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LARGE EIGENVALUES AND TRACES OF STURM-LIOUVILLE EQUATIONS ON STAR-SHAPED GRAPHS*

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Abstract. In this paper, we consider the spectral problem of small vibrations of a graph consisting of d, $d \geq 2, d \in \mathbf{N}$, joint inhomogeneous smooth strings which can be reduced to the Sturm-Liouville boundary value problem on a graph. This problem occurs also in quantum mechanics. For the Sturm-Liouville problem on the compact metric graph consisting of d segments of equal length with the Dirichlet or Neumann boundary conditions at the pendant vertices and Kirchhoff condition at the central vertex, we first derive the asymptotic expressions of its large eigenvalues and obtain precise descriptions for the formulae of the square root of the large eigenvalues up to the $O(1/n)$-term. In addition, regularized trace formulae of operators are established with residue techniques in complex analysis.

Key words. Compact metric graph, Sturm-Liouville problem, Kirchhoff condition, eigenvalue asymptotics, trace formula

AMS subject classifications. 34B24, 34L20, 47E05

1. Setting of the problem. A Quantum graph is given by a differential (self-adjoint) operator on a metric graph, i.e., the domain of the operator is a function space, each element in the space satisfying certain boundary conditions at the vertices. Differential operator on a metric graph(quantum graph) is a rather new and rapidly-developing area of modern mathematical physics. Such operators can be used to describe the motion of quantum particles confined to certain low dimensional structures. Spectral and scattering properties of Schrödinger operator in such structures have attracted considerable attention during past years. Many relevant models of nano-structure have been put out, see [27, 28, 30, 31, 32, 33]. An extensive survey of physical systems, giving rise to boundary value problems on graphs, can be found in the bibliography. Second order boundary value problems on finite graphs arise naturally from quantum mechanics and circuit theory [4, 20].

Recently, the spectral problems of quantum graphs have become a rapidly-developing field of mathematics and mathematical physics, and spectral properties of quantum graphs and different inverse problems have been studied in both forward [9, 14, 30, 31, 32, 33, 34, 44, 46, 50], and inverse [3, 7, 11, 35, 45, 48, 49, 53, 55, 56], etc. Some results on trace formulae and inverse scattering problems for Laplacians on metric graphs have appeared [6, 20, 26, 29, 36, 42, 51, 54], etc.

In this paper, we consider the following boundary value problems for the Sturm-Liouville equations on star-shaped metric graphs (i.e., a tree domain with exactly one central vertex) consisting of d segments of equal length:

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 1, 2, \cdots, d;\ d \geq 2, d \in \mathbf{N}, \tag{1.1}$$

which are subject to the boundary conditions

$$y_j(0) = 0,\ j = 1, 2, \cdots, d \tag{1.2}$$

or

$$y_j'(0) = 0,\ j = 1, 2, \cdots, d, \tag{1.3}$$

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at the pendant vertices 0, and

$$y_1(\lambda, \pi) = y_2(\lambda, \pi) = \cdots = y_d(\lambda, \pi), \tag{1.4}$$

$$y_1'(\lambda, \pi) + y_2'(\lambda, \pi) + \cdots + y_d'(\lambda, \pi) = 0, \tag{1.5}$$

at the central vertex π. In the equation (1.1), $q_j \in C^1[0, \pi]$, $j = 1, 2, \cdots, d$, are real-valued functions. (1.4) is called a continuity condition, and (1.5) is called a Kirchhoff condition. This problem occurs in the small vibrations of a graph with d inhomogeneous smooth strings, each having one end joint, and a quantum particle moving in a quasi-one-dimensional graph domain.

In the space

$$L_d^2[0, \pi] =: \bigoplus_{i=1}^{d} L^2[0, \pi],$$

define the inner product and norm by

$$(f, g) = \sum_{j=1}^{d} \int_0^{\pi} f_j(x)\overline{g_j}(x)dx, \quad ||f||^2 = \sum_{j=1}^{d} \int_0^{\pi} |f_j(x)|^2 dx, \ \forall f, g \in L_d^2[0, \pi],$$

where $f = (f_1, \cdots, f_d)^T, g = (g_1, \cdots, g_d)^T$.

For convenience, we denote by A_1, A_2 the operator acting in Hilbert space $L_d^2[0, \pi]$ for the problem (1.1), (1.2), (1.4) and (1.5) or (1.1), (1.3), (1.4) and (1.5), respectively.

It is easy to verify that operators A_1 and A_2 are both self-adjoint, and each operator's spectrum, which consists of only normal eigenvalues, is real and lower bounded, and can be determined by the variational principle. Counting multiplicities of the eigenvalues, we can arrange those eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ in an ascending order as

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \to +\infty.$$

In this paper, the first problem is to describe the asymptotic behavior of the eigenvalue sequence $\{\lambda_n\}_{n=1}^{\infty}$ as $n \to \infty$. The second problem is to present regularized trace formulae for operators A_1 and A_2.

In a finite space, an operator has a finite trace. But in a infinite space, ordinary differential operators do not have a finite trace(the sum of all eigenvalues). But Gelfand and Levitan [19] observed that the sum $\sum_n(\lambda_n - \mu_n)$ often makes sense, where $\{\lambda_n\}$ and $\{\mu_n\}$ are the eigenvalues of two differential operators. The sum $\sum_n(\lambda_n - \mu_n)$ is called a regularized trace. For the operator A_1 or A_2, the regularized trace $\sum_n(\lambda_n - \mu_n)$ may be finite where $\{\lambda_n\}$ are the eigenvalues of operator A_1 or A_2, and $\{\mu_n\}$ are the eigenvalues of the "unperturbed" differential operators. For regularized trace of Sturm-Liouville problem with their applications, see the bibliography therein. The most important application is in solving inverse problems [18, 22, 41, 52], i.e., given some spectral-related data, how to reconstruct the unknown potential function.

2. Main results. Applying asymptotic estimates for solutions to the initial value problem for (1.1) which may be established with well-known techniques in the scalar case [17, 19, 25, 40, 43], the eigenvalues for operator A_1 or A_2 may be identified with the zeros of an entire function. The asymptotic expressions of eigenvalues and trace formulae for the operators A_1 and A_2 are established with residue techniques in complex analysis.

In the case $q_j = 0, j = 1, 2, \cdots, d$, in (1.1), we can calculate the eigenvalues of operators A_1 and A_2(for the detail, see the proofs of Theorems 2.1 and 2.2 in section 3). Denote by $\mu^D_{n,j}, j = 1, 2, \cdots, d, n = 1, 2, \cdots$, the spectrum of self-adjoint operator A_1, then

$$\mu^D_{n,d} = (n - \frac{1}{2})^2 \tag{2.1}$$

and

$$\mu^D_{n,j} = n^2,\ j = 1, 2 \cdots, d-1,\ n = 1, 2, \cdots . \tag{2.2}$$

Each of the eigenvalues $(n - \frac{1}{2})^2$ is simple, and n^2 is of multiplicity $d - 1$.

