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POINTWISE CONVERGENCE OF THE BOUNDARY LAYER OF THE BOLTZMANN EQUATION FOR THE CUTOFF HARD POTENTIAL*

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Abstract. In this paper, we consider the nonlinear stability of a boundary layer of the Boltzmann equation with the cutoff hard potential when Mach number at far-field is greater than 1. Based on the Green's function for the Cauchy problem constructed in [10] and the weighted energy method, we obtain the estimates for the Green's function of the initial boundary problem and use it to obtain the nonlinear stability with an almost exponential convergent rate to the nonlinear Knudsen layer.

Key words. Boltzmann equation, cutoff hard potential, boundary layer, the Green's function, pointwise estimate.

AMS subject classifications. 82C40

1. Introduction. There are many interesting physics phenomenon related to the Boltzmann boundary layers such as thermo-creep flows, vaporization-condensation, ghost-effect, etc. [16]. The boundary layers are nonlinear stationary solutions of the Boltzmann equation with imposed incoming boundary data $\mathsf{F}(0,t,\xi)|_{\xi^1>0}$:

$$\begin{cases}\partial_t\mathsf{F}(x,t,\xi)+\xi^1\partial_x\mathsf{F}(x,t,\xi)=\dfrac{1}{Kn}B(\mathsf{F},\mathsf{F}),\ (x,t,\xi)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^3,\\ \mathsf{F}(0,t,\xi)|_{\xi^1>0},\ \ \mathsf{F}(x,0,\xi):\ \text{imposed.}\end{cases}$$

Here, the parameter $Kn>0$ is the Knudsen number which is the ratio of the mean free path to the physical dimension. The boundary layer $\bar{\mathsf{F}}$ which we will discuss in this paper satisfies the incoming data $\mathsf{F}_b(\xi)|_{\xi^1>0}$ at the boundary $x=0$ and equals to a global Maxwellian state $\mathsf{M}(\xi)$ at $x=\infty$, i.e.

$$\begin{cases}\xi^1\partial_x\bar{\mathsf{F}}=B(\bar{\mathsf{F}},\bar{\mathsf{F}}), & (x,t,\xi)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^3,\\ \bar{\mathsf{F}}(0,\xi)|_{\xi^1>0}=\mathsf{F}_b(\xi)|_{\xi^1>0}, & \\ \lim\limits_{x\to\infty}\bar{\mathsf{F}}(x,\xi)=\mathsf{M}(\xi), &\end{cases}$$

where the global Maxwellian states $\mathsf{M}=\mathsf{M}_{[\rho,\mathsf{u},T]}$, (ρ,u,T) constant,

$$\begin{cases}\mathsf{M}_{[\rho,\mathsf{u},T]}= & \dfrac{\rho}{(2\pi RT)^{3/2}}\,e^{-\frac{|\xi-\mathsf{u}|^2}{2RT}},\\ \rho>0: & \text{bulk density},\\ u=(u^1,0,0): & \text{bulk fluid velocity},\\ T>0: & \text{bulk temperature},\end{cases}$$

satisfy $B(\mathsf{M},\mathsf{M})=0$ and are constant solutions of the Boltzmann equation. Thus, the sound speed and the Mach number of the far field equilibrium state are given

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respectively by

$$c = \sqrt{\frac{5}{3}T}, \quad \mathcal{M} = \frac{u^1}{c}.$$

Here, the collision operator B is given by

$$B(F, F) = \int\int_{\mathbb{R}^3 \times \mathbb{S}^2} (F(\xi')F(\xi'_*) - F(\xi)F(\xi_*))q(V, \theta)d\xi_* d\omega,$$

where

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega$$

represent the relations between the velocities of two particles before and after an elastic collision. $V = \xi - \xi_*$ is the relative velocity and the collision angle $\theta = \cos^{-1}(\langle V, \omega\rangle / |V|)$ for $\omega \in \mathbb{S}^2$.

In this paper, the collision operator is assumed to be the model for the cutoff hard potential. As a result, $q(V, \theta)$ is not given by $|V \cdot \omega|$ like the hard sphere case. Instead it satisfies (2.5) and (2.6). The details can be found in next section.

Set $\bar{\mathsf{F}} = \mathsf{M} + \mathsf{M}^{\frac{1}{2}}\bar{\mathsf{f}}$. Then $\bar{\mathsf{f}}$ satisfies

$$\begin{cases} \xi^1 \partial_x \bar{\mathsf{f}} = L\bar{\mathsf{f}} + Q(\bar{\mathsf{f}}), & (x, t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3, \\ \bar{\mathsf{f}}(0, \xi)|_{\xi^1 > 0} = \mathsf{f}_b(\xi) = \mathsf{M}^{\frac{1}{2}}(\mathsf{F}_b - \mathsf{M}), \\ \lim_{x \to \infty} \bar{\mathsf{f}}(x, \xi) = 0, \end{cases} \tag{1.1}$$

where

$$L\bar{\mathsf{f}} = 2\mathsf{M}^{-\frac{1}{2}}B(\mathsf{M}, \mathsf{M}^{\frac{1}{2}}\bar{\mathsf{f}}), \quad Q(\bar{\mathsf{f}}) = \mathsf{M}^{-\frac{1}{2}}B(\mathsf{M}^{\frac{1}{2}}\bar{\mathsf{f}}, \mathsf{M}^{\frac{1}{2}}\bar{\mathsf{f}}).$$

The existence theories of boundary layers for the hard sphere model are obtained by energy methods by [1, 5] for the linear case. The energy methods are also applicable to the nonlinear time-asymptotic stability problem when the Mach number is less than -1, [8, 17, 18]. When the Mach number is greater than -1, the energy method alone is not sufficient to yield the nonlinear stability only for the linear stability. With the motivation for the nonlinear time-asymptotic stability of the Boundary layer with Mach number greater than -1, the development of the Green's function was initiated in [13] for planar wave solution and in [14] for 3-dimensional perturbations. The Green's function gives a quantitative prescription of the solutions and it does not require regularity information on the solution as the energy method required. (The energy method will not work for the nonlinear stability is due to that the solutions of an initial boundary value is not necessary continuous at $x = 0$). With the Green's function for the Cauchy problem as a primitive tool, the Green's function for the half-space problem is developed in [15], and this tools eventually lead to prove nonlinear stability of the boundary layer for Mach number greater than -1 in [7] for the hard sphere collision model. In the development [15], a parallel procedure consistent of energy methods and pointwise estimates to form a geometric sequence of function with an exponential decaying to zero is designed to approximate the solutions of the initial boundary value problem for the hard sphere model. This parallel procedure requires a separation of wave patterns in order to implement the vague energy methods and detailed pointwise estimates together.

For the angular cut-off hard potential case, the existence theory of nonlinear boundary layer is obtained by energy method in [6]. The energy method is also applied to prove the nonlinear time-asymptotic stability for Mach number less than -1 in [19]. In [10] the Green's function for the Cauchy problem for a hard potential model has been obtained, and it also revealed that the solution for the hard potential model with angular cut-off is rather complicated than the hard sphere model. The rate of decaying in space is much weaker than that for the hard sphere model. The effect of wave patterns separation becomes weaker. The nonlinear stability of the boundary layer for the hard potential becomes not clear.

In this paper, we aim at a simple but not trivial case the Mach number greater than 1. Since all the wave patterns decay less than those in the hard sphere case (the solution doesn't have exponential or "almost exponential" decaying rate), and sequence of geometric function to approximate the solution of an initial boundary value problem no more decay exponentially in time and space. Thus, one needs to verify the estimates obtained by the energy-pointwise estimates is sufficient to close the nonlinearity.

We consider the nonlinear time-asymptotic stability of a boundary layer $\overline{\mathsf{f}}$ connecting a boundary data $\mathsf{f}_b|_{\xi^1>0}$ at $x=0$ to a Maxwellian with Mach number great than 1 at $x=\infty$:

$$\begin{cases} \partial_t \mathsf{f} + \xi^1 \partial_x \mathsf{f} = L\mathsf{f} + Q(\mathsf{f}), & (x,t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathsf{f}(x,0,\xi) = \mathsf{w}_0(x,\xi) + \overline{\mathsf{f}}(x,\xi), \\ \mathsf{f}(0,t,\xi)|_{\xi^1>0} = \mathsf{f}_b(\xi), \\ \lim\limits_{x\to\infty} \mathsf{f}(x,t,\xi) = 0 \end{cases} \tag{1.2}$$

with a small initial perturbation $\mathsf{w}_0 = \mathsf{f} - \overline{\mathsf{f}}$. The main results are as follows

THEOREM 1.1. *Assume that the boundary data,* $\mathsf{F}_b = \mathsf{M} + \mathsf{M}^{1/2}\mathsf{f}_b$*, and the Maxwellian at* $x=\infty$ *with Mach number greater than 1,* $\mathsf{M}|_{\xi^1>0}$*, are sufficiently close. There exists* $\varsigma_0 > 0$ *such that for any* $0<\varsigma<\varsigma_0$ *the solution* $\mathsf{f}(x,t,t)$ *of* (1.2) *satisfies*

$$\begin{aligned} \|(\mathsf{f}-\overline{\mathsf{f}})(x,t)\|_{L^\infty_{\xi,3}} \le{}& C\varsigma \sum_{i=1}^{3} \frac{1}{\sqrt{1+t}} \left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N} \\ &+ C\varsigma \begin{cases} \displaystyle\sum_{l=1}^{2} \frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x \in (\lambda_1 t, \lambda_3 t), \\ 0 \text{ for } x \in (0,\lambda_1 t)\cup(\lambda_3 t,\infty), \end{cases} \end{aligned} \tag{1.3}$$

whenever the initial data $\mathsf{w}_0(x,\xi)$ *satisfies*

$$|\mathsf{w}_0(x,\xi)| \le \varsigma e^{-|x|} \mathsf{M}^{\frac{1}{2}}.$$

Here, C *is a generic constant and* N *is an arbitrary positive constant.*

THEOREM 1.2. *Suppose that* F *is a solution of a Cauchy problem for the Boltzmann equation with a hard potential with angular cut-off,* $\mathsf{F}_t + \xi^1 \mathsf{F}_x - B(\mathsf{F},\mathsf{F}) = 0.$

There exists $\varsigma_0 > 0$ such that for any $0 < \varsigma < \varsigma_0$ the solution F satisfies

$$\|(\mathsf{F}-\mathsf{M})(x,t)\|_{L^\infty_{\xi,3}} \le C\varsigma \sum_{i=1}^{3} \frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$$
$$+ C\varsigma \begin{cases} \displaystyle\sum_{l=1}^{2} \frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t), \\ 0 \text{ for } x \in (0,\lambda_1 t)\cup(\lambda_3 t,\infty), \end{cases} \tag{1.4}$$

whenever the initial data $\mathsf{F}(x,0,\xi)$ satisfies

$$|\mathsf{F}(x,\xi,0)-\mathsf{M}(\xi)| \le \varsigma e^{-|x|}\mathsf{M}^{\frac{1}{2}}.$$

Remark 1.1. *This theorem is a corollary of the estimates in obtaining the nonlinear coupling for Theorem 1.1 through the Green's function. The proof of this theorem is omitted.*

This paper is organized as follows. In Section 2, we introduce basic results on the Green's functions, on the existence theory of boundary layers and on the collision operators for the Boltzmann equation. In Section 3, we estimate the Green's function for the initial boundary value problem; while in Section 4, we obtain estimates of nonlinear wave couplings. Finally, we prove Theorem 1.1 in the last section.

The difficulty for the hard potential with angular cut-off is caused by that the rate of the collision frequency is slower than the particle velocity $|\xi|$. It results in the space decaying rate is not exponentially fast for particle with high velocity. It also indicated in the structure of a Boltzmann shock layer in Caflisch-Nicolaenko, [2]. The defect in the collision frequency also prevents the semi-group $e^{(-i\xi\eta+L)t}$ from being analytic for the long wave variable $\eta\in\mathbb{C}$ as mentioned in [10].

2. Preliminary.

In this section we will summarize the Green's function for initial value problem developed in [10] for linearized Boltzmann equation around a global Maxwellian $\mathsf{M}=\mathsf{M}_{[1,u,1]}$, $u=(u^1,0,0)$:

$$\begin{cases} \partial_t \mathsf{h} + \xi^1\partial_x \mathsf{h} = L\mathsf{h} \text{ for } x\in\mathbb{R}, t>0, \\ \mathsf{h}(x,0)=\mathsf{h}_0(x), \end{cases} \tag{2.1}$$

2.1. Basic notions and the macro-micro decomposition.

We denote L^2_ξ the restricted Hilbert spaces with given inner products:

$$L^2_\xi \equiv \{\mathsf{h}\in L^2(\mathbb{R}^3)|\mathsf{h}(\xi) : \text{an even function in } \xi^2 \text{ and } \xi^3.\}$$
$$\begin{cases} (\mathsf{h},\mathsf{g}) \equiv \displaystyle\int_{\mathbb{R}^3} \mathsf{h}(\xi)\mathsf{g}(\xi)d\xi \text{ for } \mathsf{h},\mathsf{g}\in L^2_\xi, \\ \|\mathsf{h}\|_{L^2_\xi} = \sqrt{(\mathsf{h},\mathsf{h})}. \end{cases}$$

Two auxiliary inner products $(\cdot,\cdot)_-$ and $(\cdot,\cdot)_+$ on L^2_ξ are introduced due to the presence of a physical boundary:

$$(\mathsf{g},\mathsf{h})_- \equiv \int_{\mathbb{R}^3\cap\{\xi^1\le 0\}} \mathsf{g}(\xi)\mathsf{h}(\xi)d\xi,\ (\mathsf{g},\mathsf{h})_+ \equiv \int_{\mathbb{R}^3\cap\{\xi^1\ge 0\}} \mathsf{g}(\xi)\mathsf{h}(\xi)d\xi,$$

with the corresponding space

$$L^2_{\xi,+} \equiv L^2(\mathbb{R}^3_+), \;\; \mathbb{R}^3_+ \equiv \{(\xi^1,\xi^2,\xi^3) : \xi^1 \geq 0, \xi^2, \xi^3 \in \mathbb{R}\}.$$

We still use the same sets of notions in [13]:

$$\text{Measures locally in } (x,t) : \begin{cases} \|\mathsf{g}\|_{L^2_\xi} \equiv \left(\int_{\mathbb{R}^3} \mathsf{g}(\xi)^2 d\xi\right)^{1/2}, \\ (L^2_\xi, \|\cdot\|_{L^2_\xi}) : \text{a Hilbert space with an inner product} \\ \qquad (\mathsf{h}_1,\mathsf{h}_2) \equiv \int_{\mathbb{R}^3} \mathsf{h}_1(\xi)\mathsf{h}_2(\xi)\, d\xi, \\ \|\mathsf{g}\|_{L^\infty_{\xi,\beta}} \equiv \sup_{\xi\in\mathbb{R}^3} |\mathsf{g}(\xi)|(1+|\xi|)^\beta, \\ \|\mathsf{g}\|_{L^\infty_{\xi,\beta+}} \equiv \sup_{\xi\in\mathbb{R}^3_+}(1+|\xi|^\beta)|\mathsf{g}(\xi)|, \end{cases}$$

$$\text{Measures locally in } t\text{:} \begin{cases} \|\mathsf{h}\|_{L^\infty_x(L^2_\xi)} & \equiv \sup_{x\in\mathbb{R}_+} \|\mathsf{h}(x,\cdot)\|_{L^2_\xi}, \\ \|\mathsf{h}\|_{L^2_x(L^2_\xi)} & \equiv \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^3} \mathsf{h}(x,\xi)^2 d\xi dx\right)^{1/2}, \\ \|\mathsf{h}\|_{H^i_x(L^2_\xi)} & \equiv \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^3} \sum_{j=0}^{i} |\partial_x^j \mathsf{h}(x,\xi)|^2 d\xi dx\right)^{1/2}, \\ \|\mathsf{h}\|_{L^\infty_x(L^\infty_{\xi,\beta})} & \equiv \sup_{x\in\mathbb{R}_+} \|\mathsf{h}(x,\cdot)\|_{L^\infty_{\xi,\beta}}, \\ |||\mathsf{h}||| & \equiv \|\mathsf{h}\|_{L^\infty_x(L^\infty_{\xi,3})}. \end{cases}$$

The null space of L in the restricted Hilbert space L^2_ξ is a three-dimensional vector space with orthogonal basis χ_i, $i = 0, 1, 4$:

$$\ker(L) \equiv \text{span}\{\chi_0,\chi_1,\chi_4\},$$
$$\begin{cases} \chi_0 \equiv \mathsf{M}^{1/2} \\ \chi_1 \equiv (\xi^1 - u^1)\mathsf{M}^{1/2}, \\ \chi_4 \equiv \frac{1}{\sqrt{6}}(|\xi - \mathsf{u}|^2 - 3)\mathsf{M}^{1/2}. \end{cases}$$

The macro-micro decomposition $(\mathsf{P}_0, \mathsf{P}_1)$ on L^2_ξ is given as follows:
for any $\mathsf{g} \in L^2_\xi$,

$$\begin{cases} \mathsf{g} \equiv \mathsf{P}_0\mathsf{g} + \mathsf{P}_1\mathsf{g}(\equiv \mathsf{g}_0 + \mathsf{g}_1), \\ \mathsf{P}_0\mathsf{g} \equiv (\chi_0,\mathsf{g})\chi_0 + (\chi_1,\mathsf{g})\chi_1 + (\chi_4,\mathsf{g})\chi_4, \\ \mathsf{P}_1\mathsf{g} \equiv \mathsf{g} - \mathsf{P}_0\mathsf{g}. \end{cases}$$

The characteristic information of the Euler equations

$$\partial_t \mathsf{P}_0\mathsf{f} + \partial_x \mathsf{P}_0(\xi^1\mathsf{P}_0\mathsf{f}) = 0 \tag{2.2}$$

is computed from the flux operator $\mathsf{P}_0\xi^1$ on $\mathsf{P}_0L^2_\xi$, [12]:

$$
\begin{aligned}
&\dim(\mathsf{P}_0L^2_\xi)=3,\\
&\mathsf{P}_0\xi^1\mathsf{E}_i=\lambda_i\mathsf{E}_i \text{ for } i=1,2,3,\\
&\left\{\lambda_1=-\mathsf{c}+u^1,\ \lambda_2=u^1,\ \lambda_3=\mathsf{c}+u^1\right\},\\
&\begin{cases}
\mathsf{E}_1\equiv\left(\sqrt{\frac{3}{2}}\chi_0-\sqrt{\frac{5}{2}}\chi_1+\chi_4\right),\\
\mathsf{E}_2\equiv\left(-\sqrt{\frac{2}{3}}\chi_0+\chi_4\right),\\
\mathsf{E}_3\equiv\left(\sqrt{\frac{3}{2}}\chi_0+\sqrt{\frac{5}{2}}\chi_1+\chi_4\right),\\
(\mathsf{E}_i,\mathsf{E}_j)=\delta^i_j \text{ (Kronecker's delta function)},
\end{cases}
\end{aligned}
\tag{2.3}
$$

where

$$\mathsf{c}=\sqrt{5/3}$$

is the speed of sound.

