

BIFURCATIONS OF WAVEFRONTS ON r -CORNERS: SEMI-LOCAL CLASSIFICATION*

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Abstract. We introduce the notion of *multi-reticular Legendrian unfoldings* in order to investigate stabilities and a genericity of bifurcations of wavefronts generated by m points of a hypersurface with a boundary, a corner, or an r -corner in a smooth n dimensional manifold. We define several stabilities of multi-reticular Legendrian unfoldings and prove that they and the stabilities of corresponding generating families are all equivalent and give the classification of all generic bifurcations of their wavefronts in the cases $r = 0, n \leq 5$ and $r = 1, n \leq 3$ respectively.

Key words. Wavefront, bifurcation, Legendrian unfolding, r -corner.

AMS subject classifications. 58K25, 58K40, 53Dxx.

1. Introduction. In [3] K.Jänich explained the wavefront propagation mechanism on a manifold which is completely described by a positive and positively homogeneous *Hamiltonian function* on the cotangent bundle and investigated the local gradient models given by the ray length function. Caustics and Wavefronts generated by an initial wavefront which is a hypersurface germ without boundary in the manifold were investigated as Lagrangian and Legendrian singularities by V.I.Arnold (cf., [1]).

In this paper and its prequel [7], we investigate the stabilities and the genericity of bifurcations of wavefronts generated by a hypersurface germ with *an r -corner*. Wavefronts generated by all edges of the hypersurface germ at a time give a *contact regular r -cubic configuration* on the 1-jet bundle. All wavefronts around a time give a one-parameter family of contact regular r -cubic configurations on the 1-jet bundle. In order to consider such families, we shall introduce the notion of *unfolded contact regular r -cubic configurations* on the big 1-jet bundle. A wavefront of an unfolded contact regular r -cubic configuration is the big front of the corresponding one-parameter family of contact regular r -cubic configurations. We shall consider their generating families and equivalence relations.

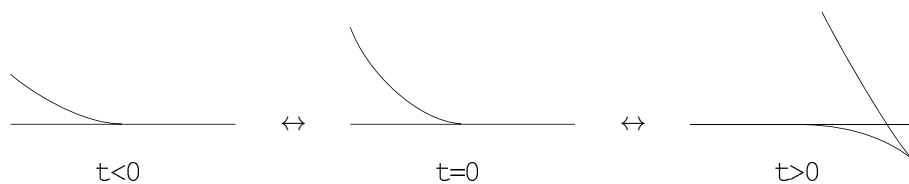


FIG. 1. *The generic bifurcation of wavefronts on a boundary 1B_3*

We also consider bifurcations of intersections of wavefronts generated by m points of the hypersurface. We introduce the notion of *multi-reticular Legendrian unfoldings* which is a generalized notion of *multi-Legendrian unfoldings* given by S.Izumiya (cf., [2]) for our situation. We shall define several stabilities of multi-reticular Legendrian

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unfoldings and prove that they and the stabilities of their generating families are all equivalent. We shall also classify generic multi-reticular Legendrian unfoldings and give all figures of their wavefronts in the case $r = 1, n \leq 3, m = 1$ and $n \leq 3, m \geq 2$.

In history, our theory in the case $r = 0$ is investigated as the theory of (multi-)Legendrian unfoldings by S.Izumiya (cf., [2]) and the classification list of generic Legendrian unfoldings is given by using V.M.Zakalyukin's theory (cf., [10]) in which he has classified generic *quasihomogeneous function germs* under *the $t\text{-}\mathcal{P}\text{-}\mathcal{K}$ -equivalence*. In [7] we have classified not only quasihomogeneous function germs but also all smooth function germs under *the reticular $t\text{-}\mathcal{P}\text{-}\mathcal{K}$ -equivalence* which is a generalized relation of the $t\text{-}\mathcal{P}\text{-}\mathcal{K}$ -equivalence.

I.G.Scherbak has studied the theory of boundary fronts in [4] and this corresponds to our theory in the case $r = m = 1$. She has introduced the notion of Legendrian pairs which is corresponding to the notion of regular 1-cubic configurations. But they are not strictly formulated and no proof is given. In this paper, we shall define and prove our theory strictly. Since the equivalence relation of function germ used in her paper is slightly different from the one which is used in this paper (see the remark in Section 2), the figures of wavefronts 0B_2 of fig.2 and 1B_3 of fig.3(cf., Figure 1), and 0B_3 of fig.4 in [4, p.365] do not coincide with our figures respectively. The classification list of function germs also different from ours (compare [4, p.371 Proposition 3] with [7, Theorem 4.7]).

For example we consider the below figures in which the initial wavefront with a boundary in a plane and wavefronts generated by the initial wavefront and the boundary to normal directions are described respectively. The generated wavefronts

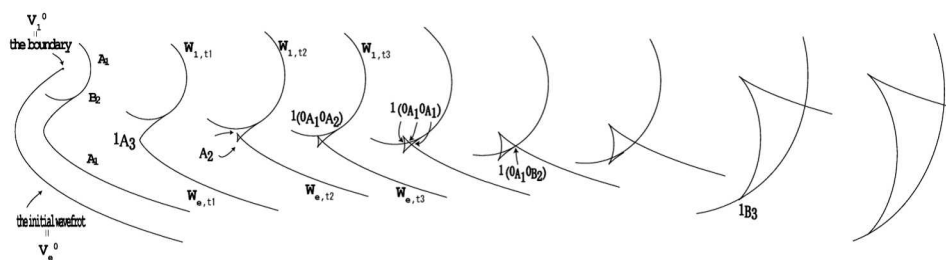


FIG. 2. The initial wavefront V^0 with a boundary and generated wavefronts ($e = \emptyset, t_1 < t_2 < t_3$)

bifurcate and typical shapes of bifurcations occur. The 1B_3 front occurs (compare this to Figure 1). The ${}^1(0A_1^0A_2)$, ${}^1(0A_1^0A_1)$, and ${}^1(0A_1^0B_2)$ fronts are intersections of fronts generated by different points of the initial wavefront. The purpose of this paper is the classification of such generic bifurcations of wavefronts and their intersections.

This paper consists of six sections. In Section 2 we give the review of stabilities under the reticular $t\text{-}(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalence relation of map germs which play important roles as generating families of multi-reticular Legendrian unfoldings (cf., [7]). We shall also give the review of the theory of contact regular r -cubic configurations. In Section 3 we shall introduce the notion of multi-reticular Legendrian unfoldings and consider their generating families. In Section 4 we shall investigate several stabilities of multi-reticular Legendrian unfoldings. In Section 5 we shall reduce our investigation to finite dimensional jet spaces and give the classification of generic multi-reticular Legendrian unfoldings in the cases $r = 0$ and 1 respectively. In Section 6 we shall

show that our method do not work well for the cases $r \geq 2$ because of modalities of generating families. All maps considered here are differentiable of class C^∞ unless stated otherwise.

2. Stabilities of unfoldings. In this section we investigate the theory of map germs with respect to the *reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalence relation* which is proved by almost parallel methods of [7].

Let $\mathbb{H}^r = \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid x_1 \geq 0, \dots, x_r \geq 0\}$ be an r -corner. We denote by $\mathcal{E}(r; k_1, r; k_2)$ the set of all germs at 0 of smooth maps $\mathbb{H}^r \times \mathbb{R}^{k_1} \rightarrow \mathbb{H}^r \times \mathbb{R}^{k_2}$ and set $\mathfrak{M}(r; k_1, r; k_2) = \{f \in \mathcal{E}(r; k_1, r; k_2) \mid f(0) = 0\}$. We denote $\mathcal{E}(r; k_1, k_2)$ for $\mathcal{E}(r; k_1, 0; k_2)$ and denote $\mathfrak{M}(r; k_1, k_2)$ for $\mathfrak{M}(r; k_1, 0; k_2)$.

If $k_2 = 1$ we write simply $\mathcal{E}(r; k)$ for $\mathcal{E}(r; k, 1)$ and $\mathfrak{M}(r; k)$ for $\mathfrak{M}(r; k, 1)$. Then $\mathcal{E}(r; k)$ is an \mathbb{R} -algebra in the usual way and $\mathfrak{M}(r; k)$ is its unique maximal ideal. We also denote by $\mathcal{E}(k)$ for $\mathcal{E}(0; k)$ and $\mathfrak{M}(k)$ for $\mathfrak{M}(0; k)$. We remark that $\mathcal{E}(r; k, p)$ is an $\mathcal{E}(r; k)$ -module generated by p -elements.

We denote by $J^l(r + k, p)$ the set of l -jets at 0 of germs in $\mathcal{E}(r; k, p)$. There are natural projections:

$$\pi_l : \mathcal{E}(r; k, p) \longrightarrow J^l(r + k, p), \quad \pi_{l_2}^{l_1} : J^{l_1}(r + k, p) \longrightarrow J^{l_2}(r + k, p) \quad (l_1 > l_2).$$

We write $j^l f(0)$ for $\pi_l(f)$ for each $f \in \mathcal{E}(r; k, p)$.

Let $(x, y) = (x_1, \dots, x_r, y_1, \dots, y_k)$ be a fixed coordinate system of $(\mathbb{H}^r \times \mathbb{R}^k, 0)$. We denote by $\mathcal{B}(r; k)$ the group of diffeomorphism germs $(\mathbb{H}^r \times \mathbb{R}^k, 0) \rightarrow (\mathbb{H}^r \times \mathbb{R}^k, 0)$ of the form:

$$\phi(x, y) = (x_1 \phi_1^1(x, y), \dots, x_r \phi_1^r(x, y), \phi_2^1(x, y), \dots, \phi_2^k(x, y)).$$

We denote by $\mathcal{B}_n(r; k + n)$ the group of diffeomorphism germs $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0) \rightarrow (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$ of the form:

$$\phi(x, y, u) = (x_1 \phi_1^1(x, y, u), \dots, x_r \phi_1^r(x, y, u), \phi_2^1(x, y, u), \dots, \phi_2^k(x, y, u), \phi_3^1(u), \dots, \phi_3^n(u)).$$

We denote $\phi(x, y, u) = (x\phi_1(x, y, u), \phi_2(x, y, u), \phi_3(u))$, $\frac{\partial f_0}{\partial y} = (\frac{\partial f_0}{\partial y_1}, \dots, \frac{\partial f_0}{\partial y_k})$, and denote other notations analogously.

LEMMA 2.1. (cf., [9, Corollary 1.8]) *Let B be a submodule of $\mathcal{E}(r; k + n + m', m)$, A_1 be a finitely generated $\mathcal{E}(m')$ -submodule of $\mathcal{E}(r; k + n + m', m)$ generated d -elements, and A_2 be a finitely generated $\mathcal{E}(n + m')$ submodule of $\mathcal{E}(r; k + n + m', m)$. Suppose*

$$\mathcal{E}(r; k + n + m', m) = B + A_2 + A_1 + \mathfrak{M}(m')\mathcal{E}(r; k + n + m', m) + \mathfrak{M}(n + m')^{d+1}\mathcal{E}(r; k + n + m', m).$$

Then

$$\mathcal{E}(r; k + n + m', m) = B + A_2 + A_1,$$

$$\mathfrak{M}(n + m')^d \mathcal{E}(r; k + n + m', m) \subset B + A_2 + \mathfrak{M}(m')\mathcal{E}(r; k + n + m', m).$$

We say that $f_0 = (f_{0,1}, \dots, f_{0,m})(x, y), g_0 = (g_{0,1}, \dots, g_{0,m})(x, y) \in \mathcal{E}(r; k, m)$ are *reticular $\mathcal{K}_{(m)}$ -equivalent* if there exist $\Phi_i \in \mathcal{B}_n(r; k + n)$ and units $a_i \in \mathcal{E}(r; k + n)$ such that $g_{0,i} = a_i \cdot f_{0,i} \circ \Phi_i$ for $i = 1, \dots, m$. We denote $\Phi = (\Phi_1, \dots, \Phi_m)$, $a = (a_1, \dots, a_m)$ and we call (Φ, a) a *reticular $\mathcal{K}_{(m)}$ -isomorphism* from f_0 to g_0 . We remark that f_0 and g_0 are *reticular $\mathcal{K}_{(m)}$ -equivalent* if and only if $f_{0,i}$ and $g_{0,i}$ are *reticular \mathcal{K} -equivalent*

for $i = 1, \dots, m$.

REMARK. The corresponding equivalence relation of function germs is given by I.G.Scherbak (see [4, p.366 §2]) as follows: Function germs $f_0, g_0 \in \mathcal{E}(1+k)$ are equivalent if there exist a diffeomorphism germ ϕ on $(\mathbb{R}^{1+k}, 0)$ of the form $\phi(x, y) = (x\phi_1(x, y), \phi_2^1(x, y), \dots, \phi_2^k(x, y))$ and a unit $a \in \mathcal{E}(r; k)$ such that $g_0 = a \cdot f_0 \circ \phi$. The variable x is defined on $(\mathbb{H}, 0)$ and $\phi_1(0) > 0$ in our equivalence relation. On the other hand x is defined on $(\mathbb{R}, 0)$ and the condition $\phi_1(0) > 0$ is not required in I.G.Scherbak's equivalence relation. These differences appear in the definition of wavefronts and consequently the figures of wavefronts in [4] and the figures in this paper are different to each other.

LEMMA 2.2. Let $f_0(x, y) \in \mathfrak{M}(r; k, m)$ and $z = j^l f_0(0)$. Let $O_{r\mathcal{K}}^l(z)$ be the submanifold of $J^l(r+k, m)$ consist of the image by π_l of the orbit under the reticular $\mathcal{K}_{(m)}$ -equivalence of f_0 . Put $z = j^l f_0(0)$. Then

$$T_z(O_{r\mathcal{K}}^l(z)) = \pi_l(\langle \langle f_{0,1}, x \frac{\partial f_{0,1}}{\partial x} \rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r; k) \langle \frac{\partial f_{0,1}}{\partial y} \rangle \rangle \times \dots \times \langle \langle f_{0,m}, x \frac{\partial f_{0,m}}{\partial x} \rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r; k) \langle \frac{\partial f_{0,m}}{\partial y} \rangle \rangle).$$

We say that a map germ $f_0 = (f_{0,1}, \dots, f_{0,m})(x, y) \in \mathfrak{M}(r; k, m)$ is *reticular $\mathcal{K}_{(m)}$ - l -determined* if all map germ in $\mathfrak{M}(r; k, m)$ which has the same l -jet of f_0 is reticular $\mathcal{K}_{(m)}$ -equivalent to f_0 .

LEMMA 2.3. (cf., [7, Lemma 2.3]) Let $f_0 = (f_{0,1}, \dots, f_{0,m})(x, y) \in \mathfrak{M}(r; k, m)$ and let

$$\mathfrak{M}(r; k)^{l+1} \subset \mathfrak{M}(r; k) \langle \langle f_{0,i}, x \frac{\partial f_{0,i}}{\partial x_1} \rangle + \mathfrak{M}(r; k) \langle \frac{\partial f_{0,i}}{\partial y} \rangle \rangle + \mathfrak{M}(r; k)^{l+2}$$

for $i = 1, \dots, m$. Then f_0 is reticular $\mathcal{K}_{(m)}$ - l -determined. Conversely if $f_0(x, y) \in \mathfrak{M}(r; k, m)$ is reticular $\mathcal{K}_{(m)}$ - l -determined, then

$$\mathfrak{M}(r; k)^{l+1} \subset \langle f_{0,i}, x \frac{\partial f_{0,i}}{\partial x_1} \rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r; k) \langle \frac{\partial f_{0,i}}{\partial y} \rangle \text{ for } i = 1, \dots, m.$$

We say that $f = (f_1, \dots, f_m)(x, y, u), g = (g_1, \dots, g_m)(x, y, u) \in \mathcal{E}(r; k+n, m)$ are *reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent* if there exist $\Phi_i \in \mathcal{B}_n(r; k+n)$ of the form:

$$\Phi_i(x, y, u) = (x\phi_1^i(x, y, t, u), \phi_2^i(x, y, t, u), \phi_3(u))$$

and a unit $a = (a_1, \dots, a_m)(x, y, u) \in \mathcal{E}(r; k+n, m)$ such that $g_i = a_i \cdot f_i \circ \Phi_i$ for $i = 1, \dots, m$. We write $\Phi = (\Phi_1, \dots, \Phi_m)$ and call (Φ, a) a reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -isomorphism from f to g .

We say that a map germ $f = (f_1, \dots, f_m)(x, y, u) \in \mathfrak{M}(r; k+n, m)$ is *reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - l -determined* if all map germ in $\mathfrak{M}(r; k+n, m)$ which has the same l -jet of f is reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent to f .

