2_{d,0}(M)$, and $U := V \oplus E_{L^2_{d,1}(M)}$. The operator U is an isometric transformation from $L^2(M)$ onto $H \oplus L^2_{d,1}(M) =: \widehat{H}$. Set

$$\widehat{A} := UQU^{-1}. \tag{23}$$

The operator $\widehat{A}$ is a self-adjoint operator in $\widehat{H}$. Notice that

$$UQU^{-1}x_{jN+k} = VQV^{-1}x_{jN+k} = VQx^j\vec{e}_k = Vx^{j+1}\vec{e}_k = x_{(j+1)N+k} = x_{jN+k+N},$$

$$0 \le j \le d-1; \quad 0 \le k \le N-1.$$

By the linearity we get

$$UQU^{-1}x = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \qquad x \in H_a,\ x = \sum_{k=0}^{dN-1} \alpha_k x_k,\ \alpha_k \in \mathbb{C}.$$

Consequently, the operator A in (13) is correctly defined, Hermitian and

$$A = \widehat{A}|_{H_a}, \tag{24}$$

i.e. A is the operator $\widehat{A}$ restricted to the subspace H_a. Since $\widehat{A}$ is self-adjoint, the operator A is Hermitian.
Consider the following operators:

$$R := \frac{2}{b-a}Q - \frac{a+b}{b-a}E_{L^2(M)}, \tag{25}$$

$$\widehat{B} := URU^{-1} = \frac{2}{b-a}\widehat{A} - \frac{a+b}{b-a}E_{\widehat{H}}. \tag{26}$$

Define an operator B by the equality (14). From (24),(26) we get

$$B = \widehat{B}|_{H_a}. \tag{27}$$

For an arbitrary $f \in D(R) = D(Q)$ we can write

$$\|Rf\|^2_{L^2(M)} = \int_a^b \left| \frac{2}{b-a}x - \frac{a+b}{b-a} \right|^2 f(x)dM(x)f^*(x) \leq \int_a^b f(x)dM(x)f^*(x) =$$

$$= \|f\|^2,$$

and therefore the operators R, $\widehat{B}$ and B are contractions. Since R is Hermitian, the operators $\widehat{B}$, B are Hermitian, as well. □

Let us continue our considerations before the statement of Proposition 2.1. In what follows we shall assume that conditions (5) are satisfied. Therefore the operators A in (13) and B in (14) are correctly defined Hermitian operators and $\|B\| \leq 1$.
Let $\widehat{B}$ be an arbitrary self-adjoint extension of B in a Hilbert space $\widehat{H} \supseteq H$. Let $R_z(\widehat{B})$ be the resolvent of $\widehat{B}$ and $\{\widehat{E}_\lambda\}_{\lambda\in\mathbb{R}}$ be an orthogonal resolution of unity of $\widehat{B}$. Recall that the operator-valued function $\mathbf{R}_z$: $\mathbf{R}_z h = P^{\widehat{H}}_H R_z(\widehat{B})h$, $h \in H$, is said to be a **generalized resolvent** of B, $z \in \mathbb{C}\backslash\mathbb{R}$. The function $\mathbf{E}_\lambda$: $\mathbf{E}_\lambda h = P^{\widehat{H}}_H \widehat{E}_\lambda h$, $\lambda \in \mathbb{R}$, $h \in H$, is said to be a **spectral function** of the symmetric operator B (e.g. [24]). There exists a bijective correspondence between generalized resolvents and (left-continuous or normalized in another way) spectral functions established by the following relation ([25]):

$$(28) \qquad (\mathbf{R}_z f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d(\mathbf{E}_\lambda f, g)_H, \qquad f, g \in H,\ z \in \mathbb{C}\backslash\mathbb{R}.$$

In order to obtain the spectral function by relation (28), one should use the Stieltjes-Perron inversion formula (e.g. [11]).
In the case when $\widehat{B}$ is a self-adjoint contraction, the corresponding generalized resolvent $\mathbf{R}_z$: $\mathbf{R}_z h = P^{\widehat{H}}_H R_z(\widehat{B})h$, $h \in H$, is said to be a **generalized sc-resolvent** of B, see [20], [21]. The corresponding spectral function of B is said to be a **sc-spectral function** of B. By Krein's theorem [22, Theorem 2, p. 440], there always exists a self-adjoint extension $\widehat{B}$ of the operator B in H. This extension has the norm $\|B\|$. Therefore the sets of generalized sc-resolvents and sc-spectral functions are non-empty.

Let $\widetilde{B}$ be an arbitrary self-adjoint contractive extension of B in a Hilbert space $\widetilde{H} \supseteq H$. Let

$$(29) \qquad \widetilde{B} = \int_{-1}^{1} \lambda d\widetilde{E}_\lambda,$$

where $\{\widetilde{E}_\lambda\}$ be the left-continuous in $[-1, 1)$, right-continuous at the point 1, constant outside $[-1, 1]$, orthogonal resolution of unity of $\widetilde{B}$.

Choose an arbitrary α, $0 \leq \alpha \leq d(N+1)-1$, $\alpha = rN+j$, $0 \leq r \leq d$, $0 \leq j \leq N-1$. Notice that

$$x_\alpha = x_{rN+j} = Ax_{(r-1)N+j} = ... = A^r x_j.$$

Then choose an arbitrary β, $0 \leq \beta \leq d(N+1)-1$, $\beta = tN+n$, $0 \leq t \leq d$, $0 \leq n \leq N-1$. Using (11) we can write

$$s^{j,n}_{r+t} = \gamma_{d;rN+j,tN+n} = (x_{rN+j}, x_{tN+n})_H = (A^r x_j, A^t x_n)_H =$$

$$= \left(\left(\frac{b-a}{2} B + \frac{a+b}{2} E_H \right)^r x_j, \left(\frac{b-a}{2} B + \frac{a+b}{2} E_H \right)^t x_n \right)_H =$$

$$= \left(\left(\frac{b-a}{2} \widetilde{B} + \frac{a+b}{2} E_{\widetilde{H}} \right)^{r+t} x_j, x_n \right)_{\widetilde{H}} = \int_{-1}^{1} \left(\frac{b-a}{2} \lambda + \frac{a+b}{2} \right)^{r+t} d(\widetilde{E}_\lambda x_j, x_n)_{\widetilde{H}}$$

$$= \int_{-1}^{1} \left(\frac{b-a}{2} \lambda + \frac{a+b}{2} \right)^{r+t} d(P_H^{\widetilde{H}} \widetilde{E}_\lambda x_j, x_n)_H.$$

Set

$$\widetilde{m}_{j,n}(x) = (P_H^{\widetilde{H}} \widetilde{E}_{\frac{2}{b-a}x - \frac{a+b}{b-a}} x_j, x_n)_H, \qquad 0 \le j, n \le N-1. \tag{30}$$

Then

$$s_{r+t}^{j,n} = \int_a^b x^{r+t} d\widetilde{m}_{j,n}(x), \qquad 0 \le j, n \le N-1,\ 0 \le r, t \le d. \tag{31}$$

By relation (31) we conclude that the matrix-valued function $\widetilde{M}(x) = (\widetilde{m}_{j,n}(x))_{j,n=0}^{N-1}$ is a solution of the matrix Hausdorff moment problem (1) (Properties of the orthogonal resolution of unity provide that $\widetilde{M}(x)$ is left-continuous in (a, b), non-decreasing and $\widetilde{M}(a) = 0$).

