METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 13, No. 4, pp. 321–336, December 2006

001

2D DENSITY-DEPENDENT LERAY PROBLEM WITH A DISCONTINUOUS DENSITY*

AMMAR KHODJA, FARID† AND MARCELO M. SANTOS‡

Abstract. We consider the existence of a solution for the stationary Navier-Stokes equations describing an inhomogeneous incompressible fluid in a two dimensional unbounded Y-shaped domain. We show the existence of a weak solution such that the density and velocity of the fluid tend to densities and parallel flows, respectively, prescribed at some 'ends' of the domain. We allow prescribed densities at different ends to have distinct values. In fact, we obtain the density in the L^∞-space.

Key words. stationary Navier-Stokes equations, incompressible flow, inhomogeneous fluid, Leray problem, discontinuous density.

AMS subject classifications. 35Q30, 76D05

1. Introduction. Inhomogeneous fluids, i.e. fluids with a variable density, are important to be investigated in both physical and mathematical aspects. They can model, for instance, stratified fluids (see e.g. [9]) and the meeting of fluids coming from different regions with distinct densities, e.g. sewerage, water-works, the junction of two or more rivers, and junctions of channels as, for instance, occurs in devices called MEMS (Micro-Electro-Mechanical Systems). In the meeting of fluids with distinct densities, we must study a fluid with a discontinuous density. The classical mathematical model is the system of Navier-Stokes equations. Many challenging open questions are related to domains with unbounded channels even for the case of constant density, as for instance, the so-called Leray problem. This problem consists of finding a solution for the incompressible stationary Navier-Stokes equations with constant density, in a domain with unbounded straight channels, such that the velocity of the fluid in each channel tends to a given Poiseuille flow (a parallel velocity field in the straight channel vanishing at the boundary of the channel; see (4) below) in the end of the channel. This problem seems to have been proposed, in the 1950s, by Jean Leray to Olga A. Ladyzhenskaya, cf. [1, p. 476]. Despite the effort of brilliant mathematicians, see e.g. [6], up to now its solution is known only in the case of Poiseuille flows with small fluxes, a result due to Charles J. Amick [1, Theorem 3.8]. Not surprisingly, the main difficulty in solving the problem is to deal with the nonlinear term in the Navier-Stokes equations. In the case of a domain with straight channels and constant density, this difficulty is overcome in [1] by seeking a solution with the velocity field $\mathbf{v}$ of the form $\mathbf{v} = \mathbf{u} + \mathbf{a}$, for a new unknown $\mathbf{u}$, where $\mathbf{a}$ is a suitable extension of the given Poiseuille flows (see Section 2). It turns out that the nonlinear term can be estimated by the fluxes of the Poiseuille flows. Thus the result comes about the restriction that these fluxes are small, in comparison with the viscosity of the fluid.

We extend Amick's theorem [1, Theorem 3.8], in the two dimensional case, to inhomogeneous fluids with the density in L^∞; see Theorem 1 below. The result for

*Received April 1, 2005; accepted for publication November 2, 2006.
†Départment de Mathématiques & UMR 6623, Université de Franche-Comté, 16, route de Gray, 25030 Besancon Cedex France (ammar@math.univ-fcomte.fr).
‡Departamento de Matemática, IMECC–UNICAMP, Cx. Postal 6065, 13.081-970 Campinas, SP, Brazil (msantos@ime.unicamp.br). Partially supported by FAPESP/07015-3 and CAPES/3344-04-5.

smooth-density solutions was obtained in [5]. It is based on the streamline formulation, an approach strictly limited to the two dimensional case. Indeed, the smooth density ρ, which we obtained in [5], is of the form $\rho = \omega(\psi)$ where ψ is the streamline function, i.e. $\nabla^\perp \psi = \mathbf{v}$ ($\nabla^\perp \psi := (-\partial_2 \psi, \partial_1 \psi)$), and ω is some scalar function connected to the given values at some ends of the channels. This approach was first used by N.N. Frolov [2] to solve a boundary value problem in a bounded domain for the stationary inhomogeneous Navier-Stokes equations. Frolov's solution does not cover the case of a discontinuous density. In fact in [2] the solution is in the class of Hölder continuous functions and the part of the boundary of the domain in which the fluid is incoming is a connected set. In [10] Frolov's result was extended to L^∞-density solutions and the part of the boundary of the domain in which the fluid is incoming is allowed to be a disconnected set. The difficulty in dealing with L^∞-density solutions is that the composition $\omega(\psi)$ is for our purposes meaningless, since for a discontinuous ω it may not yield a measurable function.

We give a new approximating scheme which permits to pass from smooth-density solutions to L^∞-density solutions. We regularize the data and take appropriate associated scalar functions ω^ϵ, $\epsilon > 0$. Then we introduce approximating solutions $\rho^\epsilon = \omega^\epsilon(\psi^\epsilon)$, $\mathbf{v}^\epsilon = \nabla^\perp \psi^\epsilon$, and we are able to pass to the limit as ϵ tends to zero due to careful (with respect to ϵ) uniform estimates derived for the smooth case. An important step in the arguments is to attain the given densities at some ends of the channels, say $|x|$ tending to infinity. Here we have a special difficulty, essentially because the density is only in L^∞ and we have to deal with the double limit of taking ϵ tending to zero and $|x|$ tending to infinity. To overcome this difficulty we use the weak formulation of the 'transport equation' $\nabla \cdot (\rho \mathbf{v}) = 0$ (the stationary equation of conservation of mass), in which the given density at infinity is taken into account, and choose special test functions. In fact, we do not get exactly the desired weak formulation. Instead, we get an approximated one–see (21) in Section 3, but with an error term that decays exponentially at infinity. This is due to the exponential decay of the approximated velocity $\mathbf{v}^\epsilon$ to the Poiseuille flow, uniformly with respect to ϵ; see Lemma 2 in Section 3.

Now, we describe our problem more precisely, state our main result–Theorem 1 below, and give more details of the ideas introduced above.

The fluid fills an open set $\Omega \subset \mathbb{R}^2$ that is simply-connected, it has a smooth boundary Γ and it is the union of four disjoint sets, $\Omega = \cup_{i=0}^3 \Omega_i$, such that Ω_0 is bounded and, in possibly different coordinate systems, $\Omega_1 = \{(x, y) \in \mathbb{R}^2 : x < 0, y \in \Sigma_1\}$, $\Omega_2 = \{(x, y) \in \mathbb{R}^2 : x < 0, y \in \Sigma_2\}$ and $\Omega_3 = \{(x, y) \in \mathbb{R}^2 : x > 0, y \in \Sigma_3\}$, with $\Sigma_i = (-d_i, d_i)$ for arbitrarily given constants $d_i > 0$, $i = 1, 2, 3$. (This kind of domain was introduced by Amick, but with two ends Ω_i only ($i \neq 0$), which he called an *admissible domain* [1, Definition 1.1].) See Figures 1 and 2 below. The Navier-Stokes equations describing a stationary inhomogeneous incompressible fluids in Ω are the following:

$$\begin{cases} \nu \Delta \mathbf{v} = \rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p \\ \nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot (\rho \mathbf{v}) = 0 \,. \end{cases} \tag{1}$$

Here, ρ, $\mathbf{v} = (v_1, v_2)$, p, and ν are, respectively, the mass density, the velocity, the pressure and the given constant viscosity of the fluid. The first equation represents the conservation of momentum and the second and third equations represent the incompressibility of the fluid and the conservation of mass, respectively. The prototype of problem we have in mind is the junction of two rivers with different densities in a stationary regime.

