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THE POINTWISE ESTIMATES OF SOLUTIONS FOR GENERAL NAVIER-STOKES SYSTEMS IN ODD MULTI-DIMENSIONS*

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In honor of the 70th birthday of Professor Joel Smoller

Abstract. This work is a continuation of [6]. We study the time-asymptotic behavior of solutions to the general Navier-Stokes equations in odd multi-dimensions. Through the pointwise estimates of the Green's function of the linearized system, we obtain explicit expressions of the time-asymptotic behavior of the solutions and exhibit the weak Huygen's principle.

Key words. compressible flow, conservation laws, general Navier-Stokes equation, multi-dimensions, Green's function, pointwise estimate

AMS subject classifications. 35L65, 76N15

1. Introduction. In this paper we derive a detailed description of the asymptotic behavior of solutions to the Cauchy problem for the general Navier-Stokes systems of conservation laws in several space dimensions

$$\begin{cases} \rho_t + \text{div } (\rho v) = 0 \\ (\rho v^j)_t + \text{div } (\rho v^j v) + P(\rho, e)_{x_j} = \varepsilon \triangle v^j + \eta \text{div} v_{x_j}, (j = 1, \cdots, n), \\ (\rho E)_t + \text{div}(\rho E v + P(\rho, e) v) = \triangle(\kappa T(e) + \frac{1}{2}\varepsilon |v|^2) \\ \quad + \varepsilon \text{div}((\nabla v) v) + (\eta - \varepsilon) \text{div}((\text{div} v) v), \end{cases} \tag{1.1}$$

with smooth initial data close to a constant state. Here $\rho(x,t), v(x,t), e(x,t), P = P(\rho, e)$ and $T(e)$ represent, respectively, the fluid density, velocity, specific internal energy, pressure and normalized temperature, and $E = e + \frac{1}{2}|v|^2$ is the specific total energy, $\kappa > 0$ is the heat conductivity, $\varepsilon > 0$ and $\eta \geq 0$ are viscosity constants, and div and $\triangle$ are the usual spatial divergence and Laplace operator. We assume throughout that the pressure $P(\rho, e)$ and the temperature $T(e)$ are smooth in a neighborhood of (ρ^*, e^*) and $p_\rho = P_\rho(\rho^*, e^*) > 0, p_e = P_e(\rho^*, e^*) > 0, p = P(\rho^*, e^*)$, and $d^2 = \kappa T'(\rho^*) > 0$.

We are interested in the time-asymptotic behavior of solutions. For the scalar space variables, Liu and Zeng (see [9]) studied general hyperbolic-parabolic systems and obtain L^p decay rates by integrating pointwise bounds. In several space variables, Kawashim [5] studied general hyperbolic-parabolic systems, and obtained L^2-estimates. The pointwise bounds for isentropic Navier-Stokes equations with effective artificial viscosity were obtained in [3]. The asymptotic behavior of solutions to the Cauchy problem for isentropic Navier-Stokes equations has been studied in [2], [6], [14] and [13]. Moreover, pointwise estimates have been obtained in [6] for the odd dimensions, and in [13] and [14] for the even dimensions. This paper is a continuation of [6]. In this paper we will give a pointwise estimate for the time-asymptotic behavior of solutions to the general Navier-Stokes equations. Here, the main difficulty is that we can't give the explicit expression for the Fourier transform of the Green's function to linearized systems of (1.1) as in [6]. Our analysis is based on the estimates for

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the Green's function of linearized systems about the constant state (ρ^*, v^*, e^*), particularly for small and large Fourier variables. For this we introduce a new set-up in next section which allows us to get the estimate without the matrix for the Fourier transform of the Green's function.

It is necessary to mention that Liu and Yu have carried out pointwise estimates for the Boltzmann equations by using Green's functions (see [7] and [8]), which is similar in some features to our present work (also see [6]), but there are the major differences in the technique because of the richer wave structures of the Boltzmann equations.

This paper is organized as follows. In section 2, we explore some important properties of the Fourier transform of the linearized system. In section 3, we list some lemmas which will be used later. In section 4, we get the estimate of Green's function which based on the analysis of section 2. In section 5 we state and prove our main theorem, we obtain the time-asymptotic behavior of solutions of (1.1).

Throughout this paper we denote the generic constant by C. For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)(\alpha_j \geq 0)$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, where $D_{x_j}^{\alpha_j} = (-\sqrt{-1})^{\alpha_j} \partial_{x_j}^{\alpha_j} = (-\sqrt{-1})^{\alpha_j} \frac{\partial^{\alpha_j}}{\partial_{x_j}}$.

2. Linearized system.

The linearized system of (1.1) about the constant state $(\rho^*, v^*, e^*)^\tau = (1, 0, e^*)^\tau$, $(e^* > 0)$ is

$$\begin{cases} \rho_t + \mathrm{div}\ v = 0 \\ v_t + p_\rho \nabla \rho + p_e \nabla e = \varepsilon \triangle v + \eta \nabla \mathrm{div}\ v \\ e_t + p \mathrm{div} v = d^2 \triangle e. \end{cases} \tag{2.1}$$

Let

$$l(\tau, \xi) = \sqrt{-1}\tau I + \sqrt{-1}A(\xi) + |\xi| B(\xi), \tag{2.2}$$

where

$$A(\xi) = \begin{pmatrix} 0 & \xi & 0 \\ p_\rho \xi^\tau & 0 & p_e \xi^\tau \\ 0 & p\xi & 0 \end{pmatrix}, \quad B(\xi) = |\xi|^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon|\xi|^2 I + \eta \xi^\tau \xi & 0 \\ 0 & 0 & d^2|\xi|^2 \end{pmatrix}.$$

It is easy to see $l(\tau, \xi)$ is a symbol of the operator in system (2.1). We also write

$$l_{\alpha,\beta}(\tau, \xi) = \sqrt{-1}(\tau I + E_{\alpha,\beta}(\xi)),$$

where $E_{\alpha,\beta}(\xi) = \beta A(\xi) - \sqrt{-1}\alpha B(\xi)$.