For $f = (f_1, \dots, f_m)(x, y, u) \in \mathcal{E}(r; k + n, m)$ we set

$$\begin{aligned} O_{r\mathcal{P}\text{-}\mathcal{K}(m)}(f) &= \langle \langle f_1, x \frac{\partial f_1}{\partial x} \rangle_{\mathcal{E}(r; k+n)} + \mathfrak{M}(r; k+n) \langle \frac{\partial f_1}{\partial y} \rangle \rangle \times \cdots \times \langle \langle f_m, x \frac{\partial f_m}{\partial x} \rangle_{\mathcal{E}(r; k+n)} \\ &\quad + \mathfrak{M}(r; k+n) \langle \frac{\partial f_m}{\partial y} \rangle \rangle + \mathfrak{M}(n) \langle \frac{\partial f}{\partial u} \rangle. \end{aligned}$$

LEMMA 2.4. *Let $f = (f_1, \dots, f_m)(x, y, u) \in \mathcal{E}(r; k + n, m)$ and $z = j^l f(0)$. We write $O_{r\mathcal{P}\text{-}\mathcal{K}(m)}^l(z)$ the submanifold which consists of l -jets of the orbit of f under the reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ equivalence relation. Then it holds that*

$$O_{r\mathcal{P}\text{-}\mathcal{K}(m)}^l(z) = \pi_l(O_{r\mathcal{P}\text{-}\mathcal{K}(m)}(f)).$$

LEMMA 2.5. (cf., [7, Lemma 3.10]) *Let $f = (f_1, \dots, f_m)(x, y, u) \in \mathfrak{M}(r; k + n, m)$ and l be a non-negative integer. If*

$$(1) \quad \mathfrak{M}(r; k + n, m)^l \subset O_{r\mathcal{P}\text{-}\mathcal{K}(m)}(f) + \mathfrak{M}(n)\mathfrak{M}(r; k + n, m)^l,$$

then f is reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - l -determined.

In convenience, we denote an unfolding of a function germ $f(x, y, u) \in \mathfrak{M}(r; k + n, m)$ by $F(x, y, t, u) \in \mathfrak{M}(r; k + m' + n, m)$.

Let $F(x, y, t, u) \in \mathfrak{M}(r; k + m'_1 + n, m)$ and $G(x, y, s, u) \in \mathfrak{M}(r; k + m'_2 + n, m)$ be unfoldings of $f(x, y, u) \in \mathfrak{M}(r; k + n, m)$.

A reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - f -morphism from F to G is a pair (Φ, a) , where $\Phi = (\Phi_1, \dots, \Phi_m)$ for $\Phi_i \in \mathfrak{M}(r; k + m'_2 + n, r; k + m'_1 + n)$ and $a = (a_1, \dots, a_m)$ is a unit of $\mathcal{E}(r; k + m'_2 + n, m)$ satisfying the following conditions:

(1) Φ_i can be written in the form:

$$\Phi_i(x, y, t, u) = (x\phi_1^i(x, y, t, u), \phi_2^i(x, y, t, u), \phi_3(t), \phi_4(t, u)),$$

(2) $\Phi_i|_{\mathbb{H}^r \times \mathbb{R}^{k+n}} = id_{\mathbb{H}^r \times \mathbb{R}^{k+n}}, a_i|_{\mathbb{H}^r \times \mathbb{R}^{k+n}} \equiv 1$,

(3) $G_i(x, y, t, u) = a_i(x, y, t, u) \cdot F_i \circ \Phi_i(x, y, t, u)$ for all $(x, y, t, u) \in (\mathbb{H}^r \times \mathbb{R}^{k+m'_2+n}, 0)$.

If there exists a reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - f -morphism from F to G , we say that G is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - f -induced from F . If $m_1 = m_2$ and Φ is invertible, we call (Φ, a) a reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - f -isomorphism from F to G and we say that F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - f -equivalent to G .

We say that $F(x, y, t, u), G(x, y, t, u) \in \mathcal{E}(r; k + m' + n, m)$ are reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent if there exist diffeomorphism germs

$$\Phi_i : (\mathbb{H}^r \times \mathbb{R}^{k+m'+n}, 0) \rightarrow (\mathbb{H}^r \times \mathbb{R}^{k+m'+n}, 0) \quad (i = 1, \dots, m)$$

of the form

$$\Phi_i(x, y, t, q, z) = (x\phi_1^i(x, y, t, u), \phi_2^i(x, y, t, u), \phi_3(t), \phi_4(t, u))$$

and a unit $a = (a_1, \dots, a_m) \in \mathcal{E}(r; k + m' + n, m)$ such that $G_i = a_i \cdot F_i \circ \Phi_i$ for $i = 1, \dots, m$. We call (Φ, a) a reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -isomorphism from F to G .

We say that function germs $F_1(x_1, \dots, x_{r_1}, y_1, \dots, y_{k_1}, t, q, z) \in \mathfrak{M}(r_1; k_1 + m' + n + 1)$ and $F_2(x_1, \dots, x_{r_2}, y_1, \dots, y_{k_2}, t, q, z) \in \mathfrak{M}(r_2; k_2 + m' + n + 1)$ are *stably reticular t - \mathcal{P} - \mathcal{K} -equivalent* if F_1 and F_2 are reticular t - \mathcal{P} - \mathcal{K} -equivalent after additions of linear forms of x of which all coefficients are not zero and non-degenerate quadratic forms in the variables y .

We define the notion of *stable reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -equivalence* in the same method of the stable reticular t - \mathcal{P} - \mathcal{K} -equivalence.

DEFINITION 2.6. We define stabilities of unfoldings. Let $F = (F_1, \dots, F_m)(x, y, t, u) \in \mathfrak{M}(r; k + m' + n, m)$ be an unfolding of $f = (f_1, \dots, f_m)(x, y, u) \in \mathfrak{M}(r; k + n, m)$.

We say that F is *reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -stable* if the following condition holds: For any neighborhood U of 0 in $\mathbb{H}^r \times \mathbb{R}^{k+m'+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R}^m)$, there exists a neighborhood $N_{\tilde{F}}$ of \tilde{F} in C^∞ -topology such that for any $\tilde{G} \in N_{\tilde{F}}$ there exist $(0, y^i, t^0, u^0) \in U$ for $i = 1, \dots, m$ such that $G = (G_1, \dots, G_m)$ and F are reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -equivalent, where $G_i \in \mathfrak{M}(r; k + m' + n)$ is defined by $G_i(x, y, t, u) = \tilde{G}_i(x, y + y^i, t + t^0, u + u^0) - \tilde{G}_i(0, y^i, t^0, u^0)$.

We say that F is a *reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -versal unfolding* of f if any unfolding of f is reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ - f -induced from F . We say that F is a *reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -universal unfolding* of f if m is minimal in reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -versal unfoldings of f .

We say that F is *reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -infinitesimally versal* if

$$\begin{aligned} \mathcal{E}(r; k + n, m) &= \langle f_1, x \frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y} \rangle_{\mathcal{E}(r; k+n)} \times \cdots \times \\ &\quad \langle f_m, x \frac{\partial f_m}{\partial x}, \frac{\partial f_m}{\partial y} \rangle_{\mathcal{E}(r; k+n)} + \langle \frac{\partial f}{\partial u} \rangle_{\mathcal{E}(n)} + \langle \frac{\partial F}{\partial t} |_{t=0} \rangle_{\mathbb{R}} \end{aligned}$$

We say that F is *reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -infinitesimally stable* if

$$\begin{aligned} \mathcal{E}(r; k + m' + n, m) &= \langle F_1, x \frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y} \rangle_{\mathcal{E}(r; k+m'+n)} \times \cdots \times \\ &\quad \langle F_m, x \frac{\partial F_m}{\partial x}, \frac{\partial F_m}{\partial y} \rangle_{\mathcal{E}(r; k+m'+n)} + \langle \frac{\partial F}{\partial u} \rangle_{\mathcal{E}(m'+n)} + \langle \frac{\partial F}{\partial t} \rangle_{\mathcal{E}(m')} \end{aligned}$$

We say that F is *reticular t - $(\mathcal{P}-\mathcal{K})_{(m)}$ -homotopically stable* if for any smooth path-germ $(\mathbb{R}, 0) \rightarrow \mathcal{E}(r; k + m' + n, m), \tau \mapsto F_\tau = (F_{1,\tau}, \dots, F_{m,\tau})$ with $F_0 = F$, there exists a smooth path-germs $(\mathbb{R}, 0) \rightarrow \mathcal{B}(r; k + m' + n) \times \mathcal{E}(r; k + m' + n), \tau \mapsto (\Phi_\tau^i, a_\tau^i)$ with $(\Phi_0^i, a_0^i) = (id, 1)$ and Φ_τ^i has the form

$$\Phi_\tau^i(x, y, t, u) = (x\phi_\tau^{i,1}(x, y, t, u), \phi_\tau^{i,2}(x, y, t, u), \phi_\tau^3(t), \phi_\tau^4(t, u))$$

such that each (Φ_τ^i, a_τ^i) is a reticular t - \mathcal{P} - \mathcal{K} -isomorphism and $F_{i,\tau} = a_\tau^i \cdot F_{i,0} \circ \Phi_\tau^i$ for $\tau \in (\mathbb{R}, 0)$ and $i = 1, \dots, m$.

Let U be a neighborhood of 0 in $\mathbb{H}^r \times \mathbb{R}^{k+m'+n}$, $\tilde{F} = (\tilde{F}^1, \dots, \tilde{F}^m) : U \rightarrow \mathbb{R}^m$ be a smooth map, and l be a non-negative integer. We choose a neighborhood U' of 0 in

$\mathbb{R}^{km+m'+n}$ such that $(0, y^i, t, u) \in U$ for any $(y^1, \dots, y^m, t, u) \in U'$ and i . We define the smooth map germ

$$j_1^l \tilde{F} : U' \longrightarrow J^l(r + k + n, m)$$

as follows: For $(y^1, \dots, y^m, t, u) \in U$ we set $j_1^l \tilde{F}(y^1, \dots, y^m, t, u)$ by the l -jet of the map germ $(f_{(y^1, t, u)}^1, \dots, f_{(y^m, t, u)}^m) \in \mathfrak{M}(r; k + n, m)$ at 0, where $f_{(y^i, t, u)}^i$ is given by $f_{(y^i, t, u)}^i(x', y', u') = \tilde{F}^i(x', y^i + y', t, u + u') - \tilde{F}^i(0, y^i, t, u)$

Let $F(x, y, t, u) \in \mathfrak{M}(r; k + m' + n, m)$ be an unfolding of $f(x, y, u) \in \mathfrak{M}(r; k + n, m)$. Let l be a non-negative integer and $z = j^l f(0)$. We say that F is *reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - l -transversal* if $j_1^l \tilde{F}$ at 0 is transversal to $O_{r\mathcal{P}\text{-}\mathcal{K}_{(m)}}^l(z)$ for a representative $\tilde{F} \in C^\infty(U, \mathbb{R}^m)$ of F .

LEMMA 2.7. (cf., [7, Lemma 3.4]) *Let $F(x, y, t, u) \in \mathfrak{M}(r; k + m' + n, m)$ be an unfolding of $f(x, y, u) \in \mathfrak{M}(r; k + n, m)$. Then F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - l -transversal unfolding of f if and only if*

$$\begin{aligned} \mathcal{E}(r; k + n, m) &= \langle f_1, x \frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y} \rangle_{\mathcal{E}(r; k+n)} \times \dots \times \langle f_m, x \frac{\partial f_m}{\partial x}, \frac{\partial f_m}{\partial y} \rangle_{\mathcal{E}(r; k+n)} \\ &\quad + \langle \frac{\partial f}{\partial u} \rangle_{\mathcal{E}(n)} + \langle \frac{\partial F}{\partial t} |_{t=0} \rangle_{\mathbb{R}} + \mathfrak{M}(r; k + n)^{l+1} \mathcal{E}(r; k + n, m). \end{aligned}$$

THEOREM 2.8. (Uniqueness of universal unfoldings) (cf., [7, Theorem 3.13]) *Let $F(x, y, u, t), G(x, y, u, t) \in \mathfrak{M}(r; k + n + m', m)$ be unfoldings of $f \in \mathfrak{M}(r; k + n, m)$. If F and G are reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -universal, then F and G are reticular t - $\mathcal{P}\text{-}\mathcal{K}$ - f -equivalent.*

The following theorem is used to prove Theorem 4.6.

THEOREM 2.9. (cf., [7, Theorem 3.14]) *Let $F(x, y, t, u) \in \mathfrak{M}(r; k + m' + n, m)$ be an unfolding of $f(x, y, u) \in \mathfrak{M}(r; k + n, m)$ and let f is an unfolding of $f_0(x, y) \in \mathfrak{M}(r; k, m)$. Then following are equivalent.*

- (1) *There exists a non-negative number l such that f_0 is reticular $\mathcal{K}_{(m)}$ - l -determined and F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ - l' -transversal for $l' \geq lm + l + m'$,*
- (2) *F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -stable,*
- (3) *F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -versal,*
- (4) *F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -infinitesimally versal,*
- (5) *F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -infinitesimally stable,*
- (6) *F is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -homotopically stable.*

3. Reticular Legendrian unfoldings. In order to consider one-parameter families of Legendrian submanifold with an r -corner, we require the notion of *reticular Legendrian unfoldings*, their equivalence relation, and generating families.

We consider the big 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on that space. Let $(t, q) = (t, q_1, \dots, q_n)$ be the canonical coordinate system on $\mathbb{R} \times \mathbb{R}^n$ and $(t, q, z, s, p) = (t, q_1, \dots, q_n, z, s, p_1, \dots, p_n)$ be the corresponding coordinate system on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then the canonical 1-form Θ is given by

$$\Theta = dz - \sum_{i=1}^n p_i dq_i - s dt = \theta - s dt,$$

where θ is the canonical 1-form on $J^1(\mathbb{R}^n, \mathbb{R})$.

Let $\{L_{\sigma,t}\}_{\sigma \subset I_r, t \in (\mathbb{R}, 0)}$ be a family of contact regular r -cubic configurations on $J^1(\mathbb{R}^n, \mathbb{R})$ defined by a family of contact embedding germs $C_t : (J^1(\mathbb{R}^n, \mathbb{R}), 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$ depending smoothly on $t \in (\mathbb{R}, 0)$ such that $\pi \circ C_0(0) = 0$ and $L_{\sigma,t} = C_t(L_\sigma^0)$ for all $\sigma \subset I_r$ and $t \in (\mathbb{R}, 0)$.

Then we consider the following contact diffeomorphism germ C on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ ($\Pi(w) = 0$):

LEMMA 3.1. *For any family of contact embedding germs $C_t : (J^1(\mathbb{R}^n, \mathbb{R}), 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$ ($\pi \circ C_0(0) = 0$) depending smoothly on $t \in (\mathbb{R}, 0)$, there exists a unique function germ h on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that the map germ $C : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ defined by*

$$C(t, q, z, s, p) = (t, q \circ C_t(q, z, p), z \circ C_t(q, z, p), h(t, q, z, s, p), p \circ C_t(q, z, p))$$

is a contact diffeomorphism.

Proof. We denote $C_t(q, z, p) = (q_t(q, z, p), z_t(z, q, p), p_t(q, z, p))$. Since C_t is a contact embedding germ for all $t \in (\mathbb{R}, 0)$, there exists a function germ $\alpha(t, q, z, p)$ around zero with $\alpha(0) \neq 0$ such that $dz_t(q, z, p) - p_t(q, z, p)dq_t(q, z, p) = \alpha(t, q, z, p)(dz - pdq)$ for all fixed t . By the direct calculation of this equation, we have that

$$\frac{\partial z_t}{\partial z} - p_t \frac{\partial q_t}{\partial z} = \alpha, \quad \frac{\partial z_t}{\partial q} - p_t \frac{\partial q_t}{\partial q} = -p\alpha, \quad \frac{\partial z_t}{\partial p} - p_t \frac{\partial q_t}{\partial p} = 0.$$

We also calculate $C^*(dz - pdq - sdt)$ by considering the above relations. Then we have that

$$\begin{aligned} & C^*(dz - pdq - sdt) \\ &= dz_t(z, q, p) - p_t(q, z, p)dq_t(q, z, p) - h(t, q, z, s, p)dt \\ &= \alpha(t, z, q, p)dz - \alpha(t, z, q, p)pdq - \left(\frac{\partial z_t}{\partial t}(q, z, p) - p_t(q, z, p)\frac{\partial q_t}{\partial t}(q, z, p) - h(t, q, z, s, p)\right)dt. \end{aligned}$$

In order to make C a contact embedding, the function $h(t, q, z, s, p)$ is uniquely determined that:

$$(2) \quad h(t, q, z, s, p) = \frac{\partial z_t}{\partial t}(q, z, p) - p_t(q, z, p)\frac{\partial q_t}{\partial t}(q, z, p) + \alpha(t, q, z, p)s.$$

□

DEFINITION 3.2. *Let C be a contact embedding germ $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. We say that C is a \mathcal{P} -contact diffeomorphism if C has the form:*

$$(3) \quad C(t, q, z, s, p) = (t, q_C(t, q, z, p), z_C(t, q, z, p), h_C(t, q, z, s, p), p_C(t, q, z, p)).$$

We remark that a \mathcal{P} -contact diffeomorphism and the corresponding one-parameter family of contact embedding germs are uniquely defined by each other.

We define that $\tilde{L}_\sigma^{r,0} = \{(t, q, z, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \mid q_\sigma = p_{I_r - \sigma} = q_{r+1} = \dots = q_n = s = z = 0, q_{I_r - \sigma} \geq 0\}$ for $\sigma \subset I_r$ and $\mathbb{L}^r = \{(t, q, z, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \mid q_1 p_1 = \dots = q_r p_r = q_{r+1} = \dots = q_n = s = z = 0, q_{I_r} \geq 0\}$ be a representative as a germ of the union of $\tilde{L}_\sigma^{r,0}$ for all $\sigma \subset I_r$. We denote that the set $\tilde{L}_\sigma^{r,0}$ is the normalization of the particles incident from the σ -corner of the initial hypersurface germ with a r -corner at time t .