Denote by $\mu^N_{n,j}, j = 1, 2, \cdots, d, n = 0, 1, 2, \cdots$, the spectrum of self-adjoint operator A_2, then

$$\mu^N_{n,d} = n^2,\ n = 0, 1, 2, \cdots \tag{2.3}$$

and

$$\mu^N_{n,j} = (n - \frac{1}{2})^2, j = 1, 2 \cdots, d-1, n = 1, 2, \cdots . \tag{2.4}$$

Each of the eigenvalues $n^2,\ n = 1, 2, \cdots$, is simple, and each of eigenvalues $(n - \frac{1}{2})^2,\ n = 1, 2, \cdots$, is of multiplicity $d - 1$.

The main results of this paper read as follows.

Theorem 2.1. *Suppose that $q_j(x) \in C^1[0, \pi]$, $j = 1,\ 2,\ \cdots,\ d$, and let $\{\lambda^D_{n,j}, j = 1, 2, \cdots, d\}^\infty_{n=1}$ be the sequence of the eigenvalues of operator A_1. Then, for sufficiently large n, the eigenvalue has the following asymptotic expression*

$$\lambda^D_{n,d} = (n - \frac{1}{2})^2 + \frac{2}{d}\sum_{j=1}^{d} \bar{q}_j + O(\frac{1}{n^2}), \tag{2.5}$$

and

$$\lambda^D_{n,j} = n^2 + 2c_{j,0} + O(\frac{1}{n^2}),\ j = 1, 2, \cdots, d-1, \tag{2.6}$$

where $\bar{q}_j = \frac{1}{2\pi}\int_0^\pi q_j(x)dx$, and $c_{j,0}$, $1 \le j \le d-1$, are the solutions of the equation for x

$$\begin{aligned} &dx^{d-1} - (d-1)\sum_{j=1}^{d} \bar{q}_j x^{d-2} + (d-2)\sum_{1\le i<j\le d} \bar{q}_i\bar{q}_j x^{d-3} \\ &+ \cdots + (-1)^{d-1}\sum_{1\le i_1<\cdots<i_{d-1}\le d} \bar{q}_{i_1}\cdots\bar{q}_{i_{d-1}} = 0. \end{aligned} \tag{2.7}$$

Remark 1. Define $f(x) = \prod_{j=1}^{d}(x - \bar{q}_j)$, where the real numbers $\bar{q}_j = \frac{1}{2\pi}\int_0^\pi q_j(x)dx$, then the zeros of $f'(x)$ are identified with all solutions to equation (2.7). By the Rolle theorem, it follows that all solutions to equation (2.7) are real.

Moreover, if the values of $\bar{q}_j, j = 1, 2, \cdots, d$, are pairwise different, then the solutions to equation (2.7) are also pairwise different. If there are k identical values among $\{\bar{q}_j\}_{j=1}^d$, then there are $k-1$ identical solutions to equation (2.7).

THEOREM 2.2. *Suppose that $q_j(x) \in C^1[0,\pi]$, $j = 1, 2, \cdots, d$, and let $\{\lambda_{n,j}^N, j = 1, 2, \cdots, d\}_{n=0}^{\infty}$ be the sequence of eigenvalues of operator A_2. Then, for sufficiently large n, the eigenvalue has the following asymptotic expression*

$$\lambda_{n,d}^N = n^2 + \frac{2}{d}\sum_{j=1}^{d} \bar{q}_j + O(\frac{1}{n^2}), \tag{2.8}$$

and

$$\lambda_{n,j}^N = (n - \frac{1}{2})^2 + 2c_{j,0} + O(\frac{1}{n^2}),\ j = 1, 2, \cdots, d-1, \tag{2.9}$$

where $\bar{q}_j = \frac{1}{2\pi}\int_0^{\pi} q_j(x)dx$, and $c_{j,0}$, $1 \le j \le d-1$, are the solutions of the equation (2.7).

THEOREM 2.3. *Suppose that $q_j(x) \in C^1[0,\pi]$, $j = 1, 2, \cdots, d$, and let $\{\lambda_{n,j}^D, j = 1, 2, \cdots, d\}_{n=1}^{\infty}$ be the sequence of the eigenvalues of operator A_1. Then*

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[\sum_{j=1}^{d}(\lambda_{n,j}^D - \mu_{n,j}^D) - 2\sum_{j=1}^{d}\bar{q}_j\right]\\ &= -\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi) + q_j(0)] + \frac{1}{2d}\sum_{j=1}^{d} q_j(\pi) + \frac{d-1}{d}\sum_{j=1}^{d}\bar{q}_j,\end{aligned} \tag{2.10}$$

where $\bar{q}_j = \frac{1}{2\pi}\int_0^{\pi} q_j(x)dx$.

THEOREM 2.4. *Suppose that $q_j(x) \in C^1[0,\pi]$, $j = 1, 2, \cdots, d$, and let $\{\lambda_{n,j}^N, j = 1, 2, \cdots, d\}_{n=0}^{\infty}$ be the sequence of the eigenvalues of operator A_2. Then*

$$\begin{aligned}&\lambda_{0,d}^N + \sum_{n=1}^{\infty}\left[\sum_{j=1}^{d}(\lambda_{n,j}^N - \mu_{n,j}^N) - 2\sum_{j=1}^{d}\bar{q}_j\right]\\ &= -\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi) - q_j(0)] + \frac{1}{2d}\sum_{j=1}^{d} q_j(\pi) + \frac{1}{d}\sum_{j=1}^{d}\bar{q}_j,\end{aligned} \tag{2.11}$$

where $\bar{q}_j = \frac{1}{2\pi}\int_0^{\pi} q_j(x)dx$.

3. The eigenvalue asymptotics. In this section, with the Gelfand-Levitan equation in [13, 38, 43], we first derive the equation for eigenvalues of operator A_1 or A_2, respectively. Then, with the help of the Rouché theorem we give the asymptotic expressions of large eigenvalues of operators A_1 and A_2. The method used here is similar to the well-known techniques in the scalar case.