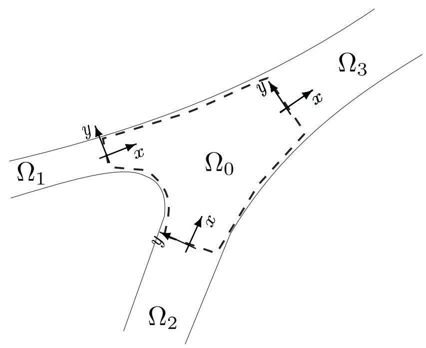

FIG. 1. *Domain Ω; Y-shaped domain.*

In this problem we have two (unbounded) channels where the fluid is incoming (Ω_1 and Ω_2 in Figure 1) and one channel where the fluid is outgoing (Ω_3 in Figure 1). See also Figure 2. Since the conservation of mass equation is a transport equation for smooth solutions, with transport vector given by $\mathbf{v}$, it is natural to give the density only at the ends of the channels where the fluid is incoming. The velocity field $\mathbf{v}$ is known to be parallel at far distances in each straight channel. Besides, we assume that the fluid does not slip on the boundary Γ of Ω, so the vector field $\mathbf{v}$ in each channel Ω_i, $i = 1, 2, 3$, is of Poiseuille type. Thus, coupled with the systems of equation (1), we have the following boundary conditions:

$$\mathbf{v} = 0 \quad \text{on} \quad \Gamma \tag{2}$$

(nonslip boundary condition) and

$$\begin{cases} \lim\limits_{|x|\to\infty}^{i} \mathbf{v} = \mathbf{v}_i, \quad i = 1, 2, 3 \\ \lim\limits_{x\to-\infty}^{i} \rho = \rho_i, \quad i = 1, 2 \end{cases} \tag{3}$$

where $\lim^i_{|x|\to\infty}$ stands for $\lim_{|x|\to\infty}$ with $(x, y) \in \Omega_i$ (similarly, $\lim^i_{x\to-\infty}$ stands for $\lim_{x\to-\infty}$ with $(x, y) \in \Omega_i$, $i = 1, 2$), $\mathbf{v}_i$ is a given Poiseuille flow in Ω_i, $i = 1, 2, 3$ (see (4) below) and ρ_i is a given nonnegative function in $C_b(\Sigma_i)$, $i = 1, 2$. Here and throughout, if X is a topological space, $C_b(X)$ will denote the space of bounded and continuous functions defined on X, endowed with the supremum norm $||f||_{C_b(X)} := \sup_{x\in X} |f(x)|$. Notice that in the second equation in (3) i varies from 1 to 2 only, i.e. we do not give ρ at the end of the outgoing channel Ω_3. We call the problem (1)-(3) *density-dependent Leray problem*.

Before stating our main result, Theorem 1 below, we need some more notations. First, let

$$\alpha_i = \int_{\Sigma_i} \mathbf{v}_i \cdot \mathbf{n}_i, \qquad i = 1, 2, 3$$

(the flux of the Poiseuille flow $\mathbf{v}_i$ in Ω_i) where $\mathbf{n}_i$ is the unit normal to Σ_i (the cross section of Ω_i) pointing toward $|x| = \infty$, i.e. pointing to the exterior of Ω_0. In the coordinates systems of Ω_i, we have $\mathbf{n}_i = (\pm 1, 0)$ and

$$\mathbf{v}_i(y) = (\theta_i(y), 0) \;\text{ for }\; \theta_i(y) = \pm\frac{3}{4d_i^3}\alpha_i(d_i^2 - y^2), \quad y \in \Sigma_i = (-d_i, d_i) \tag{4}$$

(cf. [1, p.485]) where the sign $\pm$ is $-$ if $i = 1, 2$ and $+$ if $i = 3$. Because the incompressibility equation $\nabla \cdot \mathbf{v} = 0$ and Divergence Theorem, we assume the compatibility condition $\alpha_1 + \alpha_2 + \alpha_3 = 0$, i.e. $\alpha_3 = -\alpha_1 - \alpha_2$. We assume also that α_1 and α_2 are negative real numbers (and so $\alpha_3 > 0$) which is in accordance with the directions of $\mathbf{n}_i$ and $\mathbf{v}_i$, i.e. the incoming given velocity $\mathbf{v}_i$ in Ω_i, $i = 1, 2$, is pointing to the opposite direction of $\mathbf{n}_i$ and $\mathbf{v}_3$ in Ω_3 is pointing to the same direction of $\mathbf{n}_3$. See Figure 2.

Let $\mathbf{H}_{k,loc}(\overline{\Omega})$ be the space of vector fields $\mathbf{v}$ in Ω such that $\mathbf{v}$ belongs to the Sobolev space $W^{k,2}(\Omega')$, for any open bounded subset Ω' of Ω, $\mathbf{v}$ is divergence free, i.e. $\nabla \cdot \mathbf{v} = 0$, and whose derivatives up to order $k - 1$ have zero trace on Γ. Let also $\mathcal{V}$ be the space of the vector fields $\mathbf{\Phi}$ in $C_0^\infty(\Omega)$ (the underscript '0' stands for compact support, i.e. the support set of $\mathbf{\Phi}$ is a compact set contained in Ω) and $\mathbf{\Phi}$ is divergence free.

Our main result is the following theorem.

THEOREM 1. *Let $l := \max_{i=1,2} ||\rho_i||_{C_b(\Sigma_i)}$ and $\alpha := \max_{i=1,2,3} |\alpha_i|$. There is a constant $c = c(\Omega) > 0$ such that if $cl\alpha < \nu$, then the problem (1)-(3) has a weak solution $(\rho, \mathbf{v}) \in L^\infty(\Omega) \times \mathbf{H}_{1,loc}(\overline{\Omega})$, in the following sense:*

i.

$$\nu \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{\Phi} dx = \int_\Omega \rho(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v} dx, \tag{5}$$

for all $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ in $\mathcal{V}$, where $\nabla \mathbf{v} \cdot \nabla \mathbf{\Phi} := \nabla v_1 \cdot \nabla \Phi_1 + \nabla v_2 \cdot \nabla \Phi_2$ and $\mathbf{v} \cdot \nabla \mathbf{\Phi} := (\mathbf{v} \cdot \nabla \Phi_1, \mathbf{v} \cdot \nabla \Phi_2)$,

ii.

$$\int_\Omega \rho \mathbf{v} \cdot \nabla \varphi dx = 0 \qquad \text{for all } \varphi \text{ in } C_0^\infty(\Omega), \tag{6}$$

iii.

$$\mathbf{v} - \mathbf{v}_i \in W^{2,2}(\Omega_t^c), \quad i = 1, 2, 3, \qquad \text{for some } t > 0, \tag{7}$$

where $\Omega_t^c := \cup_{i=1}^3 \Omega_{i,t}^c$, $\quad \Omega_{i,t}^c := \{(x, y) \in \Omega_i\,;\, |x| > t\}$; and

iv.

$$* - \lim_{x \to -\infty, a.e.}{}^{i} \rho(x, \cdot) = \rho_i, \qquad i = 1, 2 \tag{8}$$

where $* - \lim_{x \to -\infty, a.e.}{}^{i}$ denotes the limit in the weak-$*$ topology of $L^\infty(\Sigma_i)$, with x tending to $-\infty$ except for a set of zero Lebesgue measure.

In addition, $0 \le \rho(x, t) \le l$ for almost every $(x, y) \in \Omega$.

Equations (5) and (6) are just the weak formulations (in the sense of distributions) of the conservation of momentum and mass equations, respectively, i.e. multiply these equations by the test functions indicated in (5) and (6) and formally integrate by parts. Cf. [1, Def. 3.1]. In equation (5) the pressure p is canceled out because the (vector valued) test functions $\mathbf{\Phi}$ are divergence free. It is classical that we can recover the pressure from (5); see e.g. [11, Propositions I.1.1 and I.1.2, p. 14]. The incompressibility equation is inserted in the space $\mathbf{H}_{1,loc}(\overline{\Omega})$. Condition (7) implies that $\mathbf{v} \in C_b(\Omega_t^c)$ and

$$\lim_{|x| \to \infty} ||\mathbf{v}(x, \cdot) - \mathbf{v}_i||_{C_b(\Sigma_i)} = 0, \quad i = 1, 2, 3. \tag{9}$$

Indeed, since $\Omega^c_{i,t}$ is bounded in one direction, from the Sobolev Imbedding Theorem, we have $\mathbf{v} - \mathbf{v}_i \in C_b(\Omega^c_{i,t})$ and there is a constant k, independent of $|x| > t+1$, such that

$$||\mathbf{v}(x,\cdot) - \mathbf{v}_i||_{C_b(\Sigma_i)} \leq ||\mathbf{v} - \mathbf{v}_i||_{C_b(\Omega^c_{i,|x|-1})} \leq k||\mathbf{v} - \mathbf{v}_i||_{W^{2,2}(\Omega^c_{i,|x|-1})};$$

thus $\lim_{|x|\to\infty} ||\mathbf{v}(x,\cdot) - \mathbf{v}_i||_{C_b(\Sigma_i)} \leq k \lim_{|x|\to\infty} ||\mathbf{v} - \mathbf{v}_i||_{W^{2,2}(\Omega^c_{i,|x|-1})} = 0.$ Equation (8) says that the given density values ρ_i at the end of the incoming channels Ω_i $(i = 1, 2)$ are attained 'in average' almost everywhere, i.e. there is a Lebesgue measurable set $E_i \subset (-\infty, 0)$ with zero Lebesgue measure such that

$$\lim_{\begin{cases} x \to -\infty \\ x \in (-\infty,0)/E_i \end{cases}} \int_{\Sigma_i} \rho(x,y)\xi(y)\,dy = \int_{\Sigma_i} \rho_i(y)\xi(y)\,dy, \qquad i = 1,2$$

for any $\xi \in L^1(\Sigma_i)$.