LEMMA 2.1. *There exist positive constants* ν_0 *such that, for any* $\mathsf{h}\in L^2_\xi$,

$$
\begin{cases}
(\mathsf{P}_0\mathsf{h},\mathsf{P}_0\mathsf{h})=\displaystyle\sum_{j=1}^{3}(\mathsf{E}_j,\mathsf{h})^2,\\
(\mathsf{P}_0\mathsf{h},\xi^1\mathsf{P}_0\mathsf{h})=\displaystyle\sum_{j=1}^{3}\lambda_j(\mathsf{E}_j,\mathsf{h})^2,\\
(\mathsf{P}_1\mathsf{h},L\mathsf{P}_1\mathsf{h})\le-\nu_0(\mathsf{P}_1\mathsf{h},\nu(\xi)\mathsf{P}_1\mathsf{h}).
\end{cases}
$$

2.2. Grad's lemma on collision operator. For the hard potential model we consider, the linearized collision operator L is of the following form, [6]:

$$
\begin{cases}
L\mathsf{g}(\xi) &=-\nu(\xi-u)\mathsf{g}(\xi)-\mathsf{K}_1\mathsf{g}(\xi)+\mathsf{K}_2\mathsf{g}(\xi),\\
\mathsf{K}_1\mathsf{g}(\xi) &\equiv\int_{\mathbb{R}^3}k_1(\xi-u,\xi_*-u)\mathsf{g}(\xi_*)\,d\xi_*,\\
k_1(\xi,\xi_*) &\le C(|\xi-\xi_*|+|\xi-\xi_*|^{-\delta_1})exp\left(-\dfrac{|\xi-u|^2}{4T}-\dfrac{|\xi_*-u|^2}{4T}\right),\\
\mathsf{K}_2\mathsf{g}(\xi) &\equiv\int_{\mathbb{R}^3}k_2(\xi-u,\xi_*-u)\mathsf{g}(\xi_*)\,d\xi_*,\\
k_2(\xi,\xi_*) &\le C|\xi-\xi_*|^{-1}exp\left(-\dfrac{(|\xi-u|^2-|\xi_*-u|^2)^2}{8T|\xi-\xi_*|^2}-\dfrac{|\xi-\xi_*|^2}{8T}\right).
\end{cases}
\tag{2.4}
$$

In the following discussion, we make the same assumptions on the collision kernel as in [6] and [19].

1. There is $0\le\delta_1<1$ such that

$$0\le q(V,\theta)\le c(|V|+|V|^{-\delta_1})|cos\theta|. \tag{2.5}$$

2. There exists a positive constant $c>0$ such that

$$\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}e^{-|\xi_*|^2/2}q(\xi-\xi_*,\theta)d\xi_*dw\left(\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}e^{-|\xi_*|^2}q(\xi-\xi_*,\theta)d\xi_*dw\right)^{-1}\le c. \tag{2.6}$$

Here, $\theta = \cos^{-1}(\langle \xi - \xi_*, \omega\rangle / |\xi - \xi_*|)$ for $\omega \in \mathbb{S}^2$ is the collision angle.

3. There are constants $0 < \gamma \le \beta \le 1$ such that

$$c_1(1+|\xi|)^\gamma \le \nu(\xi) \le c_2(1+|\xi|)^\beta. \tag{2.7}$$

This lemma follows from direct computations and Carleman's theory, [3], on the negative definiteness of the operator L on $Range(\mathsf{P}_1)$.

LEMMA 2.2 (Grad, [9]). *For any given $\beta \ge 0$ there exist positive constants $C(\beta)$ and C_1 such that*

$$\begin{cases} \|\mathsf{K}\,\mathsf{j}\|_{L^\infty_{\xi,\beta+1}} \le C(\beta)\|\mathsf{j}\|_{L^\infty_{\xi,\beta}}, \\ \|\mathsf{K}\mathsf{j}\|_{L^\infty_{\xi,0}} \le C_1\|\mathsf{j}\|_{L^2_\xi}, \end{cases}$$

where

$$\mathsf{K}j(\xi) \equiv \int_{\mathbb{R}^3} K(\xi,\xi_*)\mathsf{j}(\xi_*)d\xi_*, \qquad K(\xi,\xi_*) = -\mathsf{K}_1(\xi,\xi_*) + \mathsf{K}_2(\xi,\xi_*).$$

The proof of this lemma in [4] for $\beta = 0$ can easily be generalized for any $\beta \in \mathbb{R}$.

LEMMA 2.3. *For any $p \in [1,\infty]$ and $\alpha \in [0,1]$, there exists a constant $C > 0$ such that*

$$\left\| \frac{\nu(\xi)^{-\alpha}\mathsf{B}(\mathsf{f},\mathsf{g})}{\mathsf{M}^{\frac12}} \right\|_{L^p_\xi} \le C\left(\left\| \frac{\nu(\xi)^{1-\alpha}\mathsf{f}}{\mathsf{M}^{\frac12}} \right\|_{L^p_\xi} \left\| \frac{\nu(\xi)^{1-\alpha}\mathsf{g}}{\mathsf{M}^{\frac12}} \right\|_{L^p_\xi} \right). \tag{2.8}$$

This lemma can be gained by direct computations.

2.3. Green's functions. Denote $\mathbb{G}_i(x,t,\xi,\xi_*)$ to be the Green's function for the initial value problem (2.1). The equation for $\mathbb{G}_i$ is

$$\begin{cases} \partial_t\mathbb{G}_i + \xi^1\partial_x\mathbb{G}_i = L\mathbb{G}_i \quad for \quad x\in\mathbb{R}, t>0 \xi,\xi_*\in\mathbb{R}^3, \\ \mathbb{G}_i(x,0,\xi,\xi_*) = \delta(x)\delta^3(\xi-\xi_*). \end{cases} \tag{2.9}$$

The Green's function in the above also satisfies the backward equation:

$$\begin{cases} (-\partial_s - \xi_*\partial_y - L)\mathbb{G}_i(x-y,t-s,\xi,\xi_*) = 0, \\ \mathbb{G}_i(x-y,0,\xi,\xi_*) = \delta^1(y-x)\delta^3(\xi_*-\xi), \quad x,y\in\mathbb{R}, \quad \xi,\xi_*\in\mathbb{R}^3. \end{cases} \tag{2.10}$$

We still keep the following abbreviations: For given $(x,t) \in \mathbb{R}\times\mathbb{R}^+$,

$$\begin{aligned} \mathbb{G}_i(x,t): &\quad \mathsf{h}\in L^2_\xi \longmapsto \mathbb{G}_i(x,t)\mathsf{h} \in L^2_\xi, \\ \mathbb{G}_i(x,t)\mathsf{h}(\xi) &\equiv \int_{\mathbb{R}^3} \mathbb{G}_i(x,t,\xi,\xi_*)\mathsf{h}(\xi_*)d\xi_*. \end{aligned}$$

Let g_{in} be a function satisfying

$$\begin{cases} \mathsf{g}_{in}(x,\xi) \equiv 0 \text{ for } |x| \ge 1, \\ |||\mathsf{g}_{in}||| \equiv \|\mathsf{g}_{in}\|_{L^\infty_x(L^\infty_{\xi,3})} < \infty. \end{cases} \tag{2.11}$$

We recall the following theorem in Lee-Liu-Yu [10] on Green's function $\mathbb{G}_i$ for the initial value problem.

THEOREM 2.1. *([10], Theorem 1) For any given positive integral number N, there exist positive constants $C_j, j = 0, 1, 2, 3, 4$ such that for g_{in} given by (2.11),*

for $|x - u^1 t| \le 2\lambda_3 t$,

$$\|\mathbb{G}_i^t \mathsf{g}_{in}(x)\|_{L^2_\xi} = O(1)\;|||\mathsf{g}_{in}|||\left(\sum_{i=1}^{3} C_0(1+t)^{-\frac{1}{2}}\left(1+\frac{|x-\lambda_i t|^2}{1+t}\right)^{-N} + e^{-(|x|+t)/C_1}\right),$$

for $|x - u^1 t| \ge 2\lambda_3 t$

$$\|\mathbb{G}_i^t \mathsf{g}_{in}(x)\|_{L^2_\xi} \le C_2\;|||\mathsf{g}_{in}|||\;\left(e^{-(\nu_0 t+\nu_1|x|^\gamma t^{1-\gamma})/C_3} + e^{-C_4(|x|+t)^{\frac{2}{3-\gamma}}}\right), \tag{2.12}$$

where

$$\mathbb{G}_i^t \mathsf{g}_{in}(x) \equiv \int_{\mathbb{R}} \mathbb{G}_i(x-y,t)\mathsf{g}_{in}(y)dy,$$

REMARK 2.1. *From Theorem 2 in [10], we can find that when the initial function is the microscopic one i.e. it satisfies $\mathsf{P}_1\mathsf{g}_{in} = \mathsf{g}_{in}$,*

for $|x - u^1 t| \le 2\lambda_3 t$,

$$\|\mathbb{G}_i^t \mathsf{g}_{in}(x)\|_{L^2_\xi} = O(1)\;|||\mathsf{g}_{in}|||\left(\sum_{i=1}^{3} C_0(1+t)^{-1}\left(1+\frac{|x-\lambda_i t|^2}{1+t}\right)^{-N} + e^{-(|x|+t)/C_1}\right). \tag{2.13}$$

2.4. Existence of the boundary layer. The boundary layer problem is given to be

$$\begin{cases} \xi^1 \mathsf{F}_x = Q(\mathsf{F}), & (x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathsf{F}|_{x=0,\xi^1>0} = \mathsf{F}_0(\xi), \\ \mathsf{F} \to \mathsf{M}(\xi)(x \to \infty). \end{cases} \tag{2.14}$$

We have already known that the existence of the boundary layer depends on the Mach number $\mathcal{M}$. Set

$$n+ = \begin{cases} 0, & \mathcal{M} \in (-\infty, -1), \\ 1, & \mathcal{M} \in (-1, 0), \\ 4, & \mathcal{M} \in (0, 1), \\ 5, & \mathcal{M} \in (1, \infty). \end{cases}$$

THEOREM 2.2. *(Theorem 1.1, [6]) Let $\mathcal{M} \ne 0, \pm 1$ and $\beta > 5/2$. Then there exist positive numbers $\epsilon, \epsilon_0, \epsilon_1$ and a C^1 map*

$$\Psi : L^2(\mathbb{R}^3_+, \bar{\sigma}_x^{\frac{1}{2}}(0,\xi)\xi^1 d\xi) \to \mathbb{R}^{n+}, \quad \Psi(0) = 0,$$

such that the following holds:

Suppose that the boundary data F_0 *satisfy*

$$|\mathsf{F}_0(\xi)-\mathsf{M}(\xi)|\le \epsilon_0\bar{\sigma}_x^{-\frac{1}{2}}(0,\xi)e^{-\epsilon\bar{\sigma}(0,\xi)}\mathsf{M}^{\frac{1}{2}}(\xi),$$

where $\bar{\sigma}(x,\xi)$ *is given by*

$$\begin{aligned}\bar{\sigma}(x,\xi)=&\,5(\delta x+l)^{\frac{2}{3-\gamma}}\left(1-\eta\left(\frac{\delta x+l}{(1+|\xi|)^{3-\gamma}}\right)\right)\\&+\left(\frac{\delta x+l}{(1+|\xi|)^{1-\gamma}}+3|\xi|^2\right)\eta\left(\frac{\delta x+l}{(1+|\xi|)^{3-\gamma}}\right),\end{aligned}$$

where $\eta:[0,\infty)\to\mathbb{R}$ *is a smooth non-increasing function,* $\eta(s)=1$ *for* $s\le 1$*,* $\eta(s)=0$ *for* $s\ge 2$*, and* $0\le\eta\le 1$*. The constant* δ *is small enough while* l *is sufficiently large. Then, the problem* (2.14) *admits a unique solution* F *in the class*

$$|\mathsf{F}(x,\xi)-\mathsf{M}(\xi)|\le \epsilon_1\bar{\sigma}_x^{-\frac{1}{2}}(x,\xi)e^{-\epsilon\bar{\sigma}(x,\xi)}\mathsf{M}^{\frac{1}{2}}(\xi), \tag{2.15}$$

if and only if F_0 *satisfies*

$$\Psi(e^{-\epsilon\bar{\sigma}(0,\xi)}\mathsf{M}^{-\frac{1}{2}}(\xi)(\mathsf{F}_0-\mathsf{M}))=0.$$

3. Construction of the Green's function for IBVP. Suppose f and $\bar{\mathsf{f}}$ are the solutions of the following equations respectively:

$$\begin{cases}\mathsf{f}_t+\xi^1\mathsf{f}_x=L\mathsf{f}+Q(\mathsf{f}), & (x,t,\xi)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^3,\\ \mathsf{f}(x,0,\xi)=\mathsf{w}_0(x,\xi)+\bar{\mathsf{f}},\\ \mathsf{f}(0,t,\xi)\big|_{\xi^1>0}=\mathsf{f}_b(\xi),\\ \lim\limits_{x\to\infty}\mathsf{f}(x,t,\xi)=0,\end{cases} \tag{3.1}$$

$$\begin{cases}\xi^1\bar{\mathsf{f}}_x=L\bar{\mathsf{f}}+Q(\bar{\mathsf{f}}), & (x,t,\xi)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^3,\\ \bar{\mathsf{f}}(0,t,\xi)\big|_{\xi^1>0}=\mathsf{f}_b(\xi),\\ \lim\limits_{x\to\infty}\bar{\mathsf{f}}(x,\xi)=0.\end{cases} \tag{3.2}$$

where

$$L\mathsf{g}=2\mathsf{M}^{-\frac{1}{2}}B(\mathsf{M},\mathsf{M}^{\frac{1}{2}}\mathsf{g}),\quad Q(\mathsf{g})=\mathsf{M}^{-\frac{1}{2}}B(\mathsf{M}^{\frac{1}{2}}\mathsf{g},\mathsf{M}^{\frac{1}{2}}\mathsf{g}).$$

Set $\mathsf{w}=\mathsf{f}-\bar{\mathsf{f}}$. Then the equation for w is

$$\begin{cases}\mathsf{w}_t+\xi^1\mathsf{w}_x=L\mathsf{w}+L_\tau\mathsf{w}+Q(\mathsf{w}), & (x,t,\xi)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^3,\\ \mathsf{w}(x,0,\xi)=\mathsf{w}_0(x,\xi),\\ \mathsf{w}(0,t,\xi)\big|_{\xi^1>0}=0,\end{cases} \tag{3.3}$$

where

$$L_\tau\mathsf{w}=2\mathsf{M}^{-\frac{1}{2}}B(\mathsf{M}^{\frac{1}{2}}\bar{\mathsf{f}},\mathsf{M}^{\frac{1}{2}}\mathsf{w}).$$

The Green's function $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y,s)$ for the initial boundary value problem(IBVP) is given by

$$\begin{cases} \partial_t \mathbb{G}_{ib} + \xi^1 \partial_x \mathbb{G}_{ib} - L\mathbb{G}_{ib} = 0, \quad x \ge 0, t > s, \\ \mathbb{G}_{ib}(x,s,\xi,\xi_*;y,s) = \delta(x-y)\delta^3(\xi-\xi_*), \\ \mathbb{G}_{ib}(0,t,\xi,\xi_*;y,s)|_{\xi^1>0} = 0. \end{cases} \tag{3.4}$$

We can prove that $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y,s)$ also satisfies the backward equation

$$\begin{cases} -\partial_s \mathbb{G}_{ib} - \xi_*^1 \partial_y \mathbb{G}_{ib} - L\mathbb{G}_{ib} = 0, \quad y \ge 0, s < t, \\ \mathbb{G}_{ib}(x,t,\xi,\xi_*;y,t) = \delta(x-y)\delta^3(\xi-\xi_*), \\ \mathbb{G}_{ib}(x,t,\xi,\xi_*;0,s)|_{\xi_*^1>0} = 0. \end{cases} \tag{3.5}$$

Another fact is that $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y,s) = \mathbb{G}_{ib}(x,t-s,\xi,\xi_*;y,0)$. Thus we will consider $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y)$ for convenience of presentation. From (3.5), we get that the solution $\mathsf{w}(x,t,\xi)$ of (3.3) can be represented by $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y)$

$$\begin{aligned} \mathsf{w}(x,t,\xi) = & \int_0^\infty \int_{\mathbb{R}^3} \mathbb{G}_{ib}(x,t,\xi,\xi_*;y)\mathsf{w}_0(y,\xi_*)d\xi_* dy \\ & + \int_0^t \int_0^\infty \int_{\mathbb{R}^3} \mathbb{G}_{ib}(x,t-s,\xi,\xi_*;y)\left[L_\tau \mathsf{w} + Q(\mathsf{w})\right](y,s,\xi_*)d\xi_* dy ds. \end{aligned} \tag{3.6}$$

It's obvious that if we get the estimate of $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y)$, the estimate of the solution $\mathsf{w}(x,t,\xi)$ is obtained.