First we study the equation for eigenvalues of operator A_1. Denote by $s_j(\lambda, x)$, $j = 1, 2, \cdots, d$, the solutions of (1.1) satisfying the conditions

$$s_j(\lambda, 0) = 0,\ s_j'(\lambda, 0) = 1, \tag{3.1}$$

then the solutions of equations (1.1) satisfying the conditions (1.2) are

$$y_j(\lambda, x) = c_j s_j(\lambda, x), \tag{3.2}$$

where c_j are constants. Substituting (3.2) into (1.4) and (1.5), we obtain the following equation for eigenvalues of operator A_1: λ is an eigenvalue if and only if

$$\begin{aligned}
&\varphi_1(\lambda) \\
&=: \begin{vmatrix}
s_1(\lambda,\pi) & -s_2(\lambda,\pi) & 0 & \cdots & 0 & 0 \\
0 & s_2(\lambda,\pi) & -s_3(\lambda,\pi) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & s_{d-1}(\lambda,\pi) & -s_d(\lambda,\pi) \\
s_1'(\lambda,\pi) & s_2'(\lambda,\pi) & s_3'(\lambda,\pi) & \cdots & s_{d-1}'(\lambda,\pi) & s_d'(\lambda,\pi)
\end{vmatrix} \\
&= \sum_{j=1}^{d} s_j'(\lambda,\pi) \prod_{j\neq l\in\{1,2,\cdots,d\}} s_l(\lambda,\pi) = 0.
\end{aligned} \tag{3.3}$$

Making use of the formulae in [13, 38, 43], we have

$$\begin{aligned}
&s_j(\lambda, x) \\
&= \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} - \frac{\cos(\sqrt{\lambda}x)}{\lambda} K_j(x,x) + \frac{1}{\lambda}\int_0^x K_{j,t}'(x,t)\cos(\sqrt{\lambda}t)dt; \\
&s_j'(\lambda, x) \\
&= \cos(\sqrt{\lambda}x) + \frac{K_j(x,x)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}x) + \frac{1}{\sqrt{\lambda}}\int_0^x K_{j,x}'(x,t)\sin(\sqrt{\lambda}t)dt,
\end{aligned} \tag{3.4}$$

where both of the first partial derivatives $K_{j,x}'(x,t)$ and $K_{j,t}'(x,t)$ of $K_j(x,t)$, $j = 1,2,\cdots,d$, exist and $K_{j,x}'(x,\cdot) \in L^2[0,\pi]$ and $K_{j,t}'(x,\cdot) \in L^2[0,\pi]$.

If for brevity, we put

$$a_j = \int_0^\pi K_{j,x}'(\pi,t)\sin(\sqrt{\lambda}t)dt, \quad b_j = \int_0^\pi K_{j,t}'(\pi,t)\cos(\sqrt{\lambda}t)dt,$$

then by the Riemann-Lebesgue lemma,

$$a_j \to 0,\ b_j \to 0 \text{ (as } \lambda \to \infty). \tag{3.5}$$

By (3.3) and (3.4), we have

$$\begin{aligned}
\varphi_1(\lambda) = \sum_{j=1}^{d} &\left[\cos(\sqrt{\lambda}\pi) + \frac{K_j}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\pi) + \frac{a_j}{\sqrt{\lambda}}\right] \\
&\times \prod_{j\neq l\in\{1,2,\cdots,d\}} \left[\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} + \frac{b_l - \cos(\sqrt{\lambda}\pi)K_l}{\lambda}\right],
\end{aligned} \tag{3.6}$$

where $K_j = K_j(\pi,\pi) = \frac{1}{2}\int_0^\pi q_j(x)dx$.

Now we try to get the equation for eigenvalues of operator A_2. Denote by $\widetilde{s}_j(\lambda,x),\ j = 1,2,\cdots,d$, the solutions of (1.1) satisfying the conditions

$$\widetilde{s}_j(\lambda, 0) = 1,\ \widetilde{s}_j'(\lambda, 0) = 0. \tag{3.7}$$

Then the solutions of equations (1.1) satisfying the conditions (1.3) are

$$y_j(\lambda, x) = \widetilde{c}_j \widetilde{s}_j(\lambda, x), \tag{3.8}$$

where $\widetilde{c}_j$ are constants. Substituting (3.8) into (1.4) and (1.5), we obtain the following equation for eigenvalues of operator A_2: λ is an eigenvalue if and only if

$$\varphi_2(\lambda) =: \sum_{j=1}^{d} \widetilde{s}_j'(\lambda, \pi) \prod_{j \neq l \in \{1,2,\cdots,d\}} \widetilde{s}_l(\lambda, \pi) = 0. \tag{3.9}$$

Using the formulae in [13, 38, 43], we have

$$\begin{aligned}
&\widetilde{s}_j(\lambda, x) \\
&= \cos(\sqrt{\lambda}x) + \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \widetilde{K}_j(x,x) - \frac{1}{\sqrt{\lambda}} \int_0^x \widetilde{K}'_{j,t}(x,t) \sin(\sqrt{\lambda}t)dt; \\
&\widetilde{s}_j'(\lambda, x) \\
&= -\sqrt{\lambda}\sin(\sqrt{\lambda}x) + \widetilde{K}_j(x,x)\cos(\sqrt{\lambda}x) + \int_0^x \widetilde{K}'_{j,x}(x,t)\cos(\sqrt{\lambda}t)dt,
\end{aligned} \tag{3.10}$$

where both of the first partial derivatives $\widetilde{K}'_{j,x}(x,t)$ and $\widetilde{K}'_{j,t}(x,t)$ of $\widetilde{K}_j(x,t)$, $j = 1, 2, \cdots, d$, exist and $\widetilde{K}'_{j,x}(x,\cdot) \in L^2[0,\pi]$ and $\widetilde{K}'_{j,t}(x,\cdot) \in L^2[0,\pi]$.

If for brevity, we put

$$c_j = -\int_0^\pi \widetilde{K}'_{j,t}(\pi,t)\sin(\sqrt{\lambda}t)dt, \quad d_j = \int_0^\pi \widetilde{K}'_{j,x}(\pi,t)\cos(\sqrt{\lambda}t)dt,$$

then by the Riemann-Lebesgue lemma,

$$c_j \to 0, \; d_j \to 0 \text{ (as } \lambda \to \infty). \tag{3.11}$$

From (3.9) and (3.10), we obtain that

$$\begin{aligned}
\varphi_2(\lambda) = &\sum_{j=1}^{d} \left[-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + K_j \cos(\sqrt{\lambda}\pi) + d_j \right] \\
&\times \prod_{j \neq l \in \{1,2,\cdots,d\}} \left[\cos(\sqrt{\lambda}\pi) + \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} K_l + \frac{c_l}{\sqrt{\lambda}} \right],
\end{aligned} \tag{3.12}$$

where $K_j = \widetilde{K}_j(\pi,\pi) = \frac{1}{2}\int_0^\pi q_j(x)dx$.