Remark 1. **(on uniqueness).** *The uniqueness of solution in Theorem 1 is not clear for us, since following the usual procedure of taking the differences* $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, $\rho = \rho_1 - \rho_2$ *of two solutions* $(\rho_i, \mathbf{v}_i)$, $i = 1, 2$, *we get stuck with the term* $\int_\Omega \rho(\mathbf{v}_i \cdot \nabla \mathbf{v}) \cdot \mathbf{v}_i$. *We conjecture that uniqueness of the velocity field is true under an assumption of smallness on the density, i.e. if we assume that* $||\rho||_{L^\infty(\Omega)}$ *is sufficiently small, but some new ingredient is necessary to improve the usual proof (or to find a new one). Regarding the uniqueness of the density, it is necessary to find new criteria to select the physically relevant solution (cf. [8, p.34]).*

Our approximating scheme to achieve Theorem 1 relies on an appropriate mollification of the data $\rho_i, \mathbf{v}_i$ and, at the approximated level, on the streamline function formulation. Let us give an outline of this scheme. We start by introducing the functions $\psi_i : \Sigma_i \to \mathbb{R}$, $i = 1, 2$, defined by

$$\psi_1(y) := -\int_{-d_1}^{y} \theta_1(\tau)\,d\tau, \quad y \in \Sigma_1, \quad \text{and} \quad \psi_2(y) := -\alpha_2 - \int_{-d_2}^{y} \theta_2(\tau)\,d\tau, \quad y \in \Sigma_2\,;$$

(ψ_i is a streamline function associated with the Poiseuille flow $\mathbf{v}_i$, i.e. $\nabla^\perp \psi_i = \mathbf{v}_i$). See Figure 2. Notice that because $\mathbf{v} = 0$ on Γ (condition (2)) and we shall obtain $\mathbf{v}$ of the form $\mathbf{v} = \nabla^\perp \psi$ (so ψ is constant on each connected component of Γ), we have defined ψ_1 and ψ_2 satisfying $\psi_1(-d_1) = \psi_2(d_2)$. Indeed, thanks to this condition we can apply Poincare's inequality on $\psi^\epsilon - \psi_i$ (ψ^ϵ is defined below) to obtain that $\psi^\epsilon(-s,\cdot)$ converges uniformly to ψ_i when s tends to infinity; see Section 3.

Let $\epsilon > 0$ be arbitrary but sufficiently small. Extend ρ_i to $(-\infty, \infty)$ by zero outside Σ_i and let $\rho^\epsilon_i = \rho_i * m^\epsilon$ be a standard mollification of this extended function, i.e. $*$ stands for convolution of functions on the real line and $m^\epsilon(y) = \epsilon^{-1} m(y/\epsilon)$, where m is a positive smooth function with support in $(-1, 1)$ and such that $\int_{-\infty}^{\infty} m(y)dy = 1$. Next, let ω^ϵ be a smooth function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$0 \leq \omega^\epsilon(s) \leq l \quad \text{for all } s \in \mathbb{R} \tag{10}$$

and

$$\omega^\epsilon(\psi_i(y)) = \rho^\epsilon_i(y) \text{ for } \ y \in \Sigma^\epsilon_i, \text{ where } \ \Sigma^\epsilon_i := (-d_i + \epsilon\,,\, d_i - \epsilon), \quad i = 1,2\,. \tag{11}$$

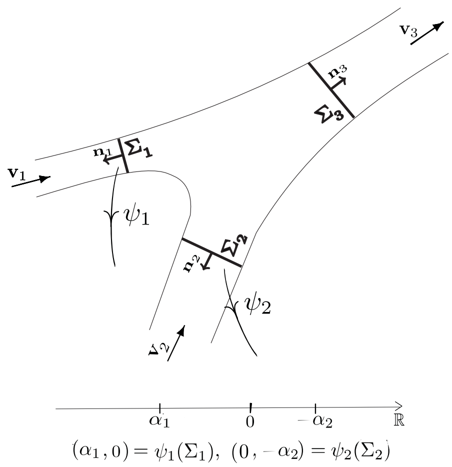

FIG. 2. *Poiseuille flows, cross sections and streamline functions of the incoming Poiseuille flows;* $(\alpha_1, 0)$ *and* $(0, -\alpha_2)$ *stand for the intervals that are the images of* Σ_1 *and* Σ_2 *under the streamline functions* ψ_1 *and* ψ_2*, respectively.*

Notice that condition (11) on ω^ϵ is reasonably imposed, since ψ_i^ϵ is an injective function. In fact, ψ_i is a decreasing function since θ_i $(i = 1, 2)$ is a positive function on Σ_i. Thus we can write

$$\omega^\epsilon(s) = \rho_i^\epsilon((\psi_i)^{-1}(s)), \quad \text{for} \quad s \in (\psi_i(d_i - \epsilon),\, \psi_i(-d_i + \epsilon)) \quad i = 1, 2.$$

Outside the set $(\psi_1(d_1-\epsilon),\, \psi_1(-d_1+\epsilon)) \cup (\psi_2(d_2-\epsilon), \psi_2(-d_2+\epsilon))$ we take ω^ϵ arbitrary but smooth (say, C^∞) and satisfying (10) for all sufficiently small $\epsilon > 0$. See Figure 3.

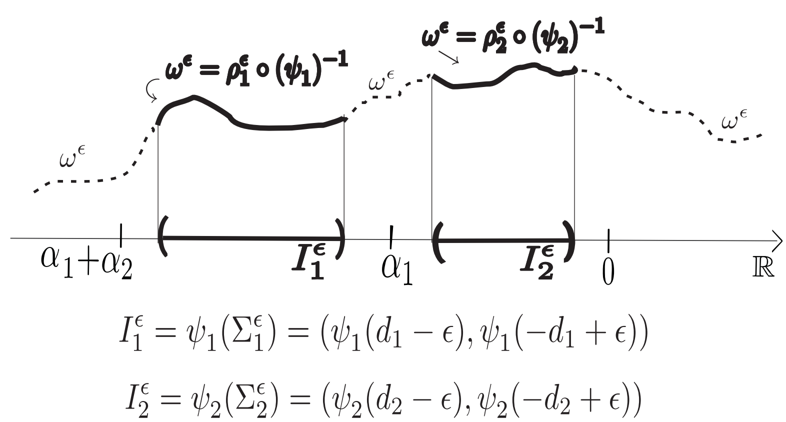

FIG. 3. *The function* ω^ϵ

Let $\mathbf{a} \in \mathbf{H}_{1,loc}(\overline{\Omega})$ be a suitable smooth vector field (see Section 2) such that $\mathbf{a}$ coincides with the Poiseuille flow $\mathbf{v}_i$ in Ω_i at large distances. Let also $\mathbf{V}$ be the closure of the space $\mathcal{V}$ endowed with the Dirichlet norm $||\mathbf{u}|| = (\int_\Omega |\nabla \mathbf{u}|^2)^{1/2}$.

Then our approximate solution $(\rho^\epsilon, \mathbf{v}^\epsilon)$ is defined as

$$\rho^\epsilon = \omega^\epsilon(\psi^\epsilon), \quad \mathbf{v}^\epsilon = \nabla^\perp \psi^\epsilon = \mathbf{u}^\epsilon + \mathbf{a}$$

where $\mathbf{u}^\epsilon \in \mathbf{V}$ is a solution of the variational problem

$$\nu \int_\Omega \nabla(\mathbf{u}^\epsilon + \mathbf{a}) \cdot \nabla \mathbf{\Phi} = \int_\Omega \omega^\epsilon(\psi^\epsilon)((\mathbf{u}^\epsilon + \mathbf{a}) \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v}^\epsilon \, dx \,, \quad \forall \; \mathbf{\Phi} \in \mathcal{V} \,. \tag{12}$$

With the aforementioned construction of ω^ϵ and the suitable extension $\mathbf{a}$ of the Poiseuille flows, we can show that the sequence $(\rho^\epsilon, \mathbf{v}^\epsilon)$ possesses some subsequence that converges to a pair $(\rho, \mathbf{v})$, in the sense explained below, satisfying all the statements in Theorem 1. This is the matter of the rest of this paper.

