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QUANTITATIVE STRONG UNIQUE CONTINUATION FOR THE LAMÉ SYSTEM WITH LESS REGULAR COEFFICIENTS*

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Abstract. In this paper we prove a quantitative form of the strong unique continuation property for the Lamé system when the Lamé coefficients μ is Lipschitz and λ is essentially bounded in dimension $n \geq 2$. This result is an improvement of our earlier result [5] in which both μ and λ were assumed to be Lipschitz.

Key words. Lame system, strong unique continuation property, Carleman estimates.

AMS subject classifications. Primary 35Q72; Secondary 35J55.

1. Introduction.

Assume that Ω is a connected open set containing 0 in $\mathbb{R}^n$ for $n \geq 2$. Let $\mu(x) \in C^{0,1}(\Omega)$ and $\lambda(x), \rho(x) \in L^\infty(\Omega)$ satisfy

$$\begin{cases} \mu(x) \geq \delta_0, & \lambda(x) + 2\mu(x) \geq \delta_0 \quad \forall \text{ a.e. } x \in \Omega, \\ \|\mu\|_{C^{0,1}(\Omega)} + \|\lambda\|_{L^\infty(\Omega)} \leq M_0, & \|\rho\|_{L^\infty(\Omega)} \leq M_0 \end{cases} \tag{1.1}$$

with positive constants δ_0, M_0, where we define

$$\|f\|_{C^{0,1}(\Omega)} = \|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^\infty(\Omega)}.$$

The isotropic elasticity system, which represents the displacement equation of equilibrium, is given by

$$\operatorname{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \operatorname{div} u) + \rho u = 0 \quad \text{in } \Omega, \tag{1.2}$$

where $u = (u_1, u_2, \cdots, u_n)^t$ is the displacement vector and $(\nabla u)_{jk} = \partial_k u_j$ for $j, k = 1, 2, \cdots, n$.

We are interested in the *strong unique continuation property* (SUCP) of (1.2). We say that the solution u of (1.2) satisfies the SUCP if u that vanishes of infinite order at $x_0 \in \Omega$ vanishes identically in Ω. In other words, if u of (1.2) satisfies that

$$\int_{|x-x_0|<R} |u|^2 dx = O(R^N) \quad \forall \quad N \in \mathbb{N}$$

for all sufficiently small R, then $u \equiv 0$ in Ω. On the other hand, we say that u of (1.2) satisfies the *unique continuation property* (UCP) if $u = 0$ in a nonempty open subset of Ω implies that $u \equiv 0$ in Ω. It is obvious that SUCP implies UCP. In this paper, we would like to show that any nontrivial solution of (1.2) can only vanish of finite order at any point of Ω. We also give an estimate of the vanishing order for u, which can be seen as a quantitative description of the SUCP for (1.2). Here we list some of the known results on the SUCP for (1.2):

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- $\lambda, \mu \in C^{1,1}$, $n \geq 2$ (quantitative): Alessandrini and Morassi [1].
- $\lambda, \mu \in C^{0,1}$, $n = 2$ (qualitative): Lin and Wang [4].
- $\lambda \in L^\infty, \mu \in C^{0,1}$, $n = 2$ (qualitative): Escauriaza [2].
- $\lambda, \mu \in C^{0,1}$, $n \geq 2$ (quantitative): Lin, Nakamura, and Wang [5].

In this paper, we relax the regularity assumption on λ in [5] to $\lambda \in L^\infty(\Omega)$. In view of counterexamples by Plis [7] or Miller [3], this regularity assumption seems to be optimal. This improvement was inspired by our recent work on the Stokes system [6]. We now state the main results of the paper. Assume that there exists $0 < R_0 \leq 1$ such that $B_{R_0} \subset \Omega$. Hereafter B_r denotes an open ball of radius $r > 0$ centered at the origin.

