ON INSTANT BLOW-UP FOR SEMILINEAR HEAT EQUATIONS WITH GROWING INITIAL DATA*

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Dedicated to Professor Neil S. Trudinger on the occasion of his 65th birthday

Abstract. For a semilinear heat equation admitting blow-up solutions a sufficient condition for nonexistence of local-in-time solutions are obtained. In particular, it is shown that if an initial data tends to infinity at space infinity then there is no local-in-time solution. As an application if the solution blows up at space infinity with least blow-up time, the solution cannot be extendable after blow-up time.

Key words. Instant blow-up, local-in-time solution, semilinear heat equation.

AMS subject classifications. 35K15, 35K55

1. Introduction and main theorems. We consider the initial value problem for a semilinear heat equation of the form

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbf{R}^n, t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{1}$$

Here we assume that the nonlinear term $f \in C^1(\mathbf{R})$ satisfies the following conditions:

$$f \text{ is positive, nondecreasing and convex in } (0, \infty) \text{ and } \int_1^\infty \frac{ds}{f(s)} < \infty. \tag{2}$$

The last condition guarantees that a positive constant solution blows up to infinity in finite time. A typical example of f is $f(u) = e^u$, $|u|^{p-1}u$ and $u_+(\log(u_+ + 1))^p$ for $p > 1$, where $u_+ = \max\{u, 0\}$.

We are interested in the problem whether there is a local-in-time solution of (1) when an initial data u_0 grows at the space infinity, for example $\lim_{|x| \to \infty} u_0(x) = \infty$.

To be precise by a solution u in $\mathbf{R}^n \times [0, T)$ of (1) we mean that $u \in C(\mathbf{R}^n \times [0, T)) \cap C^{2,1}(\mathbf{R}^n \times (0, T))$ satisfies (1). For a given initial data u_0 let $T^* = T^*(u_0)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If there are several solutions with the same initial data, we interpret that T^* is the supremum of all existence times of these solutions. If $T^* \in (0, \infty)$, we often say that the solution blows up in finite time.

In this paper among other results we shall prove that $T^* = 0$ when the initial data u_0 is growing at the space infinity. In other words there is even no local-in-time solution. We say this phenomenon $T^* = 0$ an *instant blow-up*. We are able to prove that the instant blow-up occurs for more general initial data u_0.

THEOREM 1. *Assume (2) and that $u_0 \in C(\mathbf{R}^n)$ is nonnegative. Assume that there are a sequence $\{x_m\} \subset \mathbf{R}^n$ with $|x_m| \to \infty$ as $m \to \infty$ and a number $r > 0$ such that*

$$\lim_{m \to \infty} b_m = \infty, \text{ with } b_m = \inf\{u_0(x) : |x - x_m| \le r\}. \tag{3}$$

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Then $T^* = 0$, i.e., the instant blow-up occurs provided that only nonnegative solutions are considered.

We may relax the assumption (3) so that $r = r_m$ depends on m provided that

$$\limsup_{m \to \infty} \frac{b_m}{r_m^2 f(b_m)} < \epsilon \qquad (4)$$

with small $\epsilon > 0$, say, $0 < \epsilon < \epsilon_0$; the smallness constant ϵ_0 depends only on the first eigenvalue of $-\Delta$ in a unit ball with the Dirichlet boundary condition so it depends only on the space dimension n. Note that this condition is automatically fulfilled if r_m is independent of m by our assumption (2), which in particular says that f is superlinear. We may relax the assumption on nonnegativity of solution by lower boundedness of the solution in $\mathbf{R}^n \times [0, T']$ for any $T' < T^*$ by a comparison principle provided that f is C^1 in $\mathbf{R}$. The condition (3) is fulfilled when $\lim_{|x| \to \infty} u_0(x) = \infty$.