The eigenvalues of $E_{\alpha,\beta} = \beta A - \sqrt{-1}\alpha B$ are

$$\begin{aligned}
\lambda_1^{\alpha,\beta} = \ & \Big(\sqrt[3]{-(G/2) + \sqrt{(G/2)^2 + (H/3)^3}} + \sqrt[3]{-(G/2) - \sqrt{(G/2)^2 + (H/3)^3}} \\
& - \tfrac{1}{3}\sqrt{-1}\alpha(\varepsilon + \eta + d^2) \Big) |\xi| \\
\lambda_2^{\alpha,\beta} = \ & \Big(\omega_+ \sqrt[3]{-(G/2) + \sqrt{(G/2)^2 + (H/3)^3}} + \omega_- \sqrt[3]{-(G/2) - \sqrt{(G/2)^2 + (H/3)^3}} \\
& - \tfrac{1}{3}\sqrt{-1}\alpha(\varepsilon + \eta + d^2) \Big) |\xi| \\
\lambda_3^{\alpha,\beta} = \ & \Big(\omega_- \sqrt[3]{-(G/2) + \sqrt{(G/2)^2 + (H/3)^3}} + \omega_+ \sqrt[3]{-(G/2) - \sqrt{(G/2)^2 + (H/3)^3}} \\
& - \tfrac{1}{3}\sqrt{-1}\alpha(\varepsilon + \eta + d^2) \Big) |\xi| \\
\lambda_4^{\alpha,\beta} = \ & -\sqrt{-1}\alpha\varepsilon|\xi|,
\end{aligned}$$

where $\omega_\pm = (-1 \pm \sqrt{-3})/2$, and

$$H = (\frac{\alpha^2}{3}((\varepsilon+\eta)^2 - d^2(\varepsilon+\eta) + d^4) - c^2\beta^2),$$

$$G = -\sqrt{-1}(\frac{\alpha^3}{27}(2(\varepsilon+\eta)^3 - 3(\varepsilon+\eta)^2 d^2 - 3(\varepsilon+\eta)d^4 + 2d^6) + \frac{\alpha\beta^2}{3}(3d^2 p_\rho - c^2(\varepsilon+\eta+d^2))).$$

Here $\lambda_1^{\alpha,\beta}(\xi), \lambda_2^{\alpha,\beta}(\xi), \lambda_3^{\alpha,\beta}(\xi), \lambda_4^{\alpha,\beta}(\xi)$ are with multiplicity $1,1,1,n-1$ respectively.

If denote $E(\xi) = E_{|\xi|,1} = |\xi| E_{1,1/|\xi|} = A(\xi) - \sqrt{-1}|\xi| B(\xi)$, the eigenvalues of $E(\xi)$ are $\tilde{\lambda}_j(\xi) = \lambda_j^{|\xi|,1}(\xi)$ or $\tilde{\lambda}_j(\xi) = |\xi|\lambda_j^{1,1/|\xi|}(\xi)$.

The following lemma is due to Shizuta and Kawashima (see Theorem 1.1 of [10]).

Lemma 2.1. *The following statements are equivalent*

(1) The system (2.1) is said to be dissipative;

(2) Any eigenvector of $A(\xi)$ is not in the null space of $B(\xi)$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$;

(3) There exists a constant $C > 0$, such that $\mathrm{Im}(\tilde{\lambda}_j(\xi)) \le -C\xi^2/(1+\xi^2)$, $j = 1,2,3,4$ *for real ξ.*

Lemma 2.2. *There exists a constant $C > 0$, such that* $\mathrm{Im}(\tilde{\lambda}_j(\xi)) \le -C|\xi|^2/(1+|\xi|^2)$, $j = 1,2,3,4$ *for real ξ.*

Proof. If $X = (x_0, x_1, \cdots, x_n, x_{n+1})^\tau$ is in the null space of $B(\xi)$, since

$$\begin{pmatrix} \varepsilon|\xi|^2 I + \eta\xi^\tau\xi & 0 \\ 0 & d^2|\xi|^2 \end{pmatrix}$$

is a positive matrix, we know that $X = (x_0, 0, \cdots, 0, 0)^\tau$ and $x_0 \ne 0$. Then for any $\mu \in \mathbb{R}$,

$$A(\xi)X + \mu X = (\mu x_0, p_\rho \xi_1 x_0, \cdots, p_\rho \xi_n x_0, 0)^\tau \ne 0.$$

Thus, we get Lemma 2.2 from Lemma 2.1. $\square$

Now we consider the spectral representation of matrices $E_{\alpha,\beta}(\xi)$.