THEOREM 2.2. *Let the matrix moment problem (1) with $\ell = 2d$, $d \in \mathbb{N}$, be given and conditions (5) be true. All solutions of the moment problem have the following form*

$$M(x) = (m_{j,n}(x))_{j,n=0}^{N-1}, \quad m_{j,n}(x) = (\mathbf{E}_{\frac{2}{b-a}x - \frac{a+b}{b-a}} x_j, x_n)_H, \qquad 0 \le j, n \le N-1, \tag{32}$$

where $\mathbf{E}_z$ is a left-continuous in $[-1, 1)$, right-continuous at the point 1, constant outside $[-1, 1]$ sc-spectral function of the operator B defined by (14).
Moreover, the correspondence between all solutions of the moment problem and left-continuous in $[-1, 1)$, right-continuous at the point 1, constant outside $[-1, 1]$ sc-spectral functions of B in (32) is one-to-one.

Proof. Choose an arbitrary left-continuous in $[-1, 1)$, right-continuous at the point 1, constant outside $[-1, 1]$ sc-spectral function $\mathbf{E}_z$ of the operator B from (14). This function corresponds to a resolution of unity $\{\widehat{E}_\lambda\}$ of a self-adjoint contraction $\widehat{B} \supseteq B$ in a Hilbert space $\widehat{H} \supseteq H$: $\mathbf{E}_z h = P_H^{\widehat{H}} \widehat{E}_z h$, $h \in H$. Considerations before the statement of the Theorem show that formula (32) defines a solution of the moment problem (1).

On the other hand, let $M(x) = (m_{k,j}(x))_{k,j=0}^{N-1}$ be an arbitrary solution of the matrix moment problem (1). Proceeding like at the beginning of the proof of Proposition 2.1, we shall construct a self-adjoint contraction $\widehat{B} \supseteq B$ in a space $\widehat{H} \supseteq H$. Repeating arguments before the statement of the Theorem, we obtain that the function $\widehat{M}(x) = (\widehat{m}_{j,n}(x))_{j,n=0}^{N-1}$, where $\widehat{m}_{j,n}(x)$ are given by

$$\widehat{m}_{j,n}(x) = (P_H^{\widehat{H}} \widehat{E}_{\frac{2}{b-a}x - \frac{a+b}{b-a}} x_j, x_n)_H, \qquad 0 \le j, n \le N-1, \tag{33}$$

is a solution of the Hausdorff matrix moment problem. Here $\{\widehat{E}_\lambda\}$ is the left-continuous in $[-1,1)$, right-continuous at the point 1, constant outside $[-1,1]$, orthogonal resolution of unity of $\widehat{B}$.
Let us check that $M(x) = \widehat{M}(x)$. Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$, $\lambda \in [-1,1]$, and k,j: $0 \leq k,j \leq N-1$. We may write

$$\int_{-1}^{1} \frac{1}{\lambda - z} d\widehat{m}_{k,j}\left(\frac{b-a}{2}\lambda + \frac{a+b}{2}\right) = \int_{-1}^{1} \frac{1}{\lambda - z} d(P_H^{\widehat{H}} \widehat{E}_\lambda x_k, x_j)_H$$

$$\int_{-1}^{1} \frac{1}{\lambda - z} d(\widehat{E}_\lambda x_k, x_j)_{\widehat{H}} = \left(\int_{-1}^{1} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, x_j\right)_{\widehat{H}}$$

$$= \left(U^{-1}\int_{-1}^{1} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, U^{-1}x_j\right)_{L^2(M)} = \left(\int_{-1}^{1} \frac{1}{\lambda - z} dU^{-1}\widehat{E}_\lambda U\vec{e}_k, \vec{e}_j\right)_{L^2(M)}$$

$$= \left(\int_{-1}^{1} \frac{1}{\lambda - z} dE_{R;\lambda}\vec{e}_k, \vec{e}_j\right)_{L^2(M)} = \left((R - zE_{L^2(M)})^{-1}\vec{e}_k, \vec{e}_j\right)_{L^2(M)}$$

$$= \int_a^b \left(\frac{2}{b-a}x - \frac{a+b}{b-a} - z\right)^{-1} \vec{e}_k dM(x)\vec{e}_j^* = \int_a^b \left(\frac{2}{b-a}x - \frac{a+b}{b-a} - z\right)^{-1} dm_{k,j}(x)$$

$$= \int_{-1}^{1} \frac{1}{\lambda - z} dm_{k,j}\left(\frac{b-a}{2}\lambda + \frac{a+b}{2}\right),$$

where $\{\widehat{E}_{R;\lambda}\}$ is an orthogonal resolution of unity of the operator R. By the Stieltjes-Perron inversion formula we conclude that

$$\widehat{M}(x) = M(x), \qquad x \in [a,b].$$

Consequently, all solutions of the truncated matrix Hausdorff moment problem are generated by left-continuous in $[-1,1)$, right-continuous at the point 1, constant outside $[-1,1]$ sc-spectral functions of B.
It remains to prove that different sc-spectral functions of the operator B produce different solutions of the moment problem (1). Suppose to the contrary that two different left-continuous in $[-1,1)$, right-continuous at the point 1, constant outside $[-1,1]$ sc-spectral functions produce the same solution of the moment problem. This means that there exist two self-adjoint contractions $B_\ell \supseteq B$, in Hilbert spaces $H_\ell \supseteq H$, such that

$$P_H^{H_1} E_{1,\lambda} \neq P_H^{H_2} E_{2,\lambda}, \tag{34}$$

$$(P_H^{H_1} E_{1,\lambda} x_k, x_j)_H = (P_H^{H_2} E_{2,\lambda} x_k, x_j)_H, \qquad 0 \leq k,j \leq N-1, \quad \lambda \in [-1,1], \tag{35}$$

where $\{E_{n,\lambda}\}_{\lambda\in\mathbb{R}}$ are orthogonal resolutions of unity of the operators B_n, $n, \ell = 1,2$. Set $L_N := \mathrm{Lin}\{x_k\}_{k=0}^{N-1}$. By the linearity we get

$$(P_H^{H_1} E_{1,\lambda} x, y)_H = (P_H^{H_2} E_{2,\lambda} x, y)_H, \qquad x, y \in L_N, \quad \lambda \in [-1,1]. \tag{36}$$

Denote by $R_{n,\lambda}$ the resolvent of B_n, and set $\mathbf{R}_{n,\lambda} = P_H^{H_n} R_{n,\lambda}|_H$, $n = 1,2$. By (36),(28) we obtain that

$$(\mathbf{R}_{1,\lambda}x, y)_H = (\mathbf{R}_{2,\lambda}x, y)_H, \qquad x, y \in L_N, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{37}$$

Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$ and consider the space $H_z := \overline{(B - zE_H)H_a}$. Since

$$R_{j,z}(B - zE_H)x = (B_j - zE_{H_j})^{-1}(B_j - zE_{H_j})x = x, \qquad x \in H_a = D(B),$$

we get

$$R_{1,z}u = R_{2,z}u \in H, \qquad u \in H_z; \tag{38}$$

$$\mathbf{R}_{1,z}u = \mathbf{R}_{2,z}u, \qquad u \in H_z, \ z \in \mathbb{C}\backslash\mathbb{R}. \tag{39}$$