This paper is organized as follows. In Section 2 we solve (12) and obtain the a priori estimates we need in Section 3. In Section 3 we prove that there exists a pair $(\rho, \mathbf{u})$ in $L^\infty(\Omega) \times \mathbf{V}$ such that, up to some subsequence of $\epsilon \to 0$, $(\mathbf{u}^\epsilon)$ converges to $\mathbf{u}$ in the weak topology of $\mathbf{V}$ and ρ^ϵ converges to ρ in the weak-$*$ topology of $L^\infty(\Omega)$ as $\epsilon \to 0$. Then we show that the pair $(\rho, \mathbf{v})$, $\mathbf{v} = \mathbf{u} + \mathbf{a}$, satisfies all the claims stated in Theorem 1.

2. Approximating solution. In this Section we solve the variational problem (12) and obtain the uniform estimates with respect to ϵ we need in Section 3 to pass to the limit as ϵ tends to zero.

We begin by describing the vector field $\mathbf{a}$ that extends the Poiseuille flows $\mathbf{v}_i$, $i = 1, 2, 3$. Given $t > 0$, let $\mathbf{a}$ be a smooth vector field that coincides with the Poiseuille flow $\mathbf{v}_i$ in $\Omega_{i,t}^c$, $i = 1, 2, 3$, and satisfies the estimate

$$||\nabla \mathbf{a}||_{L^2(\Omega_t)} \leq c\alpha \tag{13}$$

for some positive constant c depending only on t and Ω, where α and $\Omega_{i,t}^c$ are defined in Theorem 1, and

$$\Omega_t := \cup_{i=1}^3 \Omega_{i,t} \cup \Omega_0, \quad \Omega_{i,t} := \{x \in \Omega_i \,;\, |x| < t\}.$$

For a construction of $\mathbf{a}$ we refer the reader to [4, Lemma XI.3.1] or [1, §3.1]. In the latter, a construction of $\mathbf{a}$ is carried out in the case of a domain with two channels that is called an *admissible domain* [1, Def. 3.1], but the same construction works for an 'admissible domain' with any finite number of straight channels.

Next we state the following lemma which will be used also in Section 3.

Lemma 1. *For any $\rho \in L^\infty(\Omega)$ and $\mathbf{u} \in \mathbf{V}$ (the closure of $\mathcal{V}$ in the norm $||\nabla \mathbf{u}||_{L^2(\Omega)}$) we have the following estimates:*

$$|(\rho \mathbf{u} \cdot \nabla \mathbf{a}, \, \mathbf{u})| \leq c\alpha ||\rho||_{L^\infty(\Omega)} ||\nabla \mathbf{u}||^2_{L^2(\Omega)},$$

$$|(\rho \mathbf{a} \cdot \nabla \mathbf{a}, \, \mathbf{u})| \leq c\alpha^2 ||\rho||_{L^\infty(\Omega)} ||\nabla \mathbf{u}||_{L^2(\Omega)},$$

where c depends only on Ω and $(\,,\,)$ stands for the inner product in $L^2(\Omega)$, i.e. if $\mathbf{f}, \mathbf{g} \in L^2(\Omega)$ then $(\mathbf{f}, \mathbf{g}) := \int_\Omega \mathbf{f} \cdot \mathbf{g}$.

Proof. These estimates, except for the term ρ, are found in [4, Lemma XI.2.2]. The extra term ρ here can be majored by $||\rho||_{L^\infty(\Omega)}$. We remark that there is no integration by parts to obtain these estimates, so no term containing $\nabla \rho$ comes up. Essentially, $\nabla \mathbf{a}$ is estimated by (13) and $\mathbf{u}$ by $\nabla \mathbf{u}$ via Poincare's inequality. □

For a fixed $\epsilon > 0$, we solve the variational problem (12) using Galerkin method: Given $\{\mathbf{\Phi}_k\} \subset \mathcal{V}$ an orthonormal basis of $\mathbf{V}$ (with respect to the inner product $(\nabla \mathbf{f}, \nabla \mathbf{g})$), consider the approximate problem

$$\begin{cases} \mathbf{u}_m = \sum_{k=1}^{m} \xi_{km} \mathbf{\Phi}_k, \qquad \mathbf{u}_m + \mathbf{a} = \nabla^{\perp} \psi_m \\ \nu \displaystyle\int_{\Omega} \nabla(\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{\Phi}_k = \int_{\Omega} \omega^{\epsilon}(\psi_m)((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{\Phi}_k) \cdot (\mathbf{u}_m + \mathbf{a}), \\ k = 1, 2, \cdots, m. \end{cases} \tag{14}$$

For each $m \in \mathbb{N}$, (14) is a system of nonlinear algebraic equations for the unknown $\xi = (\xi_{1m}, \cdots, \xi_{1m}) \in \mathbb{R}^m$. Its solution is guaranteed by Brouwer's Fixed Point Theorem. Indeed, if we set

$$F_k(\xi) = \nu(\nabla(\mathbf{u}_m + \mathbf{a}), \nabla \mathbf{\Phi}_k) - \int_{\Omega} \omega^{\epsilon}(\psi_m)((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{\Phi}_k) \cdot (\mathbf{u}_m + \mathbf{a}),$$

for $k = 1, \cdots, m$, and $\mathbf{F} = (F_1, \cdots, F_m)$, then (14) becomes the equation $\mathbf{F}(\xi) = 0$. Besides, for $l_\epsilon := ||\omega^\epsilon||_{L^\infty(\mathbb{R})}$ and using Lemma 1 together with Hölder inequalities, the estimate (13), the identity

$$\begin{aligned} & \int_{\Omega} \omega^{\epsilon}(\psi_m)\,((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{u}_m) \cdot \mathbf{u}_m \, = \int_{\Omega} \omega^{\epsilon}(\psi_m)(\mathbf{u}_m + \mathbf{a}) \cdot \nabla \tfrac{1}{2} |\mathbf{u}_m|^2 \\ = \quad & -\int_{\Omega} \nabla \cdot (\omega^{\epsilon}(\psi_m)(\mathbf{u}_m + \mathbf{a}))\, \tfrac{1}{2} |\mathbf{u}_m|^2 \, = 0\,, \end{aligned}$$

and

$$\int_{\Omega} \nabla \mathbf{a} \cdot \nabla \mathbf{u}_m = -\int_{\Omega} \Delta \mathbf{a} \cdot \mathbf{u}_m = -\int_{\Omega_t} \Delta \mathbf{a} \cdot \mathbf{u}_m$$

(since $\int_{\Omega_{i,t}^c} \Delta \mathbf{a} \cdot \mathbf{u}_m = \int_{\Omega_{i,t}^c} \Delta \mathbf{v}_i \cdot \mathbf{u}_m = -\int_{\pm t}^{\pm\infty} \int_{-d_i}^{d_i} \frac{3\alpha_i}{2d_i^3} \mathbf{u}_m \cdot \mathbf{n}_i \, dydx$
$= \frac{3\alpha_i}{2d_i^3} \int_{\pm t}^{\pm\infty} \Big(\int_{\Omega_{i,y}^c} \nabla \cdot \mathbf{u}_m \Big) \, dx = 0)$ we obtain

$$\begin{aligned} \mathbf{F}(\xi) \cdot \xi = \; & \nu \int_{\Omega} \nabla \mathbf{u}_m \cdot \nabla \mathbf{u}_m + \nu \int_{\Omega} \nabla \mathbf{a} \cdot \nabla \mathbf{u}_m \\ & - \int_{\Omega} \omega^{\epsilon}(\psi_m)((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{u}_m) \cdot \mathbf{a} \\ = \; & \nu ||\nabla \mathbf{u}_m||_{L^2(\Omega)}^2 - \nu \int_{\Omega_t} \Delta \mathbf{a} \cdot \mathbf{u}_m \\ & + \int_{\Omega} \omega^{\epsilon}(\psi_m)((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{a}) \cdot \mathbf{u}_m \\ & \text{(notice that div} \, (\omega^{\epsilon}(\psi_m)(\mathbf{u}_m + \mathbf{a})) = 0) \\ = \; & \nu ||\nabla \mathbf{u}_m||_{L^2(\Omega)}^2 + \int_{\Omega} \omega^{\epsilon}(\psi_m)(\mathbf{u}_m \cdot \nabla \mathbf{a}) \cdot \mathbf{u}_m \\ & -\nu \int_{\Omega_t} \Delta \mathbf{a} \cdot \mathbf{u}_m \quad + \int_{\Omega} \omega^{\epsilon}(\psi_m)(\mathbf{a} \cdot \nabla \mathbf{a}) \cdot \mathbf{u}_m \\ \geq \; & \nu ||\nabla \mathbf{u}_m||_{L^2(\Omega)}^2 - c\alpha l_\epsilon ||\nabla \mathbf{u}_m||_{L^2(\Omega)}^2 \\ & -\nu c ||\nabla \mathbf{u}_m||_{L^2(\Omega)} - c\alpha^2 l_\epsilon ||\nabla \mathbf{u}_m||_{L^2(\Omega)} \\ = \; & (\nu - c\alpha l_\epsilon)|\xi|_{\mathbb{R}^m}^2 - c(\nu + \alpha^2 l_\epsilon)|\xi|_{\mathbb{R}^m} \end{aligned}$$

i.e.

$$\mathbf{F}(\xi) \cdot \xi \geq (\nu - c\alpha l_\epsilon)|\xi|_{\mathbb{R}^m}^2 - c(\nu + \alpha^2 l_\epsilon)|\xi|_{\mathbb{R}^m} \,. \tag{15}$$

Here, and from now on, c denotes some constant not depending on $m \in \mathbb{N}$ and $\epsilon > 0$.