Let the left and right eigenvectors associated with $\lambda_j^{\alpha,\beta}$ be $l_{j,i}^{\alpha,\beta}$ and $r_{j,i}^{\alpha}(j = 1,2,3,4;\, i = 1,\cdots,m_j)$,

$$E_{\alpha,\beta} r_{j,i}^{\alpha,\beta} = \lambda_j^{\alpha,\beta} r_{j,i}^{\alpha,\beta},\ l_{j,i}^{\alpha,\beta} E_{\alpha,\beta} = \lambda_j^{\alpha,\beta} l_{j,i}^{\alpha,\beta},\ l_{j,i}^{\alpha,\beta} r_{j',i'}^{\alpha,\beta} = \delta_{j,j'}\delta_{i,i'}. \tag{2.4}$$

Set

$$\begin{aligned} l_j^{\alpha,\beta} = (l_{j,1}^{\alpha,\beta}, l_{j,2}^{\alpha,\beta}, \cdots, l_{j,m_j}^{\alpha,\beta})^\tau, &\quad r_j^{\alpha,\beta} = (r_{j,1}^{\alpha,\beta}, r_{j,2}^{\alpha,\beta}, \cdots, r_{j,m_j}^{\alpha,\beta}), \\ P_j^{\alpha,\beta} = r_j^{\alpha,\beta} l_j^{\alpha,\beta}, &\quad (j = 1,2,3,4), \\ L^{\alpha,\beta} = (l_1^{\alpha,\beta}, l_2^{\alpha,\beta}, \lambda_3^{\alpha,\beta}, l_4^{\alpha,\beta})^\tau, &\quad R^{\alpha,\beta} = (r_1^{\alpha,\beta}, r_2^{\alpha,\beta}, \gamma_3^{\alpha,\beta}, r_4^{\alpha,\beta}). \end{aligned} \tag{2.5}$$

Thus we have

$$E_{\alpha,\beta}(\xi) = \sum_{j=1}^4 \lambda_j^{\alpha,\beta}(\xi) P_j^{\alpha,\beta}(\xi) \tag{2.6}$$

$$P_j^{\alpha,\beta} P_k^{\alpha,\beta} = \delta_{j,k} P_j^{\alpha,\beta},\quad \sum_{j=1}^4 P_j^{\alpha,\beta} = I. \tag{2.7}$$

Here $P_j^{\alpha,\beta}$ is the eigenprojection of $E_{\alpha,\beta}(\xi)$. Taking $\alpha = |\xi|$ and $\beta = 1$ in (2.6), we have

$$E(\xi) = \sum_{j=1}^{4} \tilde{\lambda}_j(\xi)\tilde{P}_j(\xi) \tag{2.8}$$

where $\tilde{\lambda}_j = \lambda_j^{|\xi|,1}(\xi), \tilde{P}_j(\xi) = P_j^{|\xi|,1}(\xi)$.

On the other hand, taking $\alpha = 0$ and $\beta = 1$ in (2.6), we have

$$A(\xi) = \sum_{j=1}^{4} \lambda_j(\xi)P_j(\xi), \tag{2.9}$$

where $\lambda_j(\xi) = \lambda_j^{0,1}(\xi)$ are eigenvalues of $A(\xi)$, $P_j(\xi) = P_j^{0,1}(\xi)$ are eigenprojections of $A(\xi)$.

Denoting by $\breve{\lambda}_j(\xi)$ and $\breve{P}_j(\xi)$ are the eigenvalues and the eigenprojections of $|\xi|B(\xi)(\xi)$ respectively, and taking $\alpha = |\xi|, \beta = 0$ in (2.6), we get

$$|\xi|B(\xi) = \sum_{j=1}^{4} \breve{\lambda}_j(\xi)\breve{P}_j(\xi). \tag{2.10}$$

Here, $\breve{\lambda}_j(\xi) = \sqrt{-1}\lambda_j^{|\xi|,0}(\xi), \breve{P}_j(\xi) = P_j^{|\xi|,0}(\xi)$.

By the above expressions, we see easily that

$$\breve{\lambda}_j(\xi) > \nu|\xi|^2 (j \geq 2),\ \breve{\lambda}_1(\xi) = 0,\ \breve{P}_1(\xi) = diag(1, 0, \cdots, 0), \tag{2.11}$$

where ν is a positive constant.

Lemma 2.3. *For sufficiently small ε and $|\xi| < \varepsilon$, we have*

$$\tilde{\lambda}_k(\xi) = \lambda_k(\xi) + |\xi|\bar{\lambda}_k(\xi) + O(|\xi|^3),\ \bar{\lambda}_j(\xi) = (\partial_\alpha \lambda_j^{\alpha,1}(\xi))_{\alpha=0}, \tag{2.12}$$

$$\tilde{P}_k(\xi) = P_k(\xi) + O(|\xi|). \tag{2.13}$$

Proof. If $|\xi| \leq 1$, $|\alpha| < 2\varepsilon$, taking the Taylor expansion at $\alpha = 0$ for $\lambda_k^{\alpha,1}(\xi)$, we get

$$\lambda_k^{\alpha,1}(\xi) = \quad \lambda_k^{0,1}(\xi) + \alpha(\partial_\alpha \lambda_k^{\alpha,1}(\xi))_{\alpha=0} + R_k(\alpha, \xi), \tag{2.14}$$

where $R_k(\alpha, \xi) \sim \alpha^2$ and $R_k(\alpha, \xi)$ is 1-homogeneous in ξ. Taking $\alpha = |\xi|$ in (2.14), we get (2.12). Since $P_k^{\alpha,1}(\xi)$ is 0-homogeneous in $|\xi|$, by the similar proof as (2.12), we see that (2.13) is valid. $\square$

Lemma 2.4. *For R sufficiently large and $|\xi| > R$, we have*

$$\tilde{\lambda}_j(\xi) = -\sqrt{-1}\breve{\lambda}_j(\xi) + \breve{\nu}_j + O(|\xi|^{-1}), \tag{2.15}$$

$$\tilde{P}_j(\xi) = \breve{P}_j(\xi) + O(|\xi|^{-1}), \tag{2.16}$$

where $\breve{\lambda}_j(\xi)$ are eigenvalues of $|\xi|B(\xi)$ and $\breve{\nu}_j$ are constants.