Our result applies to an extension problem of the solution after it blows up. For example it was proved in [5], [6] that the solution $w(x, t)$ blows up at $T^* = T^*(M) > 0$ when $f(u) = |u|^{p-1} u$ or e^u and the initial data w_0 satisfies $\lim_{|x| \to \infty} w_0(x) = M > 0$. Moreover, blow-up occurs only at the space infinity. This implies that the profile function $w(x, T^*)$ at at $t = T^*$ is continuous (in fact C^2) and positive by a standard linear regularity theory [4] (which has an elliptic counterpart [7]). As in [5], [6] $\lim_{|x| \to \infty} w(x, T^*) = \infty$. If we interpret $w(x, T^*)$ as a new initial data, because of Theorem 1 it is impossible to extend the solution for $T > T^*$. We call this phenomenon non extendable blow-up. This is clearly related to the notion of 'complete blow-up' ([17], [13]), but it is not the same. If w_0 has a direction of mean convergence which is equivalent to say the solution has a least blow-up time, (i.e., the blow-up time T^* agrees with the blow-up time of a spatially constant solution with an initial data $\sup w_0$), then the blow-up profile $w(x, T^*)$ has a blow-up direction (see [4]). As remarked in Appendix the $w(x, T^*)$ fulfills (3). Thus we always observe the non extendable blow-up.

For the space infinity blow-up the reader is refereed to papers [11], [8] [5], [6], [16] [18], [17], [15] and a review paper [4]. There is a nice book [13] for overview of blow up problems.

The reader may wonder whether there is a local-in-time solution with singularity. We say that $u \in C([0, T), L^1_{loc}(\mathbf{R}^n))$ is a strong L^1_{loc} solution of (1) with an initial data $u_0 \in L^1_{loc}$ if all terms in (1) are in $C((0, T); L^1_{loc}(\mathbf{R}^n))$ and satisfy (1) in distribution sense. Here L^1_{loc} is a Frechet space equipped with seminorm $|h|_R = \int_{B(0,R)} |h|(x) dx$, where $B(x, R)$ is an open ball of radius R centered at x. We may replace a solution in Theorem 1 by a strong L^1_{loc} solution. The strong L^1_{loc} solution may have singularity. For example if $f(u) = u^p$ with $p > n/(n-2)$ for $n \geq 3$ then $u(x) = C_p |x|^{-2/(p-1)}$ with $C_p = 2((n-2)p - n)/(p-1)^2$ is a stationary strong L^1_{loc} solution of (1).

The proof of Theorem 1 depends on a classical Kaplan's argument [10] to show the existence of blow-up which uses principal eigenfunctions of the Laplace operator with the Dirichlet condition. We give another argument based on the energy principle developed by Ball [2] and Levine [12] together with a comparison argument. We need not assume positivity of the solution. We assume that a solution is bounded from below.

We assume that

$$\text{there exists } \delta > 0 \text{ such that } s f(s) \geq (2 + \delta) \int_0^s f(\xi) d\xi \text{ for } s > 0, \qquad (5)$$

and

$$(sf(s))^{1/2} \text{ is convex for } s > 0, \quad \int_1^\infty \frac{d\xi}{(\xi f(\xi))^{1/2}} < \infty. \tag{6}$$

Let u_0 satisfy that there exists sequence $\{w_{0,m}\}_{m=1}^\infty$ with

$$w_{0,m} \in L^\infty \cap H_0^1(B) \text{ for } m = 1, 2, \ldots, \tag{7}$$

such that for some $C \in \mathbf{R}$

$$u_0(x + x_m) \geq w_{0,m}(x) \text{ for } x \in B, \tag{8}$$

$$w_{0,m}(x) = C \text{ for } x \in \partial B, \tag{9}$$

and

$$\lim_{m \to \infty} \left\{ \int_B \left(\int_C^{w_{0,m}} f(s)ds - \frac{|\nabla w_{0,m}|^2}{2} \right) dx \right\} = +\infty, \tag{10}$$

where $B = B(0, r)$ with some $r > 0$.

We are now in position to state an instant blow-up result under a different setting.

