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C^1 MEASURE RESPECTING MAPS PRESERVE BV IFF THEY HAVE BOUNDED DERIVATIVE*

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Abstract. If $\Omega_j \in \mathbb{R}^d$ ($d \geq 2$) are bounded open subsets and $\Phi \in C^1(\Omega_1\,;\,\Omega_2)$ respects Lebesgue measure and satisfies $F \circ \Phi \in BV(\Omega_1)$ for all $F \in BV(\Omega_2)$ then Φ is uniformly Lipschitzean.

Key words. C^1 measure respecting maps, BV regularity, Uniformly Lipschitzean.

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The problem addressed in this note is motivated by the study of the propagation of regularity in the transport by vector fields with bounded divergence,

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} a_j(t,x)\frac{\partial u}{\partial x_j} \;=\; 0, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad t > 0, \tag{1}$$

where $x = (x_1, x_2, \cdots, x_d)$ and,
(2)

$$\mathbf{a} := (a_1, \cdots, a_d) \;\in\; L^\infty\big([0,T]\times\mathbf{R}^d\big), \quad \mathrm{div}_x\mathbf{a} \;=\; \sum_{j=1}^{d} \partial_{x_j} a_j(t,x) \;\in\; L^\infty\big([0,T]\times\mathbf{R}^d\big).$$

The recent result of [Am] shows that this suffices to guarantee the uniqueness of L^∞ solutions of Cauchy problem if the vector field $\mathbf{a}$ is of BV regularity.

Then, for arbitrary initial data $u_0(x) \in L^\infty(\mathbf{R}^d)$ there is a unique solution $u(t,x) \in L^\infty\big([0,T]\times\mathbf{R}^d\big)$ with $u|_{t=0} = u_0$. With the same hypotheses, there is a well defined flow Φ_t and the solution is given by $u(t) = u_0 \circ \Phi_{-t}$. The flow respects Lebesgue measure in the sense of (3) below.

We have given examples [CLR2] which show that such transport equations do not in general propagate either Hölder or BV regularity. The counterexamples had flows which were mostly smooth with small singular sets. Thus there were large open sets on which the flows were C^1 maps. On those sets, the following result shows that BV preservation implies that the flow must of necessity be uniformly Lipschitzean. In the examples in [CLR2], it is easily seen that the the flows are not uniformly Lipschitzean.

The example (shown to us by L. Ambrosio) of a measure preserving $\Phi :]0,2[\to]-1,1[$

$$\Phi(x) = x \quad \text{for} \quad 0 < x < 1\,, \quad \Phi(x) = x - 2 \quad \text{for} \quad 1 < x < 2\,,$$

shows that measure preserving maps which are smooth except for jumps, can preserve BV without being Lipschitzean. The following result shows that this cannot happen for C^1 maps. The result applies as well to maps which respect but do not preserve measure.

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THEOREM 1. *Suppose that Ω_j are bounded open subsets of $\mathbf{R}^d$ $(d \geq 2)$ and $\Phi \in C^1(\Omega_1\,;\,\Omega_2)$ has the following two properties;*

$$\exists \gamma > 0,\ \forall \text{ Borel subsets } A \subset \Omega_2, \qquad \frac{1}{\gamma}\left|\Phi^{-1}(A)\right| \ \leq\ \left|A\right| \leq\ \gamma\left|\Phi^{-1}(A)\right|, \tag{3}$$

where $\left|\cdot\right|$ *denotes Lebesgue measure, and,*

$$\forall F \in BV(\Omega_2), \qquad F\circ\Phi\ \in\ BV(\Omega_1)\,. \tag{4}$$

Then $\Phi \in W^{1,\infty}(\Omega_1)$.

The proof of Theorem 1 consists of two lemmas.

LEMMA 2. *If $\Phi \in C^1$ but not in $W^{1,\infty}$ then for any positive number M, there exists an $F \in C_0^\infty(\Omega_2)$ such that*

$$||\big(F\circ\Phi\big)'||_{L^1(\Omega_1)}\ \geq\ M\,||F'||_{L^1(\Omega_2)}. \tag{5}$$

Proof. The chain rule implies that for any $F \in C_0^1$ and $1 \leq i \leq d$,

$$\int_{\Omega_1}\Big|\frac{\partial(F\circ\Phi)(x)}{\partial x_i}\Big|\,dx = \int_{\Omega_1}\Big|\sum_{j=1}^{d}\frac{\partial F}{\partial y_j}\big(\Phi(x)\big)\ \frac{\partial \Phi_j(x)}{\partial x_i}\Big|\,dx. \tag{6}$$