DEFINITION 3.3. We say that a map germ $\mathcal{L} : (\mathbb{L}^r, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ ($\Pi(w) = 0$) is a *reticular Legendrian unfolding* if \mathcal{L} is the restriction of a \mathcal{P} -contact diffeomorphism. We call $\{\mathcal{L}(\tilde{L}_\sigma^{r,0})\}_{\sigma \subset I_r}$ the *unfolded contact regular r -cubic configuration of \mathcal{L}* .

We note that: Let $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ be an unfolded contact regular r -cubic configuration associated with a one-parameter family of contact regular r -cubic configurations $\{L_{\sigma,t}\}_{\sigma \subset I_r, t \in (\mathbb{R}, 0)}$. Then there is the following relation between the wavefront $W_\sigma = \Pi(\tilde{L}_\sigma)$ and the family of wavefronts $W_{\sigma,t} = \pi(L_{\sigma,t})$:

$$W_\sigma = \bigcup_{t \in (\mathbb{R}, 0)} \{t\} \times W_{\sigma,t} \quad \text{for all } \sigma \subset I_r.$$

In order to study bifurcations of wavefronts of unfolded contact regular r -cubic configurations we introduce the following equivalence relation. Let Ψ be a contact diffeomorphism germ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$. We say that Ψ is a *reticular $\mathcal{P}_{(r)}$ -diffeomorphism* if $\pi_t \circ \Psi$ depends only on t and Ψ preserves $\tilde{L}_\sigma^{r,0}$ for all $\sigma \subset I_r$.

Let $\{\tilde{L}_\sigma^i\}_{\sigma \subset I_r}$ ($i = 1, 2$) be unfolded contact regular r -cubic configurations on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_i)$ ($\Pi(w_i) = 0$). We say that a contact diffeomorphism germ $K : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_1) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_2)$ is a *\mathcal{P} -Legendrian equivalence* if K has the form:

$$(4) \quad K(t, q, z, s, p) = (\phi_1(t), \phi_2(t, q, z), \phi_3(t, q, z), \phi_4(t, q, z, s, p), \phi_5(t, q, z, s, p)).$$

We say that they are *\mathcal{P} -Legendrian equivalent* if there exist a \mathcal{P} -Legendrian equivalence K such that $\tilde{L}_\sigma^2 = K(\tilde{L}_\sigma^1)$ for all $\sigma \subset I_r$.

In order to understand the meaning of \mathcal{P} -Legendrian equivalence, we observe the following: Let $\{\tilde{L}_\sigma^i\}_{\sigma \subset I_r}$ ($i = 1, 2$) be unfolded contact regular r -cubic configurations on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_i)$ ($\Pi(w_i) = 0$) and $\{L_{\sigma,t}^i\}_{\sigma \subset I_r, t \in (\mathbb{R}, 0)}$ be the corresponding one-parameter families of contact regular r -cubic configurations on $J^1(\mathbb{R}^n, \mathbb{R})$ respectively. We take the smooth path germs $w_i : (\mathbb{R}, 0) \rightarrow (J^1(\mathbb{R}^n, \mathbb{R}), 0)$ such that $\{L_{\sigma,t}^i\}_{\sigma \subset I_r}$ are defined at $w_i(t)$ for $i = 1, 2$. Suppose that there exists a \mathcal{P} -Legendrian equivalence K from $\{\tilde{L}_\sigma^1\}_{\sigma \subset I_r}$ to $\{\tilde{L}_\sigma^2\}_{\sigma \subset I_r}$ of the form (4). We set $W_{\sigma,t}^i$ be the wavefront of $L_{\sigma,t}^i$ for $\sigma \subset I_r$, $t \in (\mathbb{R}, 0)$ and $i = 1, 2$. We define the family of diffeomorphism $g_t : (\mathbb{R}^n \times \mathbb{R}, \pi(w_1(t))) \rightarrow (\mathbb{R}^n \times \mathbb{R}, \pi(w_2(t)))$ by $g_t(q, z) = (\phi_2(t, q, z), \phi_3(t, q, z))$. Then we have that $g_t(W_{\sigma,t}^1) = W_{\sigma, \phi_1(t)}^2$ for all $\sigma \subset I_r$, $t \in (\mathbb{R}, 0)$.

We also define the equivalence relation among reticular Legendrian unfoldings. Let $\mathcal{L}_i : (\mathbb{L}^r, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_i)$, ($i = 1, 2$) be reticular Legendrian unfoldings. We say that \mathcal{L}_1 and \mathcal{L}_2 are *\mathcal{P} -Legendrian equivalent* if there exist a \mathcal{P} -Legendrian equivalence K and a reticular r -diffeomorphism Ψ such that $K \circ \mathcal{L}_1 = \mathcal{L}_2 \circ \Psi$.

We remark that two reticular Legendrian unfoldings are \mathcal{P} -Legendrian equivalent if and only if the corresponding unfolded contact regular r -cubic configurations are \mathcal{P} -Legendrian equivalent.

By the same proof of Lemma 5.3 in [6], we have the following:

LEMMA 3.4. Let $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ be an unfolded contact regular r -cubic configuration on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ ($\Pi(w) = 0$). Then there exists a \mathcal{P} -contact diffeomorphism germ C on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that C defines $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ and preserves the canonical 1-form.

By this lemma we may assume that all reticular Legendrian unfoldings (and all unfolded contact regular r -cubic configurations) are defined by \mathcal{P} -contact diffeomorphism germs which preserve the canonical 1-form.

We can construct generating families of reticular Legendrian unfoldings. A function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; k + 1 + n + 1)$ is said to be \mathcal{P} - C -non-degenerate if $\frac{\partial F}{\partial x}(0) = \frac{\partial F}{\partial y}(0) = 0$ and $x, t, F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are independent on $(\mathbb{H}^k \times \mathbb{R}^{k+1+n+1}, 0)$.

A \mathcal{P} - C -non-degenerate function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; k + 1 + n + 1)$ is called a *generating family* of a reticular Legendrian unfoldings \mathcal{L} if

$$\mathcal{L}(\tilde{L}_\sigma^{r,0}) = \{(t, q, z, \frac{\partial F}{\partial t} / (-\frac{\partial F}{\partial z}), \frac{\partial F}{\partial q} / (-\frac{\partial F}{\partial z})) \in (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w) \mid x_\sigma = F = \frac{\partial F}{\partial x_{I_r - \sigma}} = \frac{\partial F}{\partial y} = 0, x_{I_r - \sigma} \geq 0\} \text{ for all } \sigma \subset I_r.$$

We remark that for a \mathcal{P} - C -non-degenerate function germ $F(x, y, t, q, z)$, the function germ $F(\cdot, \cdot, t, \cdot, \cdot)$ is C -non-degenerate (see [6, p.111]).

LEMMA 3.5. *Let C be a \mathcal{P} -contact diffeomorphism germ $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ ($\Pi(w) = 0$) which preserves the canonical 1-form. If the map germ*

$$(T, Q, Z, S, P) \rightarrow (T, Q, Z, s_C(T, Q, Z, S, P), p_C(T, Q, Z, S, P))$$

is diffeomorphism, then there exists a function germ $H(T, Q, p) \in \mathfrak{M}(1 + n + n)^2$ such that the canonical relation P_C associated with C has the form:

$$(5) \quad P_C = \{(T, Q, Z, -\frac{\partial H}{\partial T}(T, Q, p) + s, -\frac{\partial H}{\partial Q}, T, -\frac{\partial H}{\partial p}, H - \langle \frac{\partial H}{\partial p}, p + p^0 \rangle + s^0 T + Z, s + s^0, p + p^0)\},$$

and the function germ $F \in \mathfrak{M}(r; n + 1 + n + 1)$ defined by $F(x, y, t, q, z) = -z + H(t, x, 0, y) + \langle y + p^0, q \rangle + s^0 t$ is a generating family of the reticular Legendrian unfolding $C|_{\mathbb{L}}$.

Proof. We have that $dz - (s + s^0)dt - (p + p^0)dq = dZ - SdT - PdQ$ on P_C . It follows that $d(z - Z) = (s + s^0)dt + (p + p^0)dq - SdT - PdQ$ and $d(z - Z - (s + s^0)t - (p + p^0)q) = -tds - qdp - SdT - PdQ$. Then there exists a function germ $H'(T, Q, s, p) \in \mathfrak{M}(1 + n + 1 + 1 + n)^2$ such that

$$z - Z - (s + s^0)t - (p + p^0)q = H'(T, Q, s, p), t = -\frac{\partial H'}{\partial s}, q = -\frac{\partial H'}{\partial p}, S = -\frac{\partial H'}{\partial T}, P = -\frac{\partial H'}{\partial Q}$$

on P_C . Since $t = T = -\frac{\partial H'}{\partial s}$ on P_C , we have that $H'(T, Q, s, p) = H(T, Q, p) - Ts$ for some $H(T, Q, p) \in \mathfrak{M}(1 + n + n)^2$. Then we have that

$$z - Z - T(s + s^0) - \langle -\frac{\partial H}{\partial p}, p + p^0 \rangle = H(T, Q, p) - Ts.$$

It follows that

$$z = H(T, Q, p) - \langle \frac{\partial H}{\partial p}, p + p^0 \rangle + s^0 T + Z.$$

Then we have the required form of P_C . By the direct calculation with the form P_C , we have that F is a generating family of $C|_{\mathbb{L}^r}$. \square

We have the following theorem which gives the relations between reticular Legendrian unfoldings and their generating families.

- THEOREM 3.6.** (1) *For any reticular Legendrian unfolding $\mathcal{L} : (\mathbb{L}^r, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ ($\Pi(w) = 0$), there exists a function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; k+1+n+1)$ which is a generating family of \mathcal{L} .*
 (2) *For any \mathcal{P} - C -non-degenerate function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; k+1+n+1)$ with $\frac{\partial F}{\partial t}(0) = \frac{\partial F}{\partial q}(0) = 0$, there exists a reticular Legendrian unfolding $\mathcal{L} : (\mathbb{L}^r, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ ($\Pi(w) = 0$) of which F is a generating family.*
 (3) *Two reticular Legendrian unfolding are \mathcal{P} -Legendrian equivalent if and only if their generating families are stably reticular t - \mathcal{P} - \mathcal{K} -equivalent.*

This theorem is proved by analogous methods of [5], [6]. We give the sketch of the proof. We may assume that $w = 0$.

(1) Let C be a \mathcal{P} -contact diffeomorphism germ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that $C|_{\mathbb{L}} = \mathcal{L}$. We may assume that $C^*\Theta = \Theta$. By taking a \mathcal{P} -Legendrian equivalence of \mathcal{L} , we may assume that the canonical relation P_C associated with C has the form (5) for the function germ $H \in \mathfrak{M}(1+n+n)^2$. Then the function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; n+1+n+1)$ defined by

$$F(x, y, t, q, z) = -z + H(t, x_1, \dots, x_r, 0, y) + \langle y, q \rangle$$

is a generating family of \mathcal{L} .

(2) Let a \mathcal{P} - C -non-degenerate function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; k+1+n+1)$ with $\frac{\partial F}{\partial t}(0) = \frac{\partial F}{\partial q}(0) = 0$ be given. By [6, Lemma 2.1], we may assume that F has the form $F(x, y, t, q, z) = -z + F_0(x, y, t, q)$ for some $F_0 \in \mathfrak{M}(r; k+1+n)$. Choose an $(n-r) \times k$ -matrix A and an $(n-r) \times n$ -matrix B such that the matrix

$$(6) \quad \begin{pmatrix} \frac{\partial^2 F_0}{\partial x \partial y} & \frac{\partial^2 F_0}{\partial x \partial q} & \frac{\partial^2 F_0}{\partial x \partial t} \\ \frac{\partial^2 F_0}{\partial y \partial y} & \frac{\partial^2 F_0}{\partial y \partial q} & \frac{\partial^2 F_0}{\partial y \partial t} \\ A & B & 0 \\ 0 & 0 & 1 \end{pmatrix}_0 \text{ is invertible.}$$

Let $F' \in \mathfrak{M}(r+k+1+n+1)$ be a function germ which is obtained by an extension the source space of F to $(\mathbb{R}^{r+k+1+n+1}, 0)$. Define the function $G(S, Q, y, t, q, z) \in \mathfrak{M}(n+1+1+k+1+n+1)$ by that

$$G(Q, Z, S, y, t, q, z) = -z + F'(Q_1, \dots, Q_r, y, t, q) +$$

$$(Q_{r+1}, \dots, Q_n)A \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} + (Q_{r+1}, \dots, Q_n)B \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} + St.$$

Then G is a generating family of the canonical relation P_C associated with some \mathcal{P} -contact diffeomorphism germ C . The function germ F is a generating family of the reticular Legendrian unfolding $C|_{\mathbb{L}}$.

(3) We need only to prove that: *If $F_1, F_2 \in \mathfrak{M}(r; k + 1 + n)$ are generating families of the same reticular Legendrian unfolding, then they are reticular $t\text{-}\mathcal{P}\text{-}\mathcal{K}$ -equivalent.*

We may reduce that F_i has the form $F_i(x, y, t, q, z) = -z + F_i^0(x, y, t, q)$ for $F_i^0 \in \mathfrak{M}(r; k + 1 + n)$, $i = 1, 2$. Then F_1^0 and F_2^0 are generating families of the same reticular Lagrangian map in the sense of [5]. By [5, p.587 the assertion (3)], there exists a reticular \mathcal{R} -equivalence from F_2^0 to F_1^0 of the form:

$$F_1^0(x, y, t, q) = F_2^0(x\phi_1(x, y, t, q), \phi_2(x, y, t, q), t, q).$$

This means that F_1 and F_2 are reticular $t\text{-}\mathcal{P}\text{-}\mathcal{K}$ -equivalent. \square

4. Multi-reticular Legendrian unfoldings. Let $\mathcal{L}_i : (\mathbb{L}^{r_i}, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_i)$ ($i = 1, \dots, m$) be reticular Legendrian unfoldings with $\Pi(w_i) = 0$ where w_1, \dots, w_m are distinct. Then we call $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ a *multi-reticular Legendrian unfolding*.

Let $(\mathcal{L}_1, \dots, \mathcal{L}_m)$ and $(\mathcal{L}'_1, \dots, \mathcal{L}'_m)$ be multi-reticular Legendrian unfoldings. We say that they are $\mathcal{P}_{(m)}$ -Legendrian equivalent if there exist contact diffeomorphism germs

$$K_i : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_i) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w'_i) \quad (i = 1, \dots, m)$$

of the forms

$$(7) \quad K_i(t, q, z, s, p) = (\phi_1(t), \phi_2(t, q, z), \phi_3(t, q, z), \phi_4^i(t, q, z, s, p), \phi_5^i(t, q, z, s, p))$$

and reticular r_i -diffeomorphisms Ψ_i on $(\mathbb{L}^{r_i}, 0)$ such that $K_i \circ \mathcal{L}_i = \mathcal{L}'_i \circ \Psi_i$ for $i = 1, \dots, m$.

Let $(\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a multi-reticular Legendrian unfolding and $F_i \in \mathfrak{M}(r_i; k_i + 1 + n + 1)$ be generating families of \mathcal{L}_i for $i = 1, \dots, m$. We call $F = (F_1, \dots, F_m)$ a *generating family* of $(\mathcal{L}_1, \dots, \mathcal{L}_m)$.

By the consideration of stable reticular $t\text{-}(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalences of generating families of multi-reticular Legendrian unfoldings, we may assume that $r = r_1 = \dots = r_m$, $k = k_1 = \dots = k_m$ by taking stable reticular $t\text{-}(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalences of them.

We say that a map germ $(F_1, \dots, F_m) \in \mathfrak{M}(r; k + 1 + n + 1, m)$ is $\mathcal{P}_{(m)}$ -*C-non-degenerate* if all F_i are \mathcal{P} -*C-non-degenerate* and $(\frac{\partial F_1}{\partial t} / (-\frac{\partial F_1}{\partial z}), \frac{\partial F_1}{\partial q} / (-\frac{\partial F_1}{\partial z}), \dots, (\frac{\partial F_m}{\partial t} / (-\frac{\partial F_m}{\partial z}), \frac{\partial F_m}{\partial q} / (-\frac{\partial F_m}{\partial z}))$ are distinct.