To construct the Green's function $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y)$, we deal with the linear problem first.

$$\begin{cases} \mathsf{h}_t + \xi^1 \mathsf{h}_x - L\mathsf{h} = 0, \quad (x,t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathsf{h}(x,0,\xi) = \mathsf{h}_0(x,\xi;z), \\ \mathsf{h}(0,t,\xi)|_{\xi^1>0} = 0. \end{cases} \tag{3.7}$$

Here, the initial function $\mathsf{h}_0(x,\xi;z)$ is a bounded function with a compact support

$$\left\|(1+|\xi|)^{2m}\mathsf{h}_0(x,\xi;z)\right\|_{L^\infty_{\xi,3}} = \begin{cases} O(1), & |x-z| \le 1, \\ 0, & |x-z| > 1. \end{cases} \tag{3.8}$$

With the Green's function $\mathbb{G}_i(x,t,\xi,\xi_*)$ for the Cauchy's problem, the solution $\mathsf{h}(x,t,\xi)$ can be represented to be

$$\begin{aligned} \mathsf{h}(x,t,\xi) = & \int_0^\infty \int_{\mathbb{R}^3} \mathbb{G}_i(x-y,t,\xi,\xi_*)\mathsf{h}_0(y,\xi_*)d\xi_* dy \\ & + \int_0^t \int_{\mathbb{R}^3} \mathbb{G}_i(x,t-s,\xi,\xi_*)\xi_*^1 \mathsf{h}(0,s,\xi_*)d\xi_* ds. \end{aligned} \tag{3.9}$$

3.1. Energy estimate. Since the representation (3.9) needs the full boundary information, we will use weighted energy estimate to get $\mathsf{h}(0,s,\xi)$. The weight function is defined by the following method:

Let $\eta : [0,\infty) \to \mathbb{R}$ be a smooth non-increasing functions, $\eta(s) = 1$ for $s \le 1$, $\eta(s) = 0$ for $s \ge 2$, and $0 \le \eta \le 1$. For $x \ge 0$, set

$$\sigma(x,\xi) = -5(\delta x + l)^{\frac{2}{3-\gamma}} \left(1 - \eta\left(\frac{\delta x + l}{(1+|\xi|)^{3-\gamma}}\right)\right) - \frac{\delta x + l}{(1+|\xi|)^{1-\gamma}} \eta\left(\frac{\delta x + l}{(1+|\xi|)^{3-\gamma}}\right). \tag{3.10}$$

Denote $\mathsf{g} = \mathsf{h}e^{\epsilon\sigma}$. Then g satisfies

$$\begin{cases} \mathsf{g}_t + \xi^1\mathsf{g}_x - \epsilon\xi^1\sigma_x\mathsf{g} - e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{g} = 0, & (x,t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathsf{g}(x,0,\xi) = e^{\epsilon\sigma}\mathsf{h}_0(x,\xi), \\ \mathsf{g}(0,t,\xi)|_{\xi^1>0} = 0, \end{cases} \tag{3.11}$$

For later use, we introduce several lemmas. The following lemma comes from straightforward calculation.

Lemma 3.1. *There exists a constant $c > 0$ such that*

$$\sigma_x(x,\xi) = \begin{cases} -\delta(1+|\xi|)^{-1+\gamma} & (x,\xi) \in \Omega_1, \\ -c\delta\left((1+|\xi|)^{-1+\gamma} + (\delta x + l)^{-\frac{1-\gamma}{3-\gamma}}\right) & (x,\xi) \in \Omega_2, \\ -\frac{10\delta}{3-\gamma}(\delta x + l)^{-\frac{1-\gamma}{3-\gamma}} & (x,\xi) \in \Omega_3, \end{cases} \tag{3.12}$$

where

$$\Omega_1 = \{(x,\xi) | \delta x + l < (1+|\xi|)^{3-\gamma}\}, \quad \Omega_3 = \{(x,\xi) | \delta x + l \geq 2(1+|\xi|)^{3-\gamma}\},$$

$$\Omega_2 = \{(x,\xi) | (1+|\xi|)^{3-\gamma} \leq \delta x + l < 2(1+|\xi|)^{3-\gamma}\}.$$

Furthermore,

$$|\sigma_{xx}(x,\xi)| \leq \begin{cases} 0 & (x,\xi) \in \Omega_1, \\ c(\delta x + l)^{-\frac{1-\gamma}{3-\gamma}-1} & (x,\xi) \in \Omega_2 \cup \Omega_3. \end{cases} \tag{3.13}$$

Lemma 3.2. *There is a constant $\epsilon_0 > 0$ such that for $0 \leq \epsilon \leq \epsilon_0$ and $\mathsf{g} = \mathsf{P}_1\mathsf{g}$,*

$$-\langle \mathsf{g}, \sigma_x^{-m/2} e^{\epsilon\sigma(x,\xi)} L e^{-\epsilon\sigma(x,\xi)} \sigma_x^{m/2} \mathsf{g} \rangle \geq \nu_1 \langle \nu(|\xi|)\mathsf{g}, \mathsf{g} \rangle, \tag{3.14}$$

for some positive constant $\nu_1 = \nu_1(\epsilon_0)$.

The proof of Lemma 3.2 is similar to that of Lemma 2.2 in [6] and Lemma 2.2 in [19].

Consider

$$\begin{aligned} 0 &= \int_0^\infty \langle \mathsf{g}, \mathsf{g}_t + \xi^1\mathsf{g}_x - \epsilon\xi^1\sigma_x\mathsf{g} - e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{g} \rangle dx \\ &= \frac{1}{2}\frac{d}{dt}\int_0^\infty \langle \mathsf{g}, \mathsf{g} \rangle dx - \frac{1}{2}\langle \mathsf{g}, \xi^1\mathsf{g} \rangle|_{x=0} + \int_0^\infty \langle \mathsf{g}, -\epsilon\xi^1\sigma_x\mathsf{g} - e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{g} \rangle dx. \end{aligned} \tag{3.15}$$

From Lemma 3.1, we get

$$\begin{aligned} \langle \mathsf{P}_0\mathsf{g}, -\xi^1\sigma_x\mathsf{P}_0\mathsf{g} \rangle &= \int_{\Omega_2} c\delta\left((1+|\xi|)^{-1+\gamma} + (\delta x + l)^{-\frac{1-\gamma}{3-\gamma}}\right)\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi \\ &+ \int_{\Omega_1} \delta(1+|\xi|)^{-1+\gamma}\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi + \int_{\Omega_3} \frac{10\delta}{3-\gamma}(\delta x + l)^{-\frac{1-\gamma}{3-\gamma}}\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi. \end{aligned} \tag{3.16}$$

Since $\mathsf{P}_0\mathsf{g}$ has exponential decay in ξ like $e^{-c|\xi|^2}$, for l sufficiently large, the integral $\int_{\Omega_3} \frac{10\delta}{3-\gamma}(\delta x + l)^{-\frac{1-\gamma}{3-\gamma}}\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi$ has the same order as $\int_{\mathbb{R}^3} \frac{10\delta}{3-\gamma}(\delta x + l)^{-\frac{1-\gamma}{3-\gamma}} \mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi$ while the value of $\left|\int_{\Omega_1} \delta(1+|\xi|)^{-1+\gamma}\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi + \int_{\Omega_2} c\delta\left((1+|\xi|)^{-1+\gamma} + (\delta x+l)^{-\frac{1-\gamma}{3-\gamma}}\right)\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi\right|$ is sufficiently small. Thus

$$\begin{aligned}\langle \mathsf{P}_0\mathsf{g}, -\xi^1\sigma_x\mathsf{P}_0\mathsf{g}\rangle &= \int_{\Omega_2} c\delta\left((1+|\xi|)^{-1+\gamma} + (\delta x+l)^{-\frac{1-\gamma}{3-\gamma}}\right)\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi \\ &+ \int_{\Omega_1} \delta(1+|\xi|)^{-1+\gamma}\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi + \int_{\mathbb{R}^3}\frac{10\delta}{3-\gamma}(\delta x+l)^{-\frac{1-\gamma}{3-\gamma}}\mathsf{g}\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g}d\xi \\ &\geq C(\delta x+l)^{-\frac{1-\gamma}{3-\gamma}}\langle \mathsf{P}_0\mathsf{g}, \mathsf{P}_0\mathsf{g}\rangle. \qquad (3.17)\end{aligned}$$

The dissipation on the non-fluid part which comes from $\mathsf{g}e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{g}$ can be estimated as follows. By using the exponential decay of g, we have the following two cases.

Case 1. When $2(1+|\xi|)^{3-\gamma} \leq \delta x + l$,

$$\begin{aligned}|L_\epsilon \mathsf{P}_0\mathsf{g}(\xi)| &\leq |L\mathsf{P}_0\mathsf{g}(\xi)| + |(\mathsf{K}_\epsilon - \mathsf{K})\mathsf{P}_0\mathsf{g}(\xi)| \\ &= 0 + \left|\int_{2(1+|\xi_*|)^{3-\gamma}\leq \delta x+l}(\mathsf{K}_\epsilon - \mathsf{K})\mathsf{P}_0\mathsf{g}(\xi)d\xi_*\right| \leq Ce^{-c(\delta x+l)^{\frac{2}{3-\gamma}}}\|\mathsf{P}_0\mathsf{g}\|_{L^2_\xi}. \qquad (3.18)\end{aligned}$$

Here, $L_\epsilon = e^{\epsilon\sigma}Le^{-\epsilon\sigma}$ and $K_\epsilon = e^{\epsilon\sigma}Ke^{-\epsilon\sigma}$.

Case 2. When $2(1+|\xi|)^{3-\gamma} \geq \delta x + l$,

$$\left|\int_{2(1+|\xi|)^{3-\gamma}\geq \delta x+l}\mathsf{P}_0\mathsf{g}L_\epsilon\mathsf{P}_0\mathsf{g}d\xi\right| \leq Ce^{-c(\delta x+l)^{\frac{2}{3-\gamma}}}\|\mathsf{P}_0\mathsf{g}\|^2_{L^2_\xi}. \qquad (3.19)$$

Thus, (3.18), (3.19) together with Lemma 3.2 yield that

$$\begin{aligned}-\langle \mathsf{g}, e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{g}\rangle &= -\langle \mathsf{P}_0\mathsf{g}, e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{P}_0\mathsf{g}\rangle - \langle \mathsf{P}_0\mathsf{g}, e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{P}_1\mathsf{g}\rangle \\ &\quad -\langle \mathsf{P}_1\mathsf{g}, e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{P}_0\mathsf{g}\rangle - \langle \mathsf{P}_1\mathsf{g}, e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{P}_1\mathsf{g}\rangle \\ &\geq -Ce^{-c(\delta x+l)^{\frac{2}{3-\gamma}}}\|\mathsf{P}_0\mathsf{g}\|^2_{L^2_\xi} + C\nu_1\langle \mathsf{P}_1\mathsf{g}, \nu(|\xi|)\mathsf{P}_1\mathsf{g}\rangle. \qquad (3.20)\end{aligned}$$

From (3.17) and (3.20),

$$\begin{aligned}&\epsilon\langle \mathsf{g}, -\xi^1\sigma_x\mathsf{g}\rangle - \langle \mathsf{g}, e^{\epsilon\sigma}Le^{-\epsilon\sigma}\mathsf{g}\rangle \\ &\quad\geq \epsilon(1-\sqrt{\epsilon})\langle \mathsf{P}_0\mathsf{g}, -\xi^1\sigma_x\mathsf{P}_0\mathsf{g}\rangle + \epsilon(1-\frac{1}{\sqrt{\epsilon}})\langle \mathsf{P}_1\mathsf{g}, -\xi^1\sigma_x\mathsf{P}_1\mathsf{g}\rangle \\ &\qquad - Ce^{-c(\delta x+l)^{\frac{2}{3-\gamma}}}\langle \mathsf{P}_0\mathsf{g}, \mathsf{P}_0\mathsf{g}\rangle + C\nu_1\langle \mathsf{P}_1\mathsf{g}, \nu(|\xi|)\mathsf{P}_1\mathsf{g}\rangle \\ &\geq C\epsilon(\delta x+l)^{-\frac{1-\gamma}{3-\gamma}}\langle \mathsf{P}_0\mathsf{g}, \mathsf{P}_0\mathsf{g}\rangle + C\nu_1\langle \mathsf{P}_1\mathsf{g}, \nu(|\xi|)\mathsf{P}_1\mathsf{g}\rangle - \epsilon(1-\frac{1}{\sqrt{\epsilon}})\langle \mathsf{P}_1\mathsf{g}, |\xi^1\sigma_x|\mathsf{P}_1\mathsf{g}\rangle \\ &\quad\geq C\epsilon(\delta x+l)^{-\frac{1-\gamma}{3-\gamma}}\langle \mathsf{P}_0\mathsf{g}, \mathsf{P}_0\mathsf{g}\rangle + C\langle \mathsf{P}_1\mathsf{g}, \nu(|\xi|)\mathsf{P}_1\mathsf{g}\rangle. \qquad (3.21)\end{aligned}$$

The last step comes from Lemma 3.1 which shows that $|\sigma_x|$ is bounded by $C(1+|\xi|)^{-1+\gamma}$.

