REGULARITY OF THE EXTREMAL SOLUTION IN A MEMS MODEL WITH ADVECTION[*]

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Abstract. We consider the regularity of the extremal solution of the nonlinear eigenvalue problem

$$(S)_\lambda \qquad \begin{cases} -\Delta u + c(x) \cdot \nabla u &=& \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $c(x)$ is a smooth bounded vector field on $\bar{\Omega}$. We show that, just like in the advection-free model ($c \equiv 0$), all semi-stable solutions are smooth if (and only if) the dimension $N \leq 7$. The novelty here comes from the lack of a suitable variational characterization for the semi-stability assumption. We overcome this difficulty by using a general version of Hardy's inequality. The same method applies for the Gelfand problem (i.e., exponential nonlinearity).

Key words. Extremal solution, regularity.

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1. Introduction. The following equation has often been used to model a simple *Micro-Electro-Mechanical System* (MEMS) device:

$$(P)_\lambda \qquad \begin{cases} -\Delta u &=& \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda > 0$ is proportional to the applied voltage and $0 < u(x) < 1$ denotes the deflection of the membrane. This model has been extensively studied, see [9], [10] in regards to the model and [6], [5], [7] for mathematical aspects of $(P)_\lambda$. It is well known (see above references) that there exists some positive finite critical parameter λ^* such that for all $0 < \lambda < \lambda^*$, the equation $(P)_\lambda$ has a smooth minimal stable (see below) solution u_λ, while for $\lambda > \lambda^*$ there are no weak solutions of $(P)_\lambda$ (see [6] for a precise definition of weak solution). Standard elliptic regularity theory yields that a solution u of $(P)_\lambda$ is smooth if and only if $\sup_\Omega u < 1$. One can also show that $\lambda \mapsto u_\lambda(x)$ is increasing and hence one can define the extremal solution

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x),$$

which can be shown to be a weak solution of $(P)_{\lambda^*}$.

Recall that a smooth solution u of $(P)_\lambda$ is said to be *minimal* if any other solution v of $(P)_\lambda$ satisfies $u \leq v$ a.e. in Ω. Such solutions are then *semi-stable* meaning that the principal eigenvalue of the linearized operator

$$L_{u,\lambda} := -\Delta - \frac{2\lambda}{(1-u)^3}$$

in $H_0^1(\Omega)$ is nonnegative. This property can be expressed variationally by the inequality

$$2\lambda \int_\Omega \frac{\psi^2}{(1-u)^3} \le \int_\Omega |\nabla\psi|^2, \qquad \forall\psi \in H_0^1(\Omega). \tag{1}$$

which can be viewed as the nonnegativeness of the second variation of the energy functional associated with $(P)_\lambda$ at u.

Now a question of interest is whether u^* is a smooth solution of $(P)_{\lambda^*}$. It is shown in [6] that this is indeed the case provided $N \le 7$. This result is optimal in the sense that u^* is singular in dimension $N \ge 8$ with Ω taken to be the unit ball.

Our main interest here will be in the regularity of the extremal solution associated with

$$(S)_\lambda \qquad \begin{cases} -\Delta u + c(x)\cdot\nabla u &= \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases}$$

where $c \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$ and where again Ω is a smooth bounded domain in $\mathbb{R}^N$. Modifying the proofs used in analyzing $(P)_\lambda$ one can again show the existence of a positive finite critical parameter λ^* such that for $0 < \lambda < \lambda^*$ there exists a smooth minimal solution u_λ of $(S)_\lambda$, while there are no smooth solutions of $(S)_\lambda$ for $\lambda > \lambda^*$. Moreover, the minimal solutions are also *semi-stable* in the sense that the principal eigenvalue of the corresponding linearized operator

$$L_{u,\lambda,c} := -\Delta + c(x)\cdot\nabla - \frac{2\lambda}{(1-u_\lambda)^3}$$

in $H_0^1(\Omega)$ is non-negative. See [3] where these results are proved for general C^1 convex nonlinearities which are superlinear at ∞. Our main result concerns the regularity of the extremal solution of $(S)_\lambda$.

THEOREM 1. *If $1 \le N \le 7$, then the extremal solution u^* of $(S)_{\lambda^*}$ is smooth.*

REMARK 1. A crucial (in fact the main) ingredient in proving the regularity of u^* in $(P)_\lambda$, is the energy inequality (1) which is used in conjunction with the equation $(P)_\lambda$, to obtain uniform (in λ) L^p-estimates on $(1-u_\lambda)^{-2}$ whenever u_λ is the minimal solution (See [6]). However, the semi-stability of u_λ in the case of $(S)_\lambda$, does not translate into an *energy inequality* which allows the use of arbitrary test functions. Overcoming this will be the major hurdle in proving Theorem 1.