Proof. For β sufficiently small and $\beta \leq R^{-1}$, taking the Taylor expansion at $\beta = 0$ for $\lambda_j^{1,\beta}(\xi)$ and $P_j^{1,\beta}(\xi)$, we get

$$\lambda_j^{1,\beta}(\xi) = \lambda_j^{1,0}(\xi) + \beta((\partial_\beta \lambda_j^{1,\beta}(\xi))_{\beta=0}) + \frac{1}{2}\beta^2(\partial_\beta^2 \lambda_j^{1,\beta}(\xi))_{\beta=0} + r_j(\beta,\xi) \tag{2.17}$$

$$P_j^{1,\beta}(\xi) = P_j^{1,0}(\xi) + R_j(\beta,\xi). \tag{2.18}$$

Since $E_{1,0}(\xi) = -\sqrt{-1}B(\xi)$, we know that $\lambda_j^{1,0}(\xi)$ are eigenvalues $-\sqrt{-1}B(\xi)$, thus $|\xi|\lambda_j^{1,0}(\xi) = -\sqrt{-1}\breve{\lambda}_j$. Moreover, we know that

$$|\xi|\lambda_j^{1,1/|\xi|}(\xi) = \tilde{\lambda}_j(\xi),\ |\xi|P_j^{1,1/|\xi|}(\xi) = \tilde{P}_j(\xi),\ |\xi|P_j^{1,0}(\xi) = \breve{P}_j(\xi).$$

Since $(\partial_\beta H)_{\beta=0} = 0, (\partial_\beta G)_{\beta=0} = 0$, multiplying $|\xi|$ and taking $\beta = |\xi|^{-1}$ in (2.17) and (2.18) respectively, we obtain (2.15) and (2.16). □

3. Some Lemma. The following lemma is the Kirchoff formulas for solutions of the standard wave equation (see [1], pp. 70-74, for example).

LEMMA 3.1. *Let $\hat{w} = (2\pi)^{-n/2}(\sin c|\xi|t)/(c|\xi|)$ and $\hat{w}_t = (2\pi)^{-n/2}\cos(c|\xi|t)$, then there are constants a_α, b_α, such that for function $f(x)$ and odd integer n*

$$w * f = \sum_{0\leq|\alpha|\leq(n-3)/2} a_\alpha t^{|\alpha|+1}\int_{|y|=1} D^\alpha f(x+cty)y^\alpha dS_y, \tag{3.1}$$

$$w_t * f = \sum_{0\leq|\alpha|\leq(n-1)/2} b_\alpha t^{|\alpha|}\int_{|y|=1} D^\alpha f(x+cty)y^\alpha dS_y, \tag{3.2}$$

The following lemmas were proved in [12], [2] and [6] (see Lemma 3.1 of [12], Section 4 of [2] and Section 2 of [6]).

LEMMA 3.2. *If $\hat{f}(\xi,t)$ has compact support in the variable ξ, and there exists a constant $b > 0$, such that $\hat{f}(\xi,t)$ satisfies*

$$|D_\xi^\beta(\xi^\alpha \hat{f}(\xi,t))| \leq C(|\xi|^{(|\alpha|+k-|\beta|)_+} + |\xi|^{|\alpha|+k}t^{|\beta|/2})(1+(t|\xi|^2))^m e^{-b|\xi|^2t/2},$$

for any multi-indexes α,β with $|\beta| \leq 2N$, then

$$|D_x^\alpha f(x,t)| \leq C_N t^{-\frac{n+|\alpha|+k}{2}} B_N(|x|,t), \tag{3.3}$$

where m and k are any fixed integers, $(a)_+ = \max(0,a)$ and

$$B_N(|x|,t) = \Big(1 + \frac{|x|^2}{1+t}\Big)^{-N}. \tag{3.4}$$

LEMMA 3.3. *(1) For $2N > n$, we have*

$$I = \Big|\int_{|y|=1} B_{2N}(|x+cty|,t)y^\alpha dS_y\Big| \leq Ct^{-(n-1)/2}B_N(|x|-ct,t). \tag{3.5}$$

(2) Let $\tilde{E}^\mu(x,t) = e^{-|x|^2/\mu t}$, *then*

$$I = \Big| \int_{|y|=1} \tilde{E}^\mu(x+cty,t) y^\alpha dS_y \Big| \le Ct^{-(n-1)/2} e^{-(|x|-ct)^2/3\mu t}. \tag{3.6}$$

LEMMA 3.4. *If* $|y| \le M, t > 4M^2, p \ge 0$, *we have*

$$(1 + (|x-y| - pt)^2/t)^{-N} \le C_N (1 + (|x| - pt)^2/t)^{-N}. \tag{3.7}$$

LEMMA 3.5. *Assume that* $\text{supp}\hat{f}(\xi) \subset O_R = \{\xi, |\xi| > R\}$, $\hat{f}(\xi) \in L^\infty \cap C^{n+1}(O_R)$ *and*

$$|\hat{f}(\xi)| \le C_0, \quad |D_\xi^\alpha(\hat{f}(\xi))| \le C_0 |\xi|^{-|\alpha|-1}, \ (|\alpha| \ge 1) \tag{3.8}$$

then $f(x) \in L^1$ *and* $\|f\|_{L^1} \le CC_0$ *for some constant* C *depending only on* n.