By Theorem 3.6, we have the following:

THEOREM 4.1. (1) *For any multi-reticular Legendrian unfolding \mathcal{L} , there exists a multi-generating family of \mathcal{L} ,*

(2) *For any $\mathcal{P}_{(m)}$ -C-non-degenerate map germ $F = (F_1, \dots, F_m) \in \mathfrak{M}(r; k + 1 + n + 1, m)$, there exists a multi-reticular Legendrian unfolding of which F is a multi-generating family,*

(3) *Let $F = (F_1, \dots, F_m)$ and $F' = (F'_1, \dots, F'_m)$ be multi-generating families of multi-reticular Legendrian unfoldings $(\mathcal{L}_1, \dots, \mathcal{L}_m)$ and $(\mathcal{L}'_1, \dots, \mathcal{L}'_m)$ respectively. Then $(\mathcal{L}_1, \dots, \mathcal{L}_m)$ and $(\mathcal{L}'_1, \dots, \mathcal{L}'_m)$ are $\mathcal{P}_{(m)}$ -Legendrian equivalent if and only if F and F' are stably reticular $t\text{-}(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent.*

Let U be an open set in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. We consider contact embedding germs $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and contact embeddings $U \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Let

(T, Q, S, Z, P) and (t, q, z, s, p) be canonical coordinates of the source space and the target space respectively. We define the following notations:

$\iota : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \cap \{Z = 0\}, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be the inclusion map on the source space,

$$\begin{aligned} C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) &= \{C \mid C \text{ is a } \mathcal{P}\text{-contact embedding germ} \\ &\quad (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})\}, \\ C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) &= \{C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \mid C^*\Theta = \Theta\}, \\ C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) &= \{C \circ \iota \mid C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)\}, \\ C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) &= \{C \circ \iota \mid C \in C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)\}. \end{aligned}$$

Let $V = U \cap \{Z = 0\}$ and $\tilde{\iota} : V \rightarrow U$ be the inclusion map.

$$\begin{aligned} C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} : U \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \mid \\ &\quad \tilde{C} \text{ is a contact embedding of the form (3)}\}, \\ C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} \in C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \mid \tilde{C}^*\Theta = \Theta\}, \\ C_T^Z(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} \circ \tilde{\iota} \mid \tilde{C} \in C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))\}, \\ C_T^{\Theta, Z}(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} \circ \tilde{\iota} \mid \tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))\}. \end{aligned}$$

DEFINITION 4.2. We define stabilities of multi-reticular Legendrian unfoldings. Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a multi-reticular Legendrian unfolding.

Stability. We say that \mathcal{L} is *stable* if the following condition holds: Let $C^{0,i} \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be \mathcal{P} -contact embedding germs such that $C^{0,i}|_{\mathbb{L}^{r_i}} = \mathcal{L}_i$ and $\tilde{C}^{0,i} \in C_T(U_i, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ be representatives of $C^{0,i}$ for $i = 1, \dots, m$. Then there exist open neighborhoods $N_{\tilde{C}^{0,i}}$ of $\tilde{C}^{0,i}$ in C^∞ -topology for $i = 1, \dots, m$ such that for any $\tilde{C}^i \in N_{\tilde{C}^{0,i}}$, there exist points $x_i = (T^i, 0, \dots, 0, P_{r+1}^i, \dots, P_n^i) \in U_i$ such that the multi-reticular Legendrian unfolding $(\mathcal{L}_{x_1}^1, \dots, \mathcal{L}_{x_m}^m)$ and \mathcal{L} are $\mathcal{P}_{(m)}$ -Legendrian equivalent, where the reticular Legendrian unfolding $\mathcal{L}_{x_i}^i$ is defined by

$$x = (T, Q, Z, S, P) \mapsto \tilde{C}^i(x_0 + x) - \tilde{C}^i(x_0) + (0, 0, P_{r+1}^i Q_{r+1} + \dots + P_n^i Q_n, 0, 0).$$

Homotopical stability. A one-parameter family of \mathcal{P} -contact embedding germs $\bar{C} : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \times \mathbb{R}, (0, 0)) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) ((T, Q, Z, S, P, \tau) \mapsto C_\tau(T, Q, Z, S, P))$ is called a \mathcal{P} -contact deformation of \mathcal{L} if $C_0|_{\mathbb{L}} = \mathcal{L}$. A map germ $\bar{\Psi} : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \times \mathbb{R}, (0, 0)) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)((T, Q, Z, S, P, \tau) \mapsto \Psi_\tau(T, Q, Z, S, P))$ is called a *one-parameter deformation of reticular r -diffeomorphisms* if $\Psi_0 = id_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}$ and Ψ_t is a reticular r -diffeomorphism for all t around 0. We say that \mathcal{L} is *homotopically stable* if for any reticular \mathcal{P} -contact deformations $\bar{C}^i = \{C_\tau^i\}$ of \mathcal{L}_i , there exist one-parameter families of \mathcal{P} -Legendrian equivalences $\bar{K}^i = \{K_\tau^i\}$ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w_i)$ with $K_0^i = id$ of the form

$$(8) \quad K_\tau^i(t, q, z, s, p) = (\phi_\tau^1(t), \phi_\tau^2(t, q, z), \phi_\tau^3(t, q, z), \phi_\tau^{4,i}(t, q, z, s, p), \phi_\tau^{5,i}(t, q, z, s, p))$$

and one-parameter deformations of reticular r_i -diffeomorphisms $\bar{\Psi}^i = \{\Psi_\tau^i\}$ such that $C_\tau^i = K_\tau^i \circ C_0^i \circ \Psi_\tau^i$ for t around 0 and $i = 1, \dots, m$.

Infinitesimal stability. Let $C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be a \mathcal{P} -contact diffeomorphism germ. We say that a vector field v on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ ($\Pi(w) = 0$) along C is an *infinitesimal \mathcal{P} -contact transformation* of C if there exists a \mathcal{P} -contact deformation $\bar{C} = \{C_\tau\}$ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that $C_0 = C$ and $\frac{dC_\tau}{d\tau}|_{\tau=0} = v$. We say that a vector field ξ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ is an *infinitesimal reticular r -diffeomorphism* if there exists a one-parameter deformation of reticular r -diffeomorphisms $\bar{\Psi} = \{\Psi_\tau\}$ such that $\frac{d\Psi_\tau}{d\tau}|_{\tau=0} = \xi$. We say that a vector field η on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w)$ is an *infinitesimal \mathcal{P} -Legendrian equivalence* if there exists a one-parameter family of \mathcal{P} -Legendrian equivalences $\bar{K} = \{K_\tau\}$ such that $K_0 = id_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}$ and $\frac{dK_\tau}{d\tau}|_{\tau=0} = \eta$. We say that \mathcal{L} is *infinitesimally stable* if for any extension C^i of \mathcal{L}_i and any infinitesimal \mathcal{P} -contact transformation v_i of C^i , there exist infinitesimal reticular r_i -diffeomorphisms ξ_i and infinitesimal \mathcal{P} -Legendrian equivalences η_i at w_i of the form

$$(9) \quad \begin{aligned} \eta_i(t, q, z, s, p) = & a_1(t) \frac{\partial}{\partial t} + a_2(t, q, z) \frac{\partial}{\partial q} + a_3(t, q, z) \frac{\partial}{\partial z} \\ & + a_4^i(t, q, z, s, p) \frac{\partial}{\partial s} + a_5^i(t, q, z, s, p) \frac{\partial}{\partial p} \end{aligned}$$

such that $v_i = C_*^i \xi_i + \eta_i \circ C^i$ for $i = 1, \dots, m$.

We denote $C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)} = C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \times \dots \times C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ (m -products) and denote other notations analogously. We call an element $C = (C_1, \dots, C_m) \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)^{(m)}$ a $\mathcal{P}_{(m)}$ -contact diffeomorphism germ if $\Pi \circ C(0) = 0$ and $C_1(0), \dots, C_m(0)$ are distinct.

We may take an extension of a reticular Legendrian unfolding \mathcal{L} by an element of $C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ by Lemma 3.4. Then as the remark after the definition of the stability of reticular Legendrian maps in [6, p.121], we may consider the following other definitions of stabilities of multi-reticular Legendrian unfoldings: (1) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and $C_T(U_i, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ to $C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and $C_T^\Theta(U_i, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ of original definition respectively. (2) The definition given by replacing to $C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and $C_T^Z(V_i, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ respectively. (3) The definition given by replacing to $C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and $C_T^{\Theta, Z}(V_i, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ respectively, where $V_i = U_i \cap \{Z = 0\}$.

Then we have the following lemma which is proved by the same method of the proof of [6, Lemma 7.2]

LEMMA 4.3. *The original definition and other three definitions of stabilities of multi-reticular Legendrian unfoldings are all equivalent.*

By this lemma, we may choose an extension of a multi-reticular Legendrian unfolding from among all of $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)^{(m)}$, $C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)^{(m)}$, $C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)^{(m)}$, and $C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)^{(m)}$.

We say that a function germ H on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ is \mathcal{P} -fiber preserving if H has the form $H(t, q, z, s, p) = \sum_{j=1}^n h_j(t, q, z) p_j + h_0(t, q, z) + a(t)s$.

LEMMA 4.4. *Let $C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$. Then the following hold: (1) A vector field germ v on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ along C is an infinitesimal \mathcal{P} -contact transformation of C if and only if there exists a function germ f on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that f*

does not depend on s and $v = X_f \circ C$.

(2) A vector field germ η on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ is an infinitesimal \mathcal{P} -Legendrian equivalence if and only if there exists a \mathcal{P} -fiber preserving function germ H on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that $\eta = X_H$.

(3) A vector field ξ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ is an infinitesimal reticular r -diffeomorphism if and only if there exists a function germ $g_i \in B^r$ such that $\xi = X_g$, where $B^r = \langle q_1 p_1, \dots, q_r p_r, q_{r+1}, \dots, q_n, z \rangle_{\mathcal{E}_{t,q,z,p}} + \langle s \rangle_{\mathcal{E}_t}$.

Let U be a neighborhood of 0 in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. We define:

$$J_{C_T^\Theta}^l(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) = \{j^l C(w_0) \in J^l(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))\}$$

$C : (U, w_0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ is a \mathcal{P} -contact embedding germ which preserves Θ .

THEOREM 4.5. *Let $Q_i, i = 1, 2, \dots$ are submanifolds of $J_{C_T^\Theta}^l(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)}$. Then the set*

$T = \{C = (C_1, \dots, C_m) \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)} \mid j^l C \text{ is transversal to } Q_i \text{ for all } i \in \mathbb{N}\}$
 is a residual set in $C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)}$.

This is proved by an analogous method of [8, p.345 Theorem 6.4].

We denote the ring $\mathcal{E}(1 + n + n, m)$ on the coordinates (t, q, p) by $\mathcal{E}_{t,q,p(m)}$ and denote other notations analogously.

THEOREM 4.6. *Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a multi-reticular Legendrian unfolding with a multi-generating family $F = (F_1, \dots, F_m)$. Let $C = (C_1, \dots, C_m) \in C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)^{(m)}$ be an extension of \mathcal{L} . Then the following are equivalent.*

(u) F is a reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -stable unfolding of $F|_{t=0}$.

(hs) \mathcal{L} is homotopically stable.

(is) \mathcal{L} is infinitesimally stable.

(a) $\mathcal{E}_{t,q,p(m)} = B_0^{r_1} \times \dots \times B_0^{r_m} + \langle 1, p_1 \circ C', \dots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} + \langle s \circ C' \rangle_{\mathcal{E}_t}$, where $C' = C|_{z=s=0}$, $B_0^{r_i} = \langle q_1 p_1, \dots, q_{r_i} p_{r_i}, q_{r_i+1}, \dots, q_n \rangle_{\mathcal{E}_{t,q,p}}$, and $(\Pi \circ C')^* \mathcal{E}_{t,q,z}$ be the $\mathcal{E}_{t,q,z}$ -submodule of $\mathcal{E}_{t,q,z(m)}$ such that $(\Pi \circ C')^* a = (a(\Pi \circ C'_1), \dots, a(\Pi \circ C'_m))$ for $a \in \mathcal{E}_{t,q,z}$.

We remark that sufficiently near multi-reticular Legendrian unfoldings of stable one are also stable by the condition (a).

Proof. (u) \Rightarrow (hs): Let \mathcal{P} -contact deformations $\{C_\tau^i\}$ of \mathcal{L}_i be given for $i = 1, \dots, m$. The homotopically stability of multi-reticular Legendrian unfoldings is invariant under $\mathcal{P}_{(m)}$ -Legendrian equivalences, we may assume that the map germs

$$(T, Q, Z, S, P) \rightarrow (T, Q, Z, s \circ C_\tau^i(T, Q, Z, S, P), p \circ C_\tau^i(T, Q, Z, S, P))$$

are diffeomorphisms for all τ and i . By Lemma 3.5, there exists a one-parameter families $H_\tau^i(T, Q, p) \in \mathfrak{M}(1 + n + n)^2$ depending smoothly on $\tau \in (\mathbb{R}, 0)$ such that the canonical relations $P_{C_\tau^i}$ associated with C_τ^i have the forms (5) for function germs $H_\tau^i(T, Q, p) \in \mathfrak{M}(1 + n + n)^2$. We set $(0, 0, 0, s^i, p^i) = \mathcal{L}_i(0)$. Then the map germs $F_\tau = (F_\tau^1, \dots, F_\tau^m) \in \mathfrak{M}(r; n + 1 + n + 1, m)$ defined by

$$F_\tau^i(x, y, t, q, z) = -z + H_\tau^i(t, x, 0, y) + \langle y + p^i, q \rangle + s^i t$$

are generating families of multi-reticular Legendrian unfoldings $\mathcal{L}_\tau := (C_\tau^1|_{\mathbb{R}^{r_1}}, \dots, C_\tau^m|_{\mathbb{R}^{r_m}})$ for $\tau \in (\mathbb{R}, 0)$. Since F_0 is a reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -stable unfolding of $F|_{t=0}$, it follows that F_0 is a reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -homotopically stable unfolding of $F|_{t=0}$ by Theorem 2.9. Therefore there exists a one-parameter family of reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -isomorphism from F_τ to F_0 depending smoothly on τ . This means that there exists a one-parameter families of \mathcal{P} -Legendrian equivalences K_τ^i depending smoothly on τ such that K_τ^i have the forms (7) and

$$C_\tau^i(L_\sigma^{r_i,0}) = K_\tau^i \circ \mathcal{L}(L_\sigma^{r_i,0}) \text{ for all } \sigma \subset I_r, \tau \in (\mathbb{R}, 0), i \in \{1, \dots, m\}.$$

Then the map germs $\Psi_\tau^i := (C_0^i)^{-1} \circ (K_\tau^i)^{-1} \circ C_\tau^i$ give one-parameter deformations of reticular r_i -diffeomorphisms on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and we have that $C_\tau^i = K_\tau^i \circ C_0^i \circ \Psi_\tau^i$. This means that \mathcal{L} is homotopically stable.

(hs) \Rightarrow (is): Let $v = (v_1, \dots, v_m)$ be an infinitesimal \mathcal{P} -contact transformation of \mathcal{L} .

Then there exist \mathcal{P} -contact deformations $\{C_\tau^i\}$ of \mathcal{L}_i such that $v_i = \frac{dC_\tau^i}{d\tau}|_{\tau=0}$ for $i = 1, \dots, m$. Then there exist one-parameters of \mathcal{P} -Legendrian equivalences $\{K_\tau^i\}$ of the form (7) and one-parameter deformations of reticular $\mathcal{P}_{(r_i)}$ -diffeomorphisms $\{\Psi_\tau^i\}$ such that $C_\tau^i = K_\tau^i \circ C_0^i \circ \Psi_\tau^i$ for $\tau \in (\mathbb{R}, 0)$ and $i \in \{1, \dots, m\}$. Then we have that

$$v_i = \frac{dC_\tau^i}{d\tau}|_{\tau=0} = \frac{dK_\tau^i}{d\tau}|_{\tau=0} \circ C_0 + (C_0)_* \left(\frac{d\Psi_\tau^i}{d\tau}|_{\tau=0} \right).$$

Since v_i has the form (9) for each i , it follows that \mathcal{L} is infinitesimally stable.

(is) \Rightarrow (a): Let a map germ $f = (f_1, \dots, f_m) \in \mathcal{E}_{t,q,p(m)}$ be given. We define the function germs f'_i on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), w'_i)$ by $f'_i(t, q, z, s, p) = f_i \circ \pi_{T,Q,P} \circ C_i^{-1}(t, q, z, a_i(t, q, z, p), p)$, where $w'_i = (0, \dots, 0, p^i)$ and the function germs a_i are defined by $\pi_S \circ C_i^{-1}(t, q, z, a_i(t, q, z, p), p) \equiv 0$. This equation can be solved by (2). Since f'_i does not depend on s , it follows that $X_{f'_i} \circ C_i$ is an infinitesimal \mathcal{P} -contact transformation of C_i . Therefore there exist an infinitesimal \mathcal{P} -Legendrian equivalence η_i of the form (9) and an infinitesimal reticular r_i -diffeomorphism ξ_i such that $X_{f'_i} \circ C_i = (C_i)_* \xi_i + \eta_i \circ C_i$. By Lemma 4.4, there exist a \mathcal{P} -fiber preserving function germ H on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and $g_i \in B^{r_i}$ such that $\xi_i = X_{g_i}$ and $\eta_i = X_H$. Then we have that $f'_i \circ C_i = g_i + H \circ C_i$. Since $f'_i \circ C(T, Q, Z, S, P) = f_i \circ \pi_{T,Q,P} \circ (C_i)^{-1}(t, q, z, a_i(T, Q, Z, 0, P), p) = f_i \circ \pi_{T,Q,P}(T, Q, Z, 0, P) = f_i(T, Q, P)$ and H has the form $H(t, q, z, s, p) = \sum_{j=1}^n h_j(t, q, z) p_j + h_0(t, q, z) + h'(t) s$, We have that

$$f_i \equiv \sum_{j=1}^n (h_j(\Pi \circ C'_i))(p_i \circ C'_i) + h_0(\Pi \circ C'_i) + (h'(t \circ C'_i))(s \circ C'_i) \text{ mod } B_0^{r_i}.$$

Since $t \circ C_i = t$, we have the required form.