Furthermore, the kernels of the transformations $K_j(x,t), \widetilde{K}_j(x,t)$, $j = 1, 2, \cdots, d$, satisfy the following partial differential equations [8, 13, 43]

$$\begin{aligned}
&K''_{j,xx} - q_j(x)K_j = K''_{j,tt}, \; K_j(x,x) = \frac{1}{2}\int_0^x q_j(x)dx, \; K_j(x,0) = 0; \\
&\widetilde{K}''_{j,xx} - q_j(x)\widetilde{K}_j = \widetilde{K}''_{j,tt}, \; \widetilde{K}_j(x,x) = \frac{1}{2}\int_0^x q_j(x)dx, \; \widetilde{K}'_{j,t}(x,0) = 0.
\end{aligned} \tag{3.13}$$

When $q_j(x) \in C^1[0,\pi]$, (3.13) can be written as Volterra integral equations

$$\begin{aligned}K_j(x,t)&=\frac{1}{2}\left[\int_0^{\frac{x+t}{2}}q_j(x)dx-\int_0^{\frac{x-t}{2}}q_j(x)dx\right]\\&\quad+\int_0^{\frac{x-t}{2}}d\tau\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q_j(\sigma+\tau)K_j(\sigma+\tau,\sigma-\tau)d\sigma,\\ \widetilde{K}_j(x,t)&=\frac{1}{2}\left[\int_0^{\frac{x+t}{2}}q_j(x)dx+\int_0^{\frac{x-t}{2}}q_j(x)dx\right]\\&\quad+\int_0^{\frac{x-t}{2}}d\tau\int_{\tau}^{\frac{x+t}{2}}q_j(\sigma+\tau)\widetilde{K}_j(\sigma+\tau,\sigma-\tau)d\sigma,\end{aligned}\tag{3.14}$$

which are solvable. By (3.14) a direct calculation yields that

$$\begin{aligned}\frac{\partial K_j(x,x)}{\partial t}&=\frac{q_j(x)+q_j(0)}{4}-\frac{[\int_0^x q_j(x)dx]^2}{8},\\ \frac{\partial K_j(x,x)}{\partial x}&=\frac{q_j(x)-q_j(0)}{4}+\frac{[\int_0^x q_j(x)dx]^2}{8};\\ \frac{\partial \widetilde{K}_j(x,x)}{\partial t}&=\frac{q_j(x)-q_j(0)}{4}-\frac{[\int_0^x q_j(x)dx]^2}{8},\\ \frac{\partial \widetilde{K}_j(x,x)}{\partial x}&=\frac{q_j(x)+q_j(0)}{4}+\frac{[\int_0^x q_j(x)dx]^2}{8}.\end{aligned}\tag{3.15}$$

When $q_j(x)\in C[0,\pi]$, by integration by parts we get

$$\begin{aligned}a_j&=-\frac{\cos(\sqrt{\lambda}\pi)K'_{j,x}(\pi,\pi)}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\int_0^\pi K''_{j,xt}(\pi,t)\cos(\sqrt{\lambda}t)dt,\\ b_j&=\frac{\sin(\sqrt{\lambda}\pi)K'_{j,t}(\pi,\pi)}{\sqrt{\lambda}}-\frac{1}{\sqrt{\lambda}}\int_0^\pi K''_{j,tt}(\pi,t)\sin(\sqrt{\lambda}t)dt\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}c_j&=\frac{\cos(\sqrt{\lambda}\pi)\widetilde{K}'_{j,t}(\pi,\pi)}{\sqrt{\lambda}}-\frac{1}{\sqrt{\lambda}}\int_0^\pi \widetilde{K}''_{j,tt}(\pi,t)\cos(\sqrt{\lambda}t)dt,\\ d_j&=\frac{\sin(\sqrt{\lambda}\pi)\widetilde{K}'_{j,x}(\pi,\pi)}{\sqrt{\lambda}}-\frac{1}{\sqrt{\lambda}}\int_0^\pi \widetilde{K}''_{j,xt}(\pi,t)\sin(\sqrt{\lambda}t)dt.\end{aligned}\tag{3.17}$$

Now we can prove the theorems in this paper.

Proof of Theorem 2.1. Write $\varphi_1(\lambda)$ as

$$\varphi_1(\lambda)=\varphi_1^{(0)}(\lambda)+\mathcal{E}_1(\lambda),\tag{3.18}$$

where

$$\varphi_1^{(0)}(\lambda)=d\cos(\sqrt{\lambda}\pi)\frac{\sin^{d-1}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d-1}}\tag{3.19}$$

and $\mathcal{E}_1(\lambda)$ is the remainder.

It is easy to obtain zeros $\mu_{n,j}^D$ of the function $\varphi_1^{(0)}(\lambda)$, counting multiplicities of zero,

$$\sqrt{\mu_{n,d}^D} = n - \frac{1}{2}, \quad \sqrt{\mu_{n,j}^D} = n, \; j = 1, 2, \cdots, d-1; n = 1, 2, \cdots, \tag{3.20}$$

where $\{(n - \frac{1}{2})^2\}_{n=1}^{\infty}$ are all simple zeros and $\{n^2\}_{n=1}^{\infty}$ are all zeros of order $d-1$.

Since the zeros of $\varphi_1(\lambda)$, the eigenvalues for self-adjoint operator A_1, are real, we may suppose $|\mathrm{Im}\lambda| < \kappa$ for some fixed constant $\kappa > 0$.

Now it follows from (3.6), (3.18) and (3.19) that there exists a constant $c > 0$ such that

$$|\mathcal{E}_1(\lambda)| = |\varphi_1(\lambda) - \varphi_1^{(0)}(\lambda)| < \frac{c}{\sqrt{\lambda}}$$

for all $|\mathrm{Im}\lambda| < \kappa$. Since

$$\varphi_1^{(0)}(\lambda) = d\cos(\sqrt{\lambda}\pi)\frac{\sin^{d-1}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d-1}}, \tag{3.21}$$

for every $r \in (0, \epsilon)$, we can find $\Lambda > 0$ such that $|\varphi_1^{(0)}(\lambda)| > \Lambda$ for all $\lambda \in \mathbf{C}\backslash \bigcup_n C_n$, where C_n are circles of radii r with the centers at the points $\mu_{n,j}^D$, $j = 1, 2, \cdots, d$. Thus, for all $\lambda \in \{\lambda | \lambda \in \mathbf{C}\backslash \bigcup_n C_n, \sqrt{\lambda} > \frac{c}{\Lambda}\}$, we have

$$|\varphi_1(\lambda) - \varphi_1^{(0)}(\lambda)| < \frac{c}{\sqrt{\lambda}} < \Lambda < |\varphi_1^{(0)}(\lambda)|. \tag{3.22}$$

Let $\lambda_{n,j}^D, j = 1, 2, \cdots, d, n = 1, 2, \cdots$, be the eigenvalues of operator A_1, i.e., zeros of $\varphi_1(\lambda)$. By the Rouché theorem and taking arbitrarily small r, we obtain the following results. For sufficiently large integer n, there lie exactly 1, $d-1$ zeros of $\varphi_1(\lambda)$ in a suitable neighborhood of $\mu_{n,d}^D$, $\mu_{n,j}^D (j \neq d)$, respectively, and denote

$$\sqrt{\lambda_{n,d}^D} = n - \frac{1}{2} + \alpha_n, \tag{3.23}$$

$$\sqrt{\lambda_{n,j}^D} = n + \beta_{n,j}, \; j = 1, 2, \cdots, d-1. \tag{3.24}$$

where $\alpha_n = o(1), \beta_{n,j} = o(1)$ as $n \to \infty$. It is not difficult to see that $\alpha_n = O(\frac{1}{n-(1/2)})$, $\beta_{n,j} = O(1/n)$. In fact, we can calculate $\lim_{n\to\infty}(n - (1/2))\alpha_n$ and $\lim_{n\to\infty} n\beta_{n,j}$.