Theorem 1.1 (Optimal three-ball inequalities). *There exists a positive number $\tilde{R} < 1$, depending only on n, M_0, δ_0, such that if $0 < R_1 < R_2 < R_3 \leq R_0$ and $R_1/R_3 < R_2/R_3 < \tilde{R}$, then*

$$\int_{|x|<R_2} |u|^2 dx \leq C \left(\int_{|x|<R_1} |u|^2 dx \right)^\tau \left(\int_{|x|<R_3} |u|^2 dx \right)^{1-\tau} \tag{1.3}$$

for $u \in H^1_{loc}(B_{R_0})$ satisfying (1.2) *in B_{R_0}, where the constant C depends on R_2/R_3, n, M_0, δ_0, and $0 < \tau < 1$ depends on R_1/R_3, R_2/R_3, n, M_0, δ_0. Moreover, for fixed R_2 and R_3, the exponent τ behaves like $1/(-\log R_1)$ when R_1 is sufficiently small.*

As in [5], we can derive the lower bound of $\int_{|x|<R} |u|^2 dx$ from Theorem 1.1. We refer the reader to [5] for more details.

Theorem 1.2. *Let $u \in H^1_{loc}(\Omega)$ be a nontrivial solution of* (1.2)*, then there exist positive constants K and m, depending on n, M_0, δ_0 and u, such that*

$$\int_{|x|<R} |u|^2 dx \geq K R^m \tag{1.4}$$

for all R sufficiently small.

Remark 1.3. *Based on Theorem 1.1, the constants K and m in* (1.4) *are explicitly given by*

$$K = \int_{|x|<R_3} |u|^2 dx$$

and

$$m = \tilde{C} + \log \Big(\frac{\int_{|x|<R_3} |u|^2 dx}{\int_{|x|<R_2} |u|^2 dx} \Big),$$

where $\tilde{C}$ is a positive constant depending on n, M_0, δ_0 and R_2/R_3.

From (1.4), we have that u can only vanish of finite order at the origin. In fact, we can prove the following stronger version of SUCP.

Corollary 1.4. *Let $u \in H^1_{loc}(\Omega)$ be a solution of* (1.2) *and for some $N > m$, where m is the constant given in Theorem 1.2,*

$$\int_{|x|<R} |u|^2 dx = O(R^N), \quad \textit{for all sufficiently small} \quad R,$$

then $u \equiv 0$ in Ω.

2. Reduced system and estimates. Here we want to find a reduced system from (1.2). This is a crucial step in our approach. Let us write (1.2) into a non-divergence form:

$$\mu\Delta u + \nabla((\lambda+\mu)\ \mathrm{div}u) + (\nabla u + (\nabla u)^t)\nabla\mu - \mathrm{div}u\nabla\mu + \rho u = 0. \tag{2.1}$$

Dividing (2.1) by μ yields

$$\begin{aligned}
&\Delta u + \frac{1}{\mu}\nabla((\lambda+\mu)\ \mathrm{div}u) + (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \mathrm{div}u\frac{\nabla\mu}{\mu} + \frac{\rho}{\mu}u \\
&= \Delta u + \nabla(\frac{\lambda+\mu}{\mu}\ \mathrm{div}u) + (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \mathrm{div}u(\frac{\nabla\mu}{\mu} + (\lambda+\mu)\nabla(\frac{1}{\mu})) \\
&\quad + \frac{\rho}{\mu}u \\
&= \Delta u + \nabla(a(x)v) + G \\
&= 0, \qquad (2.2)
\end{aligned}$$

where

$$a(x) = \frac{\lambda+\mu}{\lambda+2\mu} \in L^\infty(\Omega), \quad v = \frac{\lambda+2\mu}{\mu}\ \mathrm{div}u$$

and

$$G = (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \mathrm{div}u(\frac{\nabla\mu}{\mu} + (\lambda+\mu)\nabla(\frac{1}{\mu})) + \frac{\rho}{\mu}u.$$

Taking the divergence on (2.2) gives

$$\Delta v + \mathrm{div}G = 0. \tag{2.3}$$

Our reduced system now consists of (2.2) and (2.3). It follows easily from (2.3) that if $u \in H^1_{loc}(\Omega)$, then $v \in H^1_{loc}(\Omega)$.

To prove the main results, we rely on suitable Carleman estimates. Denote $\varphi_\beta = \varphi_\beta(x) = \exp(-\beta\tilde{\psi}(x))$, where $\beta > 0$ and $\tilde{\psi}(x) = \log|x| + \log((\log|x|)^2)$. Note that φ_β is less singular than $|x|^{-\beta}$. We use the notation $X \lesssim Y$ or $X \gtrsim Y$ to mean that $X \le CY$ or $X \ge CY$ with some constant C depending only on n.