LEMMA 3.6. *Assume that* $\text{supp}\hat{g}(\xi,t) \subset O_R = \{\xi, |\xi| > R\} \times (0,\infty)$, $\hat{f}(\xi,t) \in (L^\infty \cap C^{n+1}(O_R)) \times L^\infty$ *and*

$$|D_\xi^\beta \hat{g}(\xi,t)| \le C_0 |\xi|^{-|\beta|} e^{-\theta|\xi|^2 t}, \ (|\alpha| \ge 1) \tag{3.9}$$

then for all α *and all* $p \in [1,\infty]$,

$$\|D_x^\alpha G(\cdot,t)\|_{L^p} \le Ct^{-\frac{n}{2}(1-\frac{1}{p}-|\alpha|)}. \tag{3.10}$$

4. Pointwise bounds for Green's function. In this section, we will study the Green's function for (2.1), i.e. , we consider the solution matrix for following initial value problem:

$$\begin{cases} (\partial_t + \tilde{A}(D_x) + \tilde{B}(D_x))G(x,t) = 0, \\ G(x,0) = \delta(x)I, \end{cases} \tag{4.1}$$

where the symbols of $\tilde{A}(D_x)$ and $\tilde{B}(D_x)$ are $\tilde{A}(\xi) = \sqrt{-1}A(\xi), \tilde{B}(\xi) = |\xi|B(\xi)$, $\delta(x)$ is the Dirac function and I is an $N \times N$ identity matrix.

As usual, we apply the Fourier transform to the variable x

$$\hat{f}(\xi,t) = \int_{\mathbb{R}^n} f(x,t) e^{-\sqrt{-1}x\xi} dx. \tag{4.2}$$

From (4.1), we deduce that

$$\begin{cases} \hat{G}_t(\xi,t) = -\sqrt{-1}E(\xi)\hat{G}(\xi,t), \\ \hat{G}(\xi,0) = I, \end{cases} \tag{4.3}$$

where $E(\xi) = A(\xi) - \sqrt{-1}B(\xi)|\xi|$.

From (4.3), we have

$$\hat{G}(\xi,t) = \sum_{j=1}^{4} e^{-\sqrt{-1}\tilde{\lambda}_j(\xi)t} \tilde{P}_j(\xi). \tag{4.4}$$

By direct calculation, we know that

$$\begin{aligned}\hat{G}(\xi,t) = &\ (e^{-\sqrt{-1}\tilde{\lambda}_1(\xi)t}\tilde{P}_1 + e^{-\sqrt{-1}\tilde{\lambda}_4(\xi)t}\tilde{P}_4) \\ &+ \cos(c|\xi|t)(e^{-\sqrt{-1}(\tilde{\lambda}_2(\xi)-c|\xi|)t}\tilde{P}_2 + e^{-\sqrt{-1}(\tilde{\lambda}_3(\xi)+c|\xi|)t}\tilde{P}_3) \\ &+ \tfrac{\sin(c|\xi|t)}{c|\xi|}(\sqrt{-1}c|\xi|)(e^{-\sqrt{-1}(\tilde{\lambda}_2(\xi)-c|\xi|)t}\tilde{P}_2 + e^{-\sqrt{-1}(\tilde{\lambda}_3(\xi)+c|\xi|)t}\tilde{P}_3) \\ \equiv &\ F_1(\xi,t) + \hat{W}_t(\xi,t)F_2(\xi,t) + \hat{W}(\xi,t)F_3(\xi,t).\end{aligned} \tag{4.5}$$

At some time, we also denote

$$\begin{aligned}\hat{G}(\xi,t) = &\ (e^{-\sqrt{-1}\tilde{\lambda}_1(\xi)t}\tilde{P}_1) + (\textstyle\sum_{j=2}^4 e^{-\sqrt{-1}\tilde{\lambda}_j(\xi)t}\tilde{P}_j) \\ \equiv &\ \hat{G}^+(\xi,t) + \hat{G}^-(\xi,t).\end{aligned} \tag{4.6}$$

LEMMA 4.1. *If $|\xi|$ small enough, there exist a constant $b > 0$, such that*

$$|D_\xi^\beta(\xi^\alpha F_j(\xi,t))| \le C(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|}t^{|\beta|/2})(1+t|\xi|^2)^{|\beta|+1}e^{-b|\xi|^2t}. \tag{4.7}$$

Proof. For $|\xi|$ small enough, (2.12) gives

$$\tilde{\lambda}_j(\xi) = \lambda_j(\xi) + |\xi|\bar{\lambda}_j(\xi) + O(|\xi|^3).$$

By direct calculation, we know that

$$\lambda_1 = 0,\ \lambda_2 = c|\xi|,\ \lambda_3 = -c|\xi|, \lambda_4 = 0.$$

Let $\bar{\mu}_j = \sqrt{-1}|\xi|\bar{\lambda}_j(\xi)t$, we have

$$\begin{aligned}F_1(\xi,t) = &\ e^{-\bar{\mu}_1}(e^{O(|\xi|^3)t}\tilde{P}_1) + e^{-\bar{\mu}_4}(e^{O(|\xi|^3)t}\tilde{P}_4) \\ = &\ e^{-\bar{\mu}_1}(P_1 + O(|\xi|^3t) + O(|\xi|)) + e^{-\bar{\mu}_4}(P_4 + O(|\xi|^3t) + O(|\xi|)).\end{aligned}$$

By partial differentiation on both sides of above formula, we obtain (4.7) for $j = 1$.

