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# ON SOME PROPERTIES OF $t^{h,1}$FUNCTIONS IN THE CALDERON-ZYGMUND THEORY*

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**Abstract.** In this paper we will present some results about functions having derivatives in the $L^1$ sense, according to the definition of Calderon-Zygmund [1]. In particular we prove that these functions behave nicely with respect to a certain non-homogeneous blow-up related to the generalized Taylor polynomial.

**Key words.** Functions with summable derivatives, nonhomogeneous blow-up of graphs.

**AMS subject classifications.** 46E35, 28A75.

**1. Introduction.** The spaces $t^{h,p}(x)$ of functions having derivative of order $h$ at $x$ in the $L^p$ sense, see Definition 2.1 below, were first introduced in [1] in order to investigate the pointwise behaviour of Sobolev functions. In particular the following remarkable facts hold, just to mention a few:

- If $u \in W^{k,p}(\mathbb{R}^n)$ and $\varepsilon > 0$ then there exists an open set $U$ with Bessel capacity $B_{k-h,p}(U)$ not exceeding $\varepsilon$ and such that $u \in t^{h,p}(\mathbb{R}^n \backslash U)$, with $h \le k$ and $(k-h)p < n$ (compare [9, Theorem 3.10.4]);
- The Whitney extension theorem in the framework of $t^{h,p}(x)$ spaces (see [9, Theorem 3.6.3] or Theorem 2.1 below);
- Lusin-type property of Sobolev functions (with $h, k, p$ as above): If $u \in W^{k,p}(\mathbb{R}^n)$ and $\varepsilon > 0$ then there exist an open set $U$ and $v \in C^h(\mathbb{R}^n)$ such that $B_{k-h,p}(U) \le \varepsilon$ and $D^\alpha v = D^\alpha u$ in $\mathbb{R}^n \backslash U$, for all $0 \le |\alpha| \le h$ (compare [9, Theorem 3.10.5]).

In this paper we will present some new results about $t^{h,1}(x)$. In particular, the theory developed in Chapter 3 is based on the following observation (compare [3]):

*Let $U$ be a neighborhood of $x \in \mathbb{R}^n$ and $u \in C^h(U)$ (with $h \ge 1$). Denote by $T_{u,x,d}$ the $d$-th degree Taylor polynomial of $u$ at $x$ (with $d \le h$) and for $r > 0$ define*

$$u_r(z) := \frac{u(x+rz) - T_{u,x,h-1}(x+rz)}{r^h}, \qquad z \in \frac{U-x}{r}.$$

*Then $r \mapsto u_r$ converges to the form*

$$H_{u,h}(z) := T_{u,x,h}(x+z) - T_{u,x,h-1}(x+z), \qquad z \in \mathbb{R}^n$$

*uniformly in the compact sets, as $r \downarrow 0$. Since one has $D_i(u_r) = (D_i u)_r$ and $D_i H_{u,h} = H_{D_i u,h-1}$, the same property yields at once the convergence of the graph of $u_r$ to the graph of $H_{u,h}$, in the sense of varifolds.*

Since $t^{h,1}(x) \subset t^{h-1,1}(x)$, by Proposition 3.1, for all $u \in t^{h,1}(x)$ one can define $u_r$ and $H_{u,h}$ in a similar way as above. The following results resemble the just mentioned properties occuring in the smooth case and are provided in Chapter 3:

- If $u \in t^{h,1}(x)$ then $r \mapsto u_r$ converges in $L^1_{\rm loc}$ to $H_{u,h}$, as $r \downarrow 0$;

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- Let $u$ be a function of class $C^1$ in a neighborhood of $x \in \mathbb{R}^n$ such that:
  (i) $D_i u \in t^{h-1,1}(x)$ for all $i = 1, \ldots, n$ e for a certain integer $h \geq 2$ (Proposition 3.4(I));
  (ii) $D_i P_{D_j u,x,h-1} = D_j P_{D_i u,x,h-1}$ for all $i, j = 1, \ldots, n$.
  Then $u \in t^{h,1}(x)$ and the graphs of $u_r$ converge to the graph of $H_{u,h}$, as $r \downarrow 0$, in the sense of varifolds (Theorem 3.1).

In Section 4 we deal with iterated derivatives in the context of $t^{h,1}(x)$. More precisely we prove some statements extending this trivial property of smooth functions: *If $u$ is of class $C^h$ (in an open set) and $D^h u$ is of class $C^k$, then $u$ is of class $C^{h+k}$.*

## 2. Notation, some well-known and preliminary results.

**2.1. Main notation.** Define

$$I(n) := \mathbb{N}^n, \qquad Q(m) := \{1, \ldots, n\}^m$$

and, for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in I(n)$:

$$|\alpha| := \sum_{i=1}^{n} \alpha_i, \qquad \alpha! := \alpha_1! \cdots \alpha_n!.$$

Consider the map

$$\mu : \bigcup_{m=1}^{+\infty} Q(m) \to I(n)$$

defined by

$$\mu(\theta)_i := \#\{j \,|\, \theta_j = i\} \qquad (i = 1, \ldots, n)$$

for all $\theta = (\theta_1, \ldots, \theta_m) \in Q(m)$.

Observe that if $\alpha \in I(n)$ then $\mu^{-1}(\alpha) \subset Q(|\alpha|)$ and

$$\#\mu^{-1}(\alpha) = \frac{|\alpha|!}{\alpha!}. \tag{2.1}$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in I(n)$, we let

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

For $i = 1, \ldots, n$, we set $D_i := \partial/\partial x_i$. Moreover define

$$D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \qquad (\text{with } \alpha = (\alpha_1, \ldots, \alpha_n) \in I(n))$$

and

$$D_\theta^m := D_{\theta_m} \cdots D_{\theta_1} \qquad (\text{with } \theta = (\theta_1, \ldots, \theta_m) \in Q(m)).$$

Observe that (on spaces of $C^m$ functions) one has

$$D_\theta^m = D^{\mu(\theta)} \qquad (\text{for all } \theta \in Q(m)).$$

In this paper, the open ball in $\mathbb{R}^n$ of radius $r$ centered at $x$ is denoted by $B(x, r)$.

**2.2. Functions of class $t^{h,p}$.** Here we adopt the notation and the statements of [9, Sect. 3.5], where a really clear treatment of this subject is provided.

DEFINITION 2.1. *Let $x \in \mathbb{R}^n$, $p \in [1, +\infty]$ and $h$ be a non-negative integer. Then $t^{h,p}(x)$ denotes the family of functions $u$ defined in a neighborhood of $x$ which are measurable and such that there exists a polynomial $P$ of degree less than or equal to $h$ satisfying*

$$\left( \fint_{B(x,r)} |u - P|^p \right)^{1/p} = o(r^h) \qquad (\text{as } r \to 0). \tag{2.2}$$

*If $A$ is any subset of $\mathbb{R}^n$ then also set*

$$t^{h,p}(A) := \left\{ u \in \bigcap_{x \in A} t^{h,p}(x) \;\middle|\; (2.2) \text{ holds uniformly in } A \right\}.$$

REMARK 2.1. The polynomial $P$ in Definition 2.1 is uniquely determined. Throughout this paper it will be denoted by $P_{u,x,h}$.