(a) \Rightarrow (u): By Lemma 3.5, there exists function germs $H_i(T, Q, p) \in \mathfrak{M}(1 + n + n)^2$ such that the function germ $H_i(T, Q, p) - Ts$ is a generating function of P_{C_i} for each i . Then the map germ $F = (F_1, \dots, F_m) \in \mathfrak{M}(r; n + 1 + n + 1, m)$ given by $F_i(x, y, t, q, z) = -z + H_i(t, x, 0, y) + s^i t + \langle y + p^i, q \rangle$ is a generating family of \mathcal{L} . Then $P'_{C_i} := P_{C_i}|_{Z=S=0}$ has the form

$$P'_{C_i} = \{(T, Q, -\frac{\partial H_i}{\partial Q}, T, -\frac{\partial H_i}{\partial p}, H_i - \langle \frac{\partial H_i}{\partial p}, p + p^i \rangle + s^i T, \frac{\partial H_i}{\partial T} + s^i, p + p^i)\}.$$

Then the map germ $P'_{C_i} \rightarrow (\mathbb{R}^{1+n+n}, 0)$, $w \mapsto \pi_{T,Q,P}(w)$ is a diffeomorphism. We set $D(F_i) = \{(x, y, t, q, z) \in (\mathbb{H}^r \times \mathbb{R}^{n+1+n+1}, 0) | F_i = x \frac{\partial F_i}{\partial x} = \frac{\partial F_i}{\partial y} = 0\}$. We also define

the map germ $D(F_i) \rightarrow P'_{C_i}$ by

$$(x, y, t, q, z) \mapsto (t, x, 0, -\frac{\partial F_i}{\partial x}, -\frac{\partial H_i}{\partial Q_{r+1}}(t, x, 0, y), \dots, -\frac{\partial H_i}{\partial Q_n}, t, q, z, \frac{\partial F_i}{\partial t}, y + p^i).$$

Then the composition of the above two map germs induces the map germ $\mathcal{E}_{T,Q,P}/B_0^{r_i} \rightarrow \mathcal{E}_{D(F_i)}$. We denote T, Q, P for the variables on the source space of this map germ. Then the correspondence is given that:

$$T \mapsto t, Q_1 \mapsto x_1, \dots, Q_r \mapsto x_r, P_1 \mapsto -\frac{\partial F_i}{\partial x_1}, \dots, P_r \mapsto -\frac{\partial F_i}{\partial x_r},$$

$$t \circ C'_i(T, Q, P) \mapsto t, q \circ C'_i(T, Q, P) \mapsto q, z \circ C'_i(T, Q, P) \mapsto z,$$

$$s \circ C'_i(T, Q, P) \mapsto \frac{\partial F_i}{\partial t}, p \circ C'_i(T, Q, P) \mapsto y + p^i, ((\Pi \circ C'_i)^* \mathcal{E}_{t,q,z}) \mapsto \mathcal{E}_{t,q,z}, ((t \circ C'_i)^* \mathcal{E}_t) \mapsto \mathcal{E}_t.$$

Then (a) is transferred that

$$\begin{aligned} \mathcal{E}(r; n+1+n+1) &= \langle F, x \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle_{\mathcal{E}(r;n+1+n+1)} \\ &+ \langle 1^{(m)} (= -\frac{\partial F}{\partial z}), (y_1 + p_1)^{(m)} (= \frac{\partial F}{\partial q_1}), \dots, (y_n + p_n)^{(m)} (= \frac{\partial F}{\partial q_n}) \rangle_{\mathcal{E}_{t,q,z}} + \langle \frac{\partial F}{\partial t} \rangle_{\mathcal{E}_t}, \end{aligned}$$

where $a^{(m)} = (a, \dots, a)$ for each function germ a and $(y_i + p_i)^{(m)} = (y_i + p_i^1, \dots, y_i + p_i^n)$. It follows that F is a reticular t - \mathcal{P} - \mathcal{K} -infinitesimal stable unfolding of $F|_{t=0}$. We have that F is a reticular t - \mathcal{P} - \mathcal{K} -stable unfolding of $F|_{t=0}$ by Theorem 2.9. \square

5. Genericity. In order to give a generic classification of multi-reticular Legendrian unfoldings, we reduce our investigation to finite dimensional jet spaces of \mathcal{P} -contact diffeomorphism germs.

DEFINITION 5.1. Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a multi-reticular Legendrian unfolding. We say that \mathcal{L} is l -determined if the following condition holds: For any extension $C_i \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ of \mathcal{L}_i , the multi-reticular Legendrian unfolding $(C'_1|_{\mathbb{L}^{r_1}}, \dots, C'_m|_{\mathbb{L}^{r_m}})$ and \mathcal{L} are $\mathcal{P}_{(m)}$ -Legendrian equivalent for all $C'_i \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ satisfying that $j^l C_i(0) = j^l C'_i(0)$ for $i = 1, \dots, m$.

As Lemma 4.3, we may consider the following other definition of finitely determinacy of reticular Legendrian maps:

- (1) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $C_T^{\Theta}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.
 - (2) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.
 - (3) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.
- Then the following holds by [8, p.341 Proposition 5.6]:

PROPOSITION 5.2. *Let \mathcal{L} be a multi-reticular Legendrian unfolding. Then*

- (A) *If \mathcal{L} is l -determined of the original definition, then \mathcal{L} is l -determined of the definition (1).*
- (B) *If \mathcal{L} is l -determined of the definition (1), then \mathcal{L} is l -determined of the definition (3).*
- (C) *If \mathcal{L} is $(l+1)$ -determined of the definition (3), then \mathcal{L} is l -determined of the definition (2).*
- (D) *If \mathcal{L} is l -determined of the definition (2), then \mathcal{L} is l -determined of the original definition.*

THEOREM 5.3. *Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a multi reticular Legendrian unfolding. If \mathcal{L} is infinitesimally stable then \mathcal{L} is $(n+5)$ -determined.*

Proof. It is enough to prove \mathcal{L} is $(n+4)$ -determined of Definition 5.1 (3). Let $C_i \in C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be extensions of \mathcal{L}_i for $i = 1, \dots, m$. Since the finite determinacy of multi-reticular Legendrian unfoldings is invariant under $\mathcal{P}_{(m)}$ -Legendrian equivalences, we may assume that P_{C_i} have the form (5) for some function germs $H_i(T, Q, p) \in \mathfrak{M}(2n+1)^2$. Then $F_i(x, y, t, q, z) = -z + H_{0,i}(x, y, t) + \langle y + p^i, q \rangle \in \mathfrak{M}(r; n+1+n+1)$ is a generating family of \mathcal{L}_i , where $(0, 0, 0, s^i, p^i) = \mathcal{L}_i(0)$ and $H_{0,i}(x, y, t) = H_i(t, x, 0, y) + s^i t \in \mathfrak{M}(r; n+1)$.

By Theorem 4.6, we have that $F = (F_1, \dots, F_m)$ is a reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -infinitesimally versal unfolding of $f := F|_{t=0}$. It follows that

$$\begin{aligned} \mathcal{E}(r; n+1+n+1, m) &= \langle F_1, x \frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y} \rangle_{\mathcal{E}(r; n+1+n+1)} \times \cdots \times \\ &\quad \langle F_m, x \frac{\partial F_m}{\partial x}, \frac{\partial F_m}{\partial y} \rangle_{\mathcal{E}(r; n+1+n+1)} + \langle \frac{\partial F}{\partial q}, \frac{\partial F}{\partial z} \rangle_{\mathcal{E}(1+n+1)} + \langle \frac{\partial F}{\partial t} \rangle_{\mathcal{E}(1)} \end{aligned}$$

By the restriction of the both side to $q = z = 0$, we have that

$$\begin{aligned} \mathcal{E}(r; n+1, m) &= \\ &\quad \langle H_{0,1}, x \frac{\partial H_{0,1}}{\partial x}, \frac{\partial H_{0,1}}{\partial y} \rangle_{\mathcal{E}(r; n+1)} \times \cdots \times \langle H_{0,m}, x \frac{\partial H_{0,m}}{\partial x}, \frac{\partial H_{0,m}}{\partial y} \rangle_{\mathcal{E}(r; n+1)} \\ (10) \quad &+ \langle 1^{(m)}, (y_1 + p_1)^{(m)}, \dots, (y_n + p_n)^{(m)}, \frac{\partial H_0}{\partial t} \rangle_{\mathcal{E}(1)}, \end{aligned}$$

where $H_0 = (H_{0,1}, \dots, H_{0,m})$. It follows that

$$\begin{aligned} \mathfrak{M}(r; n+1, m)^{n+2} \subset \\ &\quad \langle H_{0,1}, x \frac{\partial H_{0,1}}{\partial x}, \frac{\partial H_{0,1}}{\partial y} \rangle_{\mathcal{E}(r; n+1)} \times \cdots \times \langle H_{0,m}, x \frac{\partial H_{0,m}}{\partial x}, \frac{\partial H_{0,m}}{\partial y} \rangle_{\mathcal{E}(r; n+1)} \\ (11) \quad &+ \mathfrak{M}(1) \mathcal{E}(r; n+1, m). \end{aligned}$$

By (10) and (11) we have that

$$\begin{aligned} \mathfrak{M}(r; n+1, m)^{n+3} \subset &(\langle H_{0,1}, x \frac{\partial H_{0,1}}{\partial x} \rangle_{\mathcal{E}(r; n+1)} + \mathfrak{M}(r; n+1) \langle \frac{\partial H_{0,1}}{\partial y} \rangle) \times \cdots \times \\ &(\langle H_{0,m}, x \frac{\partial H_{0,m}}{\partial x} \rangle_{\mathcal{E}(r; n+1)} + \mathfrak{M}(r; n+1) \langle \frac{\partial H_{0,m}}{\partial y} \rangle) \\ &+ \mathfrak{M}(1) \langle 1^{(m)}, (y_1 + p_1)^{(m)}, \dots, (y_n + p_n)^{(m)}, \frac{\partial H_0}{\partial t} \rangle. \end{aligned}$$

Let $C'_i \in C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ satisfying $j^{n+4}C'_i(0) = j^{n+4}C_i(0)$ be given for $i = 1, \dots, m$. There exist function germs $H'_i(T, Q, p) \in \mathfrak{M}(2n+1)^2$ such that

$$\begin{aligned} P_{C'_i} &= \{(T, Q, -\frac{\partial H'_i}{\partial T}(T, Q, p) + s, -\frac{\partial H'_i}{\partial Q}, T, -\frac{\partial H'_i}{\partial p}, \\ &\quad H'_i - \langle \frac{\partial H'_i}{\partial p}, p + p^i \rangle + s^i T, s + s^i, p + p^i)\}. \end{aligned}$$

Since $H_i = z - qp$ on P_{C_i} and $H'_i = z - qp$ on $P_{C'_i}$, we have that $j^{n+4}H_{0,i}(0) = j^{n+4}H'_{0,i}(0)$, where $H'_{0,i}(x, y, t) = H'_i(t, x, 0, y) + s^i t \in \mathfrak{M}(r; n+1)$. By (11) we have

that

$$\begin{aligned} \mathfrak{M}(r; n, m)^{n+2} &\subset \langle H_{0,1}(x, y, 0), x \frac{\partial H_{0,1}}{\partial x}(x, y, 0), \frac{\partial H_{0,1}}{\partial y}(x, y, 0) \rangle_{\mathcal{E}(r;n)} \\ &\times \cdots \times \langle H_{0,m}(x, y, 0), x \frac{\partial H_{0,m}}{\partial x}(x, y, 0), \frac{\partial H_{0,m}}{\partial y}(x, y, 0) \rangle_{\mathcal{E}(r;n)} \end{aligned}$$

and this means that $H_{0,i}(x, y, 0)$ is reticular \mathcal{K} -($n+3$)-determined by Lemma 2.3 and hence $H_0(x, y, 0)$ is also reticular $\mathcal{K}_{(m)}$ -($n+3$)-determined. So we may assume that $H_0 - H'_0 \in \mathfrak{M}(1)\mathfrak{M}(r; n+1, m)^{n+3}$. Then the function germs

$$G_i(x, y, t, q, z) = -z + H'_{0,i}(x, y, t) + \langle y + p^i, q \rangle \in \mathfrak{M}(r; n+1+n+1)$$

are generating families of $C'_i|_{\mathbb{L}^{r_i}}$ for $i = 1, \dots, m$.

We define the function germs $E_{\tau_0,i}(x, y, t, \tau) \in \mathcal{E}(r; n+1+1)$ by $E_{\tau_0,i}(x, y, t, \tau) = (1 - \tau - \tau_0)H_{0,i}(x, y, t) + (\tau + \tau_0)H'_{0,i}(x, y, t)$ for $\tau_0 \in [0, 1]$ and define $E_{\tau_0}(x, y, t, \tau) \in \mathcal{E}(r; n+1+1, m)$ by $E_{\tau_0} = (E_{\tau_0,1}, \dots, E_{\tau_0,m})$. By an analogous method of the proof of [7, Lemma 5.10], we have that

$$\begin{aligned} \frac{\partial E_{\tau_0}}{\partial \tau} &\in \mathfrak{M}_t(\langle E_{\tau_0,1}, x \frac{\partial E_{\tau_0,1}}{\partial x} \rangle_{\mathcal{E}(r;n+2)} + \mathfrak{M}(r; n+2) \langle \frac{\partial E_{\tau_0,1}}{\partial y} \rangle) \times \cdots \times \\ &(\langle E_{\tau_0,m}, x \frac{\partial E_{\tau_0,m}}{\partial x} \rangle_{\mathcal{E}(r;n+2)} + \mathfrak{M}(r; n+2) \langle \frac{\partial E_{\tau_0,m}}{\partial y} \rangle) \\ &+ \mathfrak{M}_t^2 \langle 1^{(m)}, (y_1 + p_1)^{(m)}, \dots, (y_n + p_n)^{(m)}, \frac{\partial E_{\tau_0}}{\partial t} \rangle, \end{aligned}$$

for $\tau_0 \in [0, 1]$. This means that there exist $\Phi_i(x, y, t) \in \mathcal{B}_1(r; n+1)$ and units $a_i \in \mathcal{E}(r; n+1)$ and $b_1(t), \dots, b_n(t), c(t) \in \mathfrak{M}(1)^2$ such that (1) Φ_i has the form

$$\Phi_i(x, y, t) = (x\phi_1^i(x, y, t, q, z), \phi_2^i(x, y, t, q, z), \phi_3(t)),$$

(2) $H'_{0,i}(x, y, t) = a_i(x, y, t) \cdot H_{0,i} \circ \Phi_i(x, y, t) + \sum_{j=1}^n (y_j + p_j^i) b_j(t) + c(t)$ for $(x, y, t) \in \mathbb{H}^r \times \mathbb{R}^{n+1}$ and $i = 1, \dots, m$,

(3) $\phi(x, y, 0) = (x, y)$, $a(x, y) = 1$ for $(x, y) \in \mathbb{H}^r \times \mathbb{R}^n$.

This means that F and $G = (G_1, \dots, G_m)$ are reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -infinitesimal versal unfoldings of f . It follows that F and G are reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent. Therefore \mathcal{L} and $(C'_1|_{\mathbb{L}^{r_1}}, \dots, C'_m|_{\mathbb{L}^{r_m}})$ are $\mathcal{P}_{(m)}$ -Legendrian equivalent. \square

Let $J^l(2n+3, 2n+3)^{(m)}$ be the set of multi- l -jets of map germs from $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $tC^l(n, m)$ be the immersed manifold in $J^l(2n+3, 2n+3)^{(m)}$ which consists of multi- l -jets of \mathcal{P} -contact embedding germs. Let $L^l(2n+3)^{(m)}$ be the Lie group which consists of multi- l -jets of diffeomorphism germs on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.

We consider the Lie subgroup $rtLe^l(n, m)$ of $L^l(2n+3)^{(m)} \times L^l(2n+3)$ which consists of multi- l -jets of reticular r_i -diffeomorphisms on the source space and l -jets of \mathcal{P} -Legendrian equivalences of Π at 0:

$$\begin{aligned} rtLe^l(n, m) &= \{(j^l \Psi_1(0), \dots, j^l \Psi_m(0), j^l K(0)) \in L^l(2n+3)^{(m)} \times L^l(2n+3) \mid \\ &\Psi_i \text{ is a reticular } r_i\text{-diffeomorphism on } (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0), \\ &K \text{ is a } \mathcal{P}\text{-Legendrian equivalence of } \Pi\}. \end{aligned}$$

The group $rtLe^l(n, m)$ acts on $J^l(2n + 3, 2n + 3)$ and $tC^l(n, m)$ is invariant under this action. Let $C = (C_1, \dots, C_m)$ be a $\mathcal{P}_{(m)}$ -contact diffeomorphism germ from $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and set $z_i = j^l C_i(0)$, $\mathcal{L}_i = C_i|_{\mathbb{L}^{r_i}}$, $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$, $z = (z_1, \dots, z_m)$. We denote the orbit $rtLe^l(n, m) \cdot z$ by $[z]$. Then

$$[z] = \{j^l C'(0) \in tC^l(n, m) \mid \mathcal{L} \text{ and } (C'_1|_{\mathbb{L}^{r_1}}, \dots, C'_m|_{\mathbb{L}^{r_m}}) \text{ are } \mathcal{P}_{(m)}\text{-Legendrian equivalent}\}.$$

We denote by VI_C the vector space consisting of infinitesimal $\mathcal{P}_{(m)}$ -contact transformation germs of C and denote by VI_C^0 the subspace of VI_C consist of germs which vanish on 0. We denote by $VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}$ by the vector space consisting of infinitesimal \mathcal{P} -Legendrian equivalences on Π and denote by $VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}^0$ by the subspace of $VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}$ consists of germs which vanish at 0.