For $j = 2$, we first write,

$$F_2(\xi,t) = (e^{-\tau_2t}\tilde{P}_2 - e^{-\tau_3t}\tilde{P}_3),$$

where $\tau_2 = \sqrt{-1}(\tilde{\lambda}_2(\xi) - c|\xi|), \tau_3 = \sqrt{-1}(\tilde{\lambda}_3(\xi) + c|\xi|)$. Since

$$e^{-\tau_jt}\tilde{P}_j = e^{-\bar{\mu}_j}e^{O(|\xi|^3)t}(P_j - O(|\xi|)) = e^{-\bar{\mu}_j}(O(|\xi|^3)t + O(|\xi|)),$$

we have

$$F_2(\xi,t) = (e^{-\bar{\mu}_2} + e^{-\bar{\mu}_3})(O(|\xi|^3)t + O(|\xi|).$$

Again by partial differentiation, we obtain (4.7) from above formula for $j = 2$. The proof of the case of $j = 3$ is the same, and we omit it. □

Let

$$\chi_1(\xi) = \begin{cases} 1, & |\xi| < \epsilon \\ 0, & |\xi| > 2\epsilon, \end{cases} \qquad \chi_3(\xi) = \begin{cases} 1, & |\xi| > R+1 \\ 0, & |\xi| < R, \end{cases}$$

be cut-off functions, where $2\epsilon < R$. Set $\chi_2 = 1 - \chi_1 - \chi_3$ and

$$\hat{F}_{j,i}(\xi,t) = \chi_i\hat{F}_j(\xi,t), \quad (j = 1,2,3; i = 1,2,3).$$

The decay property is related to the behavior for $|\xi| < \epsilon$.

LEMMA 4.2. *For sufficiently small* ϵ,

$$|D_x^\alpha F_{j,1}| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_N(|x|,t). \tag{4.8}$$

Proof. We just need to prove the case of $j=1$ for (4.8), since the proofs of the others are very similar. First, we have

$$|D_\xi^\beta(\chi_1 \xi^\alpha(\hat{F}_j))| \le \sum_{|\beta_1|+|\beta_2|=|\beta|} |D_\xi^{\beta_1}\chi_1 D_\xi^{\beta_2}(\xi^\alpha(\hat{F}_j))|.$$

Since $|D_\xi^{\beta_1}\chi_1| \le C$ and $|\xi|^{-|\beta_2|} \le |\xi|^{-|\beta|}$, by (4.7), we also have

$$|D_\xi^\beta(\chi_1\xi^\alpha(\hat{F}_j)| \le C(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|}t^{|\beta|/2})(1+(|\xi|^2 t))^{|\beta|+1} e^{-b|\xi|^2 t/2}.$$

Using Lemma 3.2, we have

$$|D_x^\alpha(F_{j,1})| \le Ct^{-\frac{n+|\alpha|}{2}} B_N(|x|,t).$$

On the other hand

$$|D_x^\alpha(F_{j,1})| \le C|\int e^{\sqrt{-1}x\xi}\chi_1(\xi)(\xi^\alpha(\hat{F}_j))d\xi| \le C\int \chi_1(\xi)d\xi \le C.$$

Thus, we get (4.8). ∎

Letting $\hat{G}_j = \chi_j(\xi)\hat{G}(\xi,t)$ and $\hat{G}_j^\pm = \chi_j(\xi)\hat{G}^\pm$, we have

$$\hat{G} = \hat{G}_1 + +\hat{G}_2 + \hat{G}_3^- + \hat{G}_3^+,$$

or

$$G = G_1 + G_2 + G_3^+ + G_3^-.$$

PROPOSITION 4.1. *For sufficiently small* ϵ, *there exist positive constants* C, *such that*

$$|D_x^\alpha G_1 x,t)| \le C(1+t)^{-(n+|\alpha|)/2}(B_N(|x|,t) + t^{-(n-1)/4}B_N(|x|-ct,t)). \tag{4.9}$$

Proof. By Lemma 4.2, we know that

$$|D_x^\alpha F_{j,1}(x,t)| \le C(1+t)^{-(n+|\alpha|)/2} B_{3N}(|x|,t). \tag{4.10}$$

By (3.2)

$$|D_x^\alpha W_t * F_{2,1}| = C\sum_{|\gamma|\le(n-1)/2} a_\gamma t^{|\gamma|}\int_{|y|=1}(1+t)^{-(n+|\alpha|)/2}t^{|\gamma|/2}B_{2N}(|x+cty|,t)dS_y.$$

By Lemma 3.3

$$|D_x^\alpha W_t * F_{2,1}| \le C\sum_{|\gamma|\le(n-1)/2} t^{|\gamma|/2}(1+t)^{-(n+|\alpha|)/2}t^{-(n-1)/2}B_N(|x|-ct,t). \tag{4.11}$$

For the same reason, we have from (3.1) and Lemma 3.3

$$|D_x^\alpha W * F_{3,1}| \le C \sum_{|\gamma|\le(n-3)/2} t^{(|\gamma|+1)/2}(1+t)^{-(n+|\alpha|)/2} t^{-(n-1)/2} B_N(|x|-ct,t). \quad (4.12)$$