(3.15) and (3.21) yields that

$$\frac{d}{dt}\int_0^\infty \langle \mathsf{g}, \mathsf{g}\rangle dx + \langle |\xi^1|\mathsf{g}, \mathsf{g}\rangle_-|_{x=0} + C\epsilon\int_0^\infty \langle (\delta x+l)^{-\frac{1-\gamma}{3-\gamma}}\mathsf{g}, \mathsf{g}\rangle dx \leq 0. \qquad (3.22)$$

To gain the decay in t, we still need more information. Set

$$\mathsf{g}_1 = \sigma_x^{-\frac{1}{2}}\mathsf{g}, \quad \mathsf{g}_2 = \sigma_x^{-1}\mathsf{g}, \quad \cdots, \mathsf{g}_m = \sigma_x^{-\frac{m}{2}}\mathsf{g}, \quad \cdots$$

Then g_m satisfies

$$\partial_t \mathsf{g}_m + \xi^1 \partial_x \mathsf{g}_m + \frac{m}{2}\frac{\sigma_{xx}}{\sigma_x}\xi^1 \mathsf{g}_m - \epsilon\xi^1\sigma_x \mathsf{g}_m - \sigma_x^{-m/2} e^{\epsilon\sigma} L e^{-\epsilon\sigma}\sigma_x^{m/2}\mathsf{g}_m = 0. \tag{3.23}$$

By a straightforward calculation similar to Lemma 2.5 in [19], we can find

$$\left|\langle \mathsf{g}_m, \frac{\sigma_{xx}}{\sigma_x}\xi^1 \mathsf{g}_m\rangle\right| \leq \varepsilon_1 \left((\delta x + l)^{-\frac{1-\gamma}{3-\gamma}} \langle \mathsf{P}_0 \mathsf{g}_m, \mathsf{P}_0 \mathsf{g}_m\rangle + \langle \mathsf{P}_1 \mathsf{g}_m, \nu(|\xi|)\mathsf{P}_1 \mathsf{g}_m\rangle \right), \tag{3.24}$$

where ϵ_1 is small enough when l is sufficient large. This inequality together with Lemma 3.2 results in

$$\frac{d}{dt}\int_0^\infty \langle \mathsf{g}_m, \mathsf{g}_m\rangle dx + \langle |\xi^1| \mathsf{g}_m, \mathsf{g}_m\rangle_-|_{x=0} + C\epsilon \int_0^\infty \langle (\delta x + l)^{-\frac{1-\gamma}{3-\gamma}} \mathsf{g}_m, \mathsf{g}_m\rangle dx \leq 0. \tag{3.25}$$

Since $|\sigma_x^{-1}| \geq C(\delta x + l)^{-\frac{1-\gamma}{3-\gamma}}$, $\int_0^\infty \langle (\delta x+l)^{-\frac{1-\gamma}{3-\gamma}} \mathsf{g}, \mathsf{g}\rangle dx > 0$ and $\langle |\xi^1| \mathsf{g}_m, \mathsf{g}_m\rangle_-|_{x=0} \geq 0$, we have

$$\frac{d}{dt}\int_0^\infty \langle \mathsf{g}, \mathsf{g}\rangle dx \leq -\langle |\xi^1| \mathsf{g}, \mathsf{g}\rangle_-|_{x=0}, \tag{3.26}$$

$$\frac{d}{dt}\int_0^\infty \langle \mathsf{g}_m, \mathsf{g}_m\rangle dx + C\epsilon \int_0^\infty \langle (\delta x + l)^{-\frac{1-\gamma}{3-\gamma}} \mathsf{g}_{m-1}, \mathsf{g}_{m-1}\rangle dx \leq 0. \tag{3.27}$$

Introduce a function $\mathbb{F}(t)$ constructed in [19],

$$\begin{aligned} \mathbb{F}(t) = \frac{m!}{m!}(1+t)^m \int_0^\infty \langle \mathsf{g}, \mathsf{g}\rangle dx + \frac{m!}{(m-1)!}\epsilon^{-1}(1+t)^{(m-1)} \int_0^\infty \langle \mathsf{g}_1, \mathsf{g}_1\rangle dx \\ + \cdots + \frac{m!}{1}\epsilon^{-m+1}(1+t)\int_0^\infty \langle \mathsf{g}, \mathsf{g}\rangle dx + \frac{m!}{0!}\epsilon^{-m}\int_0^\infty \langle \mathsf{g}_m, \mathsf{g}_m\rangle dx. \end{aligned} \tag{3.28}$$

It's straightforward to obtain

$$\frac{d}{dt}\mathbb{F}(t) \leq -(1+t)^m \langle |\xi^1| \mathsf{g}, \mathsf{g}\rangle_-|_{x=0}.$$

Thus, $\mathbb{F}(t) + \int_0^t (1+s)^m \langle |\xi^1| \mathsf{g}, \mathsf{g}\rangle_-|_{x=0} ds \leq \mathbb{F}(0)$ which implies that there exists a constant $c_{m,\epsilon} > 0$ such that

$$\int_0^t (1+s)^m \langle |\xi^1| \mathsf{g}, \mathsf{g}\rangle_-|_{x=0} ds \leq c_{m,\epsilon}\int_0^\infty \langle \mathsf{g}_m, \mathsf{g}_m\rangle|_{t=0} dx. \tag{3.29}$$

3.2. Estimate for the linearized problem. With the full boundary data, we can study how it propagates into the interior region through the integration

$$\mathbb{I}(x,t,\xi) = \int_0^t \int_{\mathbb{R}^3} \mathbb{G}_i(x,t-s,\xi,\xi_*)\xi_*^1 \mathsf{h}(0,s,\xi_*)d\xi_* ds. \tag{3.30}$$

Similar to the decomposition of $\mathbb{G}$ in [13], the Green's function $\mathbb{G}_i(x,t,\xi,\xi_*)$ for the Cauchy problem can be divided into singular parts and nonsingular part

$$\mathbb{G}_i(x,t,\xi,\xi_*) = \delta^1(x-\xi_*^1 t)\delta^3(\xi-\xi_*)e^{-\nu(\xi_*)t} + \mathsf{j}^1(x,t,\xi,\xi_*) + \mathsf{j}^2(x,t,\xi,\xi_*) + \mathsf{r}(x,t,\xi,\xi_*), \tag{3.31}$$

where

$$\mathsf{j}^1(x,t,\xi,\xi_*) = \begin{cases} \frac{\mathsf{K}(\xi,\xi_*)}{\xi^1-\xi_*^1} e^{-\nu(\xi)(t-t_1)-\nu(\xi_*)t_1}, & (x-\xi^1 t)(x-\xi_*^1 t) \le 0, \\ 0, & otherwise, \end{cases} \tag{3.32}$$

$$\text{where } 0 \le t_1 = \frac{x-\xi^1 t}{\xi_*^1 - \xi^1} \le t, \tag{3.33}$$

$$\left|\mathsf{j}^2(x,t,\xi,\xi_*)\right| \le Ce^{-C(t+x^\gamma)}|\xi-\xi_*|^{-1}(1+|\xi|)^{-1}, \tag{3.34}$$

$$\|\mathsf{r}(x,t)\|_{L^2_\xi} = \sum_{i=1}^3 \frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}. \tag{3.35}$$

Here the decaying rate e^{-Cx^γ} comes from the following discussion. From (3.33),

$$|x| = |\xi^1(t-t_1)+\xi_*^1 t_1| \le |\xi|(t-t_1) + |\xi_*|t_1. \tag{3.36}$$

When $t-t_1 > t_1$(i.e. $t-t_1 > \frac{t}{2}$), if $|\xi|(t-t_1) \ge \frac{1}{2}|\xi_*|t_1$,

$$\begin{aligned} &\nu(\xi)(t-t_1)+\nu(\xi_*)t_1 \ge \nu_0(t+|\xi|^\gamma(t-t_1)+|\xi_*|^\gamma t_1) \\ &\qquad \ge \nu_0(t+\frac{1}{2}|\xi|^\gamma(t-t_1)^\gamma + \frac{1}{2}(\frac{1}{2})^\gamma|\xi_*|^\gamma t_1^\gamma) \ge C(t+|x|^\gamma). \end{aligned} \tag{3.37}$$

Thus

$$e^{-\nu(\xi)(t-t_1)-\nu(\xi_*)t_1} \le Ce^{-C(t+|x|^\gamma)}.$$

If $|\xi|(t-t_1) \le \frac{1}{2}|\xi_*|t_1$, then $|\xi| \ge \frac{1}{2}|\xi_*|$. Since

$$|\mathsf{K}(\xi,\xi_*)| \le p(\xi,\xi_*)e^{-C||\xi|^2-|\xi_*|^2|} \le p(\xi,\xi_*)e^{-C|\xi_*|^2-C|\xi|^2},$$

thus we have

$$\begin{aligned} &\nu(\xi)(t-t_1)+\nu(\xi_*)t_1 \\ &\ge \nu_0(t+|\xi|^\gamma(t-t_1)+|\xi_*|^\gamma t_1+|\xi|^2+|\xi_*|^2) \ge \nu_0(t+\frac{|x|}{|\xi_*|^{1-\gamma}}+|\xi_*|^2) \ge C(t+|x|^\gamma). \end{aligned} \tag{3.38}$$

The case when $t-t_1<t_1$ is similar. As a result, we get the decaying rate e^{-Cx^γ}.

According to (3.31), $\mathbb{I}(x,t,\xi)$ can be decomposed into four parts

$$\mathbb{I}(x,t,\xi)=\mathbb{I}_0(x,t,\xi)+\mathbb{I}_1(x,t,\xi)+\mathbb{I}_2(x,t,\xi)+\mathsf{R}(x,t,\xi). \tag{3.39}$$

For $x,t>0$,

$$\begin{aligned}
\mathbb{I}_0(x,t,\xi) &\equiv 0\\
\mathbb{I}_1(x,t,\xi) &\equiv \int_0^t\int_{\mathbb{R}^3}\mathsf{j}^1(x,t-s,\xi,\xi_*)\xi_*^1\mathsf{h}(0,s,\xi_*)d\xi_*ds\\
\mathbb{I}_2(x,t,\xi) &\equiv \int_0^t\int_{\mathbb{R}^3}\mathsf{j}^2(x,t-s,\xi,\xi_*)\xi_*^1\mathsf{h}(0,s,\xi_*)d\xi_*ds\\
\mathsf{R}(x,t,s,\xi) &\equiv \int_0^t\int_{\mathbb{R}^3}\mathsf{r}(x,t-s,\xi,\xi_*)\xi_*^1\mathsf{h}(0,s,\xi_*)d\xi_*ds.
\end{aligned} \tag{3.40}$$

From (3.40),

$$\begin{aligned}
&\mathbb{I}_1(x,t,\xi)\\
&\quad=\int_0^t\int_{(x-\xi^1(t-s))(x-\xi_*^1(t-s))<0}\frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}\mathsf{K}(\xi,\xi_*)}{\xi^1-\xi_*^1}\xi_*^1\mathsf{f}(0,s,\xi_*)d\xi_*ds,
\end{aligned} \tag{3.41}$$

where

$$t_1=t-s-\frac{x-\xi_*^1(t-s)}{\xi^1-\xi_*^1}. \tag{3.42}$$

In the special domain for integration, $\left|\xi^1-\xi_*^1\right|$ has a non-zero lower bound:

$$\left|\xi^1-\xi_*^1\right|\geq\frac{\left|x-\xi^1(t-s)\right|}{t-s}=\left|\frac{x}{t-s}-\xi^1\right|. \tag{3.43}$$

We divide the domain $\{(\xi_*,s):s\in(0,t),(x-\xi^1(t-s))(x-\xi_*^1(t-s))<0\}$ into two regions:

$$\mathsf{H}_1=\left\{(\xi_*,s):s\in(0,t),(x-\xi^1(t-s))(x-\xi_*^1(t-s))<0,|\xi^1-\xi_*^1|\geq\frac{1}{2}|\xi_*^1|\right\},$$

$$\mathsf{H}_2=\left\{(\xi_*,s):s\in(0,t),(x-\xi^1(t-s))(x-\xi_*^1(t-s))<0,|\xi^1-\xi_*^1|<\frac{1}{2}|\xi_*^1|\right\},$$

and denote

$$\mathbb{I}_{11}=\int\int_{\mathsf{H}_1}\frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}\mathsf{K}(\xi,\xi_*)}{\xi^1-\xi_*^1}\xi_*^1\mathsf{h}(0,s,\xi_*)d\xi_*ds,$$

$$\mathbb{I}_{12}=\int\int_{\mathsf{H}_2}\frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}\mathsf{K}(\xi,\xi_*)}{\xi^1-\xi_*^1}\xi_*^1\mathsf{h}(0,s,\xi_*)d\xi_*ds.$$

Then

$$|\mathbb{I}_1(x,t,\xi)|\leq|\mathbb{I}_{11}(x,t,\xi)|+|\mathbb{I}_{12}(x,t,\xi)|. \tag{3.44}$$

In H_1, $\frac{|\xi_*^1|}{|\xi^1-\xi_*^1|} \le 2$,

$$\begin{aligned}\mathbb{I}_{11}(x,t,\xi)^2 &\le \left(\int\int_{\mathsf{H}_1} \frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}\mathsf{K}(\xi,\xi_*)}{\xi^1-\xi_*^1}\xi_*^1\mathsf{h}(0,s,\xi_*)d\xi_* ds\right)^2 \\ &\le \left(\int\int_{\mathsf{H}_1} \frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}}{|\xi^1-\xi_*^1|^{\frac{3}{2}}}|\mathsf{K}(\xi,\xi_*)\xi_*^1|d\xi_* ds\right) \\ &\quad \cdot \left(\int\int_{\mathsf{H}_1} \frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}|\mathsf{K}(\xi,\xi_*)|}{|\xi^1-\xi_*^1|^{\frac{1}{2}}}|\xi_*^1\mathsf{h}^2(0,s,\xi_*)|d\xi_* ds\right). \quad (3.45)\end{aligned}$$

After a straightforward calculation, we can find that $\frac{\mathsf{K}(\xi,\xi_*)e^{\varepsilon||\xi|^2-|\xi_*|^2|}}{\sqrt{|\xi^1-\xi_*^1|}} \in L^1_{\xi_*}$ when ε is sufficiently small. Thus

$$\begin{aligned}&\int\int_{\mathsf{H}_1} \frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}}{|\xi^1-\xi_*^1|^{\frac{3}{2}}}|\mathsf{K}(\xi,\xi_*)\xi_*^1|d\xi_* ds \\ &\qquad \le \int\int_{\mathsf{H}_1} e^{-\nu_0[(t-s)+|x|^\gamma]}\frac{|\mathsf{K}(\xi,\xi_*)e^{\varepsilon||\xi|^2-|\xi_*|^2|}|}{|\xi^1-\xi_*^1|^{\frac{1}{2}}}d\xi_* ds \le Ce^{-C|x|^\gamma}. \quad (3.46)\end{aligned}$$

The L^1 property of $\frac{\mathsf{K}(\xi,\xi_*)e^{\varepsilon||\xi|^2-|\xi_*|^2|}}{\sqrt{|\xi^1-\xi_*^1|}}$ together with (3.45), (3.46) and (3.29) yields that

$$\begin{aligned}\int_{\mathbb{R}^3} \mathbb{I}_{11}(x,t,\xi)^2 d\xi &\le C\int_0^t e^{-\nu_0(t-s+|x|^\gamma)}\left\|\sqrt{|\xi^1|}\mathsf{h}(0,s)\right\|_{L^2_\xi}^2 ds \\ &\le C(1+t)^{-m}e^{-\nu_0|x|^\gamma}\int_0^\infty \langle e^{\epsilon\sigma(x,\xi)}\sigma_x^{-m/2}(x,\xi)\mathsf{h}_0, e^{\epsilon\sigma(x,\xi)}\sigma_x^{-m/2}(x,\xi)\mathsf{h}_0\rangle dx \\ &\le C(1+t)^{-m}(1+z)^{-m}e^{-\nu_0|x|^\gamma}, \quad (3.47)\end{aligned}$$

and so

$$\|\mathbb{I}_{11}(x,t)\|_{L^2_\xi} \le C(1+t)^{-m/2}(1+z)^{-m/2}e^{-|x|^\gamma/C}. \quad (3.48)$$

In the domain H_2, $|\xi_*^1|-|\xi^1| \le |\xi^1-\xi_*^1| \le \frac{1}{2}|\xi_*^1| \Rightarrow |\xi_*^1| \le 2|\xi^1|$. Since K gains decaying rate in ξ variable,

$$\begin{aligned}&\int\int_{\mathsf{H}_2} \frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}}{|\xi^1-\xi_*^1|^{\frac{3}{2}}}|\mathsf{K}(\xi,\xi_*)\xi_*^1|d\xi_* ds \\ &\qquad \le \int_{\{s|s\in(0,t)\cap|x-\xi^1(t-s)|\le 1\}} e^{-\nu_0(t-s+|x|^\gamma)}\frac{2|\xi^1|}{|x-\xi^1(t-s)|^{\frac{1}{2}}}ds \\ &\qquad + \int_{\{s|s\in(0,t)\cap|x-\xi^1(t-s)|\ge 1\}} e^{-\nu_0(t-s+|x|^\gamma)}\frac{1}{|x-\xi^1(t-s)|^{\frac{1}{2}}}ds \\ &\qquad \le \int_{\{s|s\in(0,t)\cap|x-\xi^1(t-s)|\le 1\}} e^{-\nu_0|x|^\gamma}\frac{2|\xi^1|}{|x-\xi^1(t-s)|^{\frac{1}{2}}}ds \\ &\qquad\qquad + \int_{\{s|s\in(0,t)\cap|x-\xi^1(t-s)|\le 1\}} e^{-\nu_0(t-s+|x|^\gamma)}ds \le Ce^{-\nu_0|x|^\gamma}. \quad (3.49)\end{aligned}$$