Since Φ' is not bounded, there is for any $M > 0$, an $\bar{x} \in \Omega_1$ such that

$$\max_{1\leq i,\ j\leq d}\ \Big|\frac{\partial \Phi_i}{\partial x_j}(\bar{x})\Big|\ \geq\ \frac{8M}{\gamma}\,. \tag{7}$$

Without loss of generality, we may assume that

$$\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big|\ =\ \max_{1\leq i,\ j\leq d}\ \Big|\frac{\partial \Phi_i}{\partial x_j}(\bar{x})\Big|\ \geq\ \frac{8M}{\gamma}\,. \tag{8}$$

Let $\bar{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_d) := \Phi(\bar{x})$. For $0 < \epsilon$ small,

$$N_\epsilon\ :=\ \Big\{y \in \mathbf{R}^d\ :\ |y_1 - \bar{y}_1| < \epsilon, \quad |y_j - \bar{y}_j| < \sqrt{\epsilon}\ \text{ for }\ 2 \leq j\Big\}\ \subset \Omega_2. \tag{9}$$

Define

$$M_\epsilon\ :=\ \Phi^{-1}\Big(N_\epsilon\Big).$$

For ϵ small and $x \in M_\epsilon$,

$$\Big|\frac{\partial \Phi_1}{\partial x_1}(x)\Big| \geq \frac{1}{2}\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big|, \quad \text{and for }\ j \geq 2, \quad \Big|\frac{\partial \Phi_1}{\partial x_j}(x)\Big| \leq 2\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big|\,. \tag{10}$$

Choose $\phi \in C_0^\infty(]-1\,,\,1[)$ satisfying

$$\int_{-\infty}^{\infty}|\phi(z)|dz\ =\ 1\,. \tag{11}$$

Define

$$F \ := \ \phi\left(\frac{y_1 - \bar{y}_1}{\epsilon}\right) \ \prod_{j=2}^{d} \phi\left(\frac{y_j - \bar{y}_j}{\sqrt{\epsilon}}\right) .$$

Then,

$$||F'||_{L^1(\Omega_2)} \ := \ \int_{\Omega_2} \sum_{j=1}^{d} \left|\partial_{y_j} F(y)\right| \, dy \ = \ \int_{N_\epsilon} \sum_{j=1}^{d} \ |\partial_{y_j} F(y)| \ dy$$

$$= \epsilon^{(d-1)/2}(1 + (d-1)\sqrt{\epsilon}) \int_{-\infty}^{\infty} |\phi'(z)| dz. \tag{12}$$

For ϵ small,

$$||F'||_{L^1(\Omega_2)} \ \le \ 2\,\epsilon^{(d-1)/2} \ \int_{-\infty}^{\infty} |\phi'(z)| dz. \tag{13}$$

In view of (6), (9) and (10), we have

$$\int_{\Omega_1} \Big|\frac{\partial(F\circ\Phi)(x)}{\partial x_1}\Big| \ dx \ = \ \int_{\Omega_1} \Big|\sum_{j=1}^{d} \frac{\partial F}{\partial y_j}(\Phi(x)) \ \frac{\partial \Phi_j(x)}{\partial x_1}\Big| \ dx$$

$$\ge \int_{\Omega_1} \Big|\frac{\partial F(\Phi(x))}{\partial y_1} \ \frac{\partial \Phi_1(x)}{\partial x_1}\Big| \ dx \ - \ \int_{\Omega_1} \sum_{j=2}^{d} \Big|\frac{\partial F(\Phi(x)}{\partial y_j} \ \frac{\partial \Phi_j}{\partial x_1}\Big| \ dx$$

$$= \int_{M_\epsilon} \Big|\frac{\partial F(\Phi(x))}{\partial y_1} \ \frac{\partial \Phi_1(x)}{\partial x_1}\Big| \ dx \ - \ \int_{M_\epsilon} \sum_{j=2}^{d} \Big|\frac{\partial F}{\partial y_j} \ \frac{\partial \Phi_j}{\partial x_1}\Big| \ dx$$

$$\ge \Big|\frac{\partial \Phi_1(\bar{x})}{\partial x_1}\Big| \Big[\frac{1}{2}\int_{M_\epsilon} \Big|\frac{\partial F\big(\Phi(x)\big)}{\partial y_1}\Big| \ dx - 2\int_{M_\epsilon} \sum_{j=2}^{d} \Big|\frac{\partial F\big(\Phi(x)\big)}{\partial y_j}\Big| dx\Big] .$$