We point out, however, that if $c(x) = \nabla\gamma$ for some smooth function γ on $\overline{\Omega}$, then the semi-stability condition on the minimal solution u_λ of $(S)_\lambda$ translates into

$$2\lambda \int_\Omega \frac{e^{-\gamma}\psi^2}{(1-u_\lambda)^3} \le \int_\Omega e^{-\gamma}|\nabla\psi|^2, \qquad \forall\psi \in H_0^1(\Omega). \tag{2}$$

Then, with slight modifications, one can use the standard approach for $(P)_\lambda$ to obtain the analogous result for $(S)_\lambda$ stated in Theorem 1.

The novel case is therefore when c is a divergence free vector field. Actually, we shall use the following version of the Hodge decomposition, in order to deal with general vector fields c.

LEMMA 1. *Any vector field $c \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$ can be decomposed as $c(x) = -\nabla\gamma + a(x)$ where γ is a smooth scalar function and $a(x)$ is a smooth bounded vector field such that $\operatorname{div}(e^\gamma a) = 0$.*

Proof. By the Krein-Rutman theory, the linear eigenvalue problem

$$\begin{cases} \Delta\alpha + \operatorname{div}(\alpha c) = \mu\alpha & \Omega, \\ (\nabla\alpha + \alpha c) \cdot n = 0 & \partial\Omega, \end{cases} \tag{3}$$

where n is the unit outer normal on $\partial\Omega$, has a positive solution α in Ω when μ is the principal eigenvalue. Integrating the equation over Ω, one sees that $\mu = 0$. The positivity of α on the boundary follows from the boundary condition and the maximum principle. In other words, we have that $\Delta\alpha + \operatorname{div}(\alpha c) = 0$ on Ω, and $\alpha > 0$ on $\overline{\Omega}$.

Now define $\gamma := \log(\alpha)$ and $a := c + \nabla\gamma$. An easy computation shows that $\operatorname{div}(e^\gamma a) = 0$. $\square$

Throughout the rest of this note c, a, γ will be defined as above.

2. A general Hardy inequality and non-selfadjoint eigenvalue problems.

Consider the linear eigenvalue problem

$$\begin{cases} -\Delta\phi + c \cdot \nabla\phi - \rho\phi = K\phi & \Omega, \\ \phi = 0 & \partial\Omega, \end{cases} \tag{4}$$

where c is a smooth bounded vector field on Ω, $\rho \in C^\infty(\overline{\Omega})$ and K is a scalar. We assume that (ϕ, K) is the principal eigenpair for (4) and that $\phi > 0$ in Ω, and $K \geq 0$. Note that elliptic regularity theory shows that ϕ is then smooth.

We shall now use a general Hardy inequality to make up for the lack of a variational characterization for the pair (ϕ, K). The following result is taken from [4], which we duplicate here for the convenience of the reader. For a complete discussion on general Hardy inequalities including best constants, attainability and improvements of, see [4]. We should point out that this approach to Hardy inequalities is not new, but it is generally restricted to specific functions E which yield known versions of Hardy inequalities; see [1] and reference within.

LEMMA 2. *Let $A(x)$ denote a uniformly positive definite $N \times N$ matrix with smooth coefficients defined on Ω. Suppose E is a smooth positive function on Ω and fix a constant β with $1 \leq \beta \leq 2$. Then, for all $\psi \in H_0^1(\Omega)$ we have*

$$\int_\Omega |\nabla\psi|_A^2 \geq \frac{\beta(2-\beta)}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2}\psi^2 + \frac{\beta}{2} \int_\Omega \frac{-\operatorname{div}(A\nabla E)}{E}\psi^2, \tag{5}$$

where $\int_\Omega |\nabla\psi|_A^2 = \int_\Omega A(x)\nabla\psi \cdot \nabla\psi$.

Proof. For simplicity we prove the case where $A(x)$ is given by the identity matrix. For the general case, we refer to [4]. Let E_0 denote a smooth positive function defined in Ω and let $\psi \in C_c^\infty(\Omega)$. Set $v := \frac{\psi}{\sqrt{E_0}}$. Then

$$|\nabla\psi|^2 = E_0|\nabla v|^2 + \frac{|\nabla E_0|^2}{4E_0^2}\psi^2 + v\nabla v \cdot \nabla E_0. \tag{6}$$

Integrating the last term by parts gives

$$\int_\Omega v\nabla v \cdot \nabla E_0 = \frac{1}{2} \int_\Omega \frac{-\Delta E_0}{E_0}\psi^2$$

and so integrating (6) gives

$$\int_\Omega |\nabla \psi|^2 \geq \frac{1}{4} \int_\Omega \frac{|\nabla E_0|^2}{E_0^2} \psi^2 + \frac{1}{2} \int_\Omega \frac{-\Delta E_0}{E_0} \psi^2, \tag{7}$$

where we dropped a nonnegative term. So we have the desired result for $\beta = 1$. When $\beta \neq 1$ one puts $E_0 := E^\beta$ into (7) and collects like terms to obtain the desired result. □

We now use the above lemma to obtain an energy inequality valid for the principal eigenpair of (4).