The L^1 property of $\frac{\mathsf{K}(\xi,\xi_*)}{|\xi^1-\xi^1_*|^{\frac{1}{2}}}$ together with (3.49) yields that

$$
\begin{aligned}
&\|\mathbb{I}_{12}(x,t)\|_{L^2_\xi} \\
&\quad= \left(\int_{\mathbb{R}^3}\left(\int\int_{\mathsf{H}_2}\frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}\mathsf{K}(\xi,\xi_*)}{\xi^1-\xi^1_*}\xi^1_*\mathsf{h}(0,s,\xi_*)d\xi_*ds\right)^2 d\xi\right)^{\frac{1}{2}} \\
&\qquad\le \left(\int_{\mathbb{R}^3}\left(\int\int_{\mathsf{H}_2}\frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}}{|\xi^1-\xi^1_*|^{\frac{3}{2}}}|\mathsf{K}(\xi,\xi_*)\xi^1_*|d\xi_*ds\right)\right. \\
&\qquad\quad \left.\cdot\left(\int\int_{\mathsf{H}_2}\frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}|\mathsf{K}(\xi,\xi_*)|}{|\xi^1-\xi^1_*|^{\frac{1}{2}}}|\xi^1_*\mathsf{h}^2(0,s,\xi_*)|d\xi_*ds\right)d\xi\right)^{\frac{1}{2}} \\
&\quad\le Ce^{-\nu_0|x|^\gamma}\left(\int_{\mathbb{R}^3}\int\int_{\mathsf{H}_2}\frac{e^{-\nu(\xi)(t-s-t_1)-\nu(\xi_*)t_1}|\mathsf{K}(\xi,\xi_*)|}{|\xi^1-\xi^1_*|^{\frac{1}{2}}}|\xi^1_*\mathsf{h}^2(0,s,\xi_*)|d\xi_*dsd\xi\right)^{\frac{1}{2}} \\
&\qquad\qquad \le C(1+t)^{-m/2}(1+z)^{-m/2}e^{-|x|^\gamma/C}. \quad (3.50)
\end{aligned}
$$

From (3.48) and (3.50),

$$
\|\mathbb{I}_1(x,t)\|_{L^2_\xi}\le\|\mathbb{I}_{11}(x,t)\|_{L^2_\xi}+\|\mathbb{I}_{12}(x,t)\|_{L^2_\xi}\le C(1+t)^{-m/2}(1+z)^{-m/2}e^{-|x|^\gamma/C}. \quad (3.51)
$$

Since j^2 doesn't contain the strong singularity $\frac{1}{|\xi^1-\xi^1_*|}$ and only contains an integrable singularity $\frac{1}{|\xi-\xi_*|}$. It also gains decay in ξ variable. By direct computations,

$$
\begin{aligned}
&\|\mathbb{I}_2(x,t)\|_{L^2_\xi} \\
&\le C\left(\int_0^t e^{-\nu_0(t-s+|x|^\gamma)}\left\|\sqrt{|\xi^1|}\mathsf{h}(0,s)\right\|^2_{L^2_\xi}ds\right)^{\frac{1}{2}}\le C(1+t)^{-m/2}(1+z)^{-m/2}e^{-|x|^\gamma/C}.
\end{aligned} \quad (3.52)
$$

r_D doesn't contain any singularity while contains extra decaying rate in ξ. Thus from (3.29) and Schwartz inequality, we get

$$
\begin{aligned}
\|\mathsf{R}(x,t)\|_{L^2_\xi}&\le C\int_0^t\sum_{i=1}^3\frac{1}{\sqrt{1+t-s}}\left(1+\frac{(x-\lambda_i(t-s))^2}{1+t-s}\right)^{-N}\left\|\sqrt{|\xi^1|}\mathsf{h}(0,s)\right\|_{L^2_\xi}ds \\
&\le C\sum_{i=1}^3\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_it)^2}{1+t}\right)^{-min\{N,\frac{m-2}{4}\}}(1+z)^{-m/2}. \quad (3.53)
\end{aligned}
$$

(3.51), (3.52) and (3.53) results the estimate of the function $\mathbb{I}(x,t)$:

$$
\begin{aligned}
\|\mathbb{I}(x,t)\|_{L^2_\xi}&=\left\|\int_0^t\mathbb{G}_i(x,t-s)[\xi^1\mathsf{h}](0,s)ds\right\|_{L^2_\xi} \\
&\le C\sum_{i=1}^3\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_it)^2}{1+t}\right)^{-min\{N,\frac{m-2}{4}\}}(1+z)^{-m/2}. \quad (3.54)
\end{aligned}
$$

This together with (3.8) yields the estimate of $\mathsf{h}(x,t,\xi)$

$$\begin{aligned}\|\mathsf{h}(x,t)\|_{L^2_\xi} &\le \left\|\int_0^\infty \int_{\mathbb{R}^3} \mathbb{G}_i(x-y,t,\xi,\xi_*)\mathsf{h}_0(y,\xi_*)d\xi_* dy\right\|_{L^2_\xi} \\ &\quad + \left\|\int_0^t \int_{\mathbb{R}^3} \mathbb{G}_i(x,t-s,\xi,\xi_*)\xi^1_*\mathsf{h}(0,s,\xi_*)d\xi_* ds\right\|_{L^2_\xi} \\ &\le C\sum_{i=1}^3 \frac{1}{\sqrt{1+t}}\left(1+\frac{(x-z-\lambda_i t)^2}{1+t}\right)^{-N} \\ &\quad + C\sum_{i=1}^3 \frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-min\{N,\frac{m-2}{4}\}}(1+z)^{-m/2}. \end{aligned} \tag{3.55}$$

3.3. Special initial data. For the Boltzmann equation, when the initial function is a microscopic one, i.e $\mathsf{P}_1\mathsf{h}_0 = \mathsf{h}_0$, the solution will gain extra decaying rate. We still suppose $h(x,t,\xi)$ satisfies (3.7). The initial function is also a bounded one with a compact support. The only difference is that we add an additional condition $\mathsf{P}_1\mathsf{h}_0 = \mathsf{h}_0$. Then repeat the same progress in Section 3.1 and 3.2,

$$\begin{aligned}\|\mathsf{h}(x,t)\|_{L^2_\xi} &\le \left\|\int_0^\infty \int_{\mathbb{R}^3} \mathbb{G}_i(x-y,t,\xi,\xi_*)\mathsf{P}_1\mathsf{h}_0(y,\xi_*)d\xi_* dy\right\|_{L^2_\xi} \\ &\quad + \left\|\int_0^t \int_{\mathbb{R}^3} \mathbb{G}_i(x,t-s,\xi,\xi_*)\xi^1_*\mathsf{h}(0,s,\xi_*)d\xi_* ds\right\|_{L^2_\xi} \\ &\le C\sum_{i=1}^3 \frac{1}{1+t}\left(1+\frac{(x-z-\lambda_i t)^2}{1+t}\right)^{-N} \\ &\quad + C\sum_{i=1}^3 \frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-min\{N,\frac{m-2}{4}\}}(1+z)^{-m/2}. \end{aligned} \tag{3.56}$$

3.4. Construction of the Green's function for IBVP. The Green's function $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y)$ is given by (3.4). Decompose it into several parts

$$\begin{aligned}&\mathbb{G}_{ib}(x,t,\xi,\xi_*;y) \\ &\quad = \mathbb{G}^0(x,t,\xi,\xi_*;y) + \mathbb{G}^1(x,t,\xi,\xi_*;y) + \cdots + \mathbb{G}^k(x,t,\xi,\xi_*;y) + \mathbb{G}^r(x,t,\xi,\xi_*;y). \end{aligned} \tag{3.57}$$

This decomposition is obtained by Picard's iteration:

$$\begin{cases}(\partial_t+\xi^1\partial_x+\nu(\xi))\mathbb{G}^0=0,\\ \mathbb{G}^0(x,0,\xi,\xi_*;y)=\delta^1(x-y)\delta^3(\xi-\xi_*),\\ \mathbb{G}^0(0,t,\xi,\xi_*;y)|_{\xi^1>0}=0,\end{cases}$$

$$\begin{cases}(\partial_t+\xi^1\partial_x+\nu(\xi))\mathbb{G}^1=\mathsf{K}\mathbb{G}^0,\\ \mathbb{G}^1(x,0,\xi,\xi_*;y)=0,\\ \mathbb{G}^1(0,t,\xi,\xi_*;y)|_{\xi^1>0}=0,\end{cases}$$

$$\begin{cases}(\partial_t+\xi^1\partial_x+\nu(\xi))\mathbb{G}^2=\mathsf{K}\mathbb{G}^1,\\ \mathbb{G}^2(x,0,\xi,\xi_*;y)=0,\\ \mathbb{G}^2(0,t,\xi,\xi_*;y)|_{\xi^1>0}=0,\end{cases}$$

$$\vdots$$

$$\begin{cases}(\partial_t+\xi^1\partial_x-L)\mathbb{G}^r=\mathsf{K}\mathbb{G}^k,\\ \mathbb{G}^r(x,0,\xi,\xi_*;y)=0,\\ \mathbb{G}^r(0,t,\xi,\xi_*;y)|_{\xi^1>0}=0,\end{cases}$$

$\mathbb{G}^k(k=0,1,\cdots)$ can be obtained by direct computations. Since K gains decay in ξ variable,

$$\mathbb{G}^0(x,t,\xi,\xi_*;y)=\delta^1(x-y-\xi_*^1t)\delta^3(\xi-\xi_*)e^{-\nu(\xi_*)t}, \tag{3.58}$$

$$\mathbb{G}^1(x,t,\xi,\xi_*;y)=\begin{cases}\frac{\mathsf{K}(\xi,\xi_*)}{\xi^1-\xi_*^1}e^{-\nu(\xi)(t-t_1)-\nu(\xi_*)t_1}, & \xi^1t\le x\le\xi_*^1t),\\ 0, & otherwise,\end{cases} \tag{3.59}$$

where

$$\text{where } 0\le t_1=\frac{x-y-\xi^1t}{\xi_*^1-\xi^1}\le t, \tag{3.60}$$

$$\left|\mathbb{G}^2(x,t,\xi,\xi_*;y)\right|\le Ce^{-C(t+|x-y|^\gamma)}|\xi-\xi_*|^{-1}(1+|\xi|)^{-1}, \tag{3.61}$$

$$\left\|\mathbb{G}^k(x,t,\xi_*;y)\right\|_{L^\infty_{\xi,k-1}}\le Ce^{-C(t+|x-y|^\gamma)},\quad k\ge3. \tag{3.62}$$

Consider $\mathbb{G}^r$. By Duhamel's principle, the source term $\mathsf{K}\mathbb{G}^k$ can be regarded to be initial data at time s. It can also be treated as $\sum_{j=1}^\infty\mathsf{K}\mathbb{G}^k(x)\chi(x-2j+1)$, where

$\chi(x) = 1$ when $-1 \le x < 1$ and equals 0 otherwise. Thus from (3.55), we get

$$\begin{aligned}
&\|\mathbb{G}^r(x,t,\xi_*;y)\|_{L^2_\xi} \\
&\le \int_0^t \Bigg(\sum_{i=1}^3 \frac{1}{\sqrt{1+t-s}} \left(1+\frac{(x-j-\lambda_i(t-s))^2}{1+t-s}\right)^{-N} \\
&+ \sum_{i=1}^3 \frac{1}{\sqrt{1+t-s}} \left(1+\frac{(x-\lambda_i(t-s))^2}{1+t-s}\right)^{-min\{N,\frac{[\frac{k-4}{1-\gamma}]-2}{4}\}} (1+j)^{-[\frac{k-4}{1-\gamma}]/2} \Bigg) \\
&\cdot \sum_{j=1}^\infty e^{-C(s+|j-y|^\gamma)} ds \\
&\le C \sum_{i=1}^3 \frac{1}{\sqrt{1+t}} \left(1+\frac{(x-y-\lambda_i t)^2}{1+t}\right)^{-N} \\
&+ C \sum_{i=1}^3 \frac{1}{\sqrt{1+t}} \left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-min\{N,\frac{[\frac{k-4}{1-\gamma}]-2}{4}\}} (1+y)^{-[\frac{k-4}{1-\gamma}]/2}, \quad (3.63)
\end{aligned}$$

where $[a] = \sup_b\{b|b \le a, b \in \mathbb{Z}\}$. Choose k such that $\frac{[\frac{k-4}{1-\gamma}]-2}{4} > N$. Thus

$$\|\mathbb{G}^r(x,t,\xi_*;y)\|_{L^2_\xi} \le C \sum_{i=1}^3 \frac{1}{\sqrt{1+t}} \left(1+\frac{(x-y-\lambda_i t)^2}{1+t}\right)^{-N}. \quad (3.64)$$

Then (3.57), (3.58), (3.59), (3.61), (3.62) and (3.64) result in

$$\left\| \mathbb{G}_{ib}(x,t,\xi_*;y) - \sum_{j=0}^2 \mathbb{G}^j(x,t,\xi_*;y) \right\|_{L^2_\xi} \le C \sum_{i=1}^3 \frac{1}{\sqrt{1+t}} \left(1+\frac{(x-y-\lambda_i t)^2}{1+t}\right)^{-N}. \quad (3.65)$$

4. Coupling Waves. We shall introduce some integrals which will be used in the following chapter. Denote

$$\mathbb{A}_1(x,t,\xi;\lambda_j) \equiv \int_0^t \frac{1}{\sqrt{s+1}} \left(1+\frac{(x-\xi^1(t-s)-\lambda_j s)^2}{1+s}\right)^{-N} e^{-\nu(\xi)(t-s)} ds, \quad (4.1)$$

$$\begin{aligned}
&\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1}) \equiv \int_{\{s|s\in(0,t),x-\xi^1(t-s)\in(\lambda_k s,\lambda_{k+1}s)\}} \\
&\frac{e^{-\nu(\xi)(t-s)}}{\sqrt{(x-\xi^1(t-s)-\lambda_k s+1)(\lambda_{k+1}s-x+\xi^1(t-s)+1)}} ds, \quad (4.2)
\end{aligned}$$

for $j = 1,2,3$ and $k = 1,2$.

LEMMA 4.1. *For $x,t > 0$,*

$$\mathbb{A}_1(x,t,\xi;\lambda_j) \le C \frac{1}{\nu(\xi)} \frac{1}{\sqrt{t+1}} \left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N}, \quad (4.3)$$

for $j = 1, 2, 3$.

LEMMA 4.2. *For* $x, t > 0$,

$$\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1}) \le C\frac{1}{\nu(\xi)}\sum_{i=1}^{3}\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N} + C\frac{1}{\nu(\xi)}\begin{cases}\displaystyle\sum_{l=1}^{2}\frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases} \tag{4.4}$$

for $k = 1, 2$.

Denote

$$J_1(x,t;\lambda_i,\lambda_j) \equiv \int_0^t\int_0^\infty \frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} \cdot\frac{1}{s+1}\left(1+\frac{(y-\lambda_j s)^2}{1+s}\right)^{-N_2} dyds, \tag{4.5}$$

$$J_2(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \equiv \int_0^t\int_{\lambda_k s+\sqrt{s+1}}^{\lambda_{k+1}s-\sqrt{s+1}} \frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_j(t-s))^2}{1+t-s}\right)^{-N_1} \cdot\frac{1}{(|y-\lambda_k s|+1)(|y-\lambda_{k+1}s|+1)}dyds, \tag{4.6}$$

$$J_3(x,t;\lambda_i,\lambda_j) \equiv \int_0^t\int_0^\infty \frac{(1+y)^{-N_1}}{\sqrt{t-s+1}}\left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} \cdot\frac{1}{s+1}\left(1+\frac{(y-\lambda_j s)^2}{1+s}\right)^{-N_2} dyds, \tag{4.7}$$

$$J_4(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \equiv \int_0^t\int_{\lambda_k s+\sqrt{s+1}}^{\lambda_{k+1}s-\sqrt{s+1}} \frac{(1+y)^{-N_1}}{\sqrt{t-s+1}}\left(1+\frac{(x-y-\lambda_j(t-s))^2}{1+t-s}\right)^{-N_1} \cdot\frac{1}{(|y-\lambda_k s|+1)(|y-\lambda_{k+1}s|+1)}dyds, \tag{4.8}$$

$$\mathbb{J}_1(x,t;\lambda_i,\lambda_j) \equiv \int_0^t\int_0^\infty \frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} \cdot\frac{e^{-C\epsilon y^\gamma}}{\sqrt{s+1}}\left(1+\frac{(y-\lambda_j s)^2}{1+s}\right)^{-N_2} dyds, \tag{4.9}$$

$$\mathbb{J}_2(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \equiv \int_0^t\int_{\lambda_k s+\sqrt{s+1}}^{\lambda_{k+1}s-\sqrt{s+1}} \frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_j(t-s))^2}{1+t-s}\right)^{-N_1} \cdot\frac{e^{-C\epsilon y^\gamma}}{\sqrt{(|y-\lambda_k s|+1)(|y-\lambda_{k+1}s|+1)}}dyds, \tag{4.10}$$

$$\mathbb{J}_3(x,t;\lambda_i,\lambda_j) \equiv \int_0^t \int_0^\infty \frac{(1+x)^{-N_1}}{\sqrt{t-s+1}} \left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} \cdot \frac{e^{-C\epsilon y^\gamma}}{\sqrt{s+1}} \left(1+\frac{(y-\lambda_j s)^2}{1+s}\right)^{-N_2} dyds, \quad (4.11)$$

$$\mathbb{J}_4(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \equiv \int_0^t \int_{\lambda_k s+\sqrt{s+1}}^{\lambda_{k+1}s-\sqrt{s+1}} \frac{(1+y)^{-N_1}}{\sqrt{t-s+1}} \left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} \cdot \frac{e^{-C\epsilon y^\gamma}}{\sqrt{(|y-\lambda_k s|+1)(|y-\lambda_{k+1}s|+1)}} dyds, \quad (4.12)$$

for $i,j=1,2,3$ and $k=1,2$. Set $N=\min\{N_1,N_2\}$. Then

Lemma 4.3.

$$J_1(x,t;\lambda_i,\lambda_j) \leq C\sum_{l=1}^3 \frac{1}{\sqrt{t+1}} \left(1+\frac{(x-\lambda_l t)^2}{1+t}\right)^{-N} + C\begin{cases} \sum_{l=1}^2 \frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} & for\ x\in(\lambda_1 t,\lambda_3 t), \\ 0 \ for\ x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty). \end{cases} \quad (4.13)$$

for $i,j=1,2,3$.