Substituting $\lambda_{n,d}^D$ into $\varphi_1(\lambda) = 0$, then, from (3.6), (3.16) and (3.23), we have

$$\sum_{j=1}^{d}\left[(-1)^n \sin(\alpha_n\pi) - \frac{(-1)^n K_j \cos(\alpha_n\pi)}{n - (1/2)} + O(1/n^2)\right]$$
$$\times \prod_{j\neq l\in\{1,2,\cdots,d\}}\left[\frac{(-1)^{n-1}\cos(\alpha_n\pi)}{n} + O(1/n^2)\right] = 0$$

which implies

$$\sum_{j=1}^{d}\left[(-1)^n\sin(\alpha_n\pi)-\frac{(-1)^nK_j\cos(\alpha_n\pi)}{n-(1/2)}+O(1/n^2)\right]$$
$$\times\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[(-1)^{n-1}\cos(\alpha_n\pi)+O(1/n)\right]=0,$$

thus,

$$\sin(\alpha_n\pi)\cos^{d-1}(\alpha_n\pi)=O(1/(n-(1/2))),$$

that is,

$$\sin(\alpha_n\pi)=O(1/(n-(1/2))).$$

Using Lagrange inversion formula, then we get

$$\alpha_n=\frac{c_0}{n-(1/2)}+\frac{\gamma_n}{n}, \tag{3.25}$$

where c_0 is a constant depending on $q_j(x)$ and $\gamma_n\to 0$ as $n\to\infty$.

Similarly,

$$\varphi_1(n+\beta_{n,i})=\sum_{j=1}^{d}\left[(-1)^n\cos(\beta_{n,i}\pi)+\frac{(-1)^nK_j\sin(\beta_{n,i}\pi)}{n}+O(1/n^2)\right]$$
$$\times\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[\frac{(-1)^n\sin(\beta_{n,i}\pi)}{n}+O(1/n^2)\right]=0$$

which implies

$$0=\sum_{j=1}^{d}\left[\cos(\beta_{n,i}\pi)+\frac{K_j\sin(\beta_{n,i}\pi)}{n}+O(1/n^2)\right]\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[\sin(\beta_{n,i}\pi)+O(1/n)\right],$$

that is,

$$\sin(\beta_{n,i}\pi)=O(1/n).$$

Thus we get

$$\beta_{n,i}=\frac{c_{i,0}}{n}+\frac{\gamma_{i,n}}{n}, \tag{3.26}$$

where $c_{i,0}, 1\le i\le d-1$, are constants depending on $q_j(x)$ and $\gamma_{i,n}\to 0$ as $n\to\infty$.

Substituting (3.23) and (3.25) into the equation $\varphi_1(\lambda)=0$, we obtain

$$\sum_{j=1}^{d}\left[(-1)^n\sin(\frac{c_0}{n-(1/2)}+o(1/n))\pi-\frac{(-1)^nK_j\cos(\frac{c_0}{n-(1/2)}+o(1/n))\pi}{n}+O(1/n^2)\right]$$
$$\times\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[(-1)^{n-1}\cos\left(\frac{c_0}{n-(1/2)}+o(1/n)\right)\pi+O(1/n)\right]=0,$$

which implies

$$\sum_{j=1}^{d}\left[n\sin\left(\frac{c_0}{n-(1/2)}+o(1/n)\right)\pi-K_j\cos\left(\frac{c_0}{n-(1/2)}+o(1/n)\right)\pi+O(1/n)\right]$$
$$\times\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[\cos\left(\frac{c_0}{n-(1/2)}+o(1/n)\right)\pi+O(1/n)\right]=0,$$

expanding the left-hand side of the resulting equation in power series, we have

$$\sum_{j=1}^{d}[c_0\pi-K_j+o(1)]\prod_{j\neq l\in\{1,2,\cdots,d\}}[1+o(1)]=0,$$

and let $n\to\infty$, we obtain

$$c_0=\frac{1}{\pi d}\sum_{j=1}^{d}K_j. \tag{3.27}$$

Define

$$\bar{q_j}=\frac{1}{2\pi}\int_0^{\pi}q_j(x)dx,\ \ j=1,2,\cdots,d, \tag{3.28}$$

then

$$c_0=\frac{1}{d}\sum_{j=1}^{d}\bar{q_j}. \tag{3.29}$$

Substituting (3.24) and (3.26) into the equation $\varphi_1(\lambda)=0$, by (3.16),

$$\begin{aligned}0&=\sum_{j=1}^{d}\left[\cos\left(\frac{c_{i,0}}{n}+o(1/n)\right)\pi+\frac{K_j\sin\left(\frac{c_{i,0}}{n}+o(1/n)\right)\pi}{n}+O(1/n^2)\right]\\&\quad\times\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[\sin\left(\frac{c_{i,0}}{n}+o(1/n)\right)\pi-\frac{K_l\cos\left(\frac{c_{i,0}}{n}+o(1/n)\right)\pi}{n}+o(1/n^2)\right]\\&=\sum_{j=1}^{d}\left[1+O(1/n^2)\right]\times\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[\frac{c_{i,0}\pi}{n}-\frac{K_l}{n}+o(1/n)\right]\end{aligned}$$

which implies

$$\sum_{j=1}^{d}\left[1+O(1/n^2)\right]\times\prod_{j\neq l\in\{1,2,\cdots,d\}}[(c_{i,0}\pi-K_l)+o(1)]=0,$$

and let $n\to\infty$, we have

$$\sum_{j=1}^{d}\prod_{j\neq l\in\{1,2,\cdots,d\}}(c_{i,0}\pi-K_l)=0,$$

that is,

$$dc_{i,0}^{d-1}-(d-1)\sum_{j=1}^{d}\bar{q}_j c_{i,0}^{d-2}+(d-2)\sum_{1\le i<j\le d}\bar{q}_i\bar{q}_j c_{i,0}^{d-3} + \cdots + (-1)^{d-1}\sum_{1\le i_1<\cdots<i_{d-1}\le d}\bar{q}_{i_1}\cdots\bar{q}_{i_{d-1}} = 0. \tag{3.30}$$