THEOREM 2. *Suppose that the principal eigenpair (ϕ, K) of (4) are such that $\phi > 0$ and $K \geq 0$. Then, for $1 \leq \beta \leq 2$ we have for all $\psi \in H_0^1(\Omega)$,*

$$\int_\Omega e^\gamma |\nabla \psi|^2 \geq \frac{\beta(2-\beta)}{4} \int_\Omega \frac{e^\gamma |\nabla \phi|^2}{\phi^2} \psi^2 + \frac{\beta}{2} \int_\Omega e^\gamma \rho(x) \psi^2 - \frac{\beta}{2} \int_\Omega \frac{e^\gamma a \cdot \nabla \phi}{\phi} \psi^2. \tag{8}$$

Proof. Note that (4) can be rewritten as

$$-\mathrm{div}(e^\gamma \nabla \phi) + e^\gamma a \cdot \nabla \phi = e^\gamma (\rho + K) \phi \qquad \text{in } \Omega,$$

where as mentioned above we are using the decomposition $c = -\nabla\gamma + a$. We now set $E := \phi$ and $A(x) = e^\gamma I$ (where I is the identity matrix) and use (5) along with the above equation to obtain the desired result. Note that we have dropped the nonnegative term involving K. □

3. Proof of theorem 1. For $0 < \lambda < \lambda^*$, we denote by u_λ the smooth minimal semi-stable solution of $(S)_\lambda$. Let (ϕ, K) denote the principal eigenpair associated with the linearization of $(S)_\lambda$ at u_λ. Then $0 < \phi$ in Ω, $0 \leq K$ and (ϕ, K) satisfy

$$\begin{cases} -\Delta\phi + c \cdot \nabla\phi & = & (\frac{2\lambda}{(1-u_\lambda)^3} + K)\phi & \Omega, \\ \phi & = & 0 & \partial\Omega. \end{cases} \tag{9}$$

Again, elliptic regularity theory shows that ϕ is smooth. Consider $c = -\nabla\gamma + a$ to be the decomposition of c described in Lemma 1. We now obtain the main estimate.

THEOREM 3. *For $0 < \lambda < \lambda^*$, $1 < \beta < 2$ and $0 < t < \beta + \sqrt{\beta^2 + \beta}$, we have the following estimate:*

$$\lambda \left(\beta - \frac{t^2}{2t+1} \right) \int_\Omega \frac{e^\gamma}{(1-u_\lambda)^{2t+3}} \leq 2\beta\lambda \int_\Omega \frac{e^\gamma}{(1-u_\lambda)^{t+3}} + \frac{\beta\|a\|_{L^\infty}^2}{4(2-\beta)} \int_\Omega \frac{e^\gamma}{(1-u_\lambda)^{2t}}.$$

Proof. Fix $0 < \beta < 2$, let $0 < t$ and u denote the minimal solution associated with $(S)_\lambda$. We shall use Theorem 2 with $\rho(x) = \frac{2\lambda}{(1-u_\lambda)^3}$. Put $\psi := \frac{1}{(1-u)^t} - 1$ into (8) to obtain

$$t^2 \int_\Omega \frac{e^\gamma |\nabla u|^2}{(1-u)^{2t+2}} \geq \beta\lambda \int_\Omega \frac{e^\gamma}{(1-u)^3} \left(\frac{1}{(1-u)^t} - 1 \right)^2$$

$$+ \frac{\beta}{2} \int_\Omega e^\gamma \left(\frac{(2-\beta)}{2} \frac{|\nabla\phi|^2}{\phi^2} - \frac{a \cdot \nabla\phi}{\phi} \right) \psi^2.$$

Now note that $(S)_\lambda$ can be rewritten as

$$-\mathrm{div}(e^\gamma \nabla u) + e^\gamma a \cdot \nabla u = \frac{\lambda e^\gamma}{(1-u)^2} \qquad \text{in } \Omega,$$

and test this on $\bar{\phi} := \frac{1}{(1-u)^{2t+1}} - 1$ to obtain

$$(2t + 1) \int_\Omega \frac{e^\gamma |\nabla u|^2}{(1 - u)^{2t+2}} + H = \lambda \int_\Omega \frac{e^\gamma}{(1 - u)^2} \left(\frac{1}{(1 - u)^{2t+1}} - 1 \right),$$

where

$$H := \int_\Omega e^\gamma a \cdot \nabla u \left(\frac{1}{(1 - u)^{2t+1}} - 1 \right).$$