We may write

$$(\mathbf{R}_{n,z}x, u)_H = (R_{n,z}x, u)_{H_n} = (x, R_{n,\overline{z}}u)_{H_n} = (x, \mathbf{R}_{n,\overline{z}}u)_H, \ x \in H, \ u \in H_{\overline{z}},$$

$$n = 1, 2, \tag{40}$$

and therefore we get

$$(\mathbf{R}_{1,z}x, u)_H = (\mathbf{R}_{2,z}x, u)_H, \qquad x \in H, \ u \in H_{\overline{z}}, \ z \in \mathbb{C}\backslash\mathbb{R}. \tag{41}$$

Choose an arbitrary $u \in H$, $u = \sum_{k=0}^{dN+N-1} c_k x_k$, $c_k \in \mathbb{C}$. Consider the following system of linear equations:

$$-\left(\frac{a+b}{b-a} + z\right) d_k = c_k, \qquad k = 0, 1, ..., N-1; \tag{42}$$

$$\frac{2}{b-a} d_{k-N} - \left(\frac{a+b}{b-a} + z\right) d_k = c_k, \qquad k = N, N+1, \dots, dN+N-1; \tag{43}$$

where $\{d_k\}_{k=0}^{dN+N-1}$ are unknown complex numbers, $z \in \mathbb{C}\backslash\mathbb{R}$ is a fixed parameter, a, b are from (1). Set

$$d_k = 0, \qquad k = dN, dN+1, ..., dN+N-1;$$

$$d_{k-N} = \frac{b-a}{2}\left(\left(\frac{a+b}{b-a} + z\right) d_k + c_k\right), \quad k = dN+N-1, dN+N-2, ..., N; \tag{44}$$

For the numbers $\{d_k\}_{k=0}^{dN+N-1}$ all equations in (43) are satisfied. Equations (42) are not necessarily satisfied. Set $v = \sum_{k=0}^{dN+N-1} d_k x_k = \sum_{k=0}^{dN-1} d_k x_k$. Notice that $v \in H_a = D(B)$. We can write

$$(B - zE_H)v = \left(\frac{2}{b-a}A - \frac{a+b}{b-a}E_H - zE_H\right) v =$$

$$= \sum_{k=0}^{dN-1} d_k \left(\frac{2}{b-a} x_{k+N} - \left(\frac{a+b}{b-a} + z\right) x_k\right) =$$

$$= \sum_{k=0}^{dN+N-1} \left(\frac{2}{b-a} d_{k-N} - \left(\frac{a+b}{b-a} + z \right) d_k \right) x_k,$$

where $d_{-1} = d_{-2} = ... = d_{-N} = 0$. By the construction of d_k we have

$$(B - zE_H)v - u = \sum_{k=0}^{N-1} \left(- \left(\frac{a+b}{b-a} + z \right) d_k - c_k \right) x_k;$$

$$u = (B - zE_H)v + \sum_{k=0}^{N-1} \left(\left(\frac{a+b}{b-a} + z \right) d_k + c_k \right) x_k, \qquad u \in H,\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{45}$$

By (45) an arbitrary element $y \in H$ can be represented as $y = y_{\overline{z}} + y'$, $y_{\overline{z}} \in H_{\overline{z}}$, $y' \in L_N$. By (37) and (41) we get

$$(\mathbf{R}_{1,z}x, y)_H = (\mathbf{R}_{1,z}x, y_{\overline{z}} + y')_H = (\mathbf{R}_{2,z}x, y_{\overline{z}} + y')_H = (\mathbf{R}_{2,z}x, y)_H,\ x \in L_N,\ y \in H.$$

Thus, we obtain

$$\mathbf{R}_{1,z}x = \mathbf{R}_{2,z}x, \qquad x \in L_N,\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{46}$$

Choose an arbitrary $x \in H$, $x = x_z + x'$, $x_z \in H_z$, $x' \in L_N$. By relations (39),(46) we obtain

$$\mathbf{R}_{1,z}x = \mathbf{R}_{1,z}(x_z + x') = \mathbf{R}_{2,z}(x_z + x') = \mathbf{R}_{2,z}x, \qquad x \in H,\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{47}$$

By (28) that means that the corresponding (left-continuous in $[-1, 1)$, right-continuous at the point 1, constant outside $[-1, 1]$) sc-spectral functions coincide and we obtained a contradiction. ▯

Let us return to the considerations before the statement of the last theorem. We shall use some known important facts about sc-resolvents, see [20], [21]. Set $\mathcal{D} = D(B)$, $\mathcal{R} = H \ominus \mathcal{D}$. A set of all self-adjoint contractive extensions of B inside H, we denote by $\mathcal{B}_H(B)$. A set of all self-adjoint contractive extensions of B in a Hilbert space $\widetilde{H} \supseteq H$, we denote by $\mathcal{B}_{\widetilde{H}}(B)$. As it was already mentioned, the set $\mathcal{B}_H(B)$ is non-empty. There are the "minimal" element B^μ and the "maximal" element B^M in this set, such that $\mathcal{B}_H(B)$ coincides with the operator segment

$$B^\mu \leq \widetilde{B} \leq B^M. \tag{48}$$

In the case $B^\mu = B^M$ the set $\mathcal{B}_H(B)$ consists of a unique element. This case is said to be **determinate**. The case $B^\mu \neq B^M$ is called **indeterminate**. The case $B^\mu x \neq B^M x$, $x \in \mathcal{R}\backslash\{0\}$, is said to be **completely indeterminate**. The indeterminate case can be always reduced to the completely indeterminate. If $\mathcal{R}_0 = \{x \in \mathcal{R} :\ B^\mu x = B^M x\}$, we may set

$$B_e x = Bx,\ x \in \mathcal{D}; \quad B_e x = B^\mu x,\ x \in \mathcal{R}_0. \tag{49}$$

The sets of generalized sc-resolvents for B and for B_e coincide ([21, p. 1039]).
Elements of $\mathcal{B}_H(B)$ are canonical (i.e. inside H) extensions of B and their resolvents are called **canonical sc-resolvents** of B. On the other hand, elements of $\mathcal{B}_{\widetilde{H}}(B)$ for

all possible $\widetilde{H} \supseteq H$ generate generalized sc-resolvents of B (we emphasize that the space $\widetilde{H}$ is not fixed). The set of all generalized sc-resolvents we denote by $\mathcal{R}^c(B)$. Set

$$C = B^M - B^\mu, \tag{50}$$

$$Q_\mu(z) = \left(C^{\frac{1}{2}} R_z^\mu C^{\frac{1}{2}} + E_H \right)\Big|_{\mathcal{R}}, \qquad z \in \mathbb{C}\backslash[-1,1], \tag{51}$$

where $R_z^\mu = (B^\mu - zE_H)^{-1}$.
An operator-valued function $k(z)$ with values in $[\mathcal{R}]$ belongs to the class $R_{\mathcal{R}}[-1,1]$ if

1) $k(z)$ is analytic in $z \in \mathbb{C}\backslash[-1,1]$ and

$$\frac{\operatorname{Im} k(z)}{\operatorname{Im} z} \le 0, \qquad z \in \mathbb{C}: \ \operatorname{Im} z \neq 0;$$

2) For $z \in \mathbb{R}\backslash[-1,1]$, $k(z)$ is a self-adjoint non-negative contraction.

In the completely indeterminate case, assuming that the restricted to $\mathcal{R}$ operator C has a bounded inverse defined on the whole $\mathcal{R}$, we come to the following theorem.