REMARK 2.2. If $\varphi$ is of class $C^h$ in a neighborhood of $x \in \mathbb{R}^n$ (with $h \geq 0$), then $\varphi \in t^{h,p}(x)$ for all $p \in [1, +\infty]$ and $P_{\varphi,x,h}$ is just the $h - th$ degree Taylor polynomial of $\varphi$ at $x$.

The following Whitney-type extension theorem holds, compare [9, Theorem 3.6.3].

THEOREM 2.1. *Let $A$ be a closed subset of $\mathbb{R}^n$ and $\widetilde{A} := \{x \in \mathbb{R}^n \mid dist(x, A) < 1\}$. If $u \in L^p(\widetilde{A}) \cap t^{h,p}(A)$, where $h$ is a positive integer and $p \in [1, +\infty]$, then there exists $\varphi \in C^h(\widetilde{A})$ such that*

$$D^\alpha \varphi(x) = (D^\alpha P_{u,x,h})(x)$$

*for all $x \in A$ and $\alpha \in I(n)$ with $0 \leq |\alpha| \leq h$.*

**2.3. Points of enhanced density.** We recall the following definition from [4, 5].

DEFINITION 2.2. *Let $A$ be a measurable subset of $\mathbb{R}^n$ and $m > 0$. Then $x \in \mathbb{R}^n$ is said to be a "$m$-density point of $A$" if $\mathcal{L}^n(B(x,r) \backslash A) = o(r^m)$, as $r \to 0$.*

Caccioppoli sets are more dense than generic measurable sets, where one has $n$-density almost everywhere. Indeed the following property holds [4, Lemma 4.1].

THEOREM 2.2. *Let $A$ be a locally finite perimeter subset of $\mathbb{R}^n$ and*

$$m := n + 1 + \frac{1}{n-1}.$$

*Then a.e. $x \in A$ is a $m$-density point of $A$.*

At a point of high density, classical results about differential maps can be generalized or stated in the context of spaces $t^{h,p}(x)$. The closure result [4, Theorem 2.1] provides an example (in Section 5 below, we state a version for Lipschitz functions). Another example is given by this theorem, proved in [5, Proposition 3.1], which generalizes the obvious statement: if $D\varphi$ is of class $C^h$ then $\varphi$ is of class $C^{h+1}$.

THEOREM 2.3. *Let $\varphi \in C^h(\Omega)$ and $\Phi \in C^h(\Omega; \mathbb{R}^n)$, where $\Omega$ is an open subset of $\mathbb{R}^n$ and $h \geq 1$. If $x \in \Omega$ is a $(n+h)$-density point of $\{y \in \Omega | \nabla \varphi(y) = \Phi(y)\}$ then $\varphi \in t^{h+1,1}(x)$.*

**3. Non-homogeneous blow-up of $t^{h,1}$ functions.** As we mentioned in the introduction, this section is devoted to investigating some properties about convergence of functions in $t^{h,1}(x)$ subjected to non-homogeneous blow-up.

**3.1. Preliminary results and further notation.**

PROPOSITION 3.1. *Let $k, h$ be integers satisfying $0 \le k \le h$ and let $u \in t^{h,1}(x)$. Then one has $u \in t^{k,1}(x)$ and*

$$P_{u,x,k}(y) = \sum_{\substack{\mu \in I(n) \\ |\mu| \le k}} \frac{1}{\mu!} (D^\mu P_{u,x,h})(x)(y-x)^\mu .$$

*Proof.* We can assume $k < h$ (for $k = h$ the statement is trivial). Then, if define

$$\sigma(r) := r^{-h-n} \int_{B(x,r)} |u - P_{u,x,h}|, \quad P(y) := \sum_{\substack{\mu \in I(n) \\ |\mu| \le k}} \frac{1}{\mu!} (D^\mu P_{u,x,h})(x)(y-x)^\mu$$

we obtain

$$\begin{aligned} \int_{B(x,r)} |u - P| &\le \int_{B(x,r)} |u - P_{u,x,h}| + \int_{B(x,r)} |P_{u,x,h} - P| \\ &\le r^{h+n}\sigma(r) + \sum_{\substack{\mu \in I(n) \\ k+1 \le |\mu| \le h}} \frac{1}{\mu!} |(D^\mu P_{u,x,h})(x)| \int_{B(x,r)} |(y-x)^\mu| \, dy. \end{aligned}$$

Thus there is a constant $C$ such that (for $r$ small enough)

$$\int_{B(x,r)} |u - P| \le r^{h+n}\sigma(r) + C r^{k+1+n}$$

hence

$$r^{-k-n} \int_{B(x,r)} |u - P| \le r^{h-k}\sigma(r) + Cr.$$

□

PROPOSITION 3.2. *Let $f$ be a nonnegative measurable function defined in a neighborhood of 0 and let $k, l > 0$. Then one has*

$$\lim_{r \downarrow 0} r^{-l} \int_{B(0,r)} f(y)|y|^{-k} dy = 0 \tag{3.1}$$

*if and only if*

$$\lim_{r \downarrow 0} r^{-l-k} \int_{B(0,r)} f = 0. \tag{3.2}$$

*Proof.* In order to prove the "if" part of the statement, define

$$\sigma(r) := \sup_{\rho \in (0,r]} \rho^{-l-k} \int_{B(0,\rho)} f$$

and

$$E_{r,i} := \{y \in \mathbb{R}^n \mid 2^{-i-1}r \le |y| < 2^{-i}r\}.$$

We get

$$\begin{aligned} r^{-l} \int_{B(0,r)} f(y)|y|^{-k} dy &= r^{-l} \sum_{i=0}^{\infty} \int_{E_{r,i}} f(y)|y|^{-k} dy \\ &\le r^{-l} \sum_{i=0}^{\infty} (2^{-i-1}r)^{-k} \int_{B(0,2^{-i}r)} f \\ &\le \sum_{i=0}^{\infty} 2^{(i+1)k} 2^{-i(l+k)} \sigma(2^{-i}r) \\ &\le 2^k \sigma(r) \sum_{i=0}^{\infty} (2^{-l})^i. \end{aligned}$$

Hence (3.1) follows at once from (3.2).

The opposite implication follows from the obvious inequality

$$r^{-k} \int_{B(0,r)} f \le \int_{B(0,r)} f(y)|y|^{-k} dy.$$

□

In the following proposition we assume $h \ge 1$ because for $h = 0$ it reduces to a trivial statement (compare Remark 2.2).