Lemma 4.4.

$$J_2(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \leq C\sum_{l=1}^3 \frac{1}{\sqrt{t+1}} \left(1+\frac{(x-\lambda_l t)^2}{1+t}\right)^{-N_1} + C\begin{cases} \sum_{l=1}^2 \frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} & for\ x\in(\lambda_1 t,\lambda_3 t), \\ 0 \ for\ x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty). \end{cases} \quad (4.14)$$

for $j=1,2,3$ *and* $k=1,2$.

Lemma 4.5. *When* $N_1 \geq N_2+2$,

$$J_3(x,t;\lambda_i,\lambda_j) \leq C\frac{1}{\sqrt{t+1}} \left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N_2} \quad (4.15)$$

for $i,j=1,2,3$.

Lemma 4.6.

$$J_4(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \leq C\sum_{l=1}^3 \frac{1}{\sqrt{t+1}} \left(1+\frac{(x-\lambda_l t)^2}{1+t}\right)^{-N_1} + C\begin{cases} \sum_{l=1}^2 \frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} & for\ x\in(\lambda_1 t,\lambda_3 t), \\ 0 \ for\ x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty). \end{cases} \quad (4.16)$$

for $j = 1, 2, 3$ *and* $k = 1, 2$.

LEMMA 4.7. *There exists some constant* $C > 0$ *which depends on* γ *such that*

$$\mathbb{J}_1(x,t;\lambda_i,\lambda_j) \le C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}, \tag{4.17}$$

for $i, j = 1, 2, 3$.

LEMMA 4.8. *There exists some constant* $C > 0$ *which depends on* γ *such that*

$$\mathbb{J}_2(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \le C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N_1} \tag{4.18}$$

for $j = 1, 2, 3$ *and* $k = 1, 2$.

LEMMA 4.9. *When* $N_1 \gg N_2$, *there exists some constant* $C > 0$ *which depends on* γ *such that*

$$\mathbb{J}_3(x,t;\lambda_i,\lambda_j) \le C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N_2}, \tag{4.19}$$

for $i, j = 1, 2, 3$.

LEMMA 4.10. *There exists some constant* $C > 0$ *which depends on* γ *such that*

$$\mathbb{J}_4(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) \le C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N_1} \tag{4.20}$$

for $j = 1, 2, 3$ *and* $k = 1, 2$.

We provide the proofs of those lemmas in the appendix.

5. Nonlinear stability. With the estimate for the Green's function $\mathbb{G}_{ib}(x,t,\xi,\xi_*;y)$, the representation (3.6) and the assumption on the initial data $\mathsf{w}_0(x,\xi)$

$$|\mathsf{w}_0(x,\xi)| \le \varsigma e^{-C|x|-C|\xi|^2}, \tag{5.21}$$

where $\varsigma \ll 1$, we find that there exists a constant $A_0 > 0$

$$\left\|\int_0^\infty\int_{\mathbb{R}^3}\mathbb{G}_{ib}(x,t,\xi,\xi_*;y)\mathsf{w}_0(y,\xi_*)d\xi_*dy\right\|_{L^2_\xi} \le A_0\varsigma\sum_{i=1}^3\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N} \tag{5.22}$$

To deal with the linear coupling term and the nonlinear term, we make the following ansatz

$$\begin{aligned}\|\mathsf{w}(x,t)\|_{L^2_\xi} \le{}& C_0\varsigma\sum_{i=1}^3\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}\\ &+C_0\varsigma\begin{cases}\displaystyle\sum_{l=1}^2\frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases}\end{aligned} \tag{5.23}$$

$$\|\mathsf{w}(x,t)\|_{L^\infty_{\xi,3}} \le C_1\varsigma\sum_{i=1}^{3}\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$$
$$+C_1\varsigma\begin{cases}\sum_{l=1}^{2}\frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases} \tag{5.24}$$

for some $C>0$, $C_0>2A_0$ and $C_1>0$. Then Lemma 2.3 yields that $\Gamma(\mathsf{w})$ and $L_\tau\mathsf{w}$ can be controlled by

$$\|Q(\mathsf{w})\|_{L^2_\xi}\le C\|\nu(|\xi|)\mathsf{w}\|^2_{L^2_\xi}\le C\varsigma^2\sum_{i=1}^{3}\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-2N}$$
$$+C\varsigma^2\begin{cases}\sum_{l=1}^{2}\frac{1}{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty),\end{cases} \tag{5.25}$$

$$\|L_\tau\mathsf{w}\|_{L^2_\xi}\le C\left(\|\nu(|\xi|)\mathsf{w}\|_{L^2_\xi}\left\|\nu(|\xi|)\bar{\mathsf{f}}\right\|_{L^2_\xi}\right)$$
$$\le C\tau\varsigma e^{-\epsilon|x|^\gamma/C}\sum_{i=1}^{3}\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$$
$$+C\tau\varsigma e^{-\epsilon|x|^\gamma/C}\begin{cases}\sum_{l=1}^{2}\frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty),\end{cases} \tag{5.26}$$

for some $C>0$ independent of ς and τ.

From (3.6) and (5.22), we just have to prove the following inequality to verify ansatz (5.23)

$$\left\|\int_0^t\int_0^\infty\int_{\mathbb{R}^3}\mathbb{G}_{ib}(x,t-s,\xi,\xi_*;y)\left[L_\tau\mathsf{w}+Q(\mathsf{w})\right](y,s,\xi_*)d\xi_*dyds\right\|_{L^2_\xi}$$
$$\ll\frac{C_0}{2}\varsigma\sum_{i=1}^{3}\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$$
$$+\frac{C_0}{2}\varsigma\begin{cases}\sum_{l=1}^{2}\frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases}. \tag{5.27}$$

From our experience, the algebraic decaying rate $\frac{1}{t+1}$ of $\mathbb{G}_i\mathsf{P}_1$ plays an important role in closing the nonlinearity in the initial value problem. We also know that for $\mathbb{G}_{ib}\mathsf{P}_1$ one can not get such a global decaying rate. Fortunately, we get extra decaying rate on the spatial valuable for $\mathbb{G}_{ib}\mathsf{P}_1$. From (3.56), the algebraic decaying rate of $\mathbb{G}_{ib}\mathsf{P}_1$ is $\frac{1}{t+1}+\frac{(1+y)^{-N_1}}{\sqrt{(t+1)}}$. Thus we substitute (3.56) and (5.25) into the nonlinear term to get:

$$
\begin{aligned}
&\left\|\int_0^t\int_0^\infty \mathbb{G}_{ib}(x,y,t-s)Q(\mathsf{w})(y,s)dyds\right\|_{L^2_\xi} \\
&\qquad = \left\|\int_0^t\int_0^\infty \mathbb{G}_{ib}(x,y,t-s)\mathsf{P}_1 Q(\mathsf{w})(y,s)dyds\right\|_{L^2_\xi} \\
&\le C\varsigma^2\sum_{i,j=1}^3\int_0^t\int_0^\infty\left[\frac{1}{t-s+1}+\frac{(1+y)^{-N_1}}{\sqrt{(t-s+1)}}\right]\left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} \\
&\qquad \frac{1}{1+s}\left(1+\frac{(y-\lambda_j s)^2}{1+s}\right)^{-2N} dyds \\
&\quad + C\varsigma^2\sum_{k=1}^2\sum_{j=1}^3\int_0^t\int_{\lambda_k s+\sqrt{s+1}}^{\lambda_{k+1}s-\sqrt{s+1}}\left[\frac{1}{t-s+1}+\frac{(1+y)^{-N_1}}{\sqrt{(t-s+1)}}\right] \\
&\quad \left(1+\frac{(x-y-\lambda_j(t-s))^2}{1+t-s}\right)^{-N_1}\frac{1}{(|y-\lambda_k s|+1)(|y-\lambda_{k+1}s|+1)}dyds \\
&\qquad \le C\varsigma^2\left(\sum_{i,j=1}^3 (J_1(x,t;\lambda_i,\lambda_j)+J_3(x,t;\lambda_i,\lambda_j))\right. \\
&\qquad \left. +\sum_{k=1}^2\sum_{j=1}^3 (J_2(x,t;\lambda_j,\lambda_k,\lambda_{k+1})+J_4(x,t;\lambda_j,\lambda_k,\lambda_{k+1}))\right). \qquad (5.28)
\end{aligned}
$$

For the linear coupling wave, we'll fix (x,t) and use the backward Green's function. By a symmetry consideration, one has

$$
\begin{aligned}
\|\mathbb{G}_{ib}(x,y,t-s)\mathsf{P}_1\|_{L^2_\xi} &= O(1)\left[\frac{1}{t-s+1}+\frac{(1+x)^{-N_1}}{\sqrt{(t+1)}}\right] \\
&\qquad \left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1}. \qquad (5.29)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left\|\int_0^t\int_0^\infty \mathbb{G}_{ib}(x,y,t-s)L_\tau \mathsf{w}(y,s)dyds\right\|_{L^2_\xi} \\
&\qquad = \left\|\int_0^t\int_0^\infty \mathbb{G}_{ib}(x,y,t-s)\mathsf{P}_1 L_\tau \mathsf{w}(y,s)dyds\right\|_{L^2_\xi} \\
&\le C\tau\varsigma\sum_{i,j=1}^{3}\int_0^t\int_0^\infty\left[\frac{1}{t-s+1}+\frac{(1+x)^{-N_1}}{\sqrt{(t-s+1)}}\right]\left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} \\
&\qquad \frac{e^{-C\epsilon y^\gamma}}{\sqrt{1+s}}\left(1+\frac{(y-\lambda_j s)^2}{1+s}\right)^{-N} dyds \\
&\quad + C\tau\varsigma\sum_{k=1}^{2}\sum_{j=1}^{3}\int_0^t\int_{\lambda_k s+\sqrt{s+1}}^{\lambda_{k+1}s-\sqrt{s+1}}\left[\frac{1}{t-s+1}+\frac{(1+x)^{-N_1}}{\sqrt{(t-s+1)}}\right] \\
&\quad \left(1+\frac{(x-y-\lambda_j(t-s))^2}{1+t-s}\right)^{-N_1}\frac{e^{-C\epsilon y^\gamma}}{\sqrt{(|y-\lambda_k s|+1)(|y-\lambda_{k+1}s|+1)}}dyds \\
&\qquad \le C\tau\varsigma\left(\sum_{i,j=1}^{3}\left(\mathbb{J}_1(x,t;\lambda_i,\lambda_j)+\mathbb{J}_3(x,t;\lambda_i,\lambda_j)\right)\right. \\
&\qquad\quad \left. +\sum_{k=1}^{2}\sum_{j=1}^{3}\left(\mathbb{J}_2(x,t;\lambda_j,\lambda_k,\lambda_{k+1})+\mathbb{J}_4(x,t;\lambda_j,\lambda_k,\lambda_{k+1})\right)\right).
\end{aligned}
\tag{5.30}
$$

Estimates	Reference
$J_1 \le C\sum_{l=1}^{3}\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_l t)^2}{1+t}\right)^{-N} + C\begin{cases}\sum_{l=1}^{2}\frac{1}{\sqrt{(\lvert x-\lambda_l t\rvert+1)(\lvert x-\lambda_{l+1}t\rvert+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases}$	Lemma4.3
$J_2 \le C\sum_{l=1}^{3}\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_l t)^2}{1+t}\right)^{-N_1} + C\begin{cases}\sum_{l=1}^{2}\frac{1}{\sqrt{(\lvert x-\lambda_l t\rvert+1)(\lvert x-\lambda_{l+1}t\rvert+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases}$	Lemma4.4
$J_3 \le C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$	Lemma4.5
$J_4 \le C\sum_{l=1}^{3}\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_l t)^2}{1+t}\right)^{-N_1} + C\begin{cases}\sum_{l=1}^{2}\frac{1}{\sqrt{(\lvert x-\lambda_l t\rvert+1)(\lvert x-\lambda_{l+1}t\rvert+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases}$	Lemma4.6

(5.31)

Estimates	Reference
$\mathbb{J}_1 \leq C\dfrac{1}{\sqrt{t+1}}\left(1+\dfrac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$	Lemma4.7
$\mathbb{J}_2 \leq C\dfrac{1}{\sqrt{t+1}}\left(1+\dfrac{(x-\lambda_j t)^2}{1+t}\right)^{-N_1}$	Lemma4.8
$\mathbb{J}_3 \leq C\dfrac{1}{\sqrt{t+1}}\left(1+\dfrac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$	Lemma4.9
$\mathbb{J}_4 \leq C\dfrac{1}{\sqrt{t+1}}\left(1+\dfrac{(x-\lambda_j t)^2}{1+t}\right)^{-N_1}$	Lemma4.10

(5.32)

(5.28), (5.30), Table (5.31) and Table (5.32) yield that (5.27) holds. Thus we justify the ansatz (5.23).

To verify (5.24), we only have to consider

$$\begin{aligned}\mathsf{w}(x,t,\xi) &= e^{-\nu(\xi)t}\mathsf{w}_0(x-\xi^1 t,\xi)\\ &\quad+\int_0^t e^{-\nu(\xi)(t-s)}\left(\mathsf{K}\mathsf{w}+L_\tau\mathsf{w}+Q(\mathsf{w})\right)(x-\xi^1(t-s),s,\xi)ds. \end{aligned}\tag{5.33}$$

Set

$$\begin{aligned}\Upsilon(x,t) &= \sum_{i=1}^{3}\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}\\ &\quad+\begin{cases}\displaystyle\sum_{l=1}^{2}\frac{1}{\sqrt{(|x-\lambda_l t|+1)(|x-\lambda_{l+1}t|+1)}} \text{ for } x\in(\lambda_1 t,\lambda_3 t),\\ 0 \text{ for } x\in(0,\lambda_1 t)\cup(\lambda_3 t,\infty).\end{cases}\end{aligned}\tag{5.34}$$

From Lemma 2.2 and the ansatz (5.23),

$$\begin{aligned}&\left|\int_0^t e^{-\nu(\xi)(t-s)}\mathsf{K}\mathsf{w}(x-\xi^1(t-s),s)ds\right| \leq C\int_0^t e^{-\nu(\xi)(t-s)}\left\|\mathsf{w}(x-\xi^1(t-s),s)\right\|_{L^2_\xi}ds\\ &\leq C\sum_{j=1}^{3}\int_0^t\frac{1}{\sqrt{s+1}}\left(1+\frac{(x-\xi^1(t-s)-\lambda_j s)^2}{C(s+1)}\right)^{-N}e^{-\nu(\xi)(t-s)}ds\\ &\qquad+C\sum_{k=1}^{2}\int_{\{s\in(0,t)\}\cap\{s|x-\xi^1(t-s)\in(\lambda_k s,\lambda_{k+1}s)\}}\\ &\qquad\frac{e^{-\nu(\xi)(t-s)}}{\sqrt{(x-\xi^1(t-s)-\lambda_k s+1)(\lambda_{k+1}s-x+\xi^1(t-s)+1)}}ds,\\ &=\sum_{j=1}^{3}\mathbb{A}_1(x,t,\xi;\lambda_j)+\sum_{k=1}^{2}\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1})\leq\frac{C}{\nu(\xi)}\Upsilon(x,t).\end{aligned}\tag{5.35}$$

Consider

$$
\begin{aligned}
\sup_{x,t}\frac{|\mathsf{w}(x,t)|}{\Upsilon(x,t)} \le \sup_{x,t}\frac{e^{-\nu(\xi)t}\left|\mathsf{w}_0(x-\xi^1 t,\xi)\right|}{\Upsilon(x,t)}
&+\sup_{x,t}\frac{1}{\Upsilon(x,t)}\left|\int_0^t e^{-\nu(\xi)(t-s)}\mathsf{K}\mathsf{w}(x-\xi^1(t-s),s)ds\right| \\
&+\sup_{x,t}\left|\int_0^t \frac{1}{\Upsilon(x,t)}e^{-\nu(\xi)(t-s)}\left(L_\tau \mathsf{w}+Q(\mathsf{w})\right)(x-\xi^1(t-s),s)ds\right| \\
&\le \frac{C\varsigma}{(1+|\xi|)^3}+\frac{C\varsigma}{1+|\xi|} \\
&+\sup_{x,t}\frac{|(\tau+\varsigma)\mathsf{w}(x,t)|}{\Upsilon(x,t)}\int_0^t e^{-\nu(\xi)(t-s)}\nu(|\xi|)\frac{\Upsilon(x-\xi^1(t-s),s)}{\Upsilon(x,t)}ds. \quad (5.36)
\end{aligned}
$$

From

$$
\begin{aligned}
\int_0^t e^{-\nu(\xi)(t-s)}\Upsilon(x-\xi^1(t-s),s)ds &= \sum_{j=1}^{3}\int_0^t \frac{1}{\sqrt{s+1}} \\
\cdot\left(1+\frac{(x-\xi^1(t-s)-\lambda_j s)^2}{C(s+1)}\right)^{-N} e^{-\nu(\xi)(t-s)}ds &+\sum_{k=1}^{2}\int_{\{s|s\in(0,t),x-\xi^1(t-s)\in(\lambda_k s,\lambda_{k+1}s)\}} \\
&\frac{e^{-\nu(\xi)(t-s)}}{\sqrt{(x-\xi^1(t-s)-\lambda_k s+1)(\lambda_{k+1}s-x+\xi^1(t-s)+1)}}ds, \\
&=\sum_{j=1}^{3}\mathbb{A}_1(x,t,\xi;\lambda_j)+\sum_{k=1}^{2}\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1})\le \frac{C}{\nu(\xi)}\Upsilon(x,t), \quad (5.37)
\end{aligned}
$$

one has

$$
\begin{aligned}
\int_0^t e^{-\nu(\xi)(t-s)}\nu(\xi)\frac{\Upsilon(x-\xi^1(t-s),s)}{\Upsilon(x,t)}ds
&= \frac{\nu(\xi)}{\Upsilon(x,t)}\int_0^t e^{-\nu(\xi)(t-s)}\Upsilon(x-\xi^1(t-s),s)ds \le C. \quad (5.38)
\end{aligned}
$$

Thus (5.36) and (5.38) yield that

$$
\sup_{x,t}\frac{|\mathsf{w}(x,t)|}{\Upsilon(x,t)} \le \frac{2C\varsigma}{(1+|\xi|)^3}+\frac{2C\varsigma}{1+|\xi|}, \quad (5.39)
$$

when $\varsigma,\tau\ll 1$. Repeat the process for another two times. Since the operator K gains decaying rate in ξ variable, we have

$$
\sup_{x,t}\frac{|\mathsf{w}(x,t)|}{\Upsilon(x,t)} \le \frac{C\varsigma}{(1+|\xi|)^3}. \quad (5.40)
$$

Thus the ansatz assumption (5.24) is justified.