Theorem 2.3. *([21, p. 1053]). The following equality:*

$$\widetilde{R}_z^c = R_z^\mu - R_z^\mu C^{\frac{1}{2}} k(z) \left(E_{\mathcal{R}} + (Q_\mu(z) - E_{\mathcal{R}}) k(z) \right)^{-1} C^{\frac{1}{2}} R_z^\mu, \tag{52}$$

where $k(z) \in R_{\mathcal{R}}[-1,1]$, $\widetilde{R}_z^c \in \mathcal{R}^c(B)$, establishes a bijective correspondence between the set $R_{\mathcal{R}}[-1,1]$ and the set $\mathcal{R}^c(B)$.

Moreover, the canonical resolvents correspond in (52) to the constant functions $k(z) \equiv K$, $K \in [0, E_{\mathcal{R}}]$.

Set $L_N = \operatorname{Lin}\{x_k\}_{k=0}^{N-1}$. Define a linear transformation G from $\mathbb{C}^N$ onto L_N by the following relation:

$$G\vec{e}_k = x_k, \qquad k = 0, 1, ..., N-1, \tag{53}$$

where $\vec{e}_k = (\delta_{0,k}, \delta_{1,k}, ..., \delta_{N-1,k})$. Using Theorem 2.2 and Theorem 2.3 we obtain the following result.

Theorem 2.4. *Let the matrix moment problem (1) with $\ell = 2d$, $d \in \mathbb{N}$, be given and conditions (5) hold. Let the operator B be defined by (14). The following statements are true:*

1) *If $B^\mu = B^M$, then the moment problem (1) has a unique solution. This solution is given by*

$$M(x) = (m_{j,n}(x))_{j,n=0}^{N-1}, \quad m_{j,n}(x) = (E^\mu_{\frac{2}{b-a}x - \frac{a+b}{b-a}} x_j, x_n)_H, \ 0 \le j,n \le N-1, \tag{54}$$

where $\{E_z^\mu\}$ is the left-continuous in $[-1,1)$, right-continuous at the point 1, constant outside $[-1,1]$, orthogonal resolution of unity of the operator B^μ.

2) *If $B^\mu \neq B^M$, define the extended operator B_e by (49); $\mathcal{R}_e = H \ominus D(B_e)$ and $Q'_\mu(z) = \left(C^{\frac{1}{2}} R_z^\mu C^{\frac{1}{2}} + E_H \right)\Big|_{\mathcal{R}_e}$, $z \in \mathbb{C}\backslash[-1,1]$. An arbitrary solution $M(\cdot)$ of the moment problem can be found by the Stieltjes-Perron inversion formula from the following relation*

$$\int_{-1}^{1} \frac{1}{t-z} dM^T \left(\frac{(b-a)t + (a+b)}{2} \right)$$

$$= \mathcal{A}(z) - \mathcal{C}(z)k(z)(E_{\mathcal{R}_e} + \mathcal{D}(z)k(z))^{-1}\mathcal{B}(z), \tag{55}$$

where $k(z) \in R_{\mathcal{R}_e}[-1,1]$, and on the left-hand side one means the matrix of the corresponding operator in $\mathbb{C}^N$. Here $\mathcal{A}(z), \mathcal{B}(z), \mathcal{C}(z), \mathcal{D}(z)$ are analytic operator-valued functions given by

$$\mathcal{A}(z) = G^* P^H_{L_N} R^\mu_z P^H_{L_N} G : \ \mathbb{C}^N \to \mathbb{C}^N, \tag{56}$$

$$\mathcal{B}(z) = C^{\frac{1}{2}} R^\mu_z P^H_{L_N} G : \ \mathbb{C}^N \to \mathcal{R}_e, \tag{57}$$

$$\mathcal{C}(z) = G^* P^H_{L_N} R^\mu_z C^{\frac{1}{2}} : \ \mathcal{R}_e \to \mathbb{C}^N, \tag{58}$$

$$\mathcal{D}(z) = Q'_\mu(z) - E_{\mathcal{R}_e} : \ \mathcal{R}_e \to \mathcal{R}_e. \tag{59}$$

Moreover, the correspondence between all solutions of the moment problem and $k(z) \in R_{\mathcal{R}_e}[-1,1]$ is one-to-one.

Proof. Consider the case 1). In this case all self-adjoint contractions $\widetilde{B} \supseteq B$ in a Hilbert space $\widetilde{H} \supseteq H$ coincide on H with B^μ, see [21, p. 1039]. Thus, the corresponding sc-spectral functions are spectral functions of the self-adjoint operator B^μ, as well. However, a self-adjoint operator has a unique (normalized) spectral function. Thus, a set of sc-spectral functions of B consists of a unique element. This element is the spectral function of B^μ.
Consider the case 2). By Theorem 2.2 and relation (28) it follows that an arbitrary solution $M(t) = (m_{j,n}(t))_{j,n=0}^{N-1}$ of the moment problem (1) can be found from the following relation:

$$\int_{-1}^{1} \frac{1}{t-z} dm_{j,n}\left(\frac{(b-a)t+(a+b)}{2}\right) = (\mathbf{R}_z x_j, x_n)_H, \quad 0 \le j,n \le N-1; \ z \in \mathbb{C}\backslash\mathbb{R}.$$

where $\mathbf{R}_z$ is a generalized sc-resolvent of B. Moreover, the correspondence between the set all generalized sc-resolvents of B (which is equal to the set of all generalized sc-resolvents of B_e) and solutions of the moment problem is bijective. Notice that $B^\mu = B^\mu_e$ and $B^M = B^M_e$. By Theorem 2.3 (for the operator B_e) we may rewrite the latter relation in the following form:

$$\int_{-1}^{1} \frac{1}{t-z} dm_{j,n}\left(\frac{(b-a)t+(a+b)}{2}\right)$$

$$= \left(\left\{ R^\mu_z - R^\mu_z C^{\frac{1}{2}} k(z)(E_{\mathcal{R}_e} + (Q'_\mu(z) - E_{\mathcal{R}_e})k(z))^{-1} C^{\frac{1}{2}} R^\mu_z \right\} x_j, x_n\right)_H$$

$$= \Big(G^* \Big\{ P^H_{L_N} R^\mu_z P^H_{L_N} - P^H_{L_N} R^\mu_z C^{\frac{1}{2}} k(z)(E_{\mathcal{R}_e}$$

$$+(Q'_\mu(z) - E_{\mathcal{R}_e})k(z))^{-1} C^{\frac{1}{2}} R^\mu_z P^H_{L_N} \Big\} G\vec{e}_j, \vec{e}_n\Big)_{\mathbb{C}^N}, \tag{60}$$

where $k(z) \in R_{\mathcal{R}_e}([-1,1])$. Introducing functions $\mathcal{A}(z), \mathcal{B}(z), \mathcal{C}(z), \mathcal{D}(z)$ by formulas (56)-(59) one easily obtains relation (55). □

Remark 2.1. Observe that the function $\mathcal{B}(z)$ in relation (57) can be not invertible for all $z \in \mathbb{C}\backslash\mathbb{R}$. The corresponding example will be given below.

Remark 2.2. Observe that the relation (55) holds for the case $B^{\mu} = B^{M}$, as well. In this case, the class $R_{\mathcal{R}_e}([-1,1])$ consists of a unique function $k(z) \equiv 0$.