Proposition 3.3. *Let $u$ be a function of class $C^1$ in a neighborhood of $x \in \mathbb{R}^n$, $h$ be a positive integer and assume that*

$$D_i u \in t^{h,1}(x) \quad (i = 1, \dots, n). \tag{3.3}$$

*Then the following conditions are equivalent:*

*(I) $D_i P_{D_j u,x,h} = D_j P_{D_i u,x,h}$ for all $i, j = 1, \dots, n$;*
*(II) $u \in t^{h+1,1}(x)$ and $D_i P_{u,x,h+1} = P_{D_i u,x,h}$ for all $i = 1, \dots, n$.*

*Proof.* The statement (I) follows from (II) trivially. In order to prove the vice versa, let us assume the closure condition (I). Then there exists a unique potential $P$ of the field $(P_{D_1 u,x,h}, \dots, P_{D_n u,x,h})$ such that $P(x) = u(x)$. Observe that $P$ has to be a $(h+1)$-degree polynomial. Now, for simplicity, set

$$v := u - P, \quad B_\rho := B(x,\rho)$$

and

$$\sigma(\rho) := \sup_{s \in (0,\rho]} \left( \sum_{i=1}^{n} s^{-h-1} \int_{B_s} |D_i u(y) - P_{D_i u,x,h}(y)| \, |y-x|^{1-n} dy \right).$$

Then, by the coarea formula [6, Sect. 3.4.4], one has (for $\rho$ small enough)

$$\begin{aligned}\int_{\partial B_\rho} |v| d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}} |v(x+\rho\nu)-v(x)|\rho^{n-1} d\mathcal{H}^{n-1}(\nu)\\ &= \rho^{n-1}\int_{\mathbb{S}^{n-1}} \left|\int_0^\rho \nabla v(x+t\nu)\cdot\nu\, dt\right| d\mathcal{H}^{n-1}(\nu)\\ &\le \rho^{n-1}\int_0^\rho \left(\int_{\mathbb{S}^{n-1}} |\nabla v(x+t\nu)|\, d\mathcal{H}^{n-1}(\nu)\right) dt\\ &= \rho^{n-1}\int_0^\rho \left(\int_{\partial B_t} |\nabla v(y)|\, d\mathcal{H}^{n-1}(y)\right) t^{1-n} dt\\ &= \rho^{n-1}\int_0^\rho \left(\int_{\partial B_t} |\nabla v(y)|\, |y-x|^{1-n}\, d\mathcal{H}^{n-1}(y)\right) dt\\ &= \rho^{n-1}\int_{B_\rho} |\nabla v(y)|\, |y-x|^{1-n} dy\\ &\le \rho^{n-1}\sum_{i=1}^n \int_{B_\rho} |D_i u(y) - D_i P(y)|\, |y-x|^{1-n} dy\\ &= \rho^{n-1}\sum_{i=1}^n \int_{B_\rho} |D_i u(y) - P_{D_i u,x,h}(y)|\, |y-x|^{1-n} dy\end{aligned}$$

whence

$$\int_{\partial B_\rho} |v| d\mathcal{H}^{n-1} \le \rho^{n+h}\sigma(\rho).$$

It follows that (for $r$ small enough)

$$\int_{B_r} |v| = \int_0^r \left(\int_{\partial B_\rho} |v|\, d\mathcal{H}^{n-1}\right) d\rho \le \int_0^r \rho^{n+h}\sigma(\rho)\, d\rho \le \sigma(r) r^{n+h+1}$$

namely

$$r^{-n-h-1}\int_{B_r} |u-P| \le \sigma(r).$$

But recalling (3.3) and using Proposition 3.2 (with $k = n-1$ and $l = h+1$) we find

$$\lim_{r\downarrow 0}\sigma(r) = 0$$

hence $u \in t^{h+1,1}(x)$ and $P_{u,x,h+1} = P$. □

### 3.2. Convergence $L^1_{\mathbf{loc}}$.

Definition 3.1. *Let $P$ be a polynomyal in $\mathbb{R}^n$ of degree $h$ and $x \in \mathbb{R}^n$. Then the "maximal form in $P$ at $x$" is defined as the homogeneous polynomial of degree $h$*

$$\mathbb{R}^n \ni z \mapsto \sum_{\substack{\mu\in I(n)\\ |\mu|=h}} \frac{1}{\mu!}(D^\mu P)(x)\, z^\mu.$$

Proposition 3.4. *Let $x \in \mathbb{R}^n$ and $u$ be a measurable function defined in a neighborhood of $x$. Then the following facts hold:*

*(I) If $u \in t^{h,1}(x)$ with $h \geq 1$, then $u \in t^{h-1,1}(x)$ and the functions*

$$\mathbb{R}^n \ni z \mapsto u_r(z) := \frac{u(x+rz) - P_{u,x,h-1}(x+rz)}{r^h} \qquad (r > 0)$$

*converge in $L^1_{loc}$, as $r \downarrow 0$, to the maximal form of $P_{u,x,h}$ at $x$, namely*

$$H(z) := \sum_{\substack{\mu \in I(n) \\ |\mu| = h}} \frac{1}{\mu!} (D^\mu P_{u,x,h})(x)\, z^\mu.$$

*One has*

$$H(z) = P_{u,x,h}(x+z) - P_{u,x,h-1}(x+z). \tag{3.4}$$

*(II) If for a certain integer $h \geq 1$ there exist a polynomial $Q$ of degree $h-1$ and a homogeneous polynomial $H$ of degree $h$ such that the functions*

$$\mathbb{R}^n \ni z \mapsto \frac{u(x+rz) - Q(x+rz)}{r^h} \qquad (r > 0) \tag{3.5}$$

*converge in $L^1_{loc}$ to $H$, as $r \downarrow 0$, then $u \in t^{h,1}(x)$.*

*Proof.* (I) Assume that $u \in t^{h,1}(x)$. Proposition 3.1 implies that $u \in t^{h-1,1}(x)$ and (3.4) holds. Now let $\Omega$ be a bounded measurable subset of $\mathbb{R}^n$ and consider $R > 0$ such that $\Omega \subset B(0,R)$. Then, for $r$ small enough, one has

$$\begin{aligned}
\int_\Omega \Big| u_r(z) - H(z) \Big| dz &\leq \int_{B(0,R)} \Big| \frac{u(x+rz) - P_{u,x,h-1}(x+rz)}{r^h} - H(z) \Big| dz \\
&= \int_{B(x,rR)} \Big| \frac{u(y) - P_{u,x,h-1}(y)}{r^h} - H\Big(\frac{y-x}{r}\Big) \Big| r^{-n} dy \\
&= r^{-h-n} \int_{B(x,rR)} |u - P_{u,x,h}| \\
&= R^{h+n} (rR)^{-h-n} \int_{B(x,rR)} |u - P_{u,x,h}|.
\end{aligned}$$

Thus

$$\lim_{r \downarrow 0} \int_\Omega |u_r(z) - H(z)|\, dz = 0.$$

(II) Let $Q$ and $H$ be, respectively, a polynomial of degree $h-1$ and a homogeneous polynomial of degree $h$ such that the functions (3.5) converge in $L^1_{\rm loc}$ to $H$, as $r \downarrow 0$. If set

$$P(y) := Q(y) + H(y-x), \qquad y \in \mathbb{R}^n$$

then

$$\begin{aligned}
\int_{B(x,r)} |u - P| &= \int_{B(x,r)} |u(y) - Q(y) - H(y-x)|\, dy \\
&= \int_{B(0,1)} |u(x+rz) - Q(x+rz) - H(rz)|\, r^n dz \\
&= r^{h+n} \int_{B(0,1)} \Big| \frac{u(x+rz) - Q(x+rz)}{r^h} - H(z) \Big|\, dz.
\end{aligned}$$

Hence $u \in t^{h,1}(x)$. ∎

**3.3. Graph convergence.** Let $\{e_1, \dots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$ and $J : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the trivial isometric immersion defined by

$$J(x_1, \dots, x_n) := (x_1, \dots, x_n, 0)$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. Let $e_{n+1} := (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ and observe that

$$\{Je_1, \dots, Je_n, e_{n+1}\}$$

is the standard orthonormal basis of $\mathbb{R}^{n+1}$. For simplicity we will identify $Je_i$ with $e_i$ $(i = 1, \dots, n)$.