THEOREM 2. *Assume* (2) *and that* $u_0 \in C(\mathbf{R}^n)$ *is nonnegative. Let* f *satisfy* (5) *and* (6)*. For* u_0 *assume that there exists a sequence* $\{w_{0,m}\}_{m=1}^\infty$ *with* $w_{0,m} \in L^\infty \cap H_0^1(B)$ *satisfying* (8)*,* (9) *and* (10)*. Then* $T^*(u_0) = 0$ *under the assumption that the solution is bounded from below in* $[0, T']$ *for any* $T' < T^*(u_0)$*.*

Note that the assumption (3) implies the existence of such w_m.

The authors were informed of a work Andreucci and Di Benedetto closely related to our present work. In [1] among other results a sufficient condition on initial data for nonexistence of a local-in-time nonnegative solution for $u_t = \Delta u^m + u^p/(1+|x|)^\alpha$ with $m \geq 1$, $p > 1$ and $\alpha \in \mathbf{R}$. In the case of $m = 1$ and $\alpha = 0$ the condition reads

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,1)} u_0(y)dy = \infty. \tag{11}$$

In [1] this is explicitly mentioned for $1 < p < 1 + 2/n$. However, their proof is still valid for all $p > 1$. By the way their main interest is the existence of solution; for example they proved the local existence when

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,1)} u_0(y)dy < \infty$$

for $1 < p < 1 + 2/n$. Our Theorem 1 is included in their results for $f(u) = u^p$. However, the condition (4) is not included in their result even for $f(u) = u^p$ for $p > 1 + 2/n$. In fact, if $u_0 \geq b_m$ on $B(x_m, r_m)$, then $\lim_{m \to \infty} b_m r_m^n = \infty$ is a sufficient condition for (11). Our condition (4) with $f(u) = u^p$ reads $\lim_{m \to \infty} r_m^2 b_m^{p-1} = \infty$. This shows that our condition for $p > 1 + 2/n$ is weaker than their condition.

In [1] they also prove the local existence for $p \geq 1 + 2/n$ when u_0 fulfills

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,1)} u_0^q(y)dy < \infty$$

for some $q > n(p-1)/2$. Our result suggests the nonexistence when u_0 satisfies

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,1)} u_0^{n(p-1)/2}(y) dy \text{ is sufficiently large.}$$

The rest of the paper is organized as follows. In Section 2 we show Theorem 1 by using the argument in [10]. Theorem 2 is proved by the method of [2] and [12] in Section 3. In Appendix we show that if the solution has a blow-up direction, the blow-up profile satisfies (3).

2. Eigenfunction method. We begin by recalling a comparison result.

LEMMA 2.1 (COMPARISON). *Assume* (2) *and that* $f \in C^1(\mathbf{R})$. *Let* $u(x,t)$ *and* $v(x,t)$ *be solutions of* (1) *with initial data* u_0 *and* v_0 *which are continuous in* $\mathbf{R}^n$. *Assume that* v *is bounded in* $\mathbf{R}^n \times [0, T')$ *for any* $T' < T$. *Assume that* u *is bounded from below in* $\{\mathbf{R}^n \times [0, T')\}$ *for any* $T' < T$. *If* $u_0 \geq v_0$ *in* $\mathbf{R}^n$, *then* $u \geq v$ *in* $\mathbf{R}^n \times (0, T)$.

Proof. The proof is based on an maximum principle for a parabolic equation and is standard (see [9], [3] and [14]). We give it for completeness.

We may assume that v is bounded and continuous in $\mathbf{R}^n \times [0, T]$ by taking T smaller. We may assume that u is continuous in $\mathbf{R}^n \times [0, T]$. Suppose that the conclusion were false. Then there would exist a point $(\hat{x}, \hat{y}) \in \mathbf{R}^n \times (0, T)$ such that $w(\hat{x}, \hat{t}) > 0$ for $w = v - u$. We set

$$Q = \{(x,t) \in \mathbf{R}^n \times (0, T) : w(x,t) > 0\}. \tag{12}$$

Since $u < v$ in Q and v is bounded, we see that u is bounded in Q since we have assumed that u is bounded from below. By our assumption Q is a bounded open set in $\mathbf{R}^n \times (0, T)$. Subtracting (1) for u from (1) for v we obtain

$$w_t = \Delta w + b(x,t)w \quad \text{in } Q$$

with $b(x,t) = \int_0^1 f'(u(x,t) + \theta(v(x,t) - u(x,t)))d\theta$. Since u is bounded in Q, we have $M_v = \sup_Q b < \infty$. Thus w solves

$$\begin{cases} w_t \leq \Delta w + M_v w, & (x,t) \in Q, \\ w = 0, & (x,t) \in \partial Q. \end{cases} \tag{13}$$