One easily sees that $H = 0$ after considering the fact H can be rewritten in the form $\int_\Omega (e^\gamma a) \cdot \nabla G(u)$ for an appropriately chosen function G with $G(0) = 0$. Combining the above two inequalities and dropping some positive terms gives

$$\lambda \left(\beta - \frac{t^2}{2t + 1} \right) \int_\Omega \frac{e^\gamma}{(1 - u)^{2t+3}} \leq 2\beta\lambda \int_\Omega \frac{e^\gamma}{(1 - u)^{3+t}}$$
$$+ \frac{\beta}{2} \int_\Omega e^\gamma \Lambda(x) \left(\frac{1}{(1 - u)^t} - 1 \right)^2$$

where

$$\Lambda(x) := \frac{a \cdot \nabla\phi}{\phi} - \frac{(2 - \beta)}{2} \frac{|\nabla\phi|^2}{\phi^2}.$$

Simple calculus shows that

$$\sup_\Omega \Lambda(x) \leq \frac{\|a\|_{L^\infty}^2}{2(2 - \beta)},$$

which, after substituting into the above inequality, completes the proof of the main estimate. $\square$

Note now that the restriction $t < \beta + \sqrt{\beta^2 + \beta}$ is needed to ensure that the coefficient $\beta - \frac{t^2}{2t+1}$ is positive. It follows then that $\frac{1}{(1-u_\lambda)^2}$ is uniformly bounded (in λ) in $L^p(\Omega)$ for all $p < p_0 := \frac{7}{2} + \sqrt{6} \approx 5.94...$ and after passing to limits we have the same result for the extremal solution u^*.

To conclude the proof of Theorem 1, it suffices to note the following result.

LEMMA 3. *Suppose* $3 \leq N \leq 7$ *and the extremal solution* u^* *satisfies* $\frac{1}{(1-u^*)^2} \in L^{\frac{3N}{4}}(\Omega)$, *then* u^* *is smooth.*

Proof. First note that by elliptic regularity one has $u^* \in W^{2,\frac{3N}{4}}(\Omega)$ and after applying the Sobolev embedding theorem one has $u^* \in C^{0,\frac{2}{3}}(\overline{\Omega})$. Now suppose $\|u\|_{L^\infty} = 1$ so that there is some $x_0 \in \Omega$ such that $u(x_0) = 1$. Then

$$\frac{1}{1 - u(x)} \geq \frac{C}{|x - x_0|^{\frac{2}{3}}}$$

and hence

$$\infty > \int_\Omega \frac{1}{((1 - u^*)^2)^{\frac{3N}{4}}} \geq C \int_\Omega \frac{1}{|x|^N} = \infty,$$

which is a contradiction. It follows that $\frac{1}{(1-u^*)^2} \in L^\infty(\Omega)$, and u^* is therefore smooth. $\square$

Proof of Theorem 1. Using this lemma and the above L^p-bound on $\frac{1}{(1-u^*)^2}$, one sees that u^* is smooth for $3 \leq N \leq 7$. To show the result in dimensions $N = 1, 2$, one needs a slight variation of the above argument. We omit the details, and the interested reader can consult [6] for the proof when $c(x) = 0$. $\blacksquare$

The same approach works on the following Gelfand problem with advection:

$$(T)_\lambda \qquad \begin{cases} -\Delta u + a(x) \cdot \nabla u &= \lambda e^u \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and where a is a smooth bounded vector field. For simplicity we assume that a is divergence free. Then one has the following result:

THEOREM 4. *Suppose $N \leq 9$. Then the extremal solution associated with $(T)_\lambda$ is smooth.*

Proof. Let $1 < t < \beta < 2$ and $0 < \lambda < \lambda^*$. Setting $\psi := e^{tu_\lambda} - 1$ and letting $\bar{\phi}$ be a suitable multiple of $e^{2tu_\lambda} - 1$, follow the proof of Theorem 1 to obtain the inequality

$$\lambda(\beta - t) \int_\Omega e^{(2t+1)u_\lambda} \leq \beta \int_\Omega \left(\frac{a \cdot \nabla\phi}{\phi} - \frac{(\beta - 2)}{2} \frac{|\nabla\phi|^2}{\phi^2} \right) (e^{tu_\lambda} - 1)^2$$
$$+ 2\beta\lambda \int_\Omega e^{(t+1)u_\lambda},$$

where $0 < \phi \in H_0^1 \cap C^\infty$ is the positive principal eigenfunction associated with the linearized problem at u_λ. As before one can easily obtain the estimate that for all $1 < t < 2$ there is some $C_t < \infty$ such that

$$\int_\Omega e^{(2t+1)u_\lambda} \leq C_t,$$

uniformly in λ. Using elliptic regularity theory along with the Sobolev imbedding theorem one sees that u_λ is uniformly bounded in $L^\infty(\Omega)$ provided $N \leq 9$ which gives the desired result after passing to limits. $\blacksquare$

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