Example 2.1. Let $\ell = 2d$, $d = 1$, $N = 2$, $a = -1$, $b = 1$, and

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the moment problem (1) with moments S_0, S_1, S_2. In this case

$$\Gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \geq 0, \quad \widetilde{\Gamma} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \geq 0,$$

and therefore the moment problem has a solution.
Choose $H = \mathbb{C}^3$, and $x_k = \vec{e}_k$, $0 \leq k \leq 2$; $x_3 = 0$. Here $\vec{e}_k = (\delta_{0,k}, \delta_{1,k}, \delta_{2,k})$. In this case, we have:

$$L_N = \mathrm{Lin}\{x_k\}_{k=0}^{N-1} = \mathrm{Lin}\{\vec{e}_0, \vec{e}_1\} = D(A) = D(B), \quad \mathcal{R} = \mathrm{Lin}\{\vec{e}_2\},$$

and the operator $B = A$ has the following matrix representation in the basis $\{\vec{e}_k\}_{k=0}^2$:

$$B = (B_{k,\ell})_{k,\ell=0}^2 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 1 & 0 & * \end{pmatrix},$$

where $*$ means that the corresponding value is not defined. If $\widetilde{B} \supset B$ is a self-adjoint contractive extension of B in H, then its matrix representation should be of the form:

$$\widetilde{B} = (\widetilde{B}_{k,\ell})_{k,\ell=0}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & w \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & w+1 \end{pmatrix} \geq 0, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1-w \end{pmatrix} \geq 0.$$

In particular, $\begin{vmatrix} 1 & 1 \\ 1 & w+1 \end{vmatrix} = w \geq 0$; $\begin{vmatrix} 1 & -1 \\ -1 & 1-w \end{vmatrix} = -w \geq 0$, and therefore $w = 0$.
Thus, there exists a unique self-adjoint contractive extension

$$B^{\mu} = B^{M} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The direct calculation shows that for $z \in \mathbb{C}\backslash\mathbb{R}$ we have

$$R_z^{\mu} = (B^{\mu} - zE_H)^{-1} = \begin{pmatrix} -z & 0 & 1 \\ 0 & -z & 0 \\ 1 & 0 & -z \end{pmatrix}^{-1} = \begin{pmatrix} \frac{z}{1-z^2} & 0 & \frac{1}{1-z^2} \\ 0 & -\frac{1}{z} & 0 \\ \frac{1}{1-z^2} & 0 & \frac{z}{1-z^2} \end{pmatrix}.$$

By (54) we may write

$$\int_{-1}^{1} \frac{1}{t-z} dm_{j,n}(t) = (R_z^{\mu} x_j, x_n)_H, \quad 0 \leq j, n \leq 1.$$

In particular we have

$$\int_{-1}^{1} \frac{1}{t-z} dm_{0,0}(t) = \frac{z}{1-z^2} = \frac{1}{2(1-z)} + \frac{1}{2((-1)-z)}, \quad \int_{-1}^{1} \frac{1}{t-z} dm_{1,1}(t) = -\frac{1}{z},$$

and $m_{0,1}(t) \equiv 0$, $m_{1,0}(t) \equiv 0$. Therefore the unique solution of the moment problem is given by

$$M(t) = \begin{pmatrix} m_{0,0}(t) & 0 \\ 0 & m_{1,1}(t) \end{pmatrix},$$

where $m_{0,0}(t), m_{1,1}(t)$ are left-continuous in $(-1,1)$ and piecewise constant, $m_{0,0}(-1) = m_{1,1}(-1) = 0$. The function $m_{0,0}(t)$ has jumps equal to $\frac{1}{2}$ at points -1 and 1. The function $m_{1,1}(t)$ has a jump equal to 1 at the point 0.

Example 2.2. Let $\ell = 2d$, $d = 1$, $N = 2$, $a = -1$, $b = 1$, and

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the moment problem (1) with moments S_0, S_1, S_2. In this case

$$\Gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \geq 0, \quad \widetilde{\Gamma} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{pmatrix} > 0,$$

and therefore the moment problem has a solution.
Choose $H = \mathbb{C}^3$, and $x_k = \vec{e}_k$, $0 \leq k \leq 1$; $x_2 = \frac{1}{\sqrt{3}} \vec{e}_2$, $x_3 = 0$. Here $\vec{e}_k = (\delta_{0,k}, \delta_{1,k}, \delta_{2,k})$. In this case, we have:

$$L_N = \mathrm{Lin}\{x_k\}_{k=0}^{N-1} = \mathrm{Lin}\{\vec{e}_0, \vec{e}_1\} = D(A) = D(B), \quad \mathcal{R} = \mathrm{Lin}\{\vec{e}_2\},$$

and the operator $B = A$ has the following matrix representation in the basis $\{\vec{e}_k\}_{k=0}^2$:

$$B = (B_{k,\ell})_{k,\ell=0}^2 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ \frac{1}{\sqrt{3}} & 0 & * \end{pmatrix},$$

where $*$ means that the corresponding value is not defined. If $\widetilde{B} \supset B$ is a self-adjoint contractive extension of B in H, then its matrix representation should be of the following form:

$$\widetilde{B} = (\widetilde{B}_{k,\ell})_{k,\ell=0}^2 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & w \end{pmatrix}, \tag{61}$$

and

$$\begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & w+1 \end{pmatrix} \geq 0, \quad \begin{pmatrix} 1 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 1-w \end{pmatrix} \geq 0. \tag{62}$$

Calculating all principal minors of the latter matrices we obtain that relation (62) holds iff $w \in [-\frac{2}{3}, \frac{2}{3}]$. Therefore

$$B^\mu = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{3} \end{pmatrix}, \quad B^M = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{3} \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix},$$

and $B_e = B$. By the direct calculation for $z \in \mathbb{C}\backslash\mathbb{R}$ we obtain

$$R_z^\mu = \begin{pmatrix} -z & 0 & \frac{1}{\sqrt{3}} \\ 0 & -z & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{3} - z \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{z+\frac{2}{3}}{(z+1)(z-\frac{1}{3})} & 0 & -\frac{1}{\sqrt{3}(z+1)(z-\frac{1}{3})} \\ 0 & -\frac{1}{z} & 0 \\ -\frac{1}{\sqrt{3}(z+1)(z-\frac{1}{3})} & 0 & -\frac{z+\frac{2}{3}}{(z+1)(z-\frac{1}{3})} \end{pmatrix}.$$

Notice that $\dim \mathbb{C}^N = 2 > \dim \mathcal{R} = 1$, and therefore the operator $\mathcal{B}(z)$ defined by (57) will be not invertible for all $z \in \mathbb{C}\backslash\mathbb{R}$. Let $\vec{u}_0 = (1,0), \vec{u}_1 = (0,1) \in \mathbb{C}^2$. Then we may calculate

$$\mathcal{B}(z)(\xi_0\vec{u}_0 + \xi_1\vec{u}_1) = -\frac{2}{3(z+1)(z-\frac{1}{3})}\xi_0\vec{e}_2, \quad \xi_0, \xi_1 \in \mathbb{C},$$

$$\mathcal{C}(z)\alpha\vec{e}_2 = -\frac{2}{3(z+1)(z-\frac{1}{3})}\alpha\vec{u}_0, \quad \alpha \in \mathbb{C},$$

$$\mathcal{A}(z) = \begin{pmatrix} -\frac{z+\frac{2}{3}}{(z+1)(z-\frac{1}{3})} & 0 \\ 0 & -\frac{1}{z} \end{pmatrix},$$

$$\mathcal{D}(z)\beta\vec{e}_2 = -\frac{4z}{3(z+1)(z-\frac{1}{3})}\beta\vec{e}_2, \quad \beta \in \mathbb{C}.$$