From (3.23), (3.24), (3.25), (3.26), (3.29) and (3.30), then the theorem follows. □

Scheme of the Proof of Theorem 2.2. Its proof is similar to that of Theorem 2.1. Write $\varphi_2(\lambda)$ as

$$\varphi_2(\lambda) = \varphi_2^{(0)}(\lambda) + \mathcal{E}_2(\lambda), \tag{3.31}$$

where

$$\varphi_2^{(0)}(\lambda) = -d\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)\cos^{d-1}(\sqrt{\lambda}\pi) \tag{3.32}$$

and $\mathcal{E}_2(\lambda)$ is the remainder. It is easy to obtain zeros $\mu_{n,j}^N$ of function $\varphi_2^{(0)}(\lambda)$:

$$\begin{gathered}\sqrt{\mu_{n,d}^N} = n,\ n = 0,1,2,\cdots;\\ \sqrt{\mu_{n,j}^N} = n-\tfrac{1}{2},\quad j=1,2,\cdots,d-1;\ n=1,2,\cdots,\end{gathered} \tag{3.33}$$

where $\{n^2\}_{n=1}^{\infty}$ are all simple zeros and $\{(n-\frac{1}{2})^2\}_{n=1}^{\infty}$ are all zeros of order $d-1$.

By the Rouché theorem we have

$$\sqrt{\lambda_{n,d}^N} = n+\theta_n, \tag{3.34}$$

$$\sqrt{\lambda_{n,j}^N} = n-\frac{1}{2}+\nu_{n,j},\ j=1,2,\cdots,d-1, \tag{3.35}$$

where $\theta_n = o(1), \nu_{n,j} = o(1)$ as $n\to\infty$. It is not difficult to see that $\theta_n = O(1/n)$ and $\nu_{n,j} = O(\frac{1}{n-(1/2)})$. In fact, we can compute $\lim_{n\to\infty} n\theta_n$ and $\lim_{n\to\infty}(n-(1/2))\nu_{n,j}$.

From (3.31) and (3.34) we get

$$\theta_n = \frac{f_0}{n} + \frac{\widehat{\gamma}_n}{n}, \tag{3.36}$$

where f_0 is a constant depending on $q_j(x)$ and $\widehat{\gamma}_n \to 0$ as $n\to\infty$.

Similarly,

$$\nu_{n,j} = \frac{g_{j,0}}{n-(1/2)} + \frac{\widehat{\gamma}_{j,n}}{n}, \tag{3.37}$$

where $g_{j,0}, 1\le j\le d-1$, are constants depending on $q_j(x)$ and $\widehat{\gamma}_{j,n}\to 0$ as $n\to\infty$.

Moreover, substituting (3.34) and (3.36) into the equation $\varphi_2(\lambda) = 0$ and expanding of the resulting equation in power series, we have

$$f_0 = \frac{1}{d}\sum_{j=1}^{d}\bar{q}_j, \tag{3.38}$$

and $g_{j,0},\ 1\le j\le d-1$, are the solutions of the equation (2.7).

By (3.34), (3.35), (3.36), (3.37) and (3.38), the theorem follows. □

4. Trace formulae. This section presents regularized trace for operator A_1 or A_2, respectively. We try to derive the precise values of those regularized traces with contour integration. Let the contour Γ_{N_0}, integer $N_0 = 0, 1, 2, \cdots \to \infty$, denote the following sequences of circular contours, traversed counterclockwise:

The contour Γ_{N_0} is the circle of radius $(N_0 + \frac{1}{4})^2$ with its center at the origin.

Obviously, $\mu_{n,j}^D, \mu_{n,j}^N$ defined in (3.20) and (3.33), which are the zeros of function $\varphi_k^{(0)}(\lambda), k = 1, 2$, don't lie on the contour Γ_{N_0}. To obtain trace formulae we need the following lemma in complex analysis.

Lemma 4.1 ([1, 8]). *Suppose $\omega(\lambda), \omega_0(\lambda)$ are two entire functions, $\omega_0(\lambda)$ has no zeros on a closed contour Γ_{N_0} of λ-complex plane. If these functions satisfy the estimate*

$$\frac{\omega(\lambda)}{\omega_0(\lambda)} = 1 + \frac{\alpha_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\alpha_2(\sqrt{\lambda})}{\lambda} + O(1/\sqrt{\lambda^3}) \quad \text{on } \Gamma_{N_0},$$

where the functions $\frac{\alpha_k(\sqrt{\lambda})}{\sqrt{\lambda^k}}$, $k = 1, 2$, are single valued and analytic on Γ_{N_0} and $\alpha_k(\sqrt{\lambda})$ are uniformly bounded on Γ_{N_0}. Then, on Γ_{N_0},

$$\begin{aligned}
&\sum_{\Gamma_{N_0}} (\lambda_n - \mu_n) \\
&= -\frac{1}{2\pi i} \oint_{\Gamma_{N_0}} \log \frac{\omega(\lambda)}{\omega_0(\lambda)} d\lambda \\
&= -\frac{1}{2\pi i} \oint_{\Gamma_{N_0}} \left[\frac{\alpha_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\alpha_2(\sqrt{\lambda}) - \alpha_1^2(\sqrt{\lambda})/2}{\lambda} \right] d\lambda + O(1/N_0),
\end{aligned} \tag{4.1}$$

where λ_n, μ_n are the zeros of entire functions $\omega(\lambda), \omega_0(\lambda)$ inside the contour Γ_{N_0} listed with multiplicity, respectively.

Proof of Theorem 2.3. The computation of trace for operator A_1 is based on Lemma 4.1.

Step 1, we give the estimate for $\frac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)}$ on the contour Γ_{N_0}.