We denote by $V_{\mathbb{L}^r}^0$ the vector space consisting of infinitesimal reticular r -diffeomorphisms on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ which vanishes at 0 and set $V_{\mathbb{L}}^0 = V_{\mathbb{L}^{r_1}}^0 \times V_{\mathbb{L}^{r_m}}^0$. We have that by Lemma 4.4:

$$VI_C^0 = \{(v_1, \dots, v_m) \mid v_i : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \rightarrow T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})), \\ v_i = X_{f_i} \circ C_i \text{ for some } f \in \mathfrak{M}_{t,q,z,p}^2\},$$

$$VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}^0 = \{\eta \in X(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0) \mid \eta = X_H \text{ for some} \\ \mathcal{P}\text{-fiver preserving function germ } H \in \mathfrak{M}_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}^2\},$$

$$V_{\mathbb{L}}^0 = \{(\xi_1, \dots, \xi_m) \mid \xi_i \in X(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0), \xi_i = X_{g_i} \text{ for some } g_i \in B^{r_i} \cap \mathfrak{M}_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}^2\}.$$

We define the homomorphism $tC : \mathfrak{M}_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})} VI_{\mathbb{L}} \rightarrow VI_C^0$ by $tC(v) = (C_{1*}v_1, \dots, C_{m*}v_m)$ and define the homomorphism $wC : VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}^0 \rightarrow VI_C^0$ by $wC(\eta) = (\eta \circ C_1, \dots, \eta \circ C_m)$.

We denote VI_C^l the subspace of VI_C consisting of infinitesimal \mathcal{P} -contact transformation germs of C whose l -jets are 0:

$$VI_C^l = \{(v_1, \dots, v_m) \in VI_C \mid j^l v_i(0) = 0\}.$$

For $\tilde{C} = (\tilde{C}_1, \dots, \tilde{C}_m) \in C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)}$, we define the continuous map $j_0^l \tilde{C} : U \rightarrow tC^l(n, m)$ by x to the l -jet of $(\tilde{C}_{1x}, \dots, \tilde{C}_{mx})$.

THEOREM 5.4. *Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a reticular Legendrian unfolding. Let C_i be an extension of \mathcal{L}_i and $l \geq (n + 2)^2$. We set $C = (C_1, \dots, C_m)$. Then the followings are equivalent:*

- (s) \mathcal{L} is stable.
- (t) $j_0^l C$ is transversal to $[j_0^l C(0)]$.
- (a') $\mathcal{E}_{t,q,p(m)} = B_0^{r_1} \times \dots \times B_0^{r_m} + \langle 1, p_1 \circ C', \dots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} + \langle s \circ C' \rangle_{\mathcal{E}_t} + \mathfrak{M}_{t,q,p(m)}^l$, where $C' = C|_{z=s=0}$ and $B_0^{r_i} = \langle q_1 p_1, \dots, q_{r_i} p_{r_i}, q_{r_i+1}, \dots, q_n \rangle_{\mathcal{E}_{t,q,p}}$,
- (a) $\mathcal{E}_{t,q,p(m)} = B_0^{r_1} \times \dots \times B_0^{r_m} + \langle 1, p_1 \circ C', \dots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} + \langle s \circ C' \rangle_{\mathcal{E}_t}$,
- (is) \mathcal{L} is infinitesimally stable,
- (hs) \mathcal{L} is homotopically stable,
- (u) A multi-generating family F of \mathcal{L} is reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -stable unfolding of $F|_{t=0}$.

Proof. (s) \Rightarrow (t): By theorem 4.5 and (s), there exists a multi- \mathcal{P} -contact embedding $\tilde{C}' = (\tilde{C}'_1, \dots, \tilde{C}'_m)$ around \tilde{C} such that $j_0^l \tilde{C}'$ is transversal to $[j_0^l C(0)]$,

and $((\tilde{C}'_1)_{x_1}|_{\mathbb{L}^{r_1}}, \dots, (\tilde{C}'_m)_{x_m}|_{\mathbb{L}^{r_m}})$ and \mathcal{L} are $\mathcal{P}_{(m)}$ -Legendrian equivalent for $x_i = (t^i, 0, \dots, 0, p_{r+1}^i, \dots, p_n^i) \in U$. This means that $[j_0^l \tilde{C}'_x(0)] = [j_0^l C(0)]$ and hence $j_0^l C$ is transversal to $[j_0^l C(0)]$ at 0 for $x = (x_1, \dots, x_m)$.

(t) \Leftrightarrow (a): This is proved by an analogous method of Theorem 4.6.

(a) \Leftrightarrow (a'): We need only to prove (a') \Rightarrow (a). By the restriction of (a') to $t = 0$ we have that:

$$\mathcal{E}_{q,p(m)} = B_1^{r_1} \times \dots \times B_1^{r_m} + \langle 1, p_1 \circ C'', \dots, p_n \circ C'' \rangle_{(\Pi \circ C'')^* \mathcal{E}_{t,q,z}} + \langle s \circ C'' \rangle_{\mathbb{R}} + \mathfrak{M}_{q,p(m)}^l,$$

where $C'' = C'|_{t=0}$ and $B_1^{r_i} = B_0^{r_i}|_{t=0}$. Then we have that

$$\mathcal{E}_{q,p(m)} = B_1^{r_1} \times \dots \times B_1^{r_m} + (\Pi \circ C'')^* \mathfrak{M}_{t,q,p} \mathcal{E}_{q,p(m)} + \langle 1, p_1 \circ C'', \dots, p_n \circ C'', s \circ C'' \rangle_{\mathbb{R}} + \mathfrak{M}_{q,p(m)}^l.$$

It follows that

$$\mathfrak{M}_{q,p(m)}^{n+2} \subset B_1^{r_1} \times \dots \times B_1^{r_m} + (\Pi \circ C'')^* \mathfrak{M}_{t,q,p} \mathcal{E}_{q,p(m)}.$$

Therefore

$$\mathfrak{M}_{t,q,p(m)}^{n+2} \subset B_0^{r_1} \times \dots \times B_0^{r_m} + (\Pi \circ C')^* \mathfrak{M}_{t,q,p} \mathcal{E}_{q,p(m)} + \mathfrak{M}_t \mathcal{E}_{t,q,p(m)},$$

and we have that

$$\mathfrak{M}_{t,q,p(m)}^l = (\mathfrak{M}_{t,q,p(m)}^{n+2})^{n+2} \subset B_0^{r_1} \times \dots \times B_0^{r_m} + (\Pi \circ C')^* \mathfrak{M}_{t,q,p}^{n+2} \mathcal{E}_{t,q,p(m)} + \mathfrak{M}_t \mathcal{E}_{t,q,p(m)}.$$

It follows that

$$\begin{aligned} \mathcal{E}_{t,q,p(m)} &= B_0^{r_1} \times \dots \times B_0^{r_m} + \langle 1, p_1 \circ C', \dots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} \\ &\quad + \langle s \circ C' \rangle_{\mathcal{E}_t} + (\Pi \circ C')^* \mathfrak{M}_{t,q,p}^{n+2} \mathcal{E}_{t,q,p(m)} + \mathfrak{M}_t \mathcal{E}_{t,q,p(m)}. \end{aligned}$$

This means (a) by Lemma 2.1.

(a) \Leftrightarrow (is) \Leftrightarrow (hs) \Leftrightarrow (u): This is proved in Theorem 4.6.

(t)&(is) \Rightarrow (s): Since $j_0^l C$ is transversal to $[j_0^l C(0)]$, it follows that there exists a neighborhood $W_{\tilde{C}}$ of \tilde{C} in $C_T^{\mathcal{O}}(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)}$ such that for any $\tilde{C}' \in W_{\tilde{C}}$ there exists $x \in U^{(m)}$ such that $j_0^l \tilde{C}'$ is transversal to $[j_0^l C(0)]$ at x . Since $j_0^l \tilde{C}'_x \in [j_0^l C(0)]$, it follows that there exists a multi- \mathcal{P} -contact embedding germ $C'' = (C''_1, \dots, C''_m) : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})^{(m)}, 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})^{(m)}$ such that \mathcal{L} and $\mathcal{L}'' := (C''_1|_{\mathbb{L}^{r_1}}, \dots, C''_m|_{\mathbb{L}^{r_m}})$ are $\mathcal{P}_{(m)}$ -Legendrian equivalent and $j_0^l C''_i(0) = j_0^l \tilde{C}'_{ix}(0)$. Since \mathcal{L} is infinitesimally stable, it follows that \mathcal{L} is $(n + 3)$ -determined by Theorem 5.3. Therefore we have that \mathcal{L}'' is also $(n + 3)$ -determined. Then \mathcal{L}'' and $((\tilde{C}'_1)_{x_1}|_{\mathbb{L}^{r_1}}, \dots, (\tilde{C}'_m)_{x_m}|_{\mathbb{L}^{r_m}})$ are \mathcal{P} -Legendrian equivalent. This means that \mathcal{L} is stable. \square

Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a stable multi-reticular Legendrian unfolding. We say that \mathcal{L} is *simple* if there exists a representative $\tilde{C} \in C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)}$ of an extension of \mathcal{L} such that $\{\tilde{C}_x|x \in U\}$ is covered by finite orbits $[C_1], \dots, [C_l]$ for some multi- \mathcal{P} -contact embedding germs $C_1, \dots, C_l \in C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)}$.

PROPOSITION 5.5. *Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ be a stable multi-reticular Legendrian unfolding. Then \mathcal{L} is simple if and only if all reticular Legendrian unfoldings \mathcal{L}_i are simple for $i = 1, \dots, m$.*

In order to classify generic multi-reticular Legendrian unfoldings, we classify stable unfoldings $F = (F_1, \dots, F_m)$ ($n \leq 2, m \geq 2$) of $f = (f_1, \dots, f_m)$ and $f_0 = (f_{0,1}, \dots, f_{0,m})$ satisfying the condition: the reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m-1)}$ -codimension of $(f_1, \dots, \check{f}_i, \dots, f_m) = 0$ for any i .

Let a stable unfoldings $F = (F_1, \dots, F_m)$ ($n \leq 2, m \geq 2$) of $f = (f_1, \dots, f_m)$ and $f_0 = (f_{0,1}, \dots, f_{0,m})$ satisfying the condition be given. Then each F_i is a reticular $t\text{-}\mathcal{P}\text{-}\mathcal{K}$ -stable unfolding of f_i , there exist monomials $\varphi_{i,1}, \dots, \varphi_{i,\mu_i} \in \mathfrak{M}(r_i; k_i)$ such that they consist a basis of $Q_{f_{0,i}} = \langle f_{0,i}, x \frac{\partial f_{0,i}}{\partial x}, \frac{\partial f_{0,i}}{\partial y} \rangle_{\mathcal{E}(r_i; k_i)}$ and $\varphi_{i,1}$ has the maximal degree, and $\varphi_{i,\mu_i} = 1$. Then we have that $\mu_1 + \dots + \mu_m \leq n + 2$. Since $\sum_{i=1}^m \mu_i \leq n + 2 \leq 4$, we have that all $f_{0,i}$ are simple singularities. Therefore f_0 is stably reticular $\mathcal{K}_{(m)}$ -equivalent to one of the multi-germs in the following list:

- $n = 1;$
- $m = 1; y^2, y^3, y^4, x^2, x^3, \pm xy + y^3,$
- $m = 2; (y^2, y^2), (y^2, y^3), (y^2, x^2),$
- $m = 3; (y^2, y^2, y^2),$
- $n = 2;$
- $m = 1; y^2, y^3, y^4, y^5, y_1^2 \pm y_2^2, x^2, x^3, x^4, \pm xy + y^3, xy + y^4, x^2 + y^3,$
- $m = 2; (y^2, y^2), (y^2, y^3), (y^2, y^4), (y^3, y^3), (y^2, x^2), (y^2, x^3), (y^2, \pm xy + y^3), (x^2, x^2),$
- $m = 3; (y^2, y^2, y^2), (y^2, y^2, y^3), (y^2, y^2, x^2),$
- $m = 4; (y^2, y^2, y^2, y^2).$

We construct a reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -versal unfolding for each germ by the usual method. Then the corresponding list is as follows:

- $n = 1;$
- (1) $(y^2 + u_{1,1}, y^2 + u_{2,1})$
- (2) $(y^2 + u_{1,1}, y^3 + u_{2,1}y + u_{2,2})$
- (3) $(y^2 + u_{1,1}, x^2 + u_{2,1}x + u_{2,2})$
- (4) $(y^2 + u_{1,1}, y^2 + u_{2,1}, y^2 + u_{3,1})$
- $n = 2;$
- (6) $(y^2 + u_{1,1}, y^2 + u_{2,1})$
- (7) $(y^2 + u_{1,1}, y^3 + u_{2,1}y + u_{2,2})$
- (8) $(y^2 + u_{1,1}, y^4 + u_{2,1}y^2 + u_{2,2}y + u_{2,3})$
- (9) $(y^3 + u_{1,1}y + u_{1,2}, y^3 + u_{2,1}y + u_{2,2})$
- (10) $(y^2 + u_{1,1}, x^2 + u_{2,1}x + u_{2,2})$
- (11) $(y^2 + u_{1,1}, x^3 + u_{2,1}x^2 + u_{2,2}x + u_{2,3})$
- (12) $(y^2 + u_{1,1}, \pm xy + y^3 + u_{2,1}y^2 + u_{2,2}y + u_{2,3})$
- (13) $(x^2 + u_{1,1}x + u_{1,2}, x^2 + u_{2,1}x + u_{2,2})$
- (14) $(y^2 + u_{1,1}, y^2 + u_{2,1}, y^2 + u_{3,1})$
- (15) $(y^2 + u_{1,1}, y^2 + u_{2,1}, y^3 + u_{3,1}y + u_{3,2})$
- (16) $(y^2 + u_{1,1}, y^2 + u_{2,1}, x^2 + u_{3,1}x + u_{3,2})$
- (17) $(y^2 + u_{1,1}, y^2 + u_{2,1}, y^2 + u_{3,1}, y^2 + u_{4,1}).$

In the case that the reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -codimension of $f = 0$, the map germ F is stably reticular $t\text{-}(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent to $G = (G_1, \dots, G_m)$, where $G_i(x, y, t, q, z) = f_{0,i}(x, y) + u_{i,1}\varphi_1(x, y) + \dots + u_{i,\mu_i}\varphi_{i,\mu_i}(x, y) - z$ for $i = 1, \dots, m-1$, $G_m(x, y, t, q, z) = f_{0,i}(x, y) + u_{m,1}\varphi_1(x, y) + \dots + u_{m,\mu_{m-1}}\varphi_{m,\mu_{m-1}}(x, y) - z$, and $(q_1, \dots, q_n, z) = (u_{1,1}, \dots, u_{1,\mu_1}, \dots, u_{m,1}, \dots, u_{m,\mu_{m-1}}, u_1, \dots, u_\mu, z)$.

In the case that the reticular $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -codimension of $f = 1$, F is stably reticular

t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent to (F'_1, \dots, F'_m) , where

- (1) $F'_1 = f_{0,1}(x, y) + (t + a(q, z))\varphi_1(x, y) + u_{1,1}\varphi_2(x, y) \cdots + u_{1,\mu_1-1}\varphi_{1,\mu_1}(x, y) - z$,
- (2) $F'_i = f_{0,i}(x, y) + u_{i,1}\varphi_1(x, y) + \cdots + u_{i,\mu_i}\varphi_{i,\mu_i}(x, y) - z$ for $1 < i < m$,
- (3) $F'_m = f_{0,m}(x, y) + u_{m,1}\varphi_{m,1}(x, y) + \cdots + u_{m,\mu_m-1}\varphi_{m,\mu_m-1}(x, y) - z$,
- (4) $(q_1, \dots, q_n, z) = (u_{1,1}, \dots, u_{1,\mu_1-1}, u_{2,1}, \dots, u_{2,\mu_2}, \dots, u_{m,1}, \dots, u_{m,\mu_m-1}, u_1, \dots, u_\mu, z)$,
- (5) $\frac{\partial a}{\partial u_{1,j}}(0) = 0$ for $j = 1, \dots, \mu_1 - 1$, $\frac{\partial a}{\partial z}(0) = 0$, and $(\frac{\partial^2 a}{\partial u_i \partial u_j}(0))_{i,j=1, \dots, \mu}$ is non-degenerate.