Then

$$\left(\left(\mathcal{A}(z) - \mathcal{C}(z)k(z)(E_{\mathcal{R}_e} + \mathcal{D}(z)k(z))^{-1}\mathcal{B}(z)\right)\vec{u}_0, \vec{u}_0\right)_{\mathbb{C}^2}$$

$$= -\frac{z+\frac{2}{3}}{(z+1)(z-\frac{1}{3})} - \frac{4}{9(z+1)^2(z-\frac{1}{3})^2}k(z)\left(1 - \frac{4z}{3(z+1)(z-\frac{1}{3})}k(z)\right)^{-1},$$

$$\left(\left(\mathcal{A}(z) - \mathcal{C}(z)k(z)(E_{\mathcal{R}_e} + \mathcal{D}(z)k(z))^{-1}\mathcal{B}(z)\right)\vec{u}_1, \vec{u}_0\right)_{\mathbb{C}^2} = 0,$$

$$\left(\left(\mathcal{A}(z) - \mathcal{C}(z)k(z)(E_{\mathcal{R}_e} + \mathcal{D}(z)k(z))^{-1}\mathcal{B}(z)\right)\vec{u}_1, \vec{u}_1\right)_{\mathbb{C}^2} = -\frac{1}{z}.$$

Therefore an arbitrary solution $M(t) = (m_{n,j}(t))_{n,j=0}^{1}$, may be obtained from the following relations:

$$\int_{-1}^{1} \frac{1}{t-z} dm_{0,0}(t)$$

$$= -\frac{z+\frac{2}{3}}{(z+1)(z-\frac{1}{3})} - \frac{4}{9(z+1)^2(z-\frac{1}{3})^2} k(z) \left(1 - \frac{4z}{3(z+1)(z-\frac{1}{3})} k(z)\right)^{-1},$$

where $k(z) \in R_{\mathcal{R}}([-1,1])$, $z \in \mathbb{C}\backslash\mathbb{R}$;

$$m_{0,1}(t) = m_{1,0}(t) \equiv 0,$$

and $m_{1,1}(t)$ is the same as in Example 2.1.

Consider the matrix moment problem (1) with $\ell = 0$. In this case the necessary and sufficient condition of the solvability is

$$S_0 \geq 0. \tag{63}$$

The necessity is obvious. On the other hand, if relation (63) is true, we can choose

$$\widehat{M}(x) = \frac{x-a}{b-a} S_0, \qquad x \in [a,b]. \tag{64}$$

This function is a solution of the moment problem. Set

$$\widetilde{M}(x) = 2\frac{x-a}{b-a} S_0, \quad x \in \left[a, \frac{a+b}{2}\right]; \quad \widetilde{M}(x) = S_0, \quad x \in \left(\frac{a+b}{2}, b\right]. \tag{65}$$

If $S_0 \neq 0$, then $\widetilde{M}$ is another solution of the moment problem. Thus, in this case the moment problem is indeterminate. On the other hand, if $S_0 = 0$, then $M(x) \equiv 0$ is the unique solution of the moment problem.

Observe that a set of all solutions consists of non-decreasing matrix functions $M(x)$ on $[a,b]$, left-continuous in (a,b), with the boundary conditions $M(a) = 0$, $M(b) = S_0$.

THEOREM 2.5. *Let the matrix moment problem (1) with $\ell = 2d$, $d \in \mathbb{Z}$, be given and conditions (5) hold where in the case $d = 0$ the second condition in (5) is redundant. In the case $d = 0$, the moment problem is determinate if and only if $S_0 = 0$. In the case $d \in \mathbb{N}$, the moment problem is determinate if and only if $B^{\mu} = B^{M}$, where the operator B is defined by (14) and B^{μ},B^{M} are the corresponding extremal extensions of B.*

Proof. The first statement of the theorem follows from the above considerations. The second statement follows from Theorem 2.4, if we take into account that the class $R_{\mathcal{R}_e}([-1,1])$, where $\dim \mathcal{R}_e > 0$, has at least two different elements. In fact, from the definition of the class $R_{\mathcal{R}_e}([-1,1])$ it follows that $k_1(z) \equiv 0$, and $k_1(z) \equiv E_{\mathcal{R}_e}$, belong to $R_{\mathcal{R}_e}([-1,1])$. ☐

3. The case of an even number of prescribed moments: a description of all solutions and the determinacy. Consider the matrix moment problem (1) with $\ell = 2d+1$, $d \in \mathbb{Z}_+$. We shall use the following well-known lemma about the block matrix (e.g. [26, p. 223]).

LEMMA 3.1. *The block complex matrix*

$$T = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \tag{66}$$

is non-negative, $T \geq 0$, (here A and C are square complex matrices) if and only if the following conditions hold:

1) $A \geq 0$;

2) there exists a matrix solution X of the equation $AX = B$;

3) for the solution X the following relation holds: $C - X^*AX \geq 0$.

If conditions 1)-3) are satisfied, then the value of X^*AX does not depend on the choice of X.

We shall need some conditions of the solvability of the moment problem which are different from conditions (7). Of course, they are equivalent (however, we do not see an easy way to show this equivalence without a reference to the moment problem (1)).

THEOREM 3.1. *Let the matrix moment problem (1) with $\ell = 2d+1$, $d \in \mathbb{Z}_+$, be given. The moment problem has a solution if and only if*

$$\Gamma_d \geq 0; \ \widetilde{\Gamma}_d \geq 0, \tag{67}$$

and there exist matrix solutions X, Y of matrix equations

$$\Gamma_d X = \begin{pmatrix} S_{d+1} \\ S_{d+2} \\ \vdots \\ S_{2d+1} \end{pmatrix}, \quad \widetilde{\Gamma}_d Y = \begin{pmatrix} -abS_d + (a+b)S_{d+1} - S_{d+2} \\ -abS_{d+1} + (a+b)S_{d+2} - S_{d+3} \\ \vdots \\ -abS_{2d-1} + (a+b)S_{2d} - S_{2d+1} \end{pmatrix}, \tag{68}$$

and for these solutions X, Y the following relation hold:

$$X^*\Gamma_d X \leq -abS_{2d} + (a+b)S_{2d+1} - Y^*\widetilde{\Gamma}_d Y. \tag{69}$$

Here, in the case $d = 0$, the second inequality in (67) and the second equality in (68) are redundant, and in (69) the last term ($Y^*\widetilde{\Gamma}_d Y$) should be removed.