LEMMA 2.1. [5, Lemma 2.4] *There exist a sufficiently small number $r_1 > 0$ depending on n and a sufficiently large number $\beta_1 > 3$ depending on n such that for all $w \in U_{r_1}$ and $f = (f_1, \cdots, f_n) \in (U_{r_1})^n$, $\beta \ge \beta_1$, we have that*

$$\begin{aligned}
&\int \varphi_\beta^2(\log|x|)^2(\beta|x|^{4-n}|\nabla w|^2 + \beta^3|x|^{2-n}|w|^2)dx \\
&\lesssim \int \varphi_\beta^2(\log|x|)^4|x|^{2-n}[(|x|^2\Delta w + |x|\mathrm{div}f)^2 + \beta^2\|f\|^2]dx, \qquad (2.4)
\end{aligned}$$

where $U_{r_1} = \{w \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \mathrm{supp}(w) \subset B_{r_0}\}$.

Next, replacing β by $\beta+1$ in (2.4) and note that

$$\varphi_{\beta+1}^2 = \varphi_\beta^2|x|^{-2}(\log|x|)^4.$$

Then, we can get another Carleman estimate.

LEMMA 2.2. *There exist a sufficiently small number $r_1 > 0$ depending on n and a sufficiently large number $\beta_1 > 2$ depending on n such that for all $w \in U_{r_1}$ and $f = (f_1, \cdots, f_n) \in (U_{r_1})^n$, $\beta \geq \beta_1$, we have that*

$$\begin{aligned} &\int \varphi_\beta^2 (\log|x|)^{-2}(\beta|x|^{2-n}|\nabla w|^2 + \beta^3|x|^{-n}|w|^2)dx \\ \lesssim &\int \varphi_\beta^2 |x|^{-n}[(|x|^2\Delta w + |x|\mathrm{div} f)^2 + \beta^2\|f\|^2]dx. \end{aligned} \tag{2.5}$$

In addition to Carleman estimates, we also need the following Caccioppoli's type inequality.

LEMMA 2.3. *Let $u \in (H^1_{loc}(\Omega))^n$ be a solution of* (1.1). *Then for any $0 < a_3 < a_1 < a_2 < a_4$ such that $B_{a_4 r} \subset \Omega$ and $|a_4 r| < 1$, we have*

$$\int_{a_1 r<|x|<a_2 r} |x|^4|\nabla v|^2 + |x|^2|v|^2 + |x|^2|\nabla u|^2 dx \leq C_0 \int_{a_3 r<|x|<a_4 r} |u|^2 dx \tag{2.6}$$

where the constant C_0 is independent of r and u. Here v is defined in (2.2).

The proof of Lemma 2.3 will be given in the next section. Here we would like to outline how to proceed the proofs of main theorems. The detailed arguments can be found in [5] or [6]. Firstly, applying (2.5) to $w = u$, $f = |x|a(x)v$ and using (2.2), we have that

$$\begin{aligned} &\int \varphi_\beta^2 (\log|x|)^{-2}(\beta|x|^{2-n}|\nabla u|^2 + \beta^3|x|^{-n}|u|^2)dx \\ \lesssim &\int \varphi_\beta^2 |x|^{-n}[\big(|x|^2\Delta u + |x|\mathrm{div}(|x|a(x)v)\big)^2 + \beta^2\||x|a(x)v\|^2]dx. \end{aligned} \tag{2.7}$$

Next, applying (2.4) to $w = v$, $f = |x|G$ and using (2.3), we get that

$$\begin{aligned} &\int \varphi_\beta^2 (\log|x|)^2(\beta|x|^{4-n}|\nabla v|^2 + \beta^3|x|^{2-n}|v|^2)dx \\ \lesssim &\int \varphi_\beta^2 (\log|x|)^4|x|^{2-n}[\big(|x|^2\Delta v + |x|\mathrm{div}(|x|G)\big)^2 + \beta^2\||x|G\|^2]dx. \end{aligned} \tag{2.8}$$

Finally, adding $\beta\times$(2.7) and (2.8) together and using (2.6), we can prove Theorem 1.1 and 1.2 as in [5] and [6].