From (15) we infer that if $c\alpha l_\epsilon < \nu$ then the vector field $\mathbf{F}$ satisfies $\mathbf{F}(\xi) \cdot \xi > 0$ for all sufficiently large ξ. Therefore, it has a singular point, i.e. $F(\xi) = 0$ for some ξ, inside any sufficiently large ball in $\mathbb{R}^m$. It is easy to obtain a proof of this fact by contradiction using the Brouwer Fixed Point Theorem; see the proofs of [11, Lemma II.1.4] or [4, Lemma VIII.3.1].

From now on we assume the condition $c\alpha l < \nu$. Then from (10) we have also $c\alpha l_\epsilon < \nu$ for all $\epsilon > 0$.

Since $|\xi|_{\mathbb{R}^m} = ||\nabla \mathbf{u}_m||_{L^2(\Omega)}$ and $F(\xi) = 0$, from (15) we get

$$||\nabla \mathbf{u}_m||_{L^2(\Omega)} \leq \frac{c(\nu + \alpha^2 l_\epsilon)}{\nu - c\alpha l_\epsilon}, \tag{16}$$

for all $m \in \mathbb{N}$ and $\epsilon > 0$. Then there is a subsequence $(u_{m'})$ that converges weakly in $\mathbf{V}$ and strongly in $L^2(\Omega)$ to some function $\mathbf{u}^\epsilon \in \mathbf{V}$. As a consequence, $(\psi_{m'})$ converges weakly to some function ψ^ϵ in $W^{2,2}_{loc}(\Omega)$, such that $\nabla^\perp \psi^\epsilon = \mathbf{u}^\epsilon + \mathbf{a}$. Using the Sobolev Imbedding Theorem, we deduce that $\omega^\epsilon(\psi_{m'})$ converges to $\omega^\epsilon(\psi)$ in $C(\overline{\Omega})$ and then, by a routine argument, that $\mathbf{u}^\epsilon = \nabla^\perp \psi^\epsilon - \mathbf{a}$ verifies (12). Moreover, from (16) and $l_\epsilon \leq l$, we have

$$||\nabla \mathbf{u}^\epsilon||_{L^2(\Omega)} \leq \frac{c(\nu + \alpha^2 l_\epsilon)}{\nu - c\alpha l_\epsilon} \leq \frac{c(\nu + \alpha^2 l)}{\nu - c\alpha l}\,, \tag{17}$$

for all $\epsilon > 0$. Next, we remark that our solution $\mathbf{u}^\epsilon$ is regular. In fact, for more general purposes in Section 3, let us point out the following result: Suppose that some $\mathbf{u} \in \mathbf{V}$ along with some pressure function $\tau \in L^2_{loc}(\Omega)$, is a solution of the Stokes equation

$$\nu \Delta \mathbf{u} = \nabla \tau + \mathbf{f}\,,$$

in the weak sense in the domain Ω^c_t for some $t > 0$, where

$$\mathbf{u} = \mathbf{v} - \mathbf{v}_i \text{ in } \Omega^c_{i,t} \text{ and } \mathbf{f} := \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \rho(\mathbf{a} \cdot \nabla)\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{a}\,,$$

with $\rho \in L^\infty(\Omega)$. Then $\mathbf{u} \in W^{2,2}(\Omega^c_{t+1})$ and we have the estimate

$$||\mathbf{u}||_{W^{2,2}(\Omega^c_{t+1})} \leq c\,||\nabla \mathbf{u}||_{L^2(\Omega^c_t)}. \tag{18}$$

where $c = c(d, \alpha, ||\rho||_{L^\infty(\Omega)}, ||\nabla \mathbf{u}||_{L^2(\Omega)})$ is increasing in each of its arguments. This can be obtained by a usual boot strap argument in the regularity theory for the Stokes' equations. For the details we refer the reader to [5].

3. Proof of Theorem 1. From (10) and (17) we have that $\rho^\epsilon = \omega^\epsilon(\psi^\epsilon)$ is a bounded sequence in $L^\infty(\Omega)$ and $(\mathbf{u}^\epsilon)_{\epsilon > 0}$ is a bounded sequence in $\mathbf{V}$. Then there exists some pair $(\rho, \mathbf{u})$ in $L^\infty(\Omega) \times \mathbf{V}$ such that, up to some subsequence of $\epsilon \to 0$, $(\mathbf{u}^\epsilon)$ converges to $\mathbf{u}$ in the weak topology of $\mathbf{V}$ and ρ^ϵ converges to ρ in the weak-$*$ topology of $L^\infty(\Omega)$ as $\epsilon \to 0$. Next, we show that the pair $(\rho, \mathbf{v})$, $\mathbf{v} = \mathbf{u} + \mathbf{a}$, satisfies all the claims stated in Theorem 1. Let $\mathbf{v}^\epsilon = \nabla^\perp \psi^\epsilon = \mathbf{u}^\epsilon + \mathbf{a}$. We note that $(\mathbf{v}^\epsilon)$ converges to

$\mathbf{v}$ in the weak topology of $W^{1,2}(\Omega')$ for any bounded domain $\Omega' \subset \Omega$, and so, strongly in $L^p(\Omega')$ for any $p \in [1, \infty)$. Multiplying the equation $\nabla \cdot (\rho^\epsilon \mathbf{v}^\epsilon) = 0$ (recall that $\rho^\epsilon = \omega^\epsilon(\psi^\epsilon)$) by $\varphi \in C_0^\infty(\Omega)$ and integrating by parts, we obtain $\int_\Omega \rho^\epsilon \mathbf{v}^\epsilon \cdot \nabla\varphi \, dx = 0$. Then taking $\Omega' \subset \Omega$ such that $\operatorname{spt}\varphi \subset \Omega'$ and using that $(\mathbf{v}^\epsilon)$ converges to $\mathbf{v}$ strongly in $L^2(\Omega')$ and (ρ^ϵ) converges to ρ in the weak-$*$ topology of $L^\infty(\Omega')$, we can easily see that $\int_\Omega \rho^\epsilon \mathbf{v}^\epsilon \cdot \nabla\varphi \, dx$ converges to $\int_\Omega \rho \mathbf{v} \cdot \nabla\phi \, dx$, then we have (6) satisfied. A similar argument shows (5) by using that $(\mathbf{v}^\epsilon)$ converges strongly to $\mathbf{v}$ in L^4, locally. That is, taking $\Omega' \subset \Omega$ such that $\operatorname{spt}\mathbf{\Phi} \subset \Omega'$, we can pass to the limit as $\epsilon \to 0$ in (12) and obtain (5). The difficult part would be to pass to the limit in the right hand side of (12), which is a nonlinear term, but then we can use the strong convergence of $(\mathbf{v}^\epsilon)$ to $\mathbf{v}$ in $L^4(\Omega')$, combined with the weak-$*$ convergence of ρ^ϵ to ρ in $L^\infty(\Omega')$, and obtain the correct term in the limit. Claim (7) was shown in (18). It remains to show claim (8).