By (3.6), (3.16) and (3.21), and integration by parts, on the contour Γ_{N_0},

$$\begin{aligned}
&\frac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)} \\
&= \frac{1}{d} \sum_{j=1}^{d} \left[1 + \frac{K_j \tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} + \frac{a_j}{\sqrt{\lambda} \cos(\sqrt{\lambda}\pi)} \right] \\
&\quad \times \prod_{j \neq l \in \{1,2,\cdots,d\}} \left[1 + \frac{b_l}{\sqrt{\lambda} \sin(\sqrt{\lambda}\pi)} - \frac{K_l \cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} \right] \\
&= \frac{1}{d} \sum_{j=1}^{d} \left[1 + \frac{K_j \tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} - \frac{K'_{j,x}(\pi,\pi)}{\lambda} + O(1/\sqrt{\lambda^3}) \right] \\
&\quad \times \prod_{j \neq l \in \{1,2,\cdots,d\}} \left[1 - \frac{K_l \cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} + \frac{K'_{l,t}(\pi,\pi)}{\lambda} + O(1/\sqrt{\lambda^3}) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d}\sum_{j=1}^{d}\left[1+\frac{K_j\tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}-\frac{K'_{j,x}(\pi,\pi)}{\lambda}+O(1/\sqrt{\lambda^3})\right]\times\left[1-\frac{\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\right.\\
&\quad\times\sum_{j\neq l\in\{1,2,\cdots,d\}}K_l+\frac{1}{\lambda}\sum_{j\neq l\in\{1,2,\cdots,d\}}K'_{l,t}(\pi,\pi)+\frac{\cot^2(\sqrt{\lambda}\pi)}{\lambda}\\
&\quad\left.\times\sum_{i_1<i_2\in\{1,2,\cdots,d\}\backslash\{j\}}K_{i_1}K_{i_2}+O(1/\sqrt{\lambda^3})\right]\\
&= \frac{1}{d}\sum_{j=1}^{d}\left[1-\frac{\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\sum_{j\neq l\in\{1,2,\cdots,d\}}K_l+\frac{K_j\tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}+\frac{1}{\lambda}\right.\\
&\quad\times\sum_{j\neq l\in\{1,2,\cdots,d\}}K'_{l,t}(\pi,\pi)+\frac{\cot^2(\sqrt{\lambda}\pi)}{\lambda}\sum_{i_1<i_2\in\{1,2,\cdots,d\}\backslash\{j\}}K_{i_1}K_{i_2}-\frac{1}{\lambda}\\
&\quad\left.\times K_j\sum_{j\neq l\in\{1,2,\cdots,d\}}K_l-\frac{K'_{j,x}(\pi,\pi)}{\lambda}+O(1/\sqrt{\lambda^3})\right]\\
&= \frac{1}{d}\left[d-\frac{(d-1)\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\sum_{j=1}^{d}K_j+\frac{\sum_{j=1}^{d}K_j\tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}+\frac{d-1}{\lambda}\sum_{j=1}^{d}K'_{j,t}(\pi,\pi)\right.\\
&\quad+\frac{(d-2)\cot^2(\sqrt{\lambda}\pi)}{\lambda}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}-\frac{2}{\lambda}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\\
&\quad\left.-\frac{\sum_{j=1}^{d}K'_{j,x}(\pi,\pi)}{\lambda}+O(1/\sqrt{\lambda^3})\right]\\
&= 1+\frac{1}{\sqrt{\lambda}}\left[-\frac{(d-1)}{d}\sum_{j=1}^{d}K_j\cot(\sqrt{\lambda}\pi)+\frac{\sum_{j=1}^{d}K_j\tan(\sqrt{\lambda}\pi)}{d}\right]+\frac{1}{\lambda}\left[\frac{d-1}{d}\right.\\
&\quad\times\sum_{j=1}^{d}K'_{j,t}(\pi,\pi)+\frac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\cot^2(\sqrt{\lambda}\pi)\\
&\quad\left.-\frac{2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}-\frac{\sum_{j=1}^{d}K'_{j,x}(\pi,\pi)}{d}\right]+O(1/\sqrt{\lambda^3}). \qquad (4.2)
\end{aligned}
$$

Next, the power series expansion tells us

$$
\begin{aligned}
&\log\frac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)}\\
&= \frac{1}{\sqrt{\lambda}}\left[-\frac{d-1}{d}\sum_{j=1}^{d}K_j\cot(\sqrt{\lambda}\pi)+\frac{\sum_{j=1}^{d}K_j\tan(\sqrt{\lambda}\pi)}{d}\right]+\frac{1}{\lambda}\left[\frac{d-1}{d}\right.\\
&\quad\times\sum_{j=1}^{d}K'_{j,t}(\pi,\pi)+\frac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\cot^2(\sqrt{\lambda}\pi)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}-\frac{\sum_{j=1}^{d}K'_{j,x}(\pi,\pi)}{d}\\
&-\frac{(d-1)^2}{2d^2}\left(\sum_{j=1}^{d}K_j\right)^2\cot^2(\sqrt{\lambda}\pi)-\frac{1}{2d^2}\left(\sum_{j=1}^{d}K_j\right)^2\tan^2(\sqrt{\lambda}\pi)\\
&\left.+\frac{d-1}{d^2}\left(\sum_{j=1}^{d}K_j\right)^2\right]+O(1/\sqrt{\lambda^3}).
\end{aligned}
\tag{4.3}
$$

From the above arguments and (3.22) it follows that the zeros $\lambda_{n,j}^D$ of $\varphi_1(\lambda)$ are the eigenvalues of operator A_1, and the zeros $\mu_{n,j}^D$ of $\varphi_1^{(0)}(\lambda)$ are the eigenvalues of problem (1.1), (1.2), (1.4) and (1.5) with $q_j=0, j=1,2,\cdots,d$. By Rouché's theorem, for sufficiently large N_0, the number of zeros of $\varphi_1(\lambda)$ and $\varphi_1^{(0)}(\lambda)$ inside the contour Γ_{N_0} is just the same.

Finally, by (4.3) and Lemma 4.1, it follows that for sufficiently large N_0,

$$
\sum_{n=1}^{N_0}\left[\lambda_{n,d}^D-(n-\frac{1}{2})^2\right]+\sum_{n=1}^{N_0}\sum_{j=1}^{d-1}(\lambda_{n,j}^D-n^2)=-\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\log\frac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)}d\lambda. \tag{4.4}
$$

Using well-known formulae

$$
\begin{aligned}
&\cot z=\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2},\quad \tan z=\sum_{n=0}^{\infty}\frac{8z}{(2n+1)^2\pi^2-4z^2},\\
&\csc^2 z=\sum_{n=-\infty}^{\infty}\frac{1}{(z+n\pi)^2},\quad \sec^2 z=\sum_{n=-\infty}^{\infty}\frac{1}{[z+\{(2n+1)\pi/2\}]^2},
\end{aligned}
\tag{4.5}
$$

we get

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}d\lambda=\frac{2N_0+1}{\pi},\quad \frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\tan\sqrt{\lambda}\pi}{\sqrt{\lambda}}d\lambda=-\frac{2N_0}{\pi},\\
&\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\cot^2\sqrt{\lambda}\pi}{\lambda}d\lambda=-1+O(1/N_0),\\
&\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\tan^2\sqrt{\lambda}\pi}{\lambda}d\lambda=-1+O(1/N_0).
\end{aligned}
\tag{4.6}
$$