Thus, summing up (4.5), (4.10)—(4.12), we obtain (4.9). ◻

Proposition 4.2. *For fixed ϵ and R, there exist positive constants b and C, such that*

$$|D_x^\alpha G_2(x,t)| \le C(1+t)^{-(n+|\alpha|)/2} e^{-bt} B_N(|x|,t). \quad (4.13)$$

Proof. Note that if $|\xi| \in (\epsilon, R+1)$, we have $\mathrm{Re}\lambda_\pm \le -2\theta|\xi|^2$ for some positive constant θ. By Theorem 3.2 of [2], it is easy to see that

$$|D_\xi^{2\beta}\xi^\alpha \hat{G}_2| \le C(1+|\xi|)^{|\alpha|}(1+t|\xi|)^{2|\beta|}e^{-2\theta|\xi|^2 t} \le C(1+|\xi|)^{|\alpha|-2|\beta|}(1+t|\xi|^2)^{2|\beta|}e^{-2bt}e^{-\theta|\xi|^2 t}.$$

Then

$$\begin{aligned} |x^{2\beta} D_x^\alpha G_2(x,t)| = &\quad C|\textstyle\int_{\mathbb{R}^n} e^{\sqrt{-1}x\xi} D_\xi^{2\beta}\xi^\alpha \hat{G}_2(\xi,t)d\xi| \\ \le &\quad Ce^{-2b}\textstyle\int_{\mathbb{R}^n}(1+|\xi|)^{|\alpha|-|2\beta|}(1+t|\xi|^2)^{2|\beta|}e^{-\theta|\xi|^2 t}d\xi \\ \le &\quad Ct^{-(n+|\alpha|)/2}e^{-bt}(1+t)^{|\beta|}. \end{aligned}$$

Taking $|\beta| = 0$ if $|x|^2 \le 1+t$ and $|\beta| = N$ if $|x|^2 > 1+t$, and noting $\mathrm{supp}\hat{G}_2(\xi,\cdot) \subset [\epsilon, R]$ and

$$\left(1+\frac{|x|^2}{1+t}\right) \le 2\begin{cases} 1, & |x|^2 \le 1+t, \\ |x|^2/(1+t), & |x|^2 > 1+t, \end{cases}$$

we know that (4.13) is valid. ◻

Now we consider $G_3^\pm$ for sufficiently large $|\xi|$. First, for G_3^- and G_3^*, we have

Proposition 4.3. *For sufficiently large R, there exist positive constants b and C, such that*

$$|D_x^\alpha G_3^-(x,t)| \le Ct^{-(n+|\alpha|)/2}e^{-bt}B_N(|x|,t). \quad (4.14)$$

Proof. Since $e^{-\sqrt{-1}\tilde{\lambda}_j(\xi)t} \le Ce^{-\nu|\xi|^2 t}(j \ge 2)$ from (2.11) and (2.15), by the definition of $\hat{G}^-$, we have

$$|D_\xi^{2\beta}\xi^\alpha \hat{G}_3^-| \le C(1+|\xi|)^{|\alpha|-2|\beta|}e^{-bt}(1+t|\xi|^2)^{2|\beta|}e^{-\theta|\xi|^2 t}.$$

Here θ is a positive constant. Using the same method as in the proofs of Proposition 3.2, we can prove (4.14) for G_3^-. ◻

For G_3^+, we first take the Taylor expansion for $\lambda_1^{1,\beta}(\xi)$ as (2.17),

$$\lambda_1^{1,\beta}(\xi) = \lambda_1^{1,0}(\xi) + \beta(\partial_\beta\lambda_1^{1,\beta}(\xi))_{\beta=0} + \cdots + \frac{1}{m!}\beta^m(\partial_\beta^m\lambda_1^{1,\beta}(\xi))_{\beta=0} + r(\beta,\xi), \quad (4.15)$$

where $r(\beta,\xi)=O(\beta^{m+1})$ is 1-homogeneous in ξ. Multiplying $|\xi|$ and taking $\beta=|\xi|^{-1}$ in (4.24), we get from (2.11) and (2.15)

$$-\sqrt{-1}\tilde{\lambda}_1(\xi)=-\sqrt{-1}\breve{\nu}_1+\sum_{j=1}^{m}a_j(\xi)|\xi|^{-j}+O(|\xi|^{-(m+1)}),$$

where $a_j(\xi)$ is 0-homogeneous in ξ.

By Lemma 2.1 we know that $\mathrm{Im}(\tilde{\lambda}_1(\xi))\le -c|\xi|^2/(1+|\xi|^2)$. So $\mathrm{Im}(\breve{\nu}_1)<0$. If $\breve{\nu}=a-\sqrt{-1}b$, a,b are two real constants, and $b>0$. Thus we can write

$$e^{-\sqrt{-1}\tilde{\lambda}_1(\xi)t}=e^{-bt}e^{-\sqrt{-1}at}\big(1+(\sum_{j=1}^{m}a_j|\xi|^{-j})t+\cdots+\frac{1}{m!}(\sum_{j=1}^{m}a_j|\xi|^{-j})^m t^m+R_1(t,\xi)\big),$$

where $R_1(t,\xi)\le C(1+t)^{m+1}(1+|\xi|)^{-(m+1)}$. Then we have

$$\hat{G}^+(\xi,t)=e^{\sqrt{-1}\tilde{\lambda}_1(\xi)t}\tilde{P}_1=\quad e^{-bt}e^{-\sqrt{-1}at}\big(p_0+\textstyle\sum_{j=1}^{m}p_j^+(t)q_j(\xi)+R(t,\xi)\big),$$

where $p_0=diag(1,0,\cdots,0),p_j(t),q_j(\xi)$ and $R(t,\xi)$ are matrices, and

$$|p_j(t)|\le C(1+t)^j,\ |q_j(\xi)|\le C(1+|\xi|)^{-j},\ |R(t,\xi)|\le C(1+t)^{m+1}(1+|\xi|)^{-(m+1)}.$$