To show (8) we start by multiplying the equation $\nabla \cdot (\rho^\epsilon \mathbf{v}^\epsilon) = 0$ by $\varphi \in C^1(\Omega)$ such that $\operatorname{spt}\varphi \subset \Omega_{i,t}^c$, $i = 1, 2$, where $i \in \{1, 2\}$ is fixed. Integrating by parts in $\{(x, y) \in \Omega_i\,;\, -s < x < -t\}$, $s > t$, and writing $\mathbf{v}^\epsilon = (v_1^\epsilon, v_2^\epsilon)$, we get

$$\int_{-s}^{-t} \int_{\Sigma_i} \rho^\epsilon \mathbf{v}^\epsilon \cdot \nabla\varphi \, dydx = -\int_{\Sigma_i} (\rho^\epsilon v_1^\epsilon \varphi)(-s, y) \, dy.$$

Since $\mathbf{v}^\epsilon = \mathbf{u}^\epsilon + \mathbf{a}$, $\mathbf{u}^\epsilon = (u_1^\epsilon, u_2^\epsilon)$, $\mathbf{a} = \mathbf{v}_i = (\theta_i(y), 0)$ in $\Omega_{i,t}^c$, it follows that

$$\begin{aligned} & \int_{-s}^{-t} \int_{\Sigma_i} \rho^\epsilon u_1^\epsilon \varphi_x \, dydx + \int_{-s}^{-t} \int_{\Sigma_i} \rho^\epsilon u_2^\epsilon \varphi_y \, dydx + \int_{-s}^{-t} \int_{\Sigma_i} \rho^\epsilon \theta_i(y) \varphi_x \, dydx \\ = \; & -\int_{\Sigma_i} (\rho^\epsilon u_1^\epsilon \varphi)(-s, y) \, dy - \int_{\Sigma_i} \rho^\epsilon(-s, y)\theta_i(y)\varphi(-s, y) \, dy, \end{aligned} \tag{19}$$

then taking the limit when $s \to \infty$ and assuming that φ_x has compact support (i.e. φ is constant with respect to x for large $|x|$) we obtain

$$\begin{aligned} & \int_{-\infty}^{-t}\int_{\Sigma_i} \rho^\epsilon u_1^\epsilon \varphi_x \, dydx + \int_{-\infty}^{-t}\int_{\Sigma_i} \rho^\epsilon \theta_i(y) \varphi_x \, dydx \\ = \; & -\int_{\Sigma_i} \omega^\epsilon(\psi_i(y))\theta_i(y)\varphi(-\infty, y) dy + \mathcal{R}(\epsilon), \end{aligned} \tag{20}$$

where $\varphi(-\infty, y) := \lim_{s\to\infty} \varphi(-s, y)$ and

$$\mathcal{R}(\epsilon) := -\lim_{s\to\infty} \int_{-s}^{-t}\int_{\Sigma_i} \rho^\epsilon u_2^\epsilon \varphi_y \, dydx.$$

We note that this limit exists due to equation (19), since all the other terms in (19) have a limit when $s \to \infty$. Besides, to arrive at (20) we used that $\lim_{s\to\infty} u_1^\epsilon(-s, y) = 0$ uniformly with respect to y; see (9) and (18). We have used also that $\rho^\epsilon \in L^\infty(\Omega)$ and $\lim_{s\to-\infty} \rho^\epsilon(-s, y) = \omega^\epsilon(\psi_i(y))$ uniformly with respect to y. To prove this last assertion we note that, since $\mathbf{u}^\epsilon = \mathbf{v}^\epsilon - \mathbf{v}_i \in W^{1,2}(\Omega_i)$, from Poincaré inequality we have that $\psi^\epsilon - \psi_i \in W^{2,2}(\Omega_i)$, thus, reasoning as in (9) we obtain $\lim_{s\to\infty} ||\psi^\epsilon(-s, \cdot) - \psi_i||_{C_b(\Sigma_i)} = 0$. Then, ψ^ϵ is bounded in Ω_i and given an arbitrary $\epsilon > 0$ there exists a $s_0 > 0$ such that $s \geq s_0$ implies

$$|\omega^\epsilon(\psi^\epsilon(-s, y)) - \omega^\epsilon(\psi_i(y))| < \epsilon,$$

for all $y \in (-d_i, d_i)$, since ω^ϵ is locally uniformly continuous.

Next, taking the limit as $\epsilon \to 0$ in (20) we infer that there exists the limit of $\mathcal{R}(\epsilon)$ when $\epsilon \to 0$ and

$$\begin{aligned} & \int_{-\infty}^{-t}\int_{\Sigma_i} \rho u_1 \varphi_x \, dydx + \int_{-\infty}^{-t}\int_{\Sigma_i} \rho \theta_i(y) \varphi_x \, dydx \\ = \quad & -\int_{\Sigma_i} \rho_i(y)\theta_i(y)\varphi(-\infty, y)dy + \lim_{\epsilon\to 0} \mathcal{R}(\epsilon). \end{aligned} \tag{21}$$

Now we choose $\varphi = \zeta_\tau(x)\xi(y)$ where $\xi \in C^1(\Sigma_i)$ and ζ_τ, $\tau > 0$, is a smooth function tending to the characteristic function $\chi_{(-\infty, x_0)}$, $x_0 < -t$, as $\tau \to 0$ ($\zeta_\tau(x)$ is equal to one if $x \le x_0$ and zero if $x \ge x_0 + \tau$). Then letting $\tau \to 0$, from (21) we obtain

$$\begin{aligned} & -\int_{\Sigma_i} (\rho u_1)(x_0, y)\xi(y) \, dy - \int_{\Sigma_i} \rho(x_0, y)\theta(y)\xi(y) \, dy \\ = \quad & -\int_{\Sigma_i} \rho_i(y)\theta(y)\xi(y) \, dy + \lim_{\tau\to 0} \lim_{\epsilon\to 0} \mathcal{R}(\epsilon), \end{aligned} \tag{22}$$

for almost every $x_0 < -t$. With the above φ's and $-s < x_0 + \tau < -t$, the following estimate is true.

$$\lim_{\tau\to 0} \lim_{\epsilon\to 0} |\mathcal{R}(\epsilon)| \le l \, ||\xi'||_{L^\infty(\Sigma_i)} \, c \, (2d_i/\sigma) e^{\sigma x_0} \tag{23}$$

where c and σ are some positive constants. To prove (23), first we estimate

$$\begin{aligned} \left| \int_{-s}^{-t}\int_{\Sigma_i} \rho^\epsilon u_2^\epsilon \varphi_y \, dydx \right| & = \left| \int_{-s}^{x_0+\tau}\int_{\Sigma_i} \rho^\epsilon u_2^\epsilon \zeta_\tau(x)\xi'(y) \, dydx \right| \\ & \le l \, ||\xi'||_{L^\infty(\Sigma_i)} \int_{-s}^{x_0+\tau}\int_{\Sigma_i} |u_2^\epsilon| \, dydx \end{aligned}$$

and then we use Lemma 2 below to obtain

$$\begin{aligned} \left| \int_{-s}^{-t}\int_{\Sigma_i} \rho^\epsilon u_2^\epsilon \varphi_y \, dydx \right| & \le l \, ||\xi'||_{L^\infty(\Sigma_i)} \, c \int_{-s}^{x_0+\tau}\int_{\Sigma_i} e^{\sigma x} \, dydx \\ & = l \, ||\xi'||_{L^\infty(\Sigma_i)} \, c \, (2d_i/\sigma)(e^{\sigma(x_0+\tau)} - e^{-\sigma s}); \end{aligned}$$

thus

$$|\mathcal{R}(\epsilon)| \le l \, ||\xi'||_{L^\infty(\Sigma_i)} \, c \, (2d_i/\sigma) e^{\sigma(x_0+\tau)},$$

so it follows (23).

Lemma 2. *There are constants $a, c, \sigma > 0$ independents of ϵ, such that*

$$|\mathbf{u}^\epsilon(x, y)| \le c e^{\sigma x}$$

for all $(x, y) \in (-\infty, -a) \times \Sigma_i$, $i = 1, 2$.

Proof. From Sobolev Imbedding Theorem, we have

$$|\mathbf{u}^\epsilon(x, y)| \le c ||\mathbf{u}^\epsilon||_{W^{2,2}(\Omega^c_{i,-x-1})} \qquad (i = 1, 2),$$

where c is a constant independent of ϵ, x and y. On the other hand, from (18) we have $||\mathbf{u}^\epsilon||_{W^{2,2}(\Omega^c_{i,-x-1})} \le c||\nabla \mathbf{u}^\epsilon||_{L^2(\Omega^c_{i,-x-2})}$ for some positive constant c, thus it is enough to prove that $||\nabla \mathbf{u}^\epsilon||_{L^2(\Omega^c_{i,-x})} \le c e^{\sigma x}$ for all $x << -1$, with c and σ independent of ϵ.