6. Appendix. Here, we provide the proofs of the lemmas in Section 4.

Proof of Lemma 4.1. Let m sufficient large and satisfy $m > \frac{1}{\gamma}$. Consider $\mathbb{A}_1(x,t,\xi;\lambda_j)$:

$$\begin{aligned}
&\mathbb{A}_1(x,t,\xi;\lambda_j)\\
&\le \frac{1}{\sqrt{1+t}}\left(\int_0^{t-t^{\frac{1}{m}}} e^{-\nu(\xi)t^{\frac{1}{m}}/2}e^{-\nu(\xi)(t-s)/2}\left(1+\frac{(x-\xi^1(t-s)-\lambda_j s)^2}{1+s}\right)^{-N} ds\right.\\
&\qquad\left.+\int_{t-t^{\frac{1}{m}}}^{t} e^{-\nu(\xi)(t-s)}\left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N} ds\right)\\
&\le \frac{1}{\sqrt{1+t}}\left(\int_{\{s||x-\lambda_j s|>2|\xi^1|(t-s)\}}\left(1+\frac{(x-\lambda_j s)^2}{1+t}\right)^{-N} e^{-\nu(\xi)(t-s)/2}ds\right.\\
&+\left.\int_{\{s||x-\lambda_j s|<2|\xi^1|(t-s)\}} e^{-C|x-\lambda_j s|^{\frac{1}{m}}}e^{-\frac{\nu(\xi)(t-s)}{2}}ds\right)+C\frac{1}{\nu(\xi)}\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N}\\
&\le C\frac{1}{\nu(\xi)}\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N}. \qquad (6.1)
\end{aligned}$$

Thus we get the estimate (4.3). □

Proof of Lemma 4.2. The estimate of $\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1})$ should be considered in different regions of x. When $\lambda_k t < x < \lambda_{k+1}t$,

$$\begin{aligned}
\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1}) &\le \int_0^{t-5logt} e^{-\nu(\xi)(t-s)}ds\\
&+\int_{t-5logt}^{t} e^{-\nu(\xi)(t-s)}\frac{1}{\sqrt{(x-\lambda_k t+1)(\lambda_{k+1}t-x+1)}}ds\\
&\le \frac{1}{\nu(\xi)}\frac{1}{\sqrt{(x-\lambda_k t+1)(\lambda_{k+1}t-x+1)}}. \qquad (6.2)
\end{aligned}$$

When $x > \lambda_{k+1}t$, in the special integral domain of $\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1})$,

$$0 > x-\xi^1(t-s)-\lambda_{k+1}s > \lambda_{k+1}(t-s)-\xi^1(t-s).$$

Thus $|x-\xi^1(t-s)-\lambda_{k+1}s| < C|\xi|(t-s)$ implies

$$\begin{aligned}
&\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1})\\
&\le \int_0^{t-t^{\frac{1}{m}}}\frac{1}{\sqrt{t+1}}e^{-\frac{\nu(\xi)t^{\frac{1}{m}}}{2}-|x-\xi^1(t-s)-\lambda_{k+1}s|^{\frac{1}{m}}/C}ds \le \frac{1}{\nu(\xi)}\frac{1}{\sqrt{t+1}}e^{-|x-\lambda_{k+1}t|^{\frac{1}{m}}}. \qquad (6.3)
\end{aligned}$$

When $x < \lambda_k t$, in the special integral domain of $\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1})$,

$$0 < x-\xi^1(t-s)-\lambda_k s < \lambda_k(t-s)-\xi^1(t-s).$$

Thus $|x-\xi^1(t-s)-\lambda_k s| < C|\xi|(t-s)$ which implies

$$\begin{aligned}&\mathbb{A}_2(x,t,\xi;\lambda_k,\lambda_{k+1})\\ &\le \int_0^{t-t^{\frac{1}{m}}} \frac{1}{\sqrt{t+1}} e^{-\frac{\nu(\xi)t^{\frac{1}{m}}}{2}-|x-\xi^1(t-s)-\lambda_k s|^{\frac{1}{m}}/C} ds \le \frac{1}{\nu(\xi)}\frac{1}{\sqrt{t+1}} e^{-|x-\lambda_k t|^{\frac{1}{m}}}. \quad (6.4)\end{aligned}$$

Then we complete the proof. □

Proof of Lemma 4.3. The proof is similar to that in [11]. If $\lambda_i=\lambda_j$,

$$\begin{aligned}J_1(x,t;\lambda_i,\lambda_i) &= O(1)\int_0^t \frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N} ds\\ &\le C\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}. \quad (6.5)\end{aligned}$$

If $\lambda_i\neq\lambda_j$, suppose $\lambda_i=0,\lambda_j=1$. Thus when $x<\sqrt{t+1}$,

$$\begin{aligned}&J_1(x,t;\lambda_i,\lambda_i) = O(1)\int_0^t \frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}\left(1+\frac{(x-s)^2}{1+t}\right)^{-N} ds\\ &\le C\frac{1}{\sqrt{1+t}}\int_0^t \frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}ds \le C\frac{1}{\sqrt{1+t}} \le C\frac{1}{\sqrt{1+t}}\left(1+\frac{x^2}{1+t}\right)^{-N}. \quad (6.6)\end{aligned}$$

When $\sqrt{t+1}<x<t-\sqrt{t+1}$, let M be a big constant

$$\begin{aligned}J_1(x,t;\lambda_i,\lambda_i) &= O(1)\int_0^t \frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}\left(1+\frac{(x-s)^2}{1+t}\right)^{-N} ds\\ &\le \int_0^{x/M}\frac{1}{1+t}\frac{1}{\sqrt{1+s}}\left(1+\frac{x^2}{1+t}\right)^{-N} ds\\ &\quad+\int_{t-(t-x)/M}^t \frac{1}{1+t}\frac{1}{\sqrt{1+t-s}}\left(1+\frac{(x-t)^2}{1+t}\right)^{-N} ds\\ &\quad+\int_{x/M}^{t-(t-x)/M}\frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{1+t-x}}\frac{1}{\sqrt{1+x}}\left(1+\frac{(x-s)^2}{1+t}\right)^{-N} ds\\ &\le C\frac{1}{\sqrt{1+t}}\left(1+\frac{x^2}{1+t}\right)^{-N}+C\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-t)^2}{1+t}\right)^{-N}+C\frac{1}{\sqrt{1+t-x}}\frac{1}{\sqrt{1+x}}. \quad (6.7)\end{aligned}$$

When $t-\sqrt{t+1}<x<t+\sqrt{t+1}$,

$$\begin{aligned}J_1(x,t;\lambda_i,\lambda_i) &= O(1)\int_0^t \frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}\left(1+\frac{(x-s)^2}{1+t}\right)^{-N} ds\\ &\le \int_0^{t/2}\frac{1}{1+t}\frac{1}{\sqrt{1+s}}t^{-N}ds+\int_{t-\sqrt{t+1}}^t \frac{1}{1+t}\frac{1}{\sqrt{1+t-s}}ds\\ &\quad+\int_{t/2}^{t-\sqrt{t+1}}\frac{1}{\sqrt{1+t}}\frac{1}{(1+t)^{\frac{3}{4}}}\left(1+\frac{(x-s)^2}{1+t}\right)^{-N} ds\\ &\le C\frac{1}{(1+t)^{\frac{3}{4}}} \le C\frac{1}{\sqrt{1+t-x}}\frac{1}{\sqrt{1+x}}. \quad (6.8)\end{aligned}$$

When $x > t + \sqrt{t+1}$,

$$J_1(x,t;\lambda_i,\lambda_i) = O(1)\int_0^t \frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}\left(1+\frac{(x-s)^2}{1+t}\right)^{-N} ds$$
$$\leq \int_0^t \frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}\left(1+\frac{(x-t)^2+(t-s)^2}{1+t}\right)^{-N} ds$$
$$\leq C\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-t)^2}{1+t}\right)^{-N}. \quad (6.9)$$

□

Proof of Lemma 4.4. Consider $J_2(x,t;\lambda_1,\lambda_1,\lambda_2)$ first. When $x < \lambda_1 t - \sqrt{t+1}$, $0 > x - \lambda_1 t > x - y - \lambda_1(t-s)$ holds in the special integration region $y \in \{\lambda_1 s + \sqrt{s+1} \leq y \leq \lambda_2 s - \sqrt{s+1}\}$. Thus

$$J_2(x,t;\lambda_1,\lambda_1,\lambda_2) \leq C\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}\int_0^t \frac{1}{1+t-s}\frac{\log(2+s)}{1+s}ds$$
$$\leq C\frac{\log^2 t}{1+t}\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}. \quad (6.10)$$

Similarly, when $x > \lambda_2 t + \sqrt{t+1}$,

$$J_2(x,t;\lambda_1,\lambda_1,\lambda_2) \leq C\left(1+\frac{(x-\lambda_2 t)^2}{1+t}\right)^{-N_1}\int_0^t \frac{1}{1+t-s}\frac{\log(2+s)}{1+s}ds$$
$$\leq C\frac{\log^2 t}{1+t}\left(1+\frac{(x-\lambda_2 t)^2}{1+t}\right)^{-N_1}. \quad (6.11)$$

When $\lambda_1 t + \sqrt{t+1} < x < \lambda_2 t - \sqrt{t+1}$, the discussion can be divided into two cases.

Case 1: $\lambda_1 t+\sqrt{t+1}<x<(\lambda_1+\lambda_2)t/2$. Set $\tau_0=\frac{x-\lambda_1 t}{\lambda_2-\lambda_1}$,

$$\begin{aligned}
J_2(x,t;\lambda_1,\lambda_1,\lambda_2) &\le \left(\int_0^{\tau_0}+\int_{\tau_0}^{t}\right) \\
&\quad \int_{\lambda_1 s+\sqrt{s+1}}^{\lambda_2 s-\sqrt{s+1}} \frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_1(t-s))^2}{1+t-s}\right)^{-N_1} \\
&\quad \frac{1}{(|y-\lambda_1 s|+1)(|y-\lambda_2 s|+1)}dyds \\
&\le \int_0^{\tau_0}\frac{1}{1+t-s}\left(1+\frac{(x-\lambda_2 s-\lambda_1(t-s))^2}{1+t-s}\right)^{-N_1}\frac{\log(2+s)}{1+s}ds \\
&\quad +\int_{\tau_0}^{t}\left(\int_{y\in(\lambda_1 s+\sqrt{s+1},\lambda_2 s-\sqrt{s+1})\cap\{|x-y-\lambda_1(t-s)|<(t-s)^{\frac{2}{3}}\}}\right. \\
&\quad \left.+\int_{y\in(\lambda_1 s+\sqrt{s+1},\lambda_2 s-\sqrt{s+1})\cap\{|x-y-\lambda_1(t-s)|\ge(t-s)^{\frac{2}{3}}\}}\right) \\
&\quad \frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_1(t-s))^2}{1+t-s}\right)^{-N_1}\frac{1}{(|y-\lambda_1 s|+1)(|y-\lambda_2 s|+1)}dyds \\
&\le \frac{1}{\sqrt{1+t}}\frac{\log(1+t)}{x-\lambda_1 t}+\int_{\tau_0}^{t}\frac{1}{\sqrt{1+t-s}}\frac{ds}{(x-\lambda_1 t+1)(x-\lambda_1 t+\lambda_1 s-\lambda_2 s+1)} \\
&\quad +\int_{\tau_0}^{t}\frac{1}{\sqrt{1+t-s}}\left(1+(t-s)^{\frac{1}{3}}\right)^{-N_1+1}\frac{1}{s\sqrt{s+1}}ds \\
&\le \frac{1}{\sqrt{(x-\lambda_1 t+1)(\lambda_2 t-x+1)}}+\frac{\log t}{(x-\lambda_1 t+1)\sqrt{t}}+t^{-3/2} \\
&\le C\frac{1}{\sqrt{(x-\lambda_1 t+1)(\lambda_2 t-x+1)}}. \qquad (6.12)
\end{aligned}$$

Case 2: $(\lambda_1+\lambda_2)t/2<x<\lambda_2 t-\sqrt{t+1}$. Then $\tau_0=\frac{x-\lambda_1 t}{\lambda_2-\lambda_1}=O(1)t$,

$$\begin{aligned}
J_2(x,t;\lambda_1,\lambda_1,\lambda_2) &\le \int_0^{\tau_0-\frac{\lambda_2 t-x}{3(\lambda_2-\lambda_1)}}\frac{1}{(1+t-s)^{1/2}(1+s)^{3/2}}(1+t-s)^{-N_1+1}ds \\
&\quad +\int_{\tau_0-\frac{\lambda_2 t-x}{3(\lambda_2-\lambda_1)}}^{\tau_0}\frac{1}{\sqrt{t-\tau_0}s^{3/2}}\left(1+\frac{(x-\lambda_2 s-\lambda_1(t-s))^2}{1+t-s}\right)^{-N_1+1} \\
&\quad +\int_{\tau_0}^{t}\left(\int_{y\in(\lambda_1 s+\sqrt{s+1},\lambda_2 s-\sqrt{s+1})\cap\{|x-y-\lambda_1(t-s)|<(t-s)^{\frac{2}{3}}\}}\right. \\
&\quad \left.+\int_{y\in(\lambda_1 s+\sqrt{s+1},\lambda_2 s-\sqrt{s+1})\cap\{|x-y-\lambda_1(t-s)|\ge(t-s)^{\frac{2}{3}}\}}\right) \\
&\quad \frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_1(t-s))^2}{1+t-s}\right)^{-N_1}\frac{1}{(|y-\lambda_1 s|+1)(|y-\lambda_2 s|+1)}dyds \\
&\le t^{-3/2}+t^{-3/2}+\frac{\log t}{\sqrt{(\lambda_2 t-x+1)t}}+t^{-3/2}\le C\frac{1}{\sqrt{(x-\lambda_1 t+1)(\lambda_2 t-x+1)}}. \qquad (6.13)
\end{aligned}$$

Thus the estimate (4.14) holds for $J_2(x,t;\lambda_1,\lambda_1,\lambda_2)$.

The estimate of $J_2(x,t;\lambda_2,\lambda_1,\lambda_2)$ can be obtained similarly.