The space of $n$-vectors in $\mathbb{R}^{n+1}$ is denoted by $\Lambda_n\mathbb{R}^{n+1}$. Set

$$e := e_1 \wedge \cdots \wedge e_n$$

and observe that

$$\{e\} \cup \{e_{n+1} \wedge (e \llcorner e_i)\}_{i=1}^n$$

form a basis of $\Lambda_n\mathbb{R}^{n+1}$. Let $\|\xi\|$ be the length of $\xi \in \Lambda_n\mathbb{R}^{n+1}$, i.e.

$$\|\xi\| = \left(\sum_{i=0}^{n+1} \xi_i^2\right)^{1/2}, \qquad \xi = \xi_0\, e + \sum_{i=1}^{n} \xi_i\, e_{n+1} \wedge (e \llcorner e_i).$$

Recall that if $U$ is an open subset of $\mathbb{R}^n$ and $f \in C^1(U)$ then $\| \wedge^n d(I \times f)e\|$ is the measure transformation factor for $I \times f$, namely the following formula holds for all measurable subsets $A$ of $U$

$$\mathcal{H}^n(\{(z, f(z)) | z \in A\}) = \int_A \| \wedge^n d(I \times f)e\|\, d\mathcal{L}^n \tag{3.6}$$

compare [7, Sect. 3.2].

One has

$$\begin{aligned} \wedge^n d(I \times f)e &= [e_1 + (D_1 f)e_{n+1}] \wedge \cdots \wedge [e_1 + (D_1 f)e_{n+1}] \\ &= e\, + \sum_{i=1}^{n} (D_i f)\, e_{n+1} \wedge (e \llcorner e_i) \end{aligned} \tag{3.7}$$

hence the expected result

$$\| \wedge^n d(I \times f)(e)\| = (1 + |\nabla f|^2)^{1/2}.$$

Now, given a measurable subset $E$ of $\mathbb{R}^n$ such that $E \subset\subset U$, we can consider the following functional:

$$G_{f,E}(\varphi) := \int_{(I\times f)(E)} \varphi(w; \eta(w))\, d\mathcal{H}^n(w), \quad \varphi \in C(\mathbb{R}^{n+1} \times \Lambda_n\mathbb{R}^{n+1})$$

where

$$\eta(z, t) := \frac{\wedge^n d(I \times f)e}{\| \wedge^n d(I \times f)e\|}(z), \qquad (z, t) \in U \times \mathbb{R}.$$

Observe that $G_{f,E}$ restricted to $C_c(\mathbb{R}^{n+1} \times \Lambda_n \mathbb{R}^{n+1})$ is the oriented rectifiable $n$-varifold naturally associated to the graph of $f|_E$ (compare [2]).

From (3.6) and (3.7) we get

$$\begin{aligned} G_{f,E}(\varphi) &= \int_E \varphi\left(z, f(z); \frac{\wedge^n d(I\times f)e}{\| \wedge^n d(I\times f)e\|}(z)\right) \| \wedge^n d(I\times f)e\|(z)\, dz \\ &= \int_E \varphi\left(z, f(z); \frac{e + \sum_{i=1}^n D_i f(z)\, e_{n+1} \wedge (e \llcorner e_i)}{(1+\|\nabla f(z)\|^2)^{1/2}}\right)(1+\|\nabla f(z)\|^2)^{1/2} dz \end{aligned}$$

for all $\varphi \in C(\mathbb{R}^{n+1} \times \Lambda_n \mathbb{R}^{n+1})$.

Let $x \in \mathbb{R}^n$ and $u \in t^{h,1}(x)$ with $h \geq 2$. By Proposition 3.1 we also have $u \in t^{h-1}(x)$ and thus, for $r > 0$, the transformation

$$\mathcal{T}_r : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \quad \mathcal{T}_r(y;t) := \left(\frac{y-x}{r}, \frac{t - P_{u,x,h-1}(y)}{r^h}\right)$$

is well-defined.

Remark 3.1. An obvious computation shows that $\mathcal{T}_r$ trasforms the graph of $u$ into the graph of the function $u_r$ introduced in Proposition 3.4(I), namely

$$u_r(z) = \frac{u(x+rz) - P_{u,x,h-1}(x+rz)}{r^h}, \quad z \in \frac{U-x}{r} \tag{3.8}$$

where $U$ denotes the domain of $u$.

Theorem 3.1. *Let $u$ be a function of class $C^1$ in a neighborhood of $x \in \mathbb{R}^n$ such that:*

*(i) $D_i u \in t^{h-1,1}(x)$ for all $i = 1, \dots, n$ e for a certain integer $h \geq 2$;*

*(ii) $D_i P_{D_j u,x,h-1} = D_j P_{D_i u,x,h-1}$ for all $i, j = 1, \dots, n$.*

*The following facts hold:*

*(I) $u \in t^{h,1}(x)$;*

*(II) Let $H$ denote the maximal form of $P_{u,x,h}$ and consider an arbitrary bounded measurable subset $E$ of $\mathbb{R}^n$. Then*

$$G_{u_r,E}(\varphi) \to G_{H,E}(\varphi)$$

*for all bounded $\varphi \in C(\mathbb{R}^{n+1} \times \Lambda_n \mathbb{R}^{n+1})$, as $r \downarrow 0$. In particular $G_{u_r,E} \to G_{H,E}$ in the sense of oriented varifolds, as $r \downarrow 0$.*

*Proof.* (I) This statement follows at once from Proposition 3.3.

(II) First step. For $i = 1, \dots, n$ and $z \in \mathbb{R}^n$, set

$$Q_i(z) := \sum_{\substack{\mu \in I(n) \\ |\mu| \leq h-2}} \frac{1}{\mu!}(D^\mu P_{D_i u,x,h-1})(x) z^\mu, \quad H_i(z) := \sum_{\substack{\mu \in I(n) \\ |\mu| = h-1}} \frac{1}{\mu!}(D^\mu P_{D_i u,x,h-1})(x) z^\mu$$

Observe that (for $i = 1, \dots, n$)

$$D_i u \in t^{h-2,1}(x)$$

and

$$P_{D_i u,x,h-2}(y) = \sum_{\substack{\mu \in I(n) \\ |\mu| \leq h-2}} \frac{1}{\mu!}(D^\mu P_{D_i u,x,h-1})(x)(y-x)^\mu = Q_i(y-x) \tag{3.9}$$

for all $y \in \mathbb{R}^n$, by Proposition 3.1. Moreover one has (for $i, j = 1, \dots, n$)

$$\frac{\partial}{\partial y_i} \sum_{\substack{\mu \in I(n) \\ |\mu| \le h-1}} \frac{1}{\mu!} (D^\mu P_{D_j u, x, h-1})(x)(y-x)^\mu = \frac{\partial}{\partial y_j} \sum_{\substack{\mu \in I(n) \\ |\mu| \le h-1}} \frac{1}{\mu!} (D^\mu P_{D_i u, x, h-1})(x)(y-x)^\mu$$

for all $y \in \mathbb{R}^n$, by the assumption (ii), namely

$$D_i Q_j + D_i H_j = D_j Q_i + D_j H_i.$$

Since $D_i Q_j$ and $D_j Q_i$ are polynomials of degree $h-3$ while $D_i H_j$ and $D_j H_i$ are homogeneous polynomials of degree $h-2$, we get $D_i Q_j = D_j Q_i$ (and $D_i H_j = D_j H_i$) i.e.

$$D_i P_{D_j u, x, h-2} = D_j P_{D_i u, x, h-2}$$

by (3.9). Hence $u \in t^{h-1,1}(x)$ and

$$D_i P_{u,x,h-1} = P_{D_i u, x, h-2} \tag{3.10}$$

by Proposition 3.3.