We denote the linear part of a by $v = v_2 + \cdots + v_m$, where v_i depends only on $u_{i,1}, \dots, u_{i,\mu_i}$. Then we may reduce a to

$$a = t \pm u_{2,1} \pm \cdots \pm u_{m,1} \pm u_1^2 \pm \cdots \pm u_\mu^2.$$

Then F is stably reticular t - $(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent to one of the following list:

- $n = 1$;
- $m = 2$;
- ${}^0({}^0A_1{}^0A_1)$: $(y^2 + q - z, y^2 - z)$;
- ${}^1({}^0A_1{}^0A_1)$: $(y^2 + t \pm q^2 - z, y^2 - z)$;
- ${}^1({}^0A_1{}^0A_2)$: $(y^2 + t \pm q - z, y^3 + qy - z)$;
- ${}^1({}^0A_1{}^0B_2)$: $(y^2 + t \pm q - z, x^2 + qx - z)$.
- $m = 3$;
- ${}^1({}^0A_1{}^0A_1{}^0A_1)$: $(y^2 + t - z, y^2 + q - z, y^2 - q - z)$.
- $n = 2$;
- $m = 2$;
- ${}^1({}^0A_1{}^0A_1)$: $(y^2 + t \pm q_1^2 \pm q_2^2 - z, y^2 - z)$;
- ${}^0({}^0A_1{}^0A_2)$: $(y^2 + t \pm q_1 - z, y^3 + q_1y - q_2 - z)$;
- ${}^1({}^0A_1{}^0A_2)$: $(y^2 + t \pm q_1 \pm q_2^2 - z, y^3 + q_1y - z)$;
- ${}^0({}^0A_1{}^0B_2)$: $(y^2 + q_1 - z, x^2 + q_2x - z)$;
- ${}^1({}^0A_1{}^0A_3)$: $(y^2 + t \pm q_1 - z, y^4 + q_1y^2 + q_2y - z)$;
- ${}^1({}^0A_2{}^0A_2)$: $(y^3 + (t \pm q_1)y + q_2 - z, y^3 + q_1y - z)$;
- ${}^1({}^0A_1{}^0B_2)$: $(y^2 + t \pm q_1 \pm q_2^2 - z, x^2 + q_1x - z)$;
- ${}^1({}^0A_1B_3)$: $(y^2 + t \pm q_1 - z, x^3 + q_1x^2 + q_2x - z)$;
- ${}^1({}^0A_1{}^0C_3^\pm)$: $(y^2 + t + q_1 - z, \pm xy + y^3 + q_1y^2 + q_2y - z)$;
- ${}^1({}^0B_2{}^0B_2)$: $(x^2 + (t \pm q_1)x + q_2 - z, x^2 + q_1x - z)$.
- $m = 3$;
- ${}^0({}^0A_1{}^0A_1{}^0A_1)$: $(y^2 + q_1 - z, y^2 + q_2 - z, y^2 - z)$;
- ${}^1({}^0A_1{}^0A_1{}^0A_1)$: $(y^2 + t \pm q_1 \pm q_2^2 - z, y^2 + q_1 - z, y^2 - z)$;
- ${}^1({}^0A_1{}^0A_1{}^0A_2)$: $(y^2 + t \pm q_1 - z, y^2 - z, y^3 + q_1y + q_2 - z)$;
- ${}^1({}^0A_1{}^0A_1{}^0B_2)$: $(y^2 + t \pm q_1 - z, y^2 - z, x^2 + q_1x + q_2 - z)$.
- $m = 4$;
- ${}^1({}^0A_1{}^0A_1{}^0A_1{}^0A_1)$: $(y^2 + t \pm q_1 \pm q_2 - z, y^2 + q_1 - z, y^2 + q_2 - z, y^2 - z)$.

THEOREM 5.6. *Let $r_i = 0$ or 1 for $i = 1, \dots, m$ and $n \leq 2$. Let U be a neighborhood of 0 in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then there exists a residual set $O \subset C_T^0(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)}$ such that for any $\tilde{C} = (\tilde{C}_1, \dots, \tilde{C}_m) \in O$ and $x = (x_1, \dots, x_m) \in U^{(m)}$, the multi-reticular Legendrian unfolding $((\tilde{C}_1)_{x_1}|_{\mathbb{L}^{r_1}}, \dots, (\tilde{C}_m)_{x_m}|_{\mathbb{L}^{r_m}})$ is stable and have*

a generating family which is stably reticular $t\text{-}(\mathcal{P}\text{-}\mathcal{K})_{(m)}$ -equivalent for one of the types in the above list.

Proof. Let $X_i = (X_{i,1}, \dots, X_{i,m})$ ($i = 1, \dots, s$) be all simple singularities with $\sum_{j=1}^m r\text{-}\mathcal{K}\text{-codim } X_{i,j} \leq n + 2$, that is each $X_{i,j}$ is one of simple singularities A, B, C . Let F_{X_i} be $\mathcal{P}_{(m)}$ - C -non-degenerate map germ which is unfolding of X_i for $i = 1, \dots, s$. We choose an extension $C_{(0X_i)}$ and $C_{(1X_i)}$ of multi-reticular Legendrian unfoldings with multi-generating families $F_{(0X_i)}$ and $F_{(1X_i)}$ respectively. Let $l > 16$. We define that

$$O' = \{ \tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)} \mid j_0^l \tilde{C} \text{ is transversal to } [(j^l C_{(jX_i)}(0))] \text{ for all } i = 1, \dots, s \text{ and } j = 0, 1 \}.$$

Let $X'_i = (X_{i,1}, \dots, X_{i,m-1})$ ($i = 1, \dots, s'$) be all simple singularities with $\sum_{j=1}^{m-1} r\text{-}\mathcal{K}\text{-codim } X_{i,j} \leq n + 1$. We choose $C_{(0X'_i)} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m-1)}$ by an analogous way. Then we define that

$$O'' = \{ \tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)} \mid (j_0^l \tilde{C}_1, \dots, j_0^l \tilde{C}_j, \dots, j_0^l \tilde{C}_m) \text{ is transversal to } [(j^l C_{(0X'_i)}(0))] \text{ for all } i = 1 \dots, s' \}.$$

Then O' and O'' are residual sets. We set

$$Y = \{ j^l C(0) \in tC^l(n, m) \mid \text{the codimension of } [j^l C(0)] > 2n + 4 \}.$$

Then Y is an algebraic set in $tC^l(n, m)$. Therefore we can define that

$$O''' = \{ \tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)} \mid j_0^l \tilde{C} \text{ is transversal to } Y \}.$$

Then Y has codimension $> 2n + 4$ because all $\mathcal{P}_{(m)}$ -contact embedding germ with $j^l C(0) \in Y$ adjoin to the above list which are simple. Therefore

$$O''' = \{ \tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))^{(m)} \mid j_0^l \tilde{C}(U^{(m)}) \cap Y = \emptyset \}.$$

Then the set $O = O' \cap O'' \cap O'''$ has the required condition. \square

6. Classification for the cases $r \geq 2$. In order to classify generic bifurcations of wavefronts in the case $r \geq 2$, our methods do not work well. Since the most simple singularity $f(x_1, x_2) = x_1^2 + ax_1x_2 \pm x_2^2(B_{2,2}^{\pm,a})$ have the reticular $\mathcal{P}\text{-}\mathcal{K}$ -codimension 4 and one modality. By the classification list in [6, p.127], we have that all singularities for $r \geq 2$ are adjacent to $B_{2,2}^{\pm,a}$ (cf., [5, p.593]). This means that all of them have modalities and hence Theorem 5.6 does not work well. We need other equivalence relations among reticular Legendrian unfoldings for the cases $r \geq 2$.

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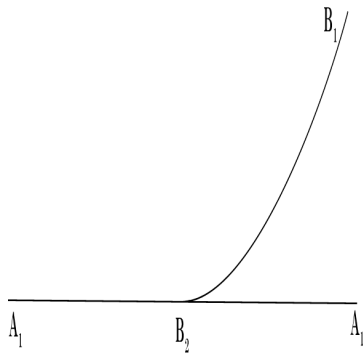