Next we'll consider $J_2(x,t;\lambda_3,\lambda_1,\lambda_2)$. When $x<\lambda_1 t-\sqrt{t+1}$,

$$\begin{aligned}J_2(x,t;\lambda_3,\lambda_1,\lambda_2) &\le C\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}\int_0^t \frac{1}{1+t-s}\frac{\log(2+s)}{1+s}ds\\ &\le C\frac{\log^2 t}{1+t}\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}. \end{aligned}\tag{6.14}$$

When $x>\lambda_3 t+\sqrt{t+1}$,

$$\begin{aligned}J_2(x,t;\lambda_3,\lambda_1,\lambda_2) &\le C\left(1+\frac{(x-\lambda_3 t)^2}{1+t}\right)^{-N_1}\int_0^t \frac{1}{1+t-s}\frac{\log(2+s)}{1+s}ds\\ &\le C\frac{\log^2 t}{1+t}\left(1+\frac{(x-\lambda_3 t)^2}{1+t}\right)^{-N_1}. \end{aligned}\tag{6.15}$$

When $\lambda_1 t+\sqrt{t+1}<x<\lambda_2 t-\sqrt{t+1}$, $\tau_0=\frac{\lambda_3 t-x}{\lambda_3-\lambda_1}=O(1)t$,

$$\begin{aligned}J_2(x,t;\lambda_3,\lambda_1,\lambda_2) \le& \int_0^{\tau_0-\frac{\lambda_2 t-x}{3(\lambda_2-\lambda_1)}}\frac{1}{(1+t-s)^{1/2}(1+s)^{3/2}}(1+t-s)^{-N_1+1}ds\\ &+\int_{\tau_0-\frac{\lambda_2 t-x}{3(\lambda_2-\lambda_1)}}^{\tau_0}\frac{1}{\sqrt{t-\tau_0}s^{3/2}}\left(1+\frac{(x-\lambda_1 t-(\lambda_3-\lambda_1)(t-s))^2}{1+t-s}\right)^{-N_1+1}\\ &+\int_{\tau_0}^t\left(\int_{y\in(\lambda_1 s+\sqrt{s+1},\lambda_2 s-\sqrt{s+1})\cap\{|x-y-\lambda_3(t-s)|<(t-s)^{\frac{2}{3}}\}}\right.\\ &\left.+\int_{y\in(\lambda_1 s+\sqrt{s+1},\lambda_2 s-\sqrt{s+1})\cap\{|x-y-\lambda_3(t-s)|\ge(t-s)^{\frac{2}{3}}\}}\right)\\ &\frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_3(t-s))^2}{1+t-s}\right)^{-N_1}\frac{1}{(|y-\lambda_1 s|+1)(|y-\lambda_2 s|+1)}dyds\\ \le& t^{-3/2}+t^{-3/2}+\frac{\log t}{\sqrt{(x-\lambda_1 t+1)(\lambda_2 t-x+1)}}+t^{-3/2}\\ \le& C\frac{1}{\sqrt{(x-\lambda_1 t+1)(\lambda_2 t-x+1)}}. \end{aligned}\tag{6.16}$$

When $\lambda_2 t+\sqrt{t+1}<x<\lambda_3 t-\sqrt{t+1}$, $\tau_0=\frac{\lambda_3 t-x}{\lambda_3-\lambda_1}$, $\tau_1=\frac{\lambda_3 t-x}{\lambda_3-\lambda_2}$. Case 1: $\lambda_2 t+$

$\sqrt{t+1} < x < (\lambda_2+\lambda_3)t/2$,

$$\begin{aligned}
&J_2(x,t;\lambda_3,\lambda_1,\lambda_2)\\
&\quad=\left(\int_0^{\tau_0}+\int_{\tau_0}^{\tau_1}+\int_{\tau_1}^{t}\right)\frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_3(t-s))^2}{1+t-s}\right)^{-N_1}\\
&\qquad\frac{1}{(|y-\lambda_1 s|+1)(|y-\lambda_2 s|+1)}\\
&\quad\le\int_0^{\tau_0}\frac{1}{\sqrt{1+t-s}s^3/2}\left(1+\frac{(x-\lambda_1 s-\lambda_3(t-s))^2}{1+t-s}\right)^{-N_1+1}ds\\
&\qquad+\int_{\tau_1}^{t}\frac{1}{\sqrt{1+t-s}s^3/2}\left(1+\frac{(x-\lambda_2 s-\lambda_3(t-s))^2}{1+t-s}\right)^{-N_1+1}ds\\
&\qquad+\int_{\tau_0}^{\tau_1}\frac{1}{\sqrt{1+t-s}}\frac{1}{(x-\lambda_3(t-s)-\lambda_1 s+1)(\lambda_2 s-x+\lambda_3(t-s)+1)}ds\\
&\qquad+\int_{\tau_0}^{\tau_1}\frac{1}{\sqrt{1+t-s}s^{3/2}}(1+(t-s)^{1/3})^{-N_1+1}\\
&\le t^{-3/2}+t^{-3/2}+\frac{1}{(\lambda_3 t-x+1)t}+t^{-3/2}\le C\frac{1}{\sqrt{(x-\lambda_2 t+1)(\lambda_3 t-x+1)}}.
\end{aligned}\tag{6.17}$$

Case 2: $(\lambda_2+\lambda_3)t/2 < x < \lambda_3 t-\sqrt{t+1}$,

$$\begin{aligned}
&J_2(x,t;\lambda_3,\lambda_1,\lambda_2)\\
&\quad=\left(\int_0^{\tau_0}+\int_{\tau_0}^{\tau_1}+\int_{\tau_1}^{t}\right)\frac{1}{t-s+1}\left(1+\frac{(x-y-\lambda_3(t-s))^2}{1+t-s}\right)^{-N_1}\\
&\qquad\frac{1}{(|y-\lambda_1 s|+1)(|y-\lambda_2 s|+1)}\\
&\quad\le\int_0^{\tau_0}\frac{\log(1+s)}{(1+t-s)(1+s)}\left(1+\frac{(x-\lambda_1 s-\lambda_3(t-s))^2}{1+t-s}\right)^{-N_1}ds\\
&\qquad+\int_{\tau_1}^{t}\frac{\log(1+s)}{(1+t-s)(1+s)}\left(1+\frac{(x-\lambda_2 s-\lambda_3(t-s))^2}{1+t-s}\right)^{-N_1}ds\\
&\qquad+\int_{\tau_0}^{\tau_1}\frac{1}{\sqrt{1+t-s}}\frac{1}{(x-\lambda_3(t-s)-\lambda_1 s+1)(\lambda_2 s-x+\lambda_3(t-s)+1)}ds\\
&\qquad+\int_{\tau_0}^{\tau_1}\frac{1}{\sqrt{1+t-s}s^{3/2}}(1+(t-s)^{1/3})^{-N_1+1}\\
&\quad\le\frac{1}{\sqrt{t}}\frac{\log(1+t)}{\lambda_3 t-x+1}+\frac{\log(1+\tau_1)}{\sqrt{t}\tau_1}+\frac{\log(1+t)}{(\lambda_3 t-x+1)\sqrt{t}}+t^{-3/2}\\
&\qquad\le C\frac{1}{\sqrt{(x-\lambda_2 t+1)(\lambda_3 t-x+1)}}.
\end{aligned}\tag{6.18}$$

Thus $J_2(x,t;\lambda_3,\lambda_1,\lambda_2)$ also satisfies (4.14).

By a similar computation, we can find that (4.14) also holds for $k=2$. Therefore we get the lemma. □

Proof of Lemma 4.5. When $|x-\lambda_i t| \geq \sqrt{t+1}$,

$$\begin{aligned}
&J_3(x,t;\lambda_i,\lambda_j)\\
&\quad = O(1)\int_0^t \frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}\left(1+\frac{(x-\lambda_j s-\lambda_i(t-s))^2}{1+t-s}\right)^{-N}(1+s)^{-N_1}ds\\
&\qquad + O(1)\int_0^t \frac{1}{\sqrt{1+t-s}}\frac{1}{1+s}\left(1+\frac{(x-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1+2}(1+s)^{-N_2}ds\\
&\qquad \leq C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N} + C\frac{1}{|x-\lambda_i t|}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}\\
&\qquad \leq C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}. \qquad (6.19)
\end{aligned}$$

If $|x-\lambda_i t| \leq \sqrt{t+1}$, $\frac{1}{\sqrt{t+1}} \leq C\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}$. Thus

$$\begin{aligned}
J_3(x,t;\lambda_i,\lambda_j) &\leq C\int_0^t \frac{1}{\sqrt{t-s+1}}\frac{1}{\sqrt{-s+1}}(1+s)^{-N_2}ds\\
&\leq C\frac{1}{\sqrt{t+1}} \leq C\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}. \qquad (6.20)
\end{aligned}$$

□

Proof of Lemma 4.6. Consider $J_4(x,t;\lambda_1,\lambda_1,\lambda_2)$ first. When $x < \lambda_1 t - \sqrt{t+1}$, $0 > x-\lambda_1 t > x-y-\lambda_1(t-s)$ holds in the special integration region $y \in \{\lambda_1 s+\sqrt{s+1} \leq y \leq \lambda_2 s - \sqrt{s+1}\}$. Thus

$$\begin{aligned}
J_4(x,t;\lambda_1,\lambda_1,\lambda_2) &\leq C\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}\int_0^t \frac{1}{\sqrt{1+t-s}}\frac{1}{(1+s)^{3/2}}ds\\
&\leq C\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}. \qquad (6.21)
\end{aligned}$$

Similarly, when $x > \lambda_2 t + \sqrt{t+1}$,

$$\begin{aligned}
J_4(x,t;\lambda_1,\lambda_1,\lambda_2) &\leq C\left(1+\frac{(x-\lambda_2 t)^2}{1+t}\right)^{-N_1}\int_0^t \frac{1}{\sqrt{1+t-s}}\frac{1}{(1+s)^{3/2}}ds\\
&\leq C\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_2 t)^2}{1+t}\right)^{-N_1}. \qquad (6.22)
\end{aligned}$$

When $\lambda_1 t + \sqrt{t+1} < x < \lambda_2 t - \sqrt{t+1}$,

$$\begin{aligned}
J_4(x,t;\lambda_1,\lambda_1,\lambda_2) &\leq \int_0^t \frac{1}{\sqrt{t-s+1}}\left(1+\frac{(x-\lambda_1(t-s))^2}{1+t-s}\right)^{-N_1+2}\frac{1}{(1+s)^{3/2}}ds\\
&\leq C\frac{1}{\sqrt{1+t}}\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1+2}. \qquad (6.23)
\end{aligned}$$

Thus the estimate (4.16) holds for $J_2(x,t;\lambda_1,\lambda_1,\lambda_2)$.

Consider $J_4(x,t;\lambda_2,\lambda_1,\lambda_2)$. When $x<\lambda_1 t-\sqrt{t+1}$ or $x>\lambda_2 t+\sqrt{t+1}$,

$$\begin{aligned} &J_4(x,t;\lambda_2,\lambda_1,\lambda_2)\\ &\le C\left(\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}+\left(1+\frac{(x-\lambda_2 t)^2}{1+t}\right)^{-N_1}\right)\int_0^t \frac{1}{\sqrt{t-s+1}}\frac{1}{(1+s)^{3/2}}ds\\ &\qquad\le C\frac{1}{\sqrt{1+t}}\left(\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}+\left(1+\frac{(x-\lambda_2 t)^2}{1+t}\right)^{-N_1}\right). \end{aligned} \tag{6.24}$$

When $\lambda_1 t+\sqrt{t+1}<x<\lambda_2 t-\sqrt{t+1}$,

$$\begin{aligned} &J_4(x,t;\lambda_2,\lambda_1,\lambda_2)\\ &\quad\le\int_0^t \frac{1}{\sqrt{t-s+1}}(1+s)^{-N_1+1}\frac{1}{\sqrt{(x-\lambda_2(t-s)-\lambda_1 s)(\lambda_2 t-x+1)}}ds\\ &\qquad\le\frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{\lambda_2 t-x+1}}. \end{aligned} \tag{6.25}$$

Thus the estimate (4.16) also holds for $J_4(x,t;\lambda_2,\lambda_1,\lambda_2)$.

Next we'll consider $J_4(x,t;\lambda_3,\lambda_1,\lambda_2)$. When $x<\lambda_1 t-\sqrt{t+1}$ or $x>\lambda_3 t+\sqrt{t+1}$,

$$\begin{aligned} J_4(x,t;\lambda_3,\lambda_1,\lambda_2)&\le C\left(\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}+\left(1+\frac{(x-\lambda_3 t)^2}{1+t}\right)^{-N_1}\right)\\ &\qquad\cdot\int_0^t \frac{1}{\sqrt{1+t-s}}\frac{1}{(1+s)^{3/2}}ds\\ &\le C\frac{1}{\sqrt{1+t}}\left(\left(1+\frac{(x-\lambda_1 t)^2}{1+t}\right)^{-N_1}+\left(1+\frac{(x-\lambda_3 t)^2}{1+t}\right)^{-N_1}\right). \end{aligned} \tag{6.26}$$

When $\lambda_1 t+\sqrt{t+1}<x<\lambda_2 t-\sqrt{t+1}$,

$$\begin{aligned} &J_4(x,t;\lambda_3,\lambda_1,\lambda_2)\\ &\quad\le\int_0^t \frac{(1+s)^{-N_1+1}}{\sqrt{t-s+1}}\frac{1}{\sqrt{(x-\lambda_3(t-s)-\lambda_1 s)(\lambda_2 s-x+\lambda_3(t-s)+1)}}ds\\ &\qquad\le\frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{\lambda_2 t-x+1}}. \end{aligned} \tag{6.27}$$

When $\lambda_2 t+\sqrt{t+1}<x<\lambda_3 t-\sqrt{t+1}$,

$$\begin{aligned} &J_4(x,t;\lambda_3,\lambda_1,\lambda_2)\\ &\quad\le\int_0^t \frac{(1+s)^{-N_1+1}}{\sqrt{t-s+1}}\frac{1}{\sqrt{(x-\lambda_3(t-s)-\lambda_1 s)(\lambda_2 s-x+\lambda_3(t-s)+1)}}ds\\ &\qquad\le\frac{1}{\sqrt{1+t}}\frac{1}{\sqrt{\lambda_3 t-x+1}}. \end{aligned} \tag{6.28}$$

Thus $J_4(x,t;\lambda_3,\lambda_1,\lambda_2)$ also satisfies (4.16).

By a similar computation, we can find that (4.16) also holds for $k=2$. Therefore we get the lemma. □

Proof of Lemma 4.7. $\mathbb{J}_1(x,t;\lambda_i,\lambda_j)$ can be rewritten to be

$$\begin{aligned}
\mathbb{J}_1(x,t;\lambda_i,\lambda_j) \le\, & C\int_0^t\int_0^{\lambda_j s/2}\frac{1}{(t-s+1)\sqrt{s+1}}e^{-C\epsilon y^\gamma} \\
& \cdot\left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} s^{-N_2}dyds \\
& + C\int_0^t\int_{\lambda_j s/2}^{\infty}\frac{1}{(t-s+1)\sqrt{s+1}}e^{-C\epsilon y^\gamma - C\epsilon s^\gamma} \\
& \cdot\left(1+\frac{(x-y-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1} dyds \\
\le\, & C\int_0^t \frac{1}{(t-s+1)\sqrt{s+1}}s^{-N_2}\left(1+\frac{(x-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1}ds \\
\le\, & C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N}. \qquad (6.29)
\end{aligned}$$

□

Proof of Lemma 4.8.

$$\begin{aligned}
\mathbb{J}_2(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) &= O(1)\int_0^t\frac{1}{1+t-s}e^{-C\epsilon s^\gamma}\left(1+\frac{(x-\lambda_j(t-s))^2}{1+t-s}\right)^{-N_1}dyds \\
&\le C\frac{1}{t+1}\left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N_1}. \qquad (6.30)
\end{aligned}$$

□

Proof of Lemma 4.9.

$$\begin{aligned}
\mathbb{J}_3(x,t;\lambda_i,\lambda_j) =\, & O(1)\int_0^t\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1+s}}(1+s)^{-N_2} \\
& \cdot\left(1+\frac{(x-\lambda_i(t-s))^2}{1+t-s}\right)^{-N_1}(1+x)^{-N_1}ds \\
\le\, & C\int_0^t\frac{1}{\sqrt{1+t-s}}\frac{1}{\sqrt{1-s}}(1+x)^{-N_1/2}(1+t-s)^{-N_1/2}(1+s)^{-N_2} \\
\le\, & C\frac{1}{\sqrt{1+t}}(1+x)^{-N_1/2}(1+t)^{-N_2} \le C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_i t)^2}{1+t}\right)^{-N_2}. \qquad (6.31)
\end{aligned}$$

□

Proof of Lemma 4.10.

$$\begin{aligned}
\mathbb{J}_4(x,t;\lambda_j,\lambda_k,\lambda_{k+1}) &= O(1)\int_0^t\frac{1}{\sqrt{1+t-s}}e^{-C\epsilon s^\gamma}\left(1+\frac{(x-\lambda_j(t-s))^2}{1+t-s}\right)^{-N_1}dyds \\
&\le C\frac{1}{\sqrt{t+1}}\left(1+\frac{(x-\lambda_j t)^2}{1+t}\right)^{-N_1}. \qquad (6.32)
\end{aligned}$$

□

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