Second step. Set

$$\xi_0 := \wedge^n d(I \times H)e, \qquad \xi_r := \wedge^n d(I \times u_r)e.$$

From (3.7) one obtains

$$\xi_0 = e + \sum_{i=1}^n (D_i H)\, e_{n+1} \wedge (e \llcorner e_i) \tag{3.11}$$

and

$$\xi_r = e + \sum_{i=1}^n (D_i u_r)\, e_{n+1} \wedge (e \llcorner e_i) \tag{3.12}$$

where

$$D_i u_r(z) = \frac{D_i u(x+rz) - P_{D_i u, x, h-2}(x+rz)}{r^{h-1}}, \qquad z \in \frac{U-x}{r} \tag{3.13}$$

by (3.8) and (3.10). Moreover, from (3.9), (3.10) and Proposition 3.3 it follows that

$$\begin{aligned} H_i(z) &= P_{D_i u, x, h-1}(x+z) - Q_i(z) = P_{D_i u, x, h-1}(x+z) - P_{D_i u, x, h-2}(x+z) \\ &= (D_i P_{u,x,h})(x+z) - (D_i P_{u,x,h-1})(x+z) \\ &= \frac{\partial}{\partial z_i} \Big( P_{u,x,h}(x+z) - P_{u,x,h-1}(x+z) \Big) \end{aligned}$$

for all $z \in \mathbb{R}^n$, hence

$$H_i = D_i H \tag{3.14}$$

by Proposition 3.1.

Third step. Let $\varphi \in C(\mathbb{R}^{n+1} \times \Lambda_n \mathbb{R}^{n+1})$ be bounded. Then, for $r$ small enough (in such a way that $E \subset\subset (U-x)/r$), we can define

$$\delta(r) := |G_{u_r,E}(\varphi) - G_{H,E}(\varphi)|.$$

From

$$\delta(r) = \left| \int_E \varphi\left(z, u_r(z); \frac{\xi_r(z)}{\|\xi_r(z)\|}\right) \|\xi_r(z)\| dz - \int_E \varphi\left(z, H(z); \frac{\xi_0(z)}{\|\xi_0(z)\|}\right) \|\xi_0(z)\| dz \right|$$

it follows that

$$\delta(r) \le \delta_1(r) + \delta_2(r)$$

with

$$\delta_1(r) := \int_E \left| \varphi\left(z, u_r(z); \frac{\xi_r(z)}{\|\xi_r(z)\|}\right) \right| \left| \|\xi_r(z)\| - \|\xi_0(z)\| \right| dz$$

$$\delta_2(r) := \int_E \left| \varphi\left(z, u_r(z); \frac{\xi_r(z)}{\|\xi_r(z)\|}\right) - \varphi\left(z, H(z); \frac{\xi_0(z)}{\|\xi_0(z)\|}\right) \right| \|\xi_0(z)\| dz.$$

Observe that

$$\left| \|\xi_r\| - \|\xi_0\| \right| \le \|\xi_r - \xi_0\| \le \sum_{i=1}^n |D_i u_r - D_i H| = \sum_{i=1}^n |D_i u_r - H_i| \tag{3.15}$$

by (3.11), (3.12) and (3.14). Thus

$$\delta_1(r) \le \|\varphi\|_\infty \sum_{i=1}^n \int_E |D_i u_r - H_i|.$$

Moreover, recalling (3.13) and using Proposition 3.4(I) with $D_i u$ in place of $u$ and $h-1$ in place of $h$, we get

$$D_i u_r \to H_i \ \text{ in } L^1_{\text{loc}}(\mathbb{R}^n) \tag{3.16}$$

as $r \downarrow 0$, hence $\delta_1(r)$ as $r \downarrow 0$. It remains to prove

$$\lim_{r \downarrow 0} \delta_2(r) = 0. \tag{3.17}$$

Recall that

$$u_r \to H \ \text{ in } L^1_{\text{loc}}(\mathbb{R}^n) \tag{3.18}$$

as $r \downarrow 0$, by Proposition 3.4(I). From (3.16) and (3.18), by a well-known result (e.g. [8, Theorem 3.12]), it follows that every sequence $\{r_j\}$ of positive numbers such that $r_j \to 0$ as $j \to \infty$ has a subsequence $\{r_{j_k}\}$ such that

$$u_{r_{j_k}} \to H, \quad D_i u_{r_{j_k}} \to H_i \,(\text{for } i = 1, \dots, n)$$

a.e. in $E$. By (3.15) we have also $\xi_{r_{j_k}} \to \xi_0$ a.e. in $E$. Hence the dominated convergence theorem yields $\delta_2(r_{j_k}) \to 0$. Finally (3.17) follows from the arbitrariness of $\{r_j\}$. □

In the special case when $h = 1$ the assumptions in Theorem 3.1 are trivially verified and (as we expect) the graph of $u_r$ converges to the tangent space to the graph of $u$ at $(x, u(x))$. In the following result we prove this fact by a straighforward adaptation of the argument above.

PROPOSITION 3.5. *Let $u$ be a function of class $C^1$ in a neighborhood of $x \in \mathbb{R}^n$ and let $L : \mathbb{R}^n \to \mathbb{R}$ be the linear functional defined by*

$$L(z) := \nabla u(x) \cdot z, \qquad z \in \mathbb{R}^n.$$

*Moreover consider an arbitrary measurable subset $E$ of $\mathbb{R}^n$. Then*

$$G_{u_r,E}(\varphi) \to G_{L,E}(\varphi)$$

*for all bounded $\varphi \in C(\mathbb{R}^{n+1} \times \Lambda_n \mathbb{R}^{n+1})$, as $r \downarrow 0$. In particular $G_{u_r,E} \to G_{L,E}$ in the sense of oriented varifolds, as $r \downarrow 0$.*

*Proof.* One has

$$P_{u,x,1}(y) = u(x) + \nabla u(x) \cdot (y - x), \qquad P_{D_i u,x,0}(y) = D_i u(x)$$

for all $y \in \mathbb{R}^n$, by Remark 2.2. Hence, if $H$ and $H_i$ are defined as in Theorem 3.1, we obtain

$$H(z) = \nabla u(x) \cdot z = L(z), \qquad H_i(z) = D_i u(x)$$

for all $z \in \mathbb{R}^n$. Then (3.14) holds trivially and the conclusion follows from the third step in the proof of Theorem 3.1. □

The following corollary also holds.