Proof. Consider the matrix moment problem (1) with $\ell = 2d+1$, $d \in \mathbb{Z}_+$. It has a solution if and only if the moment problem with an odd number of moments

$$\int_a^b x^n dM(x) = S_n, \ n = 0, 1, ..., 2d+1; \quad \int_a^b x^{2d+2} dM(x) = S_{2d+2}, \tag{70}$$

with some complex ($N \times N$) matrix S_{2d+2} has a solution. By (5) the solvability of the moment problem (70) is equivalent to the matrix inequalities

$$\Gamma_{d+1} \geq 0, \ \widetilde{\Gamma}_{d+1} \geq 0. \tag{71}$$

If we apply to the latter inequalities Lemma 3.1, we obtain that solvability of (70) is equivalent to the condition (67), existence of solutions X, Y of (68) and inequalities

$$S_{2d+2} \geq X^*\Gamma_d X, \quad S_{2d+2} \leq -abS_{2d} + (a+b)S_{2d+1} - Y^*\widetilde{\Gamma}_d Y. \tag{72}$$

Consequently, we obtain that the statement of the Theorem is true. □

THEOREM 3.2. *Let the matrix moment problem (1) with $\ell = 2d+1$, $d \in \mathbb{Z}_+$, be given and conditions (67),(68) and (69) hold. An arbitrary solution $M(\cdot)$ of the moment problem can be found by the Stieltjes-Perron inversion formula from the following relation*

$$\int_{-1}^{1} \frac{1}{t-z} dM^T \left(\frac{(b-a)t + (a+b)}{2} \right)$$

$$= \mathcal{A}(z;S) - \mathcal{C}(z;S)k(z;S)(E_{\mathcal{R}(\mathcal{S})} + \mathcal{D}(z;S)k(z;S))^{-1}\mathcal{B}(z;S), \tag{73}$$

where $k(z;S) \in R_{\mathcal{R}(\mathcal{S})}[-1,1]$, $S \in [X^*\Gamma_d X, -abS_{2d} + (a+b)S_{2d+1} - Y^*\widetilde{\Gamma}_d Y]$, and on the left-hand side one means the matrix of the corresponding operator in $\mathbb{C}^N$. Here $\mathcal{A}(z;S), \mathcal{B}(z;S), \mathcal{C}(z;S), \mathcal{D}(z;S)$ are analytic operator-valued functions given by relations (56)-(59) for the moment problem (1) with moments $\{S_k\}_{k=0}^{2d+1}$, $S_{2d+2} = S$. Also, $\mathcal{R}(S)$ is the subspace $\mathcal{R}$ calculated for the latter moment problem.
Moreover, the correspondence between all solutions of the moment problem and pairs $k(z;S) \in R_{\mathcal{R}(\mathcal{S})}[-1,1]$, $S \in [X^*\Gamma_d X, -abS_{2d} + (a+b)S_{2d+1} - Y^*\widetilde{\Gamma}_d Y]$, is one-to-one.

Proof. The proof follows easily from the considerations in the proof of Theorem 3.1 and by applying Theorem 2.4 and Remark 2.2. □

THEOREM 3.3. *Let the matrix moment problem (1) with $\ell = 2d+1$, $d \in \mathbb{Z}_+$, be given and conditions (67),(68) and (69) hold. The moment problem is determinate if and only if the following two conditions hold:*

1) $X^*\Gamma_d X = -abS_{2d} + (a+b)S_{2d+1} - Y^*\widetilde{\Gamma}_d Y$;
2) $B^\mu = B^M$, *where B^μ, B^M are extremal extensions of the operator B defined by (14) for the moment problem (1) with moments $\{S_k\}_{k=0}^{2d+1}$, $S_{2d+2} = X^*\Gamma_d X$.*

Proof. The proof follows directly from Theorem 2.4 and Theorem 3.2. □

EXAMPLE 3.1. Let $\ell = 2d+1$, $d = 0$, $N = 1$, $a = -1$, $b = 1$, and

$$S_0 = 1,\ S_1 = 0.$$

Consider the moment problem (1) with moments S_0, S_1. In this case $\Gamma_0 = S_0 = 1 > 0$. There exists a solution of $\Gamma_0 X = S_1 = 0$: $X = 0$. Condition (69) is satisfied, as well. Therefore the moment problem has a solution.
Choose an arbitrary $S \in [0,1]$ and consider the moment problem (1) with moments $S_0, S_1, S_2 = S$. It holds

$$\Gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \geq 0, \quad \widetilde{\Gamma}_1 = 1 - S \geq 0,$$

and therefore this extended moment problem has a solution.

Case $S=0$. In this case we may choose $H=\mathbb{C}$, $x_0=\vec{e}_0=(1)$, $x_1=0$. Then $L_N=H$, $D(A)=D(B)=H$, $B=0$. Therefore $B^\mu=B^M=B=0$, and the moment problem is determinate. Then $R^\mu_z=-\frac{1}{z}E_{\mathbb{C}}$,

$$\int_{-1}^{1}\frac{1}{t-z}dM(t)=-\frac{1}{z},$$

and therefore $M(t)$ coincides with the function $m_{1,1}(t)$ in Example 2.1.
Case $0<S\leq 1$. In this case we may choose $H=\mathbb{C}^2$, $x_0=\vec{e}_0=(1,0)$, $x_1=\sqrt{S}\vec{e}_1=\sqrt{S}(0,1)$. Then $L_N=\mathrm{Lin}\{\vec{e}_0\}$, $D(A)=D(B)=\mathrm{Lin}\{\vec{e}_0\}$, $\mathcal{R}=\mathrm{Lin}\{\vec{e}_1\}$. The operator $B=A$ has the following matrix representation in the basis $\{\vec{e}_k\}_{k=0}^1$:

$$B=(B_{k,\ell})_{k,\ell=0}^2=\begin{pmatrix}0 & *\\ \sqrt{S} & *\end{pmatrix},$$

where $*$ means that the corresponding value is not defined. If $\widetilde{B}\supset B$ is a self-adjoint contractive extension of B in H, then its matrix representation should be of the form:

$$\widetilde{B}=(\widetilde{B}_{k,\ell})_{k,\ell=0}^2=\begin{pmatrix}0 & \sqrt{S}\\ \sqrt{S} & w\end{pmatrix},$$

and

$$\begin{pmatrix}1 & \sqrt{S}\\ \sqrt{S} & w+1\end{pmatrix}\geq 0,\quad \begin{pmatrix}1 & -\sqrt{S}\\ -\sqrt{S} & 1-w\end{pmatrix}\geq 0.$$

The latter relations hold iff

$$-(1-S)\leq w\leq 1-S.$$

Suppose that $S=1$. In this case

$$B^\mu=B^M=B_e=\begin{pmatrix}0 & \sqrt{S}\\ \sqrt{S} & 0\end{pmatrix};$$

$$R^\mu_z=\begin{pmatrix}-z & \sqrt{S}\\ \sqrt{S} & -z\end{pmatrix}^{-1}=\frac{1}{z^2-1}\begin{pmatrix}-z & -1\\ -1 & -z\end{pmatrix},\qquad z\in\mathbb{C}\backslash\mathbb{R}.$$

Then

$$\int_{-1}^{1}\frac{1}{t-z}dM(t)=-\frac{z}{z^2-1},$$

and therefore $M(t)$ coincides with the function $m_{0,0}(t)$ from Example 2.1.