3. Proof of Lemma 2.3. Define $b_1 = (a_1 + a_3)/2$ and $b_2 = (a_2 + a_4)/2$. Let $X = B_{a_4 r}\backslash\bar{B}_{a_3 r}$, $Y = B_{b_2 r}\backslash\bar{B}_{b_1 r}$ and $Z = B_{a_2 r}\backslash\bar{B}_{a_1 r}$. Let $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \xi(x) \leq 1$ and

$$\xi(x) = \begin{cases} 0, & |x| \leq a_3 r, \\ 1, & b_1 r < |x| < b_2 r, \\ 0, & |x| \geq a_4 r. \end{cases} \tag{3.1}$$

From (1.2), we have that

$$
\begin{aligned}
0 &= -\int [\mathrm{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \mathrm{div} u) + \rho u] \cdot (\xi^2 \bar{u}) dx \\
&= \int \sum_{ijkl=1}^{n} [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \partial_{x_j} (\xi^2 \bar{u}_i) dx - \int \rho \xi^2 |u|^2 dx \\
&= \int \xi^2 \sum_{ijkl=1}^{n} [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \partial_{x_j} \bar{u}_i dx \\
&\quad + \int \sum_{ijkl=1}^{n} \partial_{x_j}(\xi^2) [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \bar{u}_i dx - \int \rho \xi^2 |u|^2 dx \\
&= I_1 + I_2, \qquad (3.2)
\end{aligned}
$$

where

$$
I_1 = \int \xi^2 [\sum_{ij=1}^{n} \lambda \partial_{x_j} u_j \partial_{x_i} \bar{u}_i + \sum_{ij=1}^{n} \mu(\partial_{x_i} u_j \partial_{x_j} \bar{u}_i + \partial_{x_j} u_i \partial_{x_j} \bar{u}_i)] dx
$$

and

$$
I_2 = \int \sum_{ijkl=1}^{n} \partial_{x_j}(\xi^2) [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \bar{u}_i dx - \int \rho \xi^2 |u|^2 dx.
$$

Observe that

$$
\begin{aligned}
&\int \xi^2 (2\mu - \frac{\delta_0}{2}) \partial_{x_i} u_j \partial_{x_j} \bar{u}_i dx \\
&= -\int \partial_{x_j} [\xi^2 (2\mu - \frac{\delta_0}{2})] \partial_{x_i} u_j \bar{u}_i dx - \int \xi^2 (2\mu - \frac{\delta_0}{2}) \partial^2_{x_i x_j} u_j \bar{u}_i dx \\
&= -\int \partial_{x_j} [\xi^2 (2\mu - \frac{\delta_0}{2})] \partial_{x_i} u_j \bar{u}_i dx + \int \partial_{x_i} [\xi^2 (2\mu - \frac{\delta_0}{2})] \partial_{x_j} u_j \bar{u}_i dx \\
&\quad + \int \xi^2 (2\mu - \frac{\delta_0}{2}) \partial_{x_j} u_j \partial_{x_i} \bar{u}_i dx. \qquad (3.3)
\end{aligned}
$$

It follows from (3.3) that

$$
\begin{aligned}
I_1 = &\int \xi^2[\sum_{ij=1}^n \lambda\partial_{x_j}u_j\partial_{x_i}\bar{u}_i + \sum_{ij=1}^n (2\mu-\frac{\delta_0}{2})(\partial_{x_i}u_j\partial_{x_j}\bar{u}_i)]dx \\
&+\int\sum_{ij=1}^n \xi^2(\mu-\frac{\delta_0}{2})(\partial_{x_j}u_i\partial_{x_j}\bar{u}_i-\partial_{x_i}u_j\partial_{x_j}\bar{u}_i)dx \\
&+\frac{\delta_0}{2}\int\sum_{ij=1}^n \xi^2\partial_{x_j}u_i\partial_{x_j}\bar{u}_i dx \\
= &\int(2\mu+\lambda-\frac{\delta_0}{2})\xi^2\sum_{ij=1}^n(\partial_{x_j}u_j\partial_{x_i}\bar{u}_i)dx \\
&+\int\sum_{ij=1}^n \xi^2(\mu-\frac{\delta_0}{2})(\partial_{x_j}u_i\partial_{x_j}\bar{u}_i-\partial_{x_i}u_j\partial_{x_j}\bar{u}_i)dx \\
&+\frac{\delta_0}{2}\int\sum_{ij=1}^n \xi^2\partial_{x_j}u_i\partial_{x_j}\bar{u}_i dx + I_3,
\end{aligned} \tag{3.4}
$$

where

$$
I_3 = \sum_{ij=1}^n\int\partial_{x_i}[\xi^2(2\mu-\frac{\delta_0}{2})]\partial_{x_j}u_j\bar{u}_i - \partial_{x_j}[\xi^2(2\mu-\frac{\delta_0}{2})]\partial_{x_i}u_j\bar{u}_i dx.
$$