COROLLARY 3.1. *Let $u$ satisfy the assumptions in Theorem 3.1 (if $h \geq 2$) or, alternatively, the assumptions in Proposition 3.5 (if $h = 1$). Moreover let $\Gamma$ and $\Gamma_r$ denote the graphs of $u$ and $u_r$, while $\Gamma_0$ be the graph of $H$ (if $h \geq 2$) or of $L$ (if $h = 1$). Then:*

*(I) For all bounded measurable subset $E$ of $\mathbb{R}^n$, the area of the graph of $u_r|_E$ converges to the area of the graph of $H|_E$ (if $h \geq 2$) or of $L|_E$ (if $h = 1$), as $r \downarrow 0$, i.e.*

$$\lim_{r \downarrow 0} \mathcal{H}^n(\Gamma_r \cap (E \times \mathbb{R})) = \mathcal{H}^n(\Gamma_0 \cap (E \times \mathbb{R}));$$

*(II) For every fixed bounded open subset $\Omega$ of $\mathbb{R}^n$, one has*

$$\mathcal{H}^n \llcorner \Gamma_r = \mathcal{H}^n \llcorner \mathcal{T}_r(\Gamma) \to \mathcal{H}^n \llcorner \Gamma_0$$

*in the weak* sense of measures in $\Omega \times \mathbb{R}$, as $r \downarrow 0$.*

*Proof.* First consider the case when $h \geq 2$.

(I) Use Theorem 3.1(II) with $\varphi \equiv 1$.

(II) Consider the immersion map $J : C_c(\Omega \times \mathbb{R}) \to C(\mathbb{R}^{n+1} \times \Lambda_n \mathbb{R}^{n+1})$ defined by

$$J\psi\,(w; \eta) := \psi(w), \qquad (w; \eta) \in \mathbb{R}^{n+1} \times \Lambda_n \mathbb{R}^{n+1}$$

and observe that $J\psi$ is bounded for all $\psi \in C_c(\Omega \times \mathbb{R})$. Then Theorem 3.1(II) yields

$$\int_{\Gamma_r} \psi \, d\mathcal{H}^n = G_{u_r,\Omega}(J\psi) \to G_{H,\Omega}(J\psi) = \int_{\Gamma_0} \psi \, d\mathcal{H}^n$$

for all $\psi \in C_c(\Omega \times \mathbb{R})$, as $r \downarrow 0$.

For $h = 1$ we repeat the previous argument with Proposition 3.5 in place of Theorem 3.1(II). □

## 4. Iterated derivatives.

### 4.1. Part I (without enhanced density assumption).

THEOREM 4.1. *Let $\varphi$ be a function of class $C^h$ in a neighborhood of $x$ (with $h \geq 1$) such that:*
*(i) $D^\beta \varphi \in t^{k,1}(x)$ for all $\beta \in I(n)$, $|\beta| = h$ (with $k \geq 1$);*
*(ii) The number $(D^{\rho-\beta} P_{D^\beta \varphi,x,k})(x)$, with*

$$h + 1 \leq |\rho| \leq h + k \quad \text{and} \quad |\beta| = h \qquad (\beta, \rho \in I(n))$$

*does not depend on $\beta$ ("compatibility condition at $x$"). It will be denoted by $d_{x,\rho}$.*

*Then $\varphi \in t^{h+k,1}(x)$ and*

$$P_{\varphi,x,h+k}(y) = P_{\varphi,x,h}(y) + \sum_{l=h+1}^{h+k} \sum_{\substack{\rho \in I(n) \\ |\rho|=l}} \frac{d_{x,\rho}}{\rho!} (y - x)^\rho.$$

*Proof.* First of all, define

$$P(y) := P_{\varphi,x,h}(y) + \sum_{l=h+1}^{h+k} \sum_{\substack{\rho \in I(n) \\ |\rho|=l}} \frac{d_{x,\rho}}{\rho!} (y - x)^\rho$$

and

$$I(r) := \int_{B(x,r)} |\varphi - P|.$$

Then, for $h \geq 1$, one has

$$I(r) \leq \int_{B(x,r)} |y - x|^m A_m(y)\, dy \qquad (m = 0, 1, \ldots, h) \tag{4.1}$$

where $A_0 := |\varphi - P|$ and (for $m = 1, \ldots, h$)

$$A_m(y) := \int_{[0,1]^m} \sum_{\theta \in Q(m)} |D^m_\theta (\varphi - P)(x + t_1 \cdots t_m(y - x))|\, dt_1 \cdots dt_m.$$

The formula (4.1) follows at once from the inequality

$$A_m(y) \leq |y - x|\, A_{m+1}(y) \qquad (m + 1 \leq h)$$

which is again an easy application of the Fundamental Theorem of Calculus and of

$$D^m_\theta \varphi(x) = (D^m_\theta P_{\varphi,x,h})(x) = D^m_\theta P(x) \qquad (m = 1, \ldots, h).$$

By (4.1) with $m = h$ and (2.1), we find

$$\begin{aligned} I(r) &\leq \int_{B(x,r)} |y - x|^h \left( \int_{[0,1]^h} \sum_{\theta \in Q(h)} \left| D^h_\theta (\varphi - P)(x + t_1 \cdots t_h(y - x)) \right| dt_1 \cdots dt_h \right) dy \\ &\leq \sum_{\substack{\beta \in I(n) \\ |\beta|=h}} \frac{r^h h!}{\beta!} \int_{(0,1)^h} \left( \int_{B(x,r)} \left| (D^\beta \varphi - D^\beta P)(x + t_1 \cdots t_h(y - x)) \right| dy \right) dt_1 \cdots dt_h. \end{aligned}$$

Hence, by recalling the formula for the change of variables in the integrals, we get

$$I(r) \le \sum_{\substack{\beta\in I(n)\\|\beta|=h}} \frac{r^h h!}{\beta!}\int_{(0,1)^h}(t_1\cdots t_h)^{-n}\left(\int_{B(x,t_1\cdots t_h r)}\left|(D^\beta\varphi - D^\beta P)(z)\right| dz\right)dt_1\cdots dt_h.$$

Observe that, for $\beta\in I(n)$ with $|\beta| = h$, the polynomial $D^\beta P$ has degree (at most equal to) $k$. Recalling (ii), it follows that

$$\begin{aligned}
D^\beta P(y) &= \sum_{\substack{\mu\in I(n)\\|\mu|\le k}} \frac{1}{\mu!} D^\mu(D^\beta P)(x)\,(y-x)^\mu\\
&= \sum_{\substack{\mu\in I(n)\\|\mu|\le k}} \frac{1}{\mu!}(D^{\mu+\beta}P)(x)\,(y-x)^\mu\\
&= D^\beta\varphi(x) \;+ \sum_{\substack{\mu\in I(n)\\1\le|\mu|\le k}} \frac{1}{\mu!} d_{x,\mu+\beta}\,(y-x)^\mu\\
&= D^\beta\varphi(x) \;+ \sum_{\substack{\mu\in I(n)\\1\le|\mu|\le k}} \frac{1}{\mu!}\left(D^{\mu+\beta-\beta}P_{D^\beta\varphi,x,k}\right)(x)\,(y-x)^\mu
\end{aligned}$$

namely

$$D^\beta P = P_{D^\beta\varphi,x,k}.$$

Substituting into the last inequality we find

$$\begin{aligned}
I(r) &\le \sum_{\substack{\beta\in I(n)\\|\beta|=h}} \frac{r^h h!}{\beta!}\int_{(0,1)^h}(t_1\cdots t_h)^{-n}\left(\int_{B(x,t_1\cdots t_h r)}\left|D^\beta\varphi - P_{D^\beta\varphi,x,k}\right|\right)dt_1\cdots dt_h\\
&\le C\, r^{n+h}\sum_{\substack{\beta\in I(n)\\|\beta|=h}}\int_{(0,1)^h}\left(⨍_{B(x,t_1\cdots t_h r)}\left|D^\beta\varphi - P_{D^\beta\varphi,x,k}\right|\right)dt_1\cdots dt_h
\end{aligned}$$

where $C$ depends only on $n$ and $h$. Hence $I(r) = o(r^{n+h+k})$, namely

$$⨍_{B(x,r)}|\varphi - P| = o(r^{h+k}).$$

□

The following uniform version of Theorem 4.1 holds (same proof).