Let

$$\mathcal{T}(x) := \{(x,y) \in \Omega_i \,;\, y \in \Sigma_i\}\,.$$

Multiplying the equation

$$\nu\Delta\mathbf{u}^\epsilon = \rho^\epsilon\mathbf{v}^\epsilon\cdot\nabla\mathbf{v}^\epsilon + \nabla p^\epsilon$$

by $\mathbf{u}^\epsilon$ and integrating by parts in $(-s,x)\times\Sigma_i$, $-s<x<<-1$, we find

$$\begin{aligned}\nu\int_{-s}^{x}\int_{\Sigma_i}|\nabla\mathbf{u}^\epsilon|^2\,dydx' = &\int_{\mathcal{T}(x)}(-p^\epsilon u_1^\epsilon + \frac{\nu}{2}\frac{\partial|\mathbf{u}^\epsilon|^2}{\partial x} - \frac{1}{2}\rho^\epsilon v_1^\epsilon|\mathbf{u}^\epsilon|^2)\,dy\\ &-\int_{\mathcal{T}(-s)}(-p^\epsilon u_1^\epsilon + \frac{\nu}{2}\frac{\partial|\mathbf{u}^\epsilon|^2}{\partial x} - \frac{1}{2}\rho^\epsilon v_1^\epsilon|\mathbf{u}^\epsilon|^2)\,dy\\ &-\int_{-s}^{x}\int_{\Sigma_i}\rho^\epsilon u_1^\epsilon u_2^\epsilon\theta_i'(y)\,dydx',\end{aligned}\tag{24}$$

we used that $\mathbf{v}^\epsilon = \mathbf{u}^\epsilon + \mathbf{v}_i = (u_1^\epsilon+\theta_i(y), u_2^\epsilon)$ in Ω_i for $|x|$ large, so $(\mathbf{v}^\epsilon\cdot\nabla\mathbf{v}^\epsilon)\cdot\mathbf{u}^\epsilon = \frac{1}{2}\mathbf{v}^\epsilon\cdot\nabla(|\mathbf{u}^\epsilon|^2) + u_1^\epsilon u_2^\epsilon\theta_i'(y)$. Next we show that

$$\lim_{n\to\infty}\int_{\mathcal{T}(-s_n)}(-p^\epsilon u_1^\epsilon + \frac{\nu}{2}\frac{\partial|\mathbf{u}^\epsilon|^2}{\partial x} - \frac{1}{2}\rho^\epsilon v_1^\epsilon|\mathbf{u}^\epsilon|^2)\,dy = 0 \tag{25}$$

where (s_n) is a sequence tending to $-\infty$ such that

$$\lim_{n\to\infty}\int_{\mathcal{T}(-s_n)}(|\nabla p^\epsilon|^2 + |\nabla\mathbf{u}^\epsilon|^2)\,dy = 0;$$

such a sequence exists because ∇p^ϵ and $\nabla\mathbf{u}^\epsilon$ are in $L^2(\Omega_c^t)$. (Use (17), (18) and $\nabla p^\epsilon = \nu\Delta\mathbf{u}^\epsilon - \rho^\epsilon\mathbf{v}^\epsilon\cdot\nabla\mathbf{v}^\epsilon$ to see that $\nabla\mathbf{u}^\epsilon$ are in $L^2(\Omega_c^t)$.) Now, notice that the flux $\int_{\mathcal{T}(x)}u_1\,dy$ is zero for any $\mathbf{u}\in\mathbf{V}$. Indeed, first, it is clear that it is constant with respect to x because div$\mathbf{u}=0$ and $\mathbf{u}|\Gamma=0$; then it is zero because it is zero for any $\mathbf{u}\in\mathcal{V}$ (in this case, to see that, take $|x|$ sufficiently large such that $\mathcal{T}(x)$ does not intercept spt$\mathbf{u}$) and because $\mathcal{V}$ is dense in $\mathbf{V}$. Therefore, setting $p_n^\epsilon = \frac{1}{d_i}\int_{\mathcal{T}(-s_n)}p^\epsilon\,dy$, we have

$$|\int_{\mathcal{T}(-s_n)}p^\epsilon u_1^\epsilon\,dy| = |\int_{\mathcal{T}(-s_n)}(p^\epsilon - p_n^\epsilon)u_1^\epsilon\,dy| \le c||\nabla p^\epsilon||_{L^2(\mathcal{T}(-s_n))}\,||u_1^\epsilon||_{C_b(\Omega_t^c)},$$

so

$$\lim_{n\to\infty}\int_{\mathcal{T}(-s_n)}p^\epsilon u_1^\epsilon\,dy = 0. \tag{26}$$

From $|\int_{\mathcal{T}(-s_n)}\frac{\partial|\mathbf{u}^\epsilon|^2}{\partial x}\,dy| = |2\int_{\mathcal{T}(-s_n)}\mathbf{u}^\epsilon\cdot\frac{\partial\mathbf{u}^\epsilon}{\partial x}\,dy| \le c||\mathbf{u}^\epsilon||_{C_b(\Omega_t^c)}||\nabla\mathbf{u}^\epsilon||_{L^2(\mathcal{T}(-s_n))}$, we also have

$$\lim_{n\to\infty}\int_{\mathcal{T}(-s_n)}\frac{\partial|\mathbf{u}^\epsilon|^2}{\partial x}\,dy = 0. \tag{27}$$

Analogously,

$$\lim_{n\to\infty}\int_{\mathcal{T}(-s_n)}\rho^\epsilon v_1^\epsilon|\mathbf{u}^\epsilon|^2\,dy = 0. \tag{28}$$

From (26) to (28), we have (25). Then, taking $s = s_n$ and letting $n \to \infty$ in (24), we obtain

$$\begin{aligned} \nu \mathcal{H}_\epsilon(x) &= \int_{\mathcal{T}(x)} (-p^\epsilon u_1^\epsilon + \frac{\nu}{2}\frac{\partial |\mathbf{u}^\epsilon|^2}{\partial x} - \frac{1}{2}\rho^\epsilon v_1^\epsilon |\mathbf{u}^\epsilon|^2)\, dy \\ &\quad - \int_{-\infty}^{x}\int_{\Sigma_i} \rho^\epsilon u_1^\epsilon u_2^\epsilon \theta_i'(y)\, dy dx', \end{aligned} \tag{29}$$

where

$$\mathcal{H}_\epsilon(x) := \int_{-\infty}^{x}\int_{\Sigma_i} |\nabla \mathbf{u}^\epsilon|^2. \tag{30}$$

Next, we shall show that

$$a\int_{-\infty}^{x} \mathcal{H}_\epsilon(x')\, dx' \le \mathcal{H}_\epsilon{}'(x) + b\mathcal{H}_\epsilon(x), \tag{31}$$

for all $x << -1$ with constants $a, b > 0$. Then, by [3, Lemma VI.2.2], we will have $\mathcal{H}_\epsilon(x) \le k\mathcal{H}_\epsilon(0)e^{\sigma x}$ where $\sigma = (\sqrt{b^2+4a} - b)/2$ and $k = \sqrt{b^2+4a}/\sigma$. We will find below such a and b independent of ϵ. Notice also that from (17) we have that $\mathcal{H}_\epsilon(0)$ is bounded from above by a constant independent of ϵ. It is easy to see that

$$|\int_{-\infty}^{x}\int_{\Sigma_i} \rho^\epsilon u_1^\epsilon u_2^\epsilon \theta_i'(y)\, dy dx'| \le \frac{3\alpha}{2d_i^2} l \int_{-\infty}^{x}\int_{\Sigma_i} |\mathbf{u}^\epsilon|^2 \le 3\alpha l \mathcal{H}_\epsilon(x),$$

then from (29) we have

$$(\nu - 3\alpha l)\mathcal{H}_\epsilon(x) \le \int_{\mathcal{T}(x)} (-p^\epsilon u_1^\epsilon + \frac{\nu}{2}\frac{\partial |\mathbf{u}^\epsilon|^2}{\partial x} - \frac{1}{2}\rho^\epsilon v_1^\epsilon |\mathbf{u}^\epsilon|^2)\, . \tag{32}$$

From (32) and using that $\lim_{x'\to\infty} \int_{\mathcal{T}(x')} |\mathbf{u}^\epsilon|^2 = 0$, we get the estimate

$$(\nu - 3\alpha l)\int_{-\infty}^{x} \mathcal{H}_\epsilon(x')\, dx' \le \frac{\nu}{2}\int_{\mathcal{T}(x)} |\mathbf{u}^\epsilon|^2 dy - \int_{-\infty}^{x}\int_{\Sigma_i} (p^\epsilon u_1^\epsilon + \frac{1}{2}\rho^\epsilon v_1^\epsilon |\mathbf{u}^\epsilon|^2)\, dy dx'\, . \tag{33}$$

From Poincaré inequality and (30) we have

$$\frac{\nu}{2}\int_{\mathcal{T}(x)} |\mathbf{u}^\epsilon|^2 dy \le \nu d_i^2 \int_{\mathcal{T}(x)} |\nabla \mathbf{u}^\epsilon|^2 dy = \nu d_i^2 \mathcal{H}_\epsilon'(x)\, . \tag{34}$$