Suppose that $0<S<1$. In this case

$$B^\mu=\begin{pmatrix}0 & \sqrt{S}\\ \sqrt{S} & S-1\end{pmatrix},\ B^M=\begin{pmatrix}0 & \sqrt{S}\\ \sqrt{S} & 1-S\end{pmatrix},\ C=\begin{pmatrix}0 & 0\\ 0 & 2(1-S)\end{pmatrix};$$

and for $z\in\mathbb{C}\backslash\mathbb{R}$ we have

$$R^\mu_z=\begin{pmatrix}-z & \sqrt{S}\\ \sqrt{S} & -z+S-1\end{pmatrix}^{-1}=\begin{pmatrix}\frac{-z+S-1}{(z-S)(z+1)} & -\frac{\sqrt{S}}{(z-S)(z+1)}\\ -\frac{\sqrt{S}}{(z-S)(z+1)} & -\frac{z}{(z-S)(z+1)}\end{pmatrix}.$$

Let $\mathbb{C} = \text{Lin}\{\vec{u}_0\}$, $\vec{u}_0 = (1)$. Then

$$\mathcal{B}(z)\xi\vec{u}_0 = -\frac{\sqrt{2(1-S)S}}{(z-S)(z+1)}\xi\vec{e}_1, \quad \xi \in \mathbb{C},$$

$$\mathcal{C}(z)\alpha\vec{e}_1 = -\frac{\sqrt{2(1-S)S}}{(z-S)(z+1)}\alpha\vec{u}_0, \quad \alpha \in \mathbb{C},$$

$$\mathcal{A}(z)\beta u_0 = -\frac{z-S+1}{(z-S)(z+1)}\beta u_0, \quad \beta \in \mathbb{C},$$

$$\mathcal{D}(z)\gamma\vec{e}_1 = -\frac{2(1-S)z}{(z-S)(z+1)}\gamma\vec{e}_1, \quad \gamma \in \mathbb{C}.$$

Then

$$\left(\left(\mathcal{A}(z) - \mathcal{C}(z)k(z)(E_{\mathcal{R}_e} + \mathcal{D}(z)k(z))^{-1}\mathcal{B}(z)\right)\vec{u}_0, \vec{u}_0\right)_{\mathbb{C}^2}$$

$$= -\frac{z-S+1}{(z-S)(z+1)} - \frac{2(1-S)S}{(z-S)^2(z+1)^2}k(z)\left(1 - \frac{2(1-S)S}{(z-S)(z+1)}k(z)\right)^{-1}.$$

Therefore an arbitrary solution $M(t)$ (in the case $0 < S < 1$), may be obtained from the following relation:

$$\int_{-1}^{1} \frac{1}{t-z} dM(t)$$

$$= -\frac{z-S+1}{(z-S)(z+1)} - \frac{2(1-S)S}{(z-S)^2(z+1)^2}k(z)\left(1 - \frac{2(1-S)S}{(z-S)(z+1)}k(z)\right)^{-1}, \tag{74}$$

where $k(z) \in R_{\mathcal{R}}([-1,1])$, $z \in \mathbb{C}\backslash\mathbb{R}$.

Finally, the set of all solutions of the moment problem (1) with moments $S_0 = 1$, $S_1 = 0$, consists of functions $m_{0,0}(t)$, $m_{1,1}(t)$ from Example 2.1, and functions given by relation (74), where $k(z) \in R_{\mathcal{R}}([-1,1])$, $0 < S < 1$.

Remark 3.1. It would be of interest to study the density questions for finite sets of matrix polynomials by means of the truncated Hausdorff moment problem. Also, the density questions for finite sets of complex polynomials on radial rays can be studied in this framework (see [27]).

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REFERENCES

[1] M. G. Krein and A. A. Nudelman, *The Markov moment problem and extremal problems*, Nauka, Moscow (1973), 552p. (Russian).

[2] M. G. Krein and M. A. Krasnoselskiy, *Basic theorems on an extension of Hermitian operators and some their applications to the theory of orthogonal polynomials and the moment problem*, Uspehi matem. nauk, 3:19 (1947), pp. 60–106 (Russian).

[3] T. Ando, *Truncated moment problems for operators*, Acta Sci. Math. (Szeged), 31 (1970), pp. 319–334.

[4] A. E. Choque Rivero, Yu. M. Dyukarev, B. Fritzsche and B. Kirstein, *A truncated matricial moment problem on a finite interval*, Operator Theory: Advances and Applications, 165 (2006), pp. 121–173.

[5] *A. E. Choque Rivero, Yu. M. Dyukarev, B. Fritzsche and B. Kirstein*, *A truncated matricial moment problem on a finite interval. The case of an odd number of prescribed moments*, Operator Theory: Advances and Applications, 176 (2007), pp. 99–164.

[6] R. E. Curto and L. A. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston Journal of Mathematics, 17:4 (1991), pp. 603–635.

[7] V. A. Derkach and M. M. Malamud, *Generalized resolvents of Hermitian operators and the truncated moment problem*, Dopovidi AN Ukrainy (1991), no. 11, pp. 34–39.

[8] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, Journal of Mathematical Sciences, 73:2 (1995), pp. 141–242.

[9] M. A. Neumark, *Spectral functions of a symmetric operator*, Izvestiya AN SSSR, 4 (1940), pp. 277–318 (Russian).

[10] M. A. Neumark, *On spectral functions of a symmetric operator*, Izvestiya AN SSSR, 7 (1943), pp. 285–296 (Russian).

[11] N. I. Akhiezer, *Classical moment problem. Fizmatlit*, Moscow (1961), 312p. (Russian).

[12] Ju. M. Berezanskii, *Expansions in eigenfunctions of selfadjoint operators*, Amer. Math. Soc., Providence, RI (1968). (Russian edition: Naukova Dumka, Kiev, 1965).

[13] B. Szökefalvi-Nagy and A. Koranyi, *Relations d'un probléme de Nevanlinna et Pick avec la théorie de l'espace hilbertien*, Acta Math. Acad. Sci. Hungar., 7 (1957), pp. 295–302.

[14] B. Szökefalvi-Nagy and A. Koranyi, *Operatortheoretische Behandlung und Verallgemeinerung eines Problemkreises in der komplexen Funktionentheorie*, Acta Math., 100 (1958), pp. 171–202.

[15] I. M. Gelfand and M. A. Naimark, *On the inclusion of a normed ring into the ring of operators in a Hilbert space*, Matem. Sb., 12 (1943), pp. 197–213 (in Russian).

[16] M. Putinar and K. Schmüdgen, *Multivariate determinateness*, Indiana Univ. Math. J., 57:6 (2008), pp. 2931–2968.

[17] Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral Methods in the Infinite-dimensional Analysis*, Naukova dumka, Kiev, 1988, (Russian).

[18] B. Fuglede, *The multidimensional moment problem*, Expo. Math., 1 (1983), pp. 47–65.

[19] S. M. Zagorodnyuk, *Positive definite kernels satisfying difference equations*, Methods Funct. Anal. Topology, 16:1 (2010), pp. 83–100.

[20] M. G. Krein and I. E. Ovcharenko, *On the theory of generalized resolvents of nondensely defined Hermitian contractions*, Dopovidi AN URSR (1976), seriya A, no. 10, pp. 881–884, (Ukrainian).

[21] M. G. Krein and I. E. Ovcharenko, *On Q-functions and sc-resolvents of nondensely defined Hermitian contractions*, Sibirskiy matem. zhurnal, XVIII:5 (1977), pp. 1032–1056 (Russian).

[22] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications*, I. Matem. sbornik, 20(62) (1947), No. 3, pp. 431–495 (Russian).

[23] M. M. Malamud and S. M. Malamud, *Operator measures in a Hilbert space*, Algebra i analiz, 15:3 (2003), pp. 1–52.

[24] A. V. Shtraus, *Generalized resolvents of symmetric operators*, Izvestiya AN SSSR, 18 (1954), pp. 51–86 (Russian).

[25] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in a Hilbert space*, Gos. izdat. teh.-teor. lit. Moscow, Leningrad (1950), 484p. (Russian).

[26] V. A. Zolotarev, *Analytic methods of spectral representations of non-selfadjoint and nonunitary operators*, KhNU. Kharkov (2003), 344p. (Russian).

[27] F. Marcellán and S. M. Zagorodnyuk, *Density of polynomials in some L^2 spaces on radial rays in the complex plane*, Linear Algebra and its Applications, 435 (2011), pp. 128–146.