THEOREM 4.2. *Let $A$ be a subset of $\mathbb{R}^n$ and $\varphi$ be a function of class $C^h$ in a neighborhood of $A$ (with $h\ge 1$). Assume that:*
*(i) $D^\beta\varphi\in t^{k,1}(A)$ for all $\beta\in I(n)$, $|\beta| = h$ (with $k\ge 1$);*
*(ii) the number $(D^{\rho-\beta}P_{D^\beta\varphi,x,k})(x)$, with $x\in A$ and*

$$h+1\le|\rho|\le h+k,\ |\beta| = h \qquad (\beta,\rho\in I(n))$$

*does not depend on $\beta$.*
*Then $\varphi\in t^{h+k,1}(A)$.*

**4.2. Part II (with enhanced density assumption).**

THEOREM 4.3. *Let be given: a measurable subset $A$ of $\mathbb{R}^n$, a point $x \in \mathbb{R}^n$, a measurable function $u$ defined in a neighborhood of $x$ and a family $\{v_\alpha \in t^{k,1}(x) \mid \alpha \in I(n), |\alpha| = h\}$ (with $h, k \geq 1$). Assume that:*

*(i) $x$ is a $(n+h+k)$-density point of $A$;*

*(ii) There is a constant $C$ such that $\int_{B(x,r)\setminus A} |u| \leq C\mathcal{L}^n(B(x,r)\setminus A)$ provided $r$ is small enough;*

*(iii) There exists a function $\varphi$ of class $C^h$ in a neighborhood of $x$ such that*

$$u = \varphi, \quad v_\beta = D^\beta \varphi$$

*a.e. in $A \cap B(x,r)$, for a certain positive $r$ and for all $\beta \in I(n)$ with $|\beta| = h$ ("jet-connectedness condition");*

*(iv) The number $(D^{\rho-\beta} P_{v_\beta,x,k})(x)$, with*

$$h+1 \leq |\rho| \leq h+k \quad \text{and} \quad |\beta| = h \qquad (\beta, \rho \in I(n)),$$

*does not depend on $\beta$ ("compatibility condition at $x$"). It will be denoted by $d_{x,\rho}$.*

*Then $u, \varphi \in t^{h+k,1}(x)$ and*

$$P_{u,x,h+k}(y) = P_{\varphi,x,h+k}(y) = P_{\varphi,x,h}(y) + \sum_{l=h+1}^{h+k} \sum_{\substack{\rho \in I(n) \\ |\rho|=l}} \frac{d_{x,\rho}}{\rho!}(y-x)^\rho.$$

*Proof.* For $r$ small enough and for all $\beta \in I(n)$ with $|\beta| = h$, one has

$$\begin{aligned}\mathcal{L}^n(B(x,r)) \fint_{B(x,r)} |D^\beta\varphi - P_{v_\beta,x,k}| &= \int_{B(x,r)\cap A} |v_\beta - P_{v_\beta,x,k}| + \int_{B(x,r)\setminus A} |D^\beta\varphi - P_{v_\beta,x,k}| \\ &\leq \int_{B(x,r)} |v_\beta - P_{v_\beta,x,k}| + C\,\mathcal{L}^n(B(x,r)\setminus A)\end{aligned}$$

where $C$ does not depend on $r$ and $\beta$. Since $v_\beta \in t^{k,1}(x)$ and by (i), we get at once

$$\fint_{B(x,r)} |D^\beta\varphi - P_{v_\beta,x,k}| = o(r^k)$$

namely $D^\beta\varphi \in t^{k,1}(x)$ and

$$P_{D^\beta\varphi,x,k} = P_{v_\beta,x,k}.$$

From Theorem 4.1 it follows that $\varphi \in t^{h+k,1}(x)$ and

$$P_{\varphi,x,h+k}(y) = P_{\varphi,x,h}(y) + \sum_{l=h+1}^{h+k} \sum_{\substack{\rho \in I(n) \\ |\rho|=l}} \frac{d_{x,\rho}}{\rho!}(y-x)^\rho.$$

Now it is easy to verify that $u \in t^{h+k,1}(x)$. Indeed this inequality holds (for $r$ small enough)

$$\begin{aligned}\int_{B(x,r)} |u - P_{\varphi,x,h+k}| &\leq \int_{B(x,r)} |u-\varphi| + \int_{B(x,r)} |\varphi - P_{\varphi,x,h+k}| \\ &= \int_{B(x,r)\setminus A} |u-\varphi| + o(r^{n+h+k}) \\ &\leq \int_{B(x,r)\setminus A} |u| + \int_{B(x,r)\setminus A} |\varphi| + o(r^{n+h+k})\end{aligned}$$

and the conclusion follows by recalling assumptions (i) and (ii). ◻

Corollary 4.1. *Let be given $u \in L^1(\mathbb{R}^n)$, a closed subset $A$ of $\mathbb{R}^n$ and a compact subset $K$ of $A$. Suppose that:*

*(i)* $u \in t^{h,1}(A)$;

*(ii)* $\mathcal{L}^n(B(x,r)\backslash A) = o(r^{n+h+k})$, *as $r \to 0$, uniformly with respect to $x \in K$ (where $h,k \in \mathbb{N}\backslash\{0\}$);*

*(iii) There are $r_0, C_0 > 0$ such that*

$$\int_{B(x,r)\backslash A} |u| \le C_0\, \mathcal{L}^n(B(x,r)\backslash A)$$

*for all $x \in K$ and $r \in [0, r_0]$;*

*(iv) For all $\beta \in I(n)$ with $|\beta| = h$, there exists $v_\beta \in t^{k,1}(K)$ such that*

$$v_\beta(x) = (D^\beta P_{u,x,h})(x)$$

*for a.e. $x \in A$. Moreover the number $(D^{\rho-\beta} P_{v_\beta,x,k})(x)$, with $x \in K$ and*

$$h+1 \le |\rho| \le h+k,\ |\beta| = h \qquad (\beta, \rho \in I(n))$$

*does not depend on $\beta$.*

*Then $u \in t^{h+k,1}(K)$, hence there exists $\psi \in C^{h+k}(\widetilde{K})$, with $\widetilde{K} := \{z \in \mathbb{R}^n \mid dist(z,K) < 1\}$, such that*

$$D^\alpha \psi(x) = (D^\alpha P_{u,x,h+k})(x)$$

*for all $x \in K$ and $\alpha \in I(n)$ with $0 \le |\alpha| \le h+k$.*

*Proof.* By Theorem 2.1 there is $\varphi \in C^h(\widetilde{A})$, with $\widetilde{A} := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x,A) < 1\}$, such that

$$D^\alpha \varphi(x) = (D^\alpha P_{u,x,h})(x) \tag{4.2}$$

for all $x \in A$ and $\alpha \in I(n)$ with $0 \le |\alpha| \le h$. Since $K$ is compact and by (iv) and (4.2), we obtain

$$\begin{aligned}\int_{B(x,r)} |D^\beta \varphi - P_{v_\beta,x,k}| &= \int_{B(x,r)\cap A} |D^\beta \varphi - P_{v_\beta,x,k}| + \int_{B(x,r)\backslash A} |D^\beta \varphi - P_{v_\beta,x,k}| \\ &\le \int_{B(x,r)} |v_\beta - P_{v_\beta,x,k}| + C\mathcal{L}^n(B(x,r)\backslash A)\end{aligned}$$

for all $x \in K$, for $r$ small enough (uniformly w.r.t. $x$) and for all $\beta \in I(n)$ with $|\beta| = h$, where $C$ does not depend on $r, x$. Hence

$$D^\beta \varphi \in t^{k,1}(K), \qquad P_{D^\beta\varphi,x,k} = P_{v_\beta,x,k} \quad (x \in K;\, \beta \in I(n), |\beta| = h).$$