We set $W(x,t) = e^{-(M_v+1)t} w(x,t)$ to set

$$\begin{cases} W_t \leq \Delta W - W, & (x,t) \in Q, \\ W = 0, & (x,t) \in \partial Q. \end{cases} \tag{14}$$

(If Q is bounded, we immediately conclude that W cannot take a possible maximum in Q. However, Q may be unbounded so we modify W.)

We set $W_\epsilon = W - \epsilon(|x|^2 + At)$ with $A = 2n+1$ and small $\epsilon > 0$ to be determined later. It follow that

$$(\partial_t - \Delta + 1)W_\epsilon < 0, \quad (x,t) \in Q. \tag{15}$$

For $(\tilde{x}, \tilde{t}) \in Q$ there exist $\alpha > 0$ such that $W(\tilde{x}, \tilde{t}) > \alpha$. Let $\epsilon > 0$ small enough. Then we see

$$W_\epsilon(\tilde{x}, \tilde{t}) = W - \epsilon(|\tilde{x}|^2 + A\tilde{t}) \geq \frac{\alpha}{2} > 0 \tag{16}$$

as well as $w = v - u$. We now put $R = \epsilon^{-1/2}(\sup W)^{1/2}$. It is easily seen that $W_\epsilon(x,t) < 0$ for $|x| > R$. Since W_ϵ is continuous in Q, There exists $(\hat{x}, \hat{t}) \in \bar{B}_R \times [0,T) \cap Q$ such that

$$W_\epsilon(\hat{x}, \hat{t}) = \sup\{W_\epsilon(x,t) \in \bar{B}_R \times [0,T] \cap Q\}.$$

It follows that

$$(W_\epsilon)_t(\hat{x}, \hat{t}) \geq 0, \ \Delta W_\epsilon(\hat{x}, \hat{t}) \leq 0, \ \text{and} \ W_\epsilon(\hat{x}, \hat{t}) \geq 0.$$

Thus we have

$$(\partial_t - \Delta + 1)W_\epsilon(\hat{x}, \hat{t}) \geq 0,$$

and we have a contradiction to (15). We thereby get

$$w(x,t) \leq 0,$$

and $u(x,t) \geq v(x,t)$. $\square$

Proof of Theorem 1. Let $\{b_m\}_{m=1}^\infty$ and $\{x_m\}_{m=1}^\infty$ be as in Theorem 1 with $r = r_m$ satisfying (4). Set $\lambda_m > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in $B_{r_m}(0)$, and let $\phi_m(x) \geq 0$ denote the corresponding positive eigenfunction normalized by $\int_{B_{r_m}(0)} \phi_m(x)dx = 1$. By scaling it is easy to observe that

$$\lambda_m = \frac{c}{r_m^2} \tag{17}$$

with some $c > 0$. Define

$$G_m(t) = \int_{B(x_m, r_m)} u(x,t)\phi_m(x - x_m)dx.$$

Let $n_m(x)$ denote the outward unit normal to $B(0, r_m)$ at $x \in \partial B(0, r_m)$. Integrating by parts, by the fact that $\phi_m = 0$ and $\partial \phi_m / \partial n_m \leq 0$ on $\partial B(0, r_m)$ with the unit normal vector n_m, and applying Green's formula and Jensen's inequality, we obtain

$$\begin{aligned}
G_m'(t) &= \int_{B(x_m, r_m)} u_t(x,t)\phi_m(x - x_m)dx \\
&\geq \int_{B(x_m, r_m)} \{(\Delta u(x,t) + f(u(x,t))\}\phi_m(x - x_m)dx \\
&\geq -\lambda_m G_m(t) + f(G_m(t)).
\end{aligned}$$