FIG. 3. 0B_2

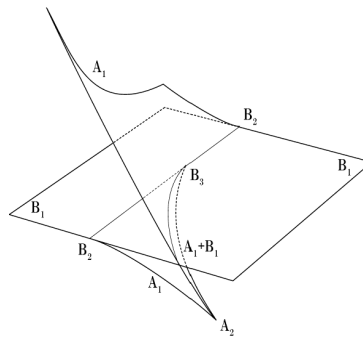


FIG. 4. 0B_3

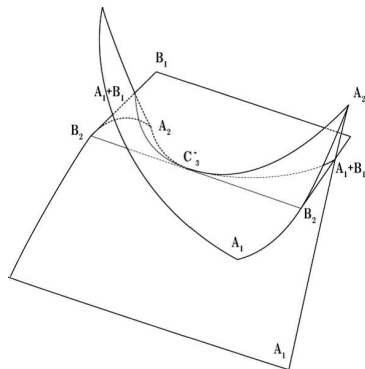


FIG. 5. ${}^0C_3^-$

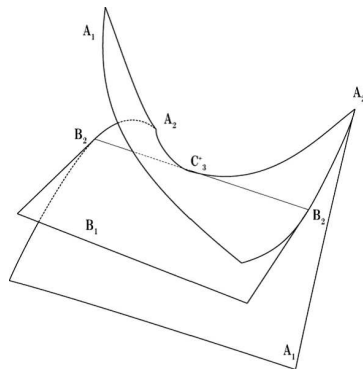


FIG. 6. ${}^0C_3^+$

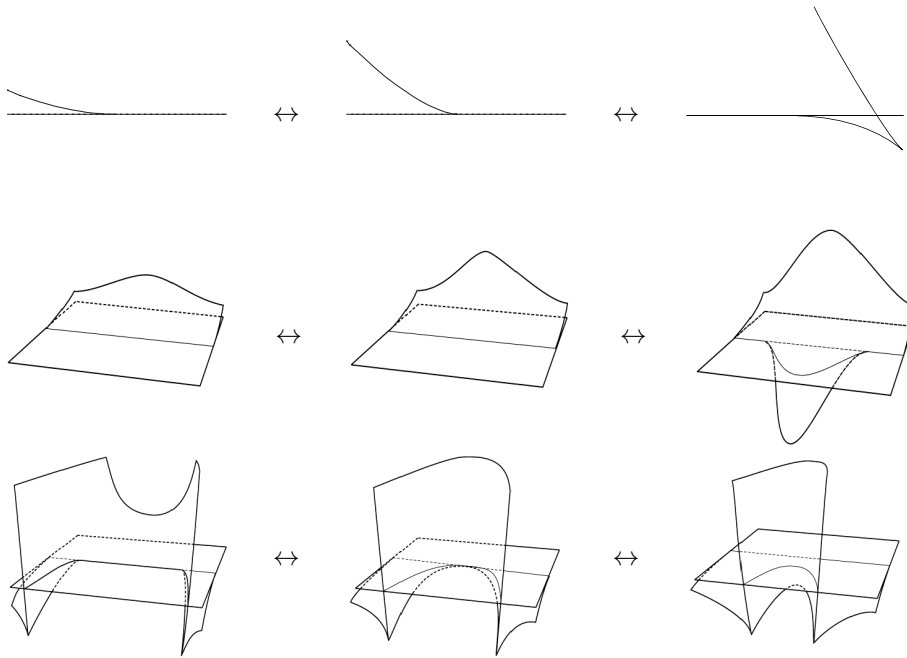


FIG. 7. 1B_3

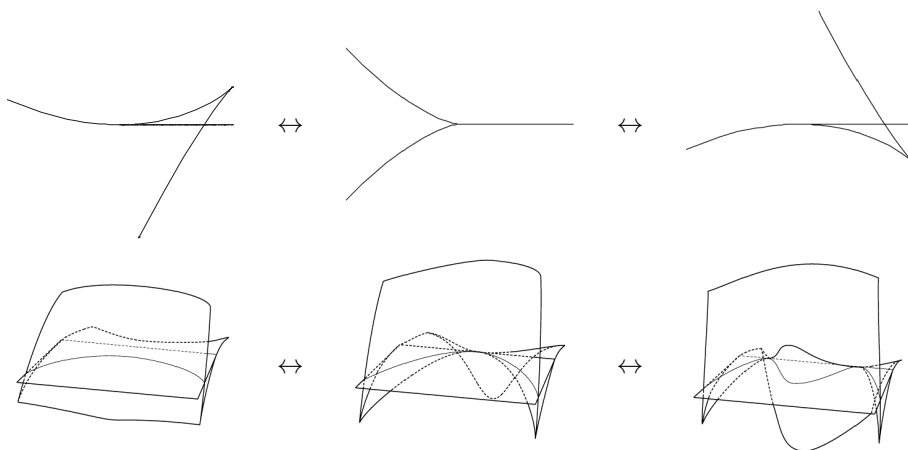


FIG. 8. ${}^1C_3^-$

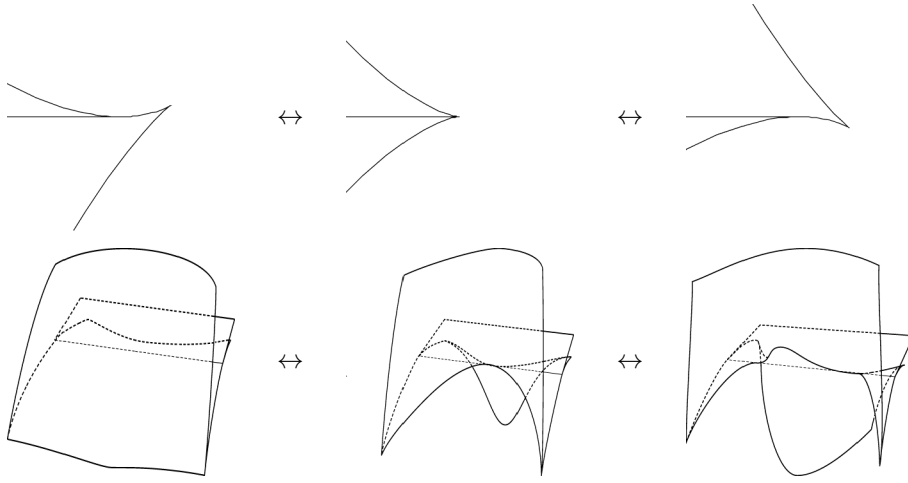


FIG. 9. ${}^1C_3^+$

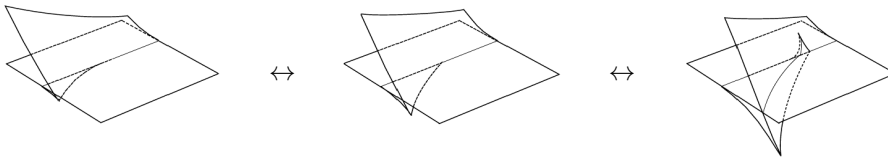


FIG. 10. 1B_4

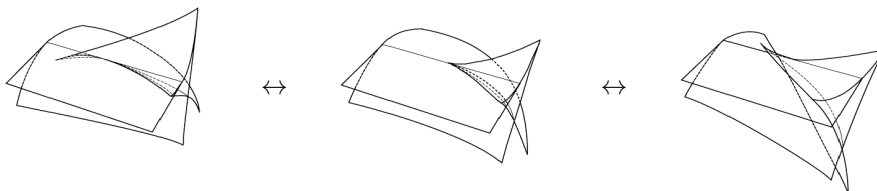


FIG. 11. 1C_4

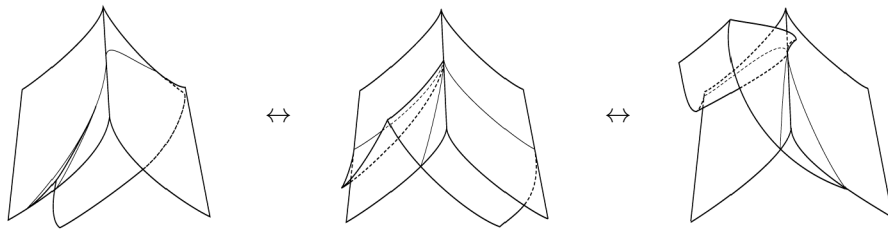


FIG. 12. 1F_4

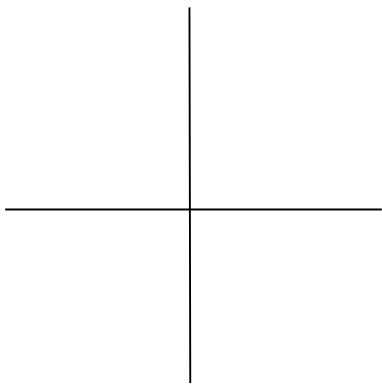


FIG. 13. ${}^0({}^0A_1{}^0A_1)$

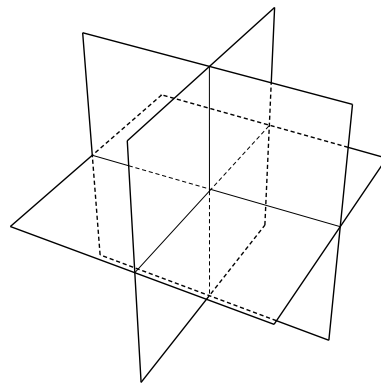


FIG. 14. ${}^0({}^0A_1{}^0A_1{}^0A_1)$

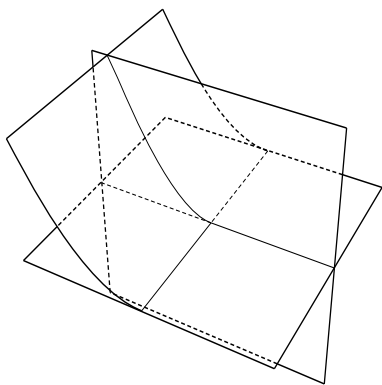


FIG. 15. ${}^0({}^0A_1{}^0B_2)$

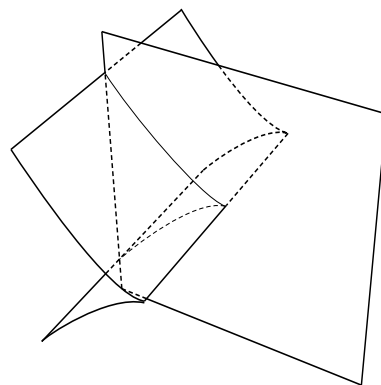


FIG. 16. ${}^0({}^0A_1{}^0A_2)$

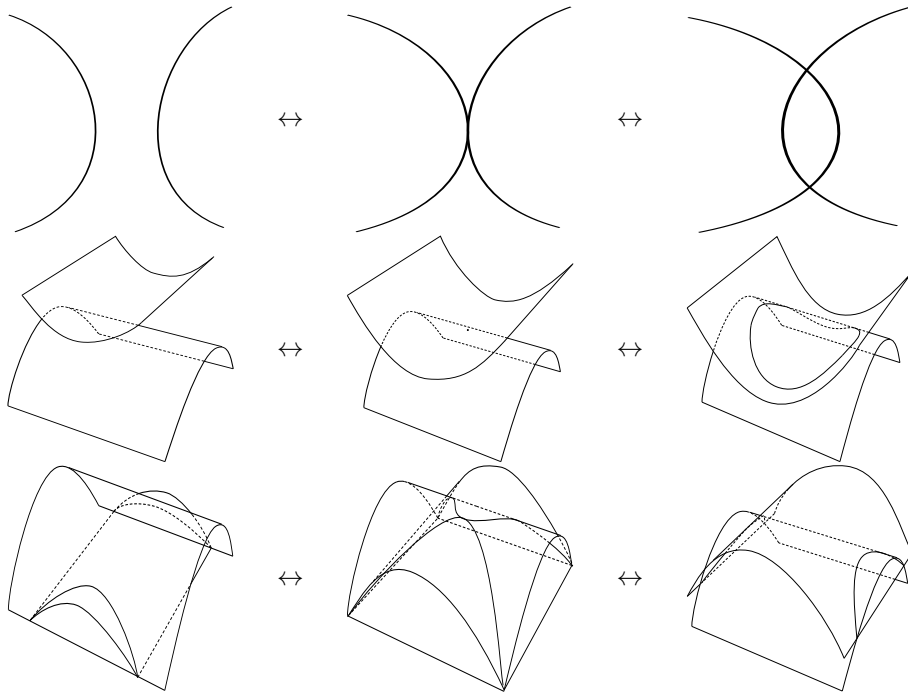


FIG. 17. ${}^1(0A_1^0A_1)$

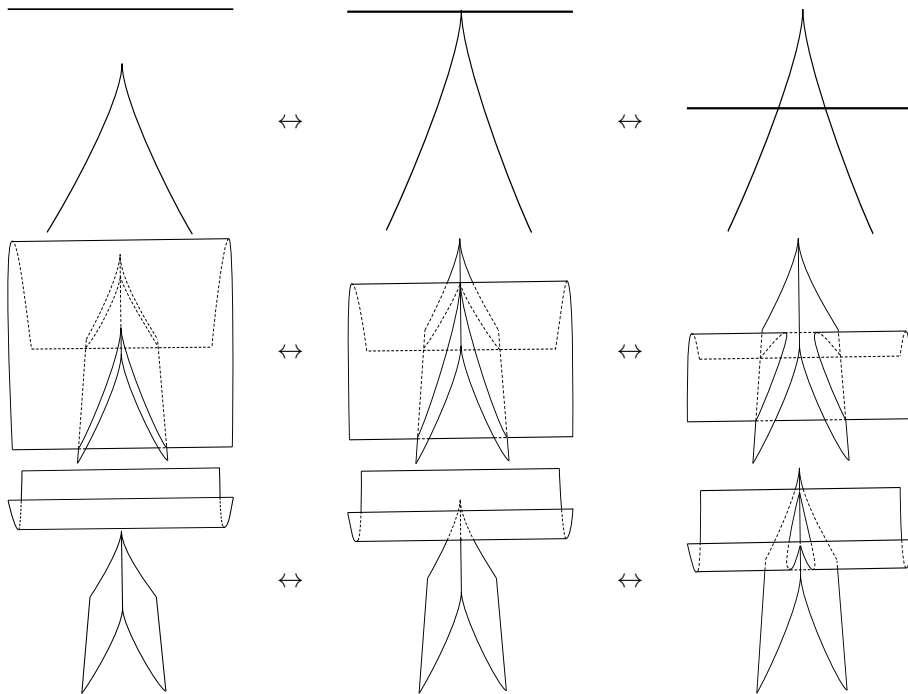


FIG. 18. ${}^1(0A_1^0A_2)$

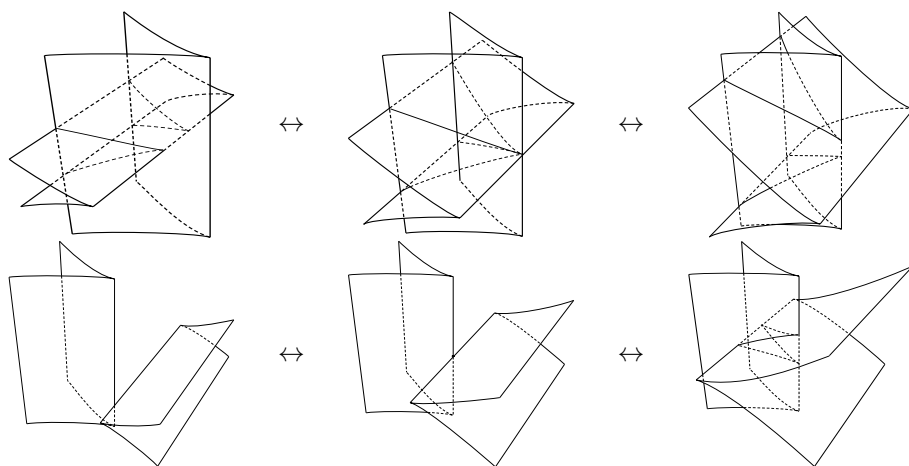


FIG. 19. ${}^1(0A_2^0A_2)$

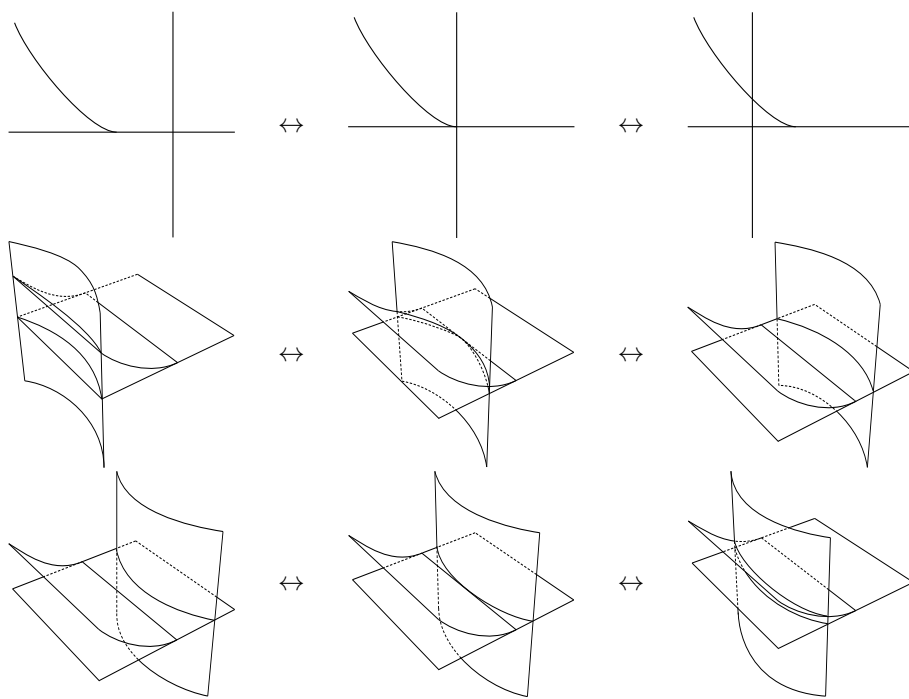


FIG. 20. ${}^1(0A_1^0B_2)$

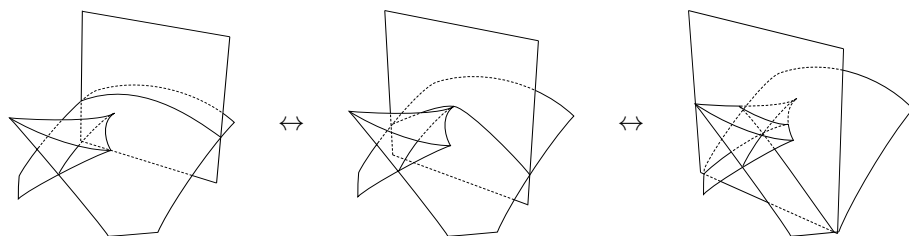


FIG. 21. ${}^1(0A_1^0A_3)$

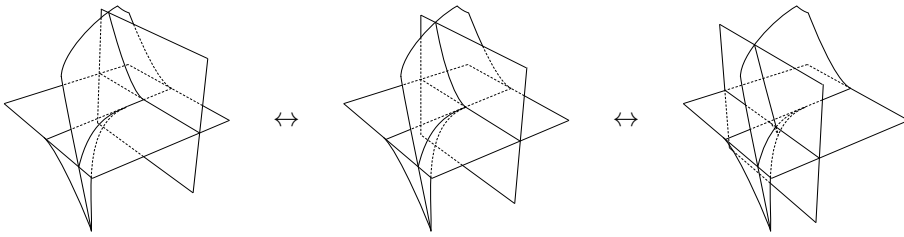


FIG. 22. $1(0A_1^0B_3)$

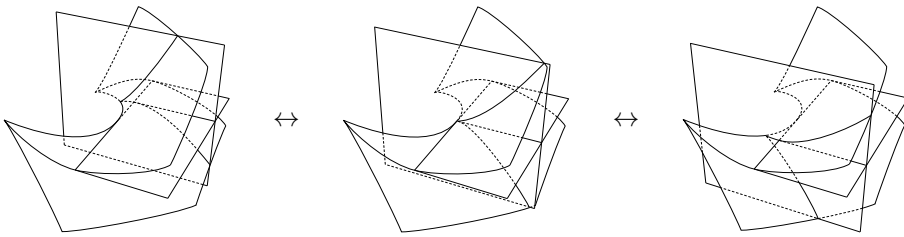


FIG. 23. $1(0A_1^0C_3^+)$

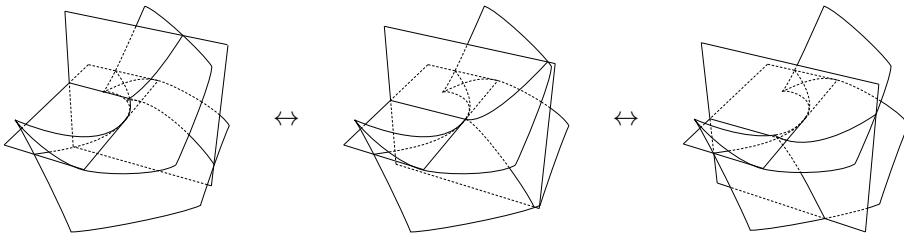


FIG. 24. $1(0A_1^0C_3^-)$

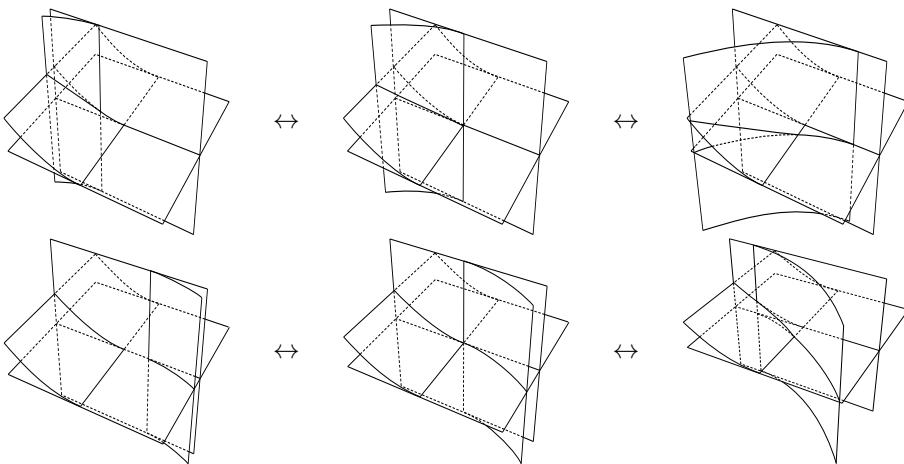


FIG. 25. $1(0B_2^0B_2)$

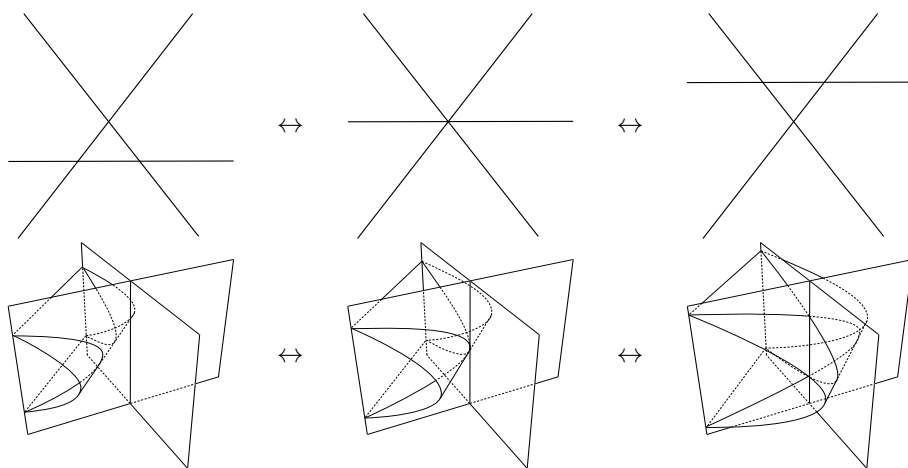


FIG. 26. ${}^1({}^0A_1{}^0A_1{}^0A_1)$

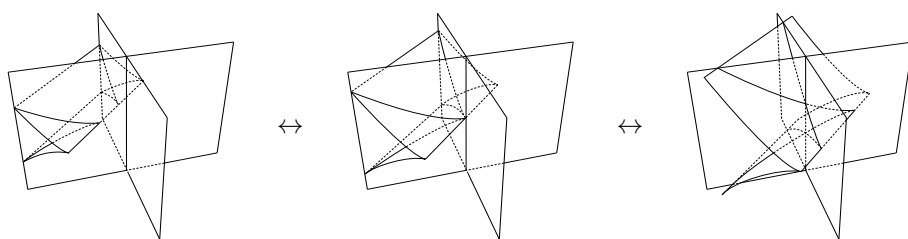


FIG. 27. ${}^1({}^0A_1{}^0A_1{}^0A_2)$

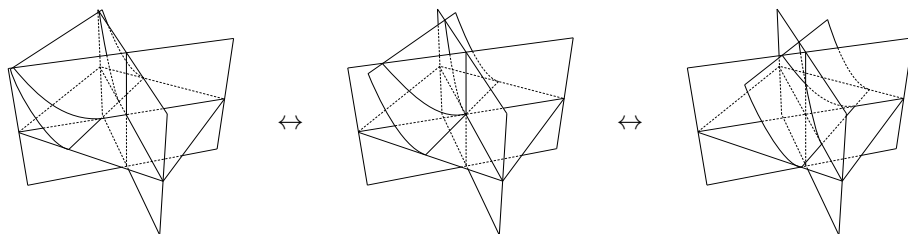


FIG. 28. ${}^1({}^0A_1{}^0A_1{}^0B_2)$

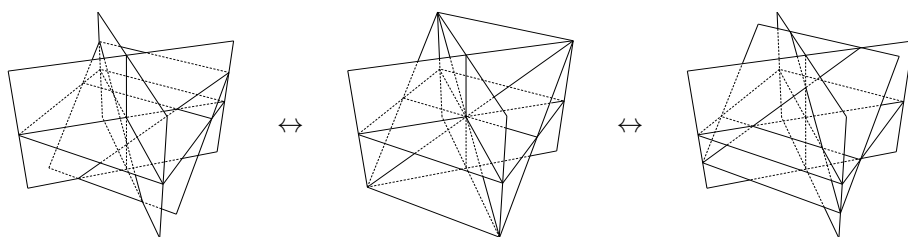


FIG. 29. ${}^1({}^0A_1{}^0A_1{}^0A_1{}^0A_1)$