It is not difficult to see that we have the estimate

$$\begin{aligned} |\int_{-\infty}^{x}\int_{\Sigma_i} \tfrac{1}{2}\rho^\epsilon v_1^\epsilon |\mathbf{u}^\epsilon|^2 &\le \tfrac{1}{2} l (\int_{-\infty}^{x}\int_{\Sigma_i} |\mathbf{u}^\epsilon|^4 + \int_{-\infty}^{x}\int_{\Sigma_i} |\theta_i(y)||\mathbf{u}^\epsilon|^2) \\ &\le \tfrac{1}{2} l c (||\nabla \mathbf{u}^\epsilon||_{L^2(\Omega)} + ||\theta_i||_{C_b(\Sigma_i)})\mathcal{H}_\epsilon(x) \\ &\le b_1 \mathcal{H}_\epsilon(x) \end{aligned} \tag{35}$$

for some constant b_1 independent of ϵ. Now, for estimating the term $\int_{-\infty}^{x}\int_{\Sigma_i} p^\epsilon u_1^\epsilon$ occurring in (33), we first write $u_1^\epsilon = \nabla \cdot \mathbf{w}$ for some $\mathbf{w}$ in $\Omega^k := (x-k-1, x-k) \times \Sigma_i$, $k \in \{0, 1, 2, \cdots\}$, such that $\mathbf{w} \in W_0^{1,2}(\Omega^k)$ and

$$||\nabla \mathbf{w}||_{L^2(\Omega^k)} \le c||u_1^\epsilon||_{L^2(\Omega^k)} \tag{36}$$

for some constant c independent of ϵ and k; see [7, (2.7)] or [3, Lemma III.3.1]. Then, writing $\int_{-\infty}^{x}\int_{\Sigma_i} = \sum_{k=0}^{\infty}\int_{\Omega^k}$, integrating by parts and using the equation $\nu\Delta\mathbf{v}^\epsilon = \rho^\epsilon(\mathbf{v}^\epsilon\cdot\nabla)\mathbf{v}^\epsilon + \nabla p^\epsilon$, we obtain

$$\begin{aligned}\int_{-\infty}^{x}\int_{\Sigma_i} p^\epsilon u_1^\epsilon &= \textstyle\sum_{k=0}^{\infty}\displaystyle\int_{\Omega^k} p^\epsilon\nabla\cdot\mathbf{w} = \textstyle\sum_{k=0}^{\infty}\displaystyle\int_{\Omega^k}\nabla p^\epsilon\cdot\mathbf{w}\\ &= \textstyle\sum_{k=0}^{\infty}\displaystyle\int_{\Omega^k}(\nu\Delta\mathbf{u}^\epsilon - \rho^\epsilon\mathbf{v}^\epsilon\cdot\nabla\mathbf{v}^\epsilon)\cdot\mathbf{w}\\ &= \textstyle\sum_{k=0}^{\infty}\displaystyle\int_{\Omega^k}(-\nu\nabla\mathbf{u}^\epsilon\cdot\nabla\mathbf{w} - \rho^\epsilon(\mathbf{v}^\epsilon\cdot\nabla\mathbf{v}^\epsilon)\cdot\mathbf{w})\,;\end{aligned}$$

next, by (36) and Hölder and Poincaré's inequalities, we have

$$\begin{aligned}|\int_{\Omega^k}\nabla\mathbf{u}^\epsilon\cdot\nabla\mathbf{w}| &\le |\nabla\mathbf{u}^\epsilon|_{L^2(\Omega^k)}|\nabla\mathbf{w}|_{L^2(\Omega^k)}\\ &\le c|\nabla\mathbf{u}^\epsilon|_{L^2(\Omega^k)}|u_1^\epsilon|_{L^2(\Omega^k)}\\ &\le c^2|\nabla\mathbf{u}^\epsilon|^2_{L^2(\Omega^k)};\end{aligned}$$

similarly,

$$\begin{aligned}|\int_{\Omega^k}\rho^\epsilon(\mathbf{v}^\epsilon\cdot\nabla\mathbf{v}^\epsilon)\cdot\mathbf{w}| &= |\int_{\Omega^k}\rho^\epsilon(\mathbf{u}^\epsilon\cdot\nabla\mathbf{a})\cdot\mathbf{w}\\ &\quad+\int_{\Omega^k}\rho^\epsilon(\mathbf{a}\cdot\nabla\mathbf{u}^\epsilon)\cdot\mathbf{w} + \int_{\Omega^k}\rho^\epsilon(\mathbf{u}^\epsilon\cdot\nabla\mathbf{u}^\epsilon\cdot\mathbf{w})|\\ &\le lc(|\nabla\mathbf{u}^\epsilon|^2_{L^2(\Omega^k)} + |\nabla\mathbf{u}^\epsilon|_{L^2(\Omega^k)}|\mathbf{u}^\epsilon|_{L^4(\Omega^k)}|\mathbf{w}|_{L^4(\Omega^k)})\\ &\le lc(1+|\nabla\mathbf{u}^\epsilon|_{L^2(\Omega)})|\nabla\mathbf{u}^\epsilon|^2_{L^2(\Omega^k)}.\end{aligned}$$

Then, using (17) to bound $|\nabla\mathbf{u}^\epsilon|_{L^2(\Omega)}$, we have

$$|\int_{-\infty}^{x}\int_{\Sigma_i} p^\epsilon u_1^\epsilon| \le b_2\mathcal{H}_\epsilon(x) \tag{37}$$

for some constant b_2 independent of ϵ. Therefore, from (33) to (37), we conclude (31) with $a = (\nu - 3\alpha l)/(\nu d_i^2)$ and $b = (b_1 + b_2)/(\nu d_i^2)$. □

Then, from (22) and (23) it follows that

$$\lim_{\substack{x_0\to-\infty\\(x_0<0\ a.e.)}}\int_{\Sigma_i}\rho(x_0,y)\theta_i(y)\xi(y)\,dy = \int_{\Sigma_i}\rho_i(y)\theta_i(y)\xi(y)\,dy, \tag{38}$$

where we used also that $\lim_{x_0\to-\infty} u_1(x_0,y) = 0$ uniformly with respect to y; see (7) and (9). Since $\theta_i > 0$, from (38) we have (8). Indeed, to show (8) it is enough to show that $\int_{\Sigma_i}\rho(x_0,y)\xi(y)\,dy$ tends to $\int_{\Sigma_i}\rho_i(y)\xi(y)\,dy$ as $x_0\to-\infty$ for any characteristic function ξ of some interval contained in Σ_i.

Finally, ρ satisfies $0\le\rho\le l$ a.e. because $\rho^\epsilon = \omega^\epsilon(\psi^\epsilon)$ satisfies $0\le\rho^\epsilon(x,y)\le l$ for every $(x,y)\in\Omega$, due to the construction of ω^ϵ (see (10)) and the fact that weak-$*$ limits preserve these inequalities. □

We have shown the existence of a solution for the systems of equations (1), coupled with the boundary conditions (2) and (3), in the domain Ω. Since we allow distinct densities ρ_1 and ρ_2 in the incoming channels Ω_1 and Ω_2, a very interesting question we leave it open is the study of the transition set inside the fluid where the fluid density ρ change its values from ρ_1 to ρ_2.

Acknowledgment. We thank Prof. Milton C. Lopes Filho for his important remarks concerning the conclusions of this work.

REFERENCES

[1] C.J. Amick, *Steady solutions of the Navier-Stokes equations in unbounded channels and pipes*, Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4), 4:3 (1977), pp. 473–513.

[2] N.N. Frolov, *Solvability of a boundary problem of motion of an inhomogeneous fluid*, Mat. Zametki, 53:06 (1993), pp. 130–140.

[3] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, v. I, Springer-Verlag (1994).

[4] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, v. II, Springer-Verlag (1994).

[5] F. Ammar-Khodja and M.M. Santos, *2d density-dependent Leray problem for inhomogeneous fluids*, Mat. Contemp., to appear.

[6] O.A. Ladyzhenskaya, *Stationary motion of a viscous incompressible fluid in a pipe*, Dokl. Akad. Nauk. SSSR, 124 (1959), pp. 551–553.

[7] O.A. Ladyzhenskaya and V.A. Solonnikov, *Determination of solutions of boundary value problems for steady-state Stokes and Navier-Stokes equations in domains having an unbounded Dirichlet integral*, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI), 96, pp. 117–160; English Transl.: J. Soviet Math., 21 (1983), pp. 728–761.

[8] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Clarendon Press-Oxford (1996).

[9] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag (1984).

[10] M.M. Santos, *Stationary solution of the Navier-Stokes equations in a 2d bounded domain for incompressible flow with discontinuous density*, ZAMP-Z. angew. Math. Phys., 53 (2002), pp. 661–675.

[11] R. Temam, *Navier-Stokes equations. Theory and Numerical Analysis.* Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford (1979)