Thus, we obtain

$$G_m'(t) \geq -\lambda_m G_m(t) + f(G_m(t)). \tag{18}$$

Let us consider the system of ordinary differential equations

$$\begin{cases} g_m'(t) = -\lambda_m g_m(t) + f(g_m(t)), \\ g_m(0) = G_m(0) \geq b_m. \end{cases} \tag{19}$$

Define $T_{g_m} = \sup\{t \geq 0 : g_m(t) < \infty\}$ and $T_{G_m} = \sup\{t \geq 0 : G_m(t) < \infty\}$. Since $G_m \geq g_m$, we obtain $T_{g_m} \geq T_{G_m}$. If r_m is a constant so that $\lambda_m = \lambda$ is depend of m, then

$$T_{g_m} \leq \int_{b_m}^{\infty} \frac{d\xi}{-\lambda\xi + f(\xi)} \to 0 \text{ as } m \to \infty.$$

This implies that $T_{G_m} \to 0$ as $m \to \infty$. In particular for sufficiently large m, $T_{G_m} < T$. This is a contradiction since u is continuous in $\mathbf{R}^n \times [0, T)$.

We shall discuss the case that $r_m \to 0$ as $m \to \infty$ satisfying (4). Consider the solutions of (1) with the initial data b_m. The maximal existence time of the solution denoted by $T^*(b_m)$ is estimated as

$$T^*(b_m) = \int_{b_m}^{\infty} \frac{d\xi}{f(\xi)}.$$

Note that $\lim_{m\to\infty} T^*(b_m) = 0$. Consider the formula;

$$\frac{T^*(b_m)}{T_{g_m}} = \frac{\int_{b_m}^{\infty} d\xi/f(\xi)}{\int_{b_m}^{\infty} d\xi/(-\lambda_m\xi + f(\xi))}. \tag{20}$$

From (4) we may assume that there exist $m_0 \geq 0$ such that

$$\frac{b_m}{r_m^2 f(b_m)} < \epsilon$$

for $m \geq m_0$. From (17) we see that

$$\lambda_m b_m < c\epsilon f(b_m).$$

Since f satisfies (2), we get

$$\lambda_m \xi < c\epsilon f(\xi) \tag{21}$$

for $\xi \geq b_m$. Substituting (21) for (20), we have

$$\frac{T^*(b_m)}{T_{g_m}} > 1 - c\epsilon$$

for $m \geq m_0$. Thus we obtain

$$\lim_{m\to\infty} \frac{T^*(b_m)}{T_{g_m}} > 1 - c\epsilon > 0.$$

Noting that $\lim_{m\to\infty} T^*(b_m) = 0$, we see that $\lim_{m\to\infty} T_{g_m} = 0$. Again we get $T_{G_m} \to 0$ as $m \to \infty$ which is a contradiction. $\blacksquare$

REMARK 2.2. *So far we did not use Lemma 2.1. Even if we consider the sign changing solution for the nonnegative initial data, by comparison (Lemma 2.1) it must be nonnegative provided that it is bounded from below in $\mathbf{R}^n \times [0, T']$ for any $T' < T$.*

3. Energy method. In this section, we prove Theorem 2. We may assume that $C = 0$ in (8), (9) and (10).

For $m = 1, 2, \ldots$, we consider a problem:

$$\begin{cases} (w_m)_t = \Delta w_m + f(w_m), & x \in B, t > 0, \\ w_m(x, 0) = w_{0,m}(x), & x \in B, \\ w_m = 0 & x \in \partial B, \end{cases} \qquad (22)$$

where $w_{0,m} \in L^\infty \cap H_0^1(B)$ satisfies (8), (9) and (10), and $B = B(0, 1)$

By comparison in B we have

$$u(x + x_m, t) \geq w_m(x, t) \quad \text{in } B$$

for $m = 1, 2, \ldots$. Put

$$\phi_m(t) = \int_B w_m^2(x, t) dx \qquad (23)$$

and

$$E_m(t) = \int_B \left(\frac{|\nabla w|^2}{2} - \int_0^w f(\xi) d\xi \right) dx. \qquad (24)$$

The ideas of the proof of next two lemmas are standard and go back to [2] and [12]. We give it for completeness.