Using (3) yields

$$\ge \Big|\frac{\partial \Phi_1(\bar{x})}{\partial x_1}\Big| \Big[\frac{\gamma}{2}\int_{N_\epsilon} \Big|\frac{\partial F(y)}{\partial y_1}\Big| \ dy - \frac{2}{\gamma}\int_{N_\epsilon} \sum_{j=2}^{d} \Big|\frac{\partial F(y)}{\partial y_j}\Big| dy\Big]$$

$$= \Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big| \left(\frac{\gamma}{2} - \frac{2}{\gamma}\,\epsilon(d-1)\right) \epsilon^{(d-1)/2} \int_{-\infty}^{\infty} |\phi'(z)| dz\,.$$

Thus, for ϵ small

$$\int_{\Omega_1} \Big|\frac{\partial(F\circ\Phi)(x)}{\partial x_1}\Big| \ dx \ \ge \ \frac{\gamma}{4} \ \Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big| \ \epsilon^{(d-1)/2} \ \int_{-\infty}^{\infty} |\phi'(z)| dz. \tag{14}$$

Estimates (13) and (14) imply

$$\int_{\Omega_1} \Big|\frac{\partial(F\circ\Phi)(x)}{\partial x_1}\Big| \ dx \ \ge \ \frac{\gamma}{8} \ \Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big| \ \big\|F'\big\|_{L^1(\Omega_2)}. \tag{15}$$

(5) follows from (8) and (15). ▯

The next lemma completes the proof.

LEMMA 3. *If $\Phi \in C^1(\Omega_1\,;\Omega_2)$ satisfies hypotheses (3) and (4) of Theorem 1, then there is a constant $C > 0$ so that for all $F \in BV(\Omega_2)$*

$$\left\|(F\circ\Phi)'\right\|_{\mathrm{Var}} \;\le\; C\,\|F'\,\|_{\mathrm{Var}}\,.$$

Proof. The space of functions H belonging to $BV(\Omega_j)$ (modulo the constants) is a Banach space normed by $\|H'\,\|_{\mathrm{Var}}$. Using the Closed Graph Theorem, it suffices to verify that the map from $BV(\Omega_2)$ to $BV(\Omega_1)$ which sends F to $F\circ\Phi$ has closed graph.

To that end, suppose that

$$F_n \quad\to\quad F \qquad \text{in} \quad BV(\Omega_2)\,,$$

and

$$F_n\circ\Phi \quad\to\quad G \quad \text{in} \quad BV(\Omega_1)\,. \tag{16}$$

It suffices to show that $G' = (F\circ\Phi)'$ in the sense of distributions.

Choose representatives $\tilde{F_n}$ of F_n and $\tilde F$ of F so that,

$$\int_{\Omega_2} \tilde{F_n}\;dy \;=\; 0\,, \qquad \int_{\Omega_2} \tilde F\;dy \;=\; 0\,.$$

This together with BV convergence implies that

$$\tilde{F_n} \quad\to\quad \tilde F \quad \text{in} \;\; L^1(\Omega_2)\,. \tag{17}$$

Since

$$|A| \;=\; \left|\Phi\big(\Phi^{-1}(A)\big)\right| \;\ge\; \gamma|\Phi^{-1}(A)|\,,$$

one sees , starting with $g=\chi_A$, that the map sending g to $g\circ\Phi$ is continuous from $L^1(\Omega_2)$ to $L^1(\Omega_1)$. Therefore,

$$\tilde{F_n}\circ\Phi \quad\to\quad \tilde F\circ\Phi \quad \text{in} \;\; L^1(\Omega_1)\,.$$

Therefore

$$(\tilde{F_n}\circ\Phi)' \;\to\; (\tilde F\circ\Phi)' \quad \text{in the sense of distributions}, \mathcal{D}'(\Omega_1)\,.$$

On the other hand (16) implies that,

$$(\tilde{F_n}\circ\Phi)' \quad\to\quad G' \quad \text{in} \;\; \mathcal{D}'(\Omega_1)\,.$$

Therefore $(F\circ\Phi)' = G'$ which completes the proof. ☐

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