From Theorem 4.2 it follows that $\varphi \in t^{h+k,1}(K)$. Finally (use (4.2) and compare the last lines in the proof of Theorem 4.3)

$$\int_{B(x,r)} |u - P_{\varphi,x,h+k}| \le \int_{B(x,r)\backslash A} |u| + \int_{B(x,r)\backslash A} |\varphi| + \int_{B(x,r)} |\varphi - P_{\varphi,x,h+k}|$$

for all $x \in K$. Hence the conclusion follows by assumptions (ii), (iii) and by the compactness of $K$. ◻

**5. Appendix.** The argument used for [4, Theorem 2.1] can be easily adapted to prove the next result. For the convenience of the reader, we provide here such a slightly modified proof.

THEOREM 5.1. *Let $U$ be an open subset of $\mathbb{R}^n$, $f \in Lip\,(U)$ and $\Psi \in Lip\,(U, \mathbb{R}^n)$. Consider a point $x_0 \in \mathbb{R}^n$ such that:*
*(i) $x_0$ is a $(n+1)$-density point of $\{x \in U \mid f$ is differentiable at $x$, $\nabla f(x) = \Psi(x)\}$;*
*(ii) $x_0$ is in the Lebesgue set of $curl\,\Psi$;*
*(iii) $\Psi$ is differentiable at $x_0$.*
*Then one has $curl\,\Psi(x_0) = 0$.*

*Proof.* Let $\Psi_i$ denote the $i$-th component of $\Psi$. Then we are reduced to prove that, for $i, j \in \{1, \ldots, n\}$ and $i \neq j$, the following crossed derivative condition holds

$$D_i\Psi_j(x_0) - D_j\Psi_i(x_0) = 0. \tag{5.1}$$

To this aim, given $\rho \in (0,1)$, consider $\varphi \in C_c^2(B(0,1))$ such that

$$0 \le \varphi \le 1, \qquad \varphi|B(0,\rho) \equiv 1$$

and

$$|D_h\varphi| \le \frac{2}{1-\rho} \qquad (h = 1, \ldots, n).$$

For $r > 0$ and $x \in \mathbb{R}^n$, define

$$\varphi_r(x) := \varphi\left(\frac{x - x_0}{r}\right)$$

and observe that

$$D_h\varphi_r(x) = \frac{1}{r} D_h\varphi\left(\frac{x - x_0}{r}\right)$$

hence

$$|D_h\varphi_r| \le \frac{2}{r(1-\rho)}. \tag{5.2}$$

If for simplicity we set

$$\Lambda := D_i\Psi_j - D_j\Psi_i, \qquad B_r := B(x_0, r)$$

then

$$\begin{aligned}
\int_{B_r} \Lambda\varphi_r &= \int_{B_r} \Psi_i D_j\varphi_r - \Psi_j D_i\varphi_r \\
&= \int_{B_r \setminus K} \Psi_i D_j\varphi_r - \Psi_j D_i\varphi_r + \int_{B_r \cap K} D_i f\, D_j\varphi_r - D_j f\, D_i\varphi_r \\
&= \int_{B_r \setminus K} (\Psi_i - D_i f) D_j\varphi_r + (D_j f - \Psi_j) D_i\varphi_r + \int_{B_r} D_i f\, D_j\varphi_r - D_j f\, D_i\varphi_r
\end{aligned}$$

where

$$\int_{B_r} D_i f\, D_j \varphi_r - D_j f\, D_i \varphi_r = - \int_{B_r} f\,(D_i D_j \varphi_r - D_j D_i \varphi_r) = 0$$

by the Schwartz theorem. Thus

$$\int_{B_r} \Lambda \varphi_r = \int_{B_r \backslash K} (\Psi_i - D_i f) D_j \varphi_r + (D_j f - \Psi_j) D_i \varphi_r .$$

It follows from (5.2) that there exists a constant $C$, not depending on $r$ and $\rho$, such that

$$\left| \int_{B_r} \Lambda \varphi_r \right| \leq \frac{C}{r(1-\rho)} \mathcal{L}^n (B_r \backslash K).$$

On the other hand

$$\left| \int_{B_r} \Lambda \varphi_r \right| \geq \left| \int_{B_{\rho r}} \Lambda \right| - \left| \int_{B_r \backslash B_{\rho r}} \Lambda \varphi_r \right|$$

hence there are constants $C_1$ and $C_2$, which do not depend on $r$ and $\rho$, such that

$$\begin{aligned} \rho^n \left| \fint_{B_{\rho r}} \Lambda \right| &\leq \left| \fint_{B_r} \Lambda \varphi_r \right| + \frac{1}{\mathcal{L}^n(B_r)} \left| \int_{B_r \backslash B_{\rho r}} \Lambda \varphi_r \right| \\ &\leq C_1 (1-\rho)^{-1} \frac{\mathcal{L}^n (B_r \backslash K)}{r^{n+1}} + C_2 \frac{r^n - (\rho r)^n}{r^n} \\ &= C_1 (1-\rho)^{-1} \frac{\mathcal{L}^n (B_r \backslash K)}{r^{n+1}} + C_2 (1 - \rho^n). \end{aligned}$$

Passing to the limit for $r \downarrow 0$, we obtain

$$\rho^n \Lambda(x_0) \leq C_2 (1 - \rho^n).$$

Finally the arbitrariness of $\rho \in (0,1)$ yields at once $\Lambda(x_0) = 0$, that is just (5.1). □

This corollary of Theorem 5.1 holds.

Corollary 5.1. *Let $U$ be an open set in $\mathbb{R}^n$, $h \geq 2$ and $\varphi \in C^{h-1,1}(U)$. Then*

$$D_i D_j (D^\beta \varphi) = D_j D_i (D^\beta \varphi) \text{ a.e. in } U$$

*for all $i, j = 1, \ldots, n$ and $\beta \in I(n)$ with $|\beta| = h - 2$.*

*Proof.* Let $\beta \in I(n)$ with $|\beta| = h-2$. Then

$$f := D^\beta \varphi \in C^{1,1}(U) \subset \mathrm{Lip}\,(U), \qquad \Psi := \nabla (D^\beta \varphi) \in \mathrm{Lip}\,(U, \mathbb{R}^n)$$

and

$$\{ x \in U \mid f \text{ is differentiable at } x,\ \nabla f(x) = \Psi(x) \} = U.$$

The conclusion follows at once from Theorem 5.1. □

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