LEMMA 3.1 (MONOTONICITY OF ENERGY). *Let $E_m(t)$ be as in (24). Then*

$$E_m'(t) \leq 0.$$

Proof. From the proof of [13, Lemma 17.5] and the fact that

$$\frac{d}{dt} \left(\int_0^w f(\xi) d\xi \right) = f(w) w_t,$$

we get

$$E_m'(t) = - \int_B w_t^2(x, t) dx \leq 0.$$

$\square$

LEMMA 3.2 (DIFFERENTIAL INEQUALITY FOR L^2 NORM). *Define ϕ_m and E_m in (23) and (24). Then*

$$\phi_m'(t) \geq -2E_m(0) + cg(\phi_m(t))$$

with $g(\xi) = (\xi f(\xi))^{1/2}$ and $c = c(\delta)$, where δ is defined in (5).

Proof. Differentiating (23) with respect to t and multiplying $1/2$, we have

$$\frac{1}{2} \phi_m'(t) = \int_B w w_t dx$$

$$= \int_B w (\Delta w + f(w)) dx$$

$$= \int_B (-|\nabla w|^2 + w f(w)) dx$$

From (5) we obtain

$$\frac{1}{2}\phi_m'(t) \geq \int_B \left(-|\nabla w|^2 + \delta w f(w) + 2 \int_0^w f(\xi)d\xi \right) dx$$

$$= -2E_m(t) + \int_B \delta w f(w)dx.$$

From Lemma 3.1 and the fact that $g(\xi) = (\xi f(\xi))^{1/2}$ is convex, we have

$$\frac{1}{2}\phi_m'(t) \geq -2E(0) + cg\left(\int_B w^2 dx \right)$$

$$= -2E(0) + cg(\phi_m)$$

by Jensen's inequality, where $c = \delta|B|$. $\square$

LEMMA 3.3. *Assume that $g \in C[0,\infty)$ is positive, nondecreasing and convex in $[0,\infty)$. Assume that $\int_1^\infty d\xi/g(\xi) \leq C < \infty$. Then there exists a sequence $\{\eta_m\}_{m=1}^\infty$ such that $\lim_{m\to\infty} \eta_m = \infty$ and*

$$\lim_{m\to\infty} \int_1^\infty \frac{d\xi}{\eta_m + g(\xi)} = 0.$$

Proof. Assume that

$$\lim_{m\to\infty} \int_1^\infty \frac{d\xi}{\eta_m + g(\xi)} \geq \epsilon > 0.$$

Then for any $M > 1$

$$\lim_{m\to\infty} \int_M^\infty \frac{d\xi}{\eta_m + g(\xi)} \geq \frac{\epsilon}{2} \quad \text{or} \quad \lim_{m\to\infty} \int_1^M \frac{d\xi}{\eta_m + g(\xi)} \geq \frac{\epsilon}{2}.$$

However, for any $\epsilon > 0$ we can take $M > 1$ large enough such that

$$\lim_{m\to\infty} \int_M^\infty \frac{d\xi}{g(\xi)} < \frac{\epsilon}{2}.$$

Thus we obtain

$$\lim_{m\to\infty} \int_M^\infty \frac{d\xi}{\eta_m + g(\xi)} \leq \lim_{m\to\infty} \int_M^\infty \frac{d\xi}{g(\xi)} < \frac{\epsilon}{2}.$$

