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003

NON RELATIVISTIC STRINGS MAY BE APPROXIMATED BY RELATIVISTIC STRINGS*

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Abstract. We show that bounded families of global classical relativistic strings that can be written as graphs are relatively compact in C^0 topology, but their accumulation points include many non relativistic strings.

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1. Some relativistic strings and their non relativistic limits.

Let us consider a graph

$$(t,s) \in \mathbf{R} \times \mathbf{R} \to (t,s,X(t,s)) \in \mathbf{R}^2 \times \mathbf{R}^d,$$

defined by a sufficiently smooth (at least locally Lipschitz continuous) function X. According to string theory (see [Po], for instance), this graph defines a global classical relativistic string if and only if, for all bounded open set $\Omega \subset \mathbf{R}^2$, X makes stationary, with respect to all perturbations, compactly supported in Ω, the Nambu-Goto Action defined by

$$\int_\Omega \sqrt{(1+\partial_s X^2)(1-\partial_t X^2)+(\partial_t X\cdot\partial_s X)^2}\ \ dtds$$

which is nothing but the area (over Ω) of the graph, in the space $\mathbf{R}^2 \times \mathbf{R}^d$, with respect to the Minkowski metric $(-1,+1,\cdots,+1)$ (for which the speed of light has unit value). Since we limit ourself to graphs, we automatically exclude many kinds of relativistic strings, in particular loops are ruled out. In this limited framework, the variational principle just means that X is a solution to the following first order partial differential system (of hyperbolic type):

$$\partial_t(B\partial_t X - C\partial_s X) - \partial_s(C\partial_t X + D\partial_s X) = 0, \tag{1}$$

where

$$B = \frac{1+\partial_s X^2}{A}, \quad C = \frac{\partial_t X\cdot\partial_s X}{A}, \quad D = \frac{1-\partial_t X^2}{A},$$

$$A = \sqrt{(1+\partial_s X^2)(1-\partial_t X^2)+(\partial_t X\cdot\partial_s X)^2}.$$

We say that such a string is global if X is a global solution, i.e. for the full range $-\infty < t < +\infty$, of (1). In the present paper, we exhibit some compactness properties of these global relativistic strings and characterize their limits.
In order to motivate this work, let us first consider, given some constant $0 < \kappa < 1$,

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the non trivial family $\mathbf{F}_\kappa$ of global solutions to the relativistic string equation (1), made of all X that satisfy, the linear wave equation:

$$\partial_{tt}X = \kappa^2\partial_{ss}X, \quad (t,s) \in \mathbf{R}^2, \tag{2}$$

together with the nonlinear constraint:

$$\kappa\partial_t X \cdot \partial_s X = 0, \quad \partial_t X^2 + \kappa^2\partial_s X^2 = 1-\kappa^2, \tag{3}$$

at time $t = 0$. Any solution of the wave equation (2) which does not satisfy (3) will be subsequently called a non relativistic string.
To check that, indeed, every $X \in \mathbf{F}_\kappa$ is a global solution to the relativistic string equation (1), let us first notice that the wave equation (2) also reads:

$$(\partial_t \mp \kappa\partial_s)(\partial_t X \pm \kappa\partial_s X) = 0, \tag{4}$$

which leads to the celebrated d'Alembert formula:

$$(\partial_t X \pm \kappa\partial_s X)(t,s) = (\partial_t X \pm \kappa\partial_s X)(0, s \pm \kappa t). \tag{5}$$

Thus, condition (3), which can be written:

$$|\partial_t X \pm \kappa\partial_s X|^2 = 1-\kappa^2, \tag{6}$$

is propagated by the wave equation (2) and, therefore, holds true at all time if it does at time 0. Finally, we get from (3), $A = \kappa(1+\partial_s X^2)$, $B = \kappa^{-1}$, $C = 0$ and $D = \kappa$, so that equation (1) reduces to (2).
Let us now study $\mathbf{F}_\kappa$ from the viewpoint of compactness and completeness:

Theorem 1.1. *The family $\mathbf{F}_\kappa$ of all X satisfying (2,3), with normalisation $X(0,0) = 0$, is a relatively compact subset of $C^0(\mathbf{R}^2;\ \mathbf{R}^d)$. The closure of $\mathbf{F}_\kappa$ is made of all functions functions X satisfying (2) and:*

$$\partial_t X^2 + \kappa^2\partial_s X^2 + 2\kappa|\partial_t X \cdot \partial_s X| \le 1-\kappa^2. \tag{7}$$

The main point of this very easy result (see the proof below) is that there are many non relativistic strings that can be uniformly approximated by relativistic strings. We will call them "subrelativistic" strings. In other words, under completion, algebraic constraints generated by relativity requirements can be relaxed as algebraic inequalities. As we will see below, the situation is very different for minimal surfaces in Riemannian geometry. This difference is, unsurprisingly, due to the hyperbolic character of the string equations, in sharp contrast with the minimal surface equations, of elliptic nature, for which elliptic regularity applies.
Before proving Theorem 1.1, let us provide an elementary example in the case $d = 3$. For each integer n, we consider the unique solution $X^{(n)}$ to the linear wave equation (2) with $\kappa = 2^{-1/2}$, and initial conditions:

$$\partial_t X^{(n)}(0,s) = 0, \quad X^{(n)}(0,s) =$$

$$(\cos s - 1, \frac{\sin(n+1)s}{2(n+1)} + \frac{\sin(n-1)s}{2(n-1)}, \frac{\cos(n+1)s - 1}{2(n+1)} + \frac{\cos(n-1)s - 1}{2(n-1)}),$$

This solution satisfies the relativistic constraints (3), since $\kappa^2 = 1/2$ and

$$\partial_s X^{(n)}(0,s) = (-\sin s, \cos s \cos ns, -\cos s \sin ns), \quad \partial_t X^{(n)}(0,s) = 0.$$

Then, as $n \to +\infty$, $X^{(n)}(t,s)$ uniformly converges toward a limit $X(t,s)$, still solution to the wave equation (2) with $\kappa = 2^{-1/2}$, but with initial conditions

$$\partial_t X(0,s) = 0, \quad X(0,s) = (\cos s - 1, 0, 0),$$

which makes $