Substituting (4.3) into (4.4), together with (3.15) and (4.6), we have

$$
\begin{aligned}
&\sum_{n=1}^{N_0}\sum_{j=1}^{d}(\lambda_{n,j}^D-\mu_{n,j}^D)\\
&=-\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi)+q_j(0)]+\frac{1}{2d}\sum_{j=1}^{d}q_j(\pi)+\frac{2N_0d+d-1}{\pi d}\sum_{j=1}^{d}K_j+O(1/N_0),
\end{aligned}
$$

that is,

$$\sum_{n=1}^{N_0}\left[\sum_{j=1}^{d}(\lambda_{n,j}^D-\mu_{n,j}^D)-\frac{2}{\pi}\sum_{j=1}^{d}K_j\right]$$
$$=-\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi)+q_j(0)]+\frac{1}{2d}\sum_{j=1}^{d}q_j(\pi)+\frac{d-1}{\pi d}\sum_{j=1}^{d}K_j+O(1/N_0). \tag{4.7}$$

Let $N_0\to\infty$ in (4.7), we have

$$\sum_{n=1}^{\infty}\left[\sum_{j=1}^{d}(\lambda_{n,j}^D-\mu_{n,j}^D)-2\sum_{j=1}^{d}\bar{q}_j\right]$$
$$=-\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi)+q_j(0)]+\frac{1}{2d}\sum_{j=1}^{d}q_j(\pi)+\frac{d-1}{d}\sum_{j=1}^{d}\bar{q}_j,$$

where $\bar{q}_j=\frac{1}{2\pi}\int_0^\pi q_j(x)dx$. The proof of theorem is completed. □

Scheme of the Proof of Theorem 2.4. Its proof is similar to that of Theorem 2.3.

Step 1, we give the estimate for $\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)}$ on the contour Γ_{N_0}.

By (3.12), (3.17) and (3.32), and integration by parts, on contour Γ_{N_0},

$$\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)}$$
$$=\frac{1}{d}\sum_{j=1}^{d}\left[1-\frac{K_j\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}-\frac{d_j}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}\right]$$
$$\times\prod_{j\neq l\in\{1,2,\cdots,d\}}\left[1+\frac{c_l}{\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)}+\frac{K_l\tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\right]$$
$$=1+\frac{1}{\sqrt{\lambda}}\left[\frac{d-1}{d}\sum_{j=1}^{d}K_j\tan(\sqrt{\lambda}\pi)-\frac{\sum_{j=1}^{d}K_j\cot(\sqrt{\lambda}\pi)}{d}\right]+\frac{1}{\lambda}\left[\frac{d-1}{d}\right.$$
$$\times\sum_{j=1}^{d}\widetilde{K}'_{j,t}(\pi,\pi)+\frac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\tan^2(\sqrt{\lambda}\pi)$$
$$\left.-\frac{2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}-\frac{\sum_{j=1}^{d}\widetilde{K}'_{j,x}(\pi,\pi)}{d}\right]+O(1/\sqrt{\lambda^3}).$$

Next, the power series expansion tells us

$$\log\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)}$$
$$=\frac{1}{\sqrt{\lambda}}\left[\frac{d-1}{d}\sum_{j=1}^{d}K_j\tan(\sqrt{\lambda}\pi)-\frac{\sum_{j=1}^{d}K_j\cot(\sqrt{\lambda}\pi)}{d}\right]+\frac{1}{\lambda}\left[\frac{d-1}{d}\right.$$
$$\times\sum_{j=1}^{d}\widetilde{K}'_{j,t}(\pi,\pi)+\frac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\tan^2(\sqrt{\lambda}\pi)$$

$$
\begin{aligned}
&-\frac{2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}} K_{i_1}K_{i_2}-\frac{\sum_{j=1}^{d}\widetilde{K}'_{j,x}(\pi,\pi)}{d}\\
&-\frac{(d-1)^2}{2d^2}\left(\sum_{j=1}^{d}K_j\right)^2\tan^2(\sqrt{\lambda}\pi)-\frac{1}{2d^2}\left(\sum_{j=1}^{d}K_j\right)^2\cot^2(\sqrt{\lambda}\pi)\\
&\left.+\frac{d-1}{d^2}\left(\sum_{j=1}^{d}K_j\right)^2\right]+O(1/\sqrt{\lambda^3}).
\end{aligned}\tag{4.8}
$$

By Rouché's theorem, for sufficiently large N_0, the number of zeros of $\varphi_2(\lambda)$ and $\varphi_2^{(0)}(\lambda)$ inside the contour Γ_{N_0} is just the same. In fact, the zeros $\lambda_{n,j}^N$ of $\varphi_2(\lambda)$ are the eigenvalues of operator A_2, and the zeros $\mu_{n,j}^N$ of $\varphi_2^{(0)}(\lambda)$ are the eigenvalues of problem (1.1), (1.3), (1.4) and (1.5) with $q_j=0, j=1,2,\cdots,d$.

By Lemma 4.1, we obtain

$$
\sum_{n=0}^{N_0}(\lambda_{n,d}^N-n^2)+\sum_{n=1}^{N_0}\left[\sum_{j=1}^{d-1}(\lambda_{n,j}^N-(n-\frac{1}{2})^2)\right]=-\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\log\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)}d\lambda. \tag{4.9}
$$

Substituting (4.8) into (4.9), together with (3.15) and (4.6), we have

$$
\begin{aligned}
&\lambda_{0,d}^N+\sum_{n=1}^{N_0}\sum_{j=1}^{d}(\lambda_{n,j}^N-\mu_{n,j}^N)\\
&=-\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi)-q_j(0)]+\frac{1}{2d}\sum_{j=1}^{d}q_j(\pi)+\frac{2N_0d+1}{\pi d}\sum_{j=1}^{d}K_j+O(1/N_0),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\lambda_{0,d}^N+\sum_{n=1}^{N_0}\left[\sum_{j=1}^{d}(\lambda_{n,j}^N-\mu_{n,j}^N)-2\sum_{j=1}^{d}\bar{q}_j\right]\\
&=-\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi)-q_j(0)]+\frac{1}{2d}\sum_{j=1}^{d}q_j(\pi)+\frac{1}{\pi d}\sum_{j=1}^{d}K_j+O(1/N_0).
\end{aligned}
$$

Let $N_0\to\infty$, then we have

$$
\begin{aligned}
&\lambda_{0,d}^N+\sum_{n=1}^{\infty}\left[\sum_{j=1}^{d}(\lambda_{n,j}^N-\mu_{n,j}^N)-2\sum_{j=1}^{d}\bar{q}_j\right]\\
&=-\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi)-q_j(0)]+\frac{1}{2d}\sum_{j=1}^{d}q_j(\pi)+\frac{1}{d}\sum_{j=1}^{d}\bar{q}_j,
\end{aligned}
$$

where $\bar{q}_j=\frac{1}{2\pi}\int_0^\pi q_j(x)dx$. The proof of theorem is finished. □

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