On the other hand, for any $M > 1$

$$\int_1^M \frac{d\xi}{\eta_m + g(\xi)} = \int_1^M \frac{g(\xi)}{\eta_m + g(\xi)} \cdot \frac{d\xi}{g(\xi)} \leq \frac{g(M)}{\eta_m + g(M)} \int_1^M \frac{d\xi}{g(\xi)}$$

$$\leq \frac{Cg(M)}{\eta_m + g(M)} \to 0 \text{ as } m \to 0.$$

Thus we have

$$\lim_{m\to\infty} \int_1^M \frac{d\xi}{\eta_m + g(\xi)} < \frac{\epsilon}{2}$$

for any $M > 1$. We thereby have a contradiction. Thus we obtain

$$\lim_{m \to \infty} \int_1^\infty \frac{d\xi}{\eta_m + g(\xi)} = 0.$$

$\square$

Proof of Theorem 2. By Lemma 2.1 we may assume that u is nonnegative. From Lemma 3.2 we have

$$\phi'_m(t) \geq -2E_m(0) + cg(\phi_m(t)).$$

If $E_m(0) < 0$ and $\phi_m(0) > 0$, then there exist a constant $T_m > 0$ such that $\lim_{t \to T_m} \phi_m(t) = \infty$, and

$$T_m \leq \int_{\phi_m(0)}^\infty \frac{d\xi}{-2E_m(0) + cg(\xi)}.$$

From (10) we see that

$$\lim_{m \to \infty} E_m(0) = -\infty.$$

Thus from Lemma 3.3 we obtain

$$\lim_{m \to \infty} T_m = 0.$$

Since $u \geq 0$, by a comparison in B we have $T^* \leq T_m$. This implies $T^*(u_0) = 0$. $\square$

4. Appendix. In this section we shall show that if u_0 is a profile at blow-up having a blow-up direction in the sense of [6] and continuous, then (3) is fulfilled. We consider the solution w of the initial value problem (1) with an initial data w_0 having a direction $\psi \in S^{n-1}$ of means convergence in [4]. One of equivalent definitions reads: there exists a positive constant M such that $0 \leq w_0 \leq M$ and

$$\inf_{x \in B(x_m, r_m)} (w_0(x) - M_m) \geq 0$$

with sequences $\{r_m\}_{m=1}^\infty \subset (0, \infty)$, $\{x_m\}_{m=1}^\infty \subset \mathbf{R}^n$, and $\{M_m\}_{m=1}^\infty$ satisfying $r_m \to \infty$, $M_m \to M$ as $m \to \infty$. It turns out that this condition is equivalent to say that the solution has a least blow-up time (see [16], [4]).

From [16, Theorem 1.5] and [4, Theorem 3.2] the solution w satisfies that for each $R > 0$

$$\lim_{m \to \infty} \sup_{x \in B(x_m, R)} (v(t) - w(x, t)) = 0, \tag{25}$$

where v is a solution of (1) with an initial data M, and the solution w has a blow-up direction at $t = T$. In other words there exist a direction $\psi \in S^{n-1}$, sequences $\{x_m\}_{m=1}^\infty$ and $\{t_m\}_{m=1}^\infty$ such that $x_m/|x_m| \to \psi$ and $w(x_m, t_m) \to \infty$ as $m \to \infty$.

If we let $u_0(x) = w(x, T^*) = \lim_{t \to T^*} w(x, t)$ with $T^* = T^*(M)$, then u_0 has a blow-up direction.

LEMMA 4.1. *Assume that $w_0 \in C(\mathbf{R}^n)$ has a direction of mean convergence. Let w be the solution of (1) with an initial data w_0. Then the blow-up profile $u_0(x) = w(x, T^*)$ fulfills the assumption (3) of Theorem 1.*

Proof. From (25) we see that there exists a sequence $\{(x_m, t_m)\}_{m=1}^{\infty}$ satisfying $t_m \to T^*$ as $m \to \infty$ such that

$$\lim_{m \to \infty} \left\{ \sup_{x \in B(x_m, R)} (v(t_m) - u(x, t_m)) \right\} = 0.$$

Thus, since $\lim_{m \to \infty} v(t_m) = \infty$, we have

$$\lim_{m \to \infty} \left\{ \inf_{x \in B(x_m, R)} u(x, t_m) \right\} = \infty$$

for any $x \in B(0, R)$. We set $b_m = \inf_{x \in B(x_m, R)} u(x, t_m)$. Then $u_0(x) = w(x, T^*)$ satisfies (3). ▯

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