Since

$$
\begin{aligned}
&\int\sum_{ij=1}^n \xi^2(\mu-\frac{\delta_0}{2})(\partial_{x_j}u_i\partial_{x_j}\bar{u}_i-\partial_{x_i}u_j\partial_{x_j}\bar{u}_i)dx \\
= &\frac{1}{2}\int\sum_{ij=1}^n \xi^2(\mu-\frac{\delta_0}{2})|\partial_{x_j}u_i-\partial_{x_i}u_j|^2 dx,
\end{aligned}
$$

we obtain that

$$
I_1 \ge \frac{\delta_0}{2}\int|\xi\nabla u|^2dx + I_3. \tag{3.5}
$$

Combining (3.2) and (3.5), we have that

$$
\int_Y|\nabla u|^2dx \le \int_X|\xi\nabla u|^2dx \le C_1\int_X|x|^{-2}|u|^2dx,
$$

which implies

$$
\int_Y|x|^2|\nabla u|^2dx \le C_2\int_X|u|^2dx. \tag{3.6}
$$

Here and below all constants $C_1, C_2, \cdots$ depend on δ_0, M_0.

To estimate ∇v, we define $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \chi(x) \le 1$ and

$$
\chi(x) = \begin{cases} 0, & |x| \le b_1 r, \\ 1, & a_1 r < |x| < a_2 r, \\ 0, & |x| \ge b_2 r. \end{cases}
$$

By (2.3), we derive that

$$\begin{aligned}
&\int |\chi(x)\nabla v|^2 dx \\
&= \int \nabla v \cdot \nabla(\chi^2 \bar{v}) dx - 2\int \chi \nabla v \cdot \bar{v} \nabla \chi dx \\
&\le |\int (\mathrm{div} G)\chi^2 \bar{v} dx| + 2\int |\chi \nabla v \cdot \bar{v} \nabla \chi| dx \\
&\le |\int (\mathrm{div} G)\chi^2 \bar{v} dx| + \frac{1}{4}\int |\chi \nabla v|^2 dx + C_3 \int_Y |x|^{-2}|v|^2 dx \\
&\le C_4 \int_Y |\nabla u|^2 dx + C_4 \int_Y |u|^2 dx + \frac{1}{2}\int |\chi\nabla v|^2 dx + C_4 \int_Y |x|^{-2}|v|^2 dx \\
&\le C_5 \int_Y |x|^{-2}|\nabla u|^2 dx + C_4 \int_Y |u|^2 dx + \frac{1}{2}\int |\chi \nabla v|^2 dx. \qquad (3.7)
\end{aligned}$$

Therefore, we get from (3.7) that

$$\int_Z |\nabla v|^2 dx \le 2C_5 \int_Y |x|^{-2}|\nabla u|^2 dx + 2C_4 \int_Y |u|^2 dx$$

and hence

$$\int_Z |x|^4 |\nabla v|^2 dx \le C_6 \int_Y |x|^2 |\nabla u|^2 dx + C_6 \int_Y |x|^4 |u|^2 dx. \qquad (3.8)$$

Putting together $K\times$(3.6) and (3.8), we have that

$$\begin{aligned}
&K\int_Y |x|^2|\nabla u|^2 dx + \int_Z |x|^4 |\nabla v|^2 dx \\
&\le KC_2 \int_X |u|^2 dx + C_6 \int_Y |x|^2 |\nabla u|^2 dx + C_6 \int_Y |x|^4 |u|^2 dx. \qquad (3.9)
\end{aligned}$$

Choosing $K = 2C_6$ in (3.9) yields

$$\begin{aligned}
&\int_Z |x|^2 |v|^2 dx + \int_Z |x|^2 |\nabla u|^2 dx + \int_Z |x|^4 |\nabla v|^2 dx \\
&\le C_7 \int_Y |x|^2 |\nabla u|^2 dx + C_7 \int_Z |x|^4 |\nabla v|^2 dx \\
&\le C_8 \int_X |u|^2 dx,
\end{aligned}$$

The proof is now complete. □

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