Let

$$L_0=e^{-\sqrt{-1}at}diag(1,0,\cdots,0),L_j(t,D_x)=e^{-\sqrt{-1}at}p_j(t)q_j(D_x),$$

where $q_j(D_x)$ is pseudo-differential operator with symbol $q_j(\xi)$.

By the definitions of G_3^+ and L_j, it is easy to see that

Proposition 4.4. *For R sufficiently large, there exist distributions*

$$K_l(x,t)=\Big(\sum_{j=0}^{n+l}L_j\delta(x)\Big)e^{-bt},$$

such that for $|\alpha|=l$

$$|D_x^\alpha(G_3^+-\chi_3(D)K_l(x,t))|\le Ce^{-bt/2}B_N(|x|,t). \tag{4.16}$$

Theorem 4.1. *For $x\in I\!R^n, t>1$ and $|\alpha|=l$, we have*

$$\begin{aligned}&|D_x^\alpha(G(x,t)-\chi_3(D)K_l(x,t))|\\ \le\ & C_\alpha t^{-(n+|\alpha|)/2}(t^{-(n-1)/4}B_N(|x|-ct,t)+B_N(|x|,t)).\end{aligned}\tag{4.17}$$

Proof. We can write

$$\begin{aligned}&(G(x,t)-\chi_3(D)K_l(x,t))\\ =\ & G_1+G_2+G_3^-+(G_3^+-\chi_3(D)K_l).\end{aligned}$$

By Propositions 4.1 to 4.4, we thus have that (4.17) is valid. ∎

5. Pointwise bounds for the non-linear system. We denote by $u = (\rho - \rho^*, v - v^*, e - e^*)^\tau = (\rho - 1, v, e - e^*)^\tau, u_0 = (\rho_0 - 1, v_0, e_0 - e^*)^\tau$ and rewrite (1.1) as

$$\partial_t u + \tilde{A}(D_x)u + \tilde{B}(D_x)u = Q(u). \tag{5.1}$$

For $|u|$ small enough, we may write

$$Q(u) = Q_1 + Q_2 = \sum_j D_{x_j} q_j(u) + \sum_{j,l} D_{x_j} D_{x_l} q_{j,l}(u)), \tag{5.2}$$

where $q_j(u) = O(|u|^2), q_{j,l}(u) = O(|u|^2)$.

In this section, we consider the Cauchy problem of (5.1)

$$\left\{ \begin{array}{l} \partial_t u + \tilde{A}(D_x)u + \tilde{B}(D_x)u = Q(u), \\ u|_{t=0} = u_0. \end{array} \right. \tag{5.3}$$

As in [6], we have

THEOREM 5.1. *Suppose that $u_0 \in H^{s+l}(I\!R^n), s = [n/2] + 1$, l is a nonnegative integer, and that $\|u_0\|_{H^{s+l}}$ is sufficiently small. Then there exist a unique, global, classical solution $u \in H^{s+l}$ of (1.1), satisfying*

$$\left. \begin{array}{rl} \|D_x^\alpha u\|_{L^2}(t), & 0 \le |\alpha| \le s + l \\ (\int_0^\infty \|D_x^\alpha u\|_{L^2}^2(t)dt)^{1/2}, & 1 \le |\alpha| \le s + l \\ \|D_x^\alpha u\|_{L^\infty}, & 0 \le |\alpha| \le l. \end{array} \right\} = O(1)\|u_0\|_{H^{s+l}}. \tag{5.4}$$

Let $E \equiv$max$\{\|u_0\|_{H^{s+l}}, \|u_0\|_{W^{1,l}}\}$, by Theorem 5.1 we have $\|u_0\|_{W^{\infty,l}} \le CE$. Using interpolation we know that $\|u_0\|_{W^{p,l}} \le CE$ $(1 \le p \le \infty)$.

Now we will give a pointwise estimate for the solution u of (5.3). Taking D_x^α on (5.1) and applying the Duhamel's principle, we obtain

$$D_x^\alpha u = \quad D_x^\alpha G(t) * u_0 + \int_0^t G(t-s) * D_x^\alpha Q(s)ds = R_1^\alpha + R_2^\alpha. \tag{5.5}$$

By the same method as in [6] and [12], we can give pointwise estimates and proved the main result in this paper.

THEOREM 5.2. *Suppose that $u_0 \in H^{s+l}(I\!R^n), s > [n/2] + 1, l > 2$ with E is small enough and $|\alpha| \le l - 2$. Then the solution $u(x,t)$ of (5.3) satisfies*

$$|D_x^\alpha u(x,t)| \le C(1+t)^{-(n+|\alpha|)/2}((1+t)^{-(n-1)/4} B_{n/2}(|x| - ct, t) + B_{n/2}(|x|, t)). \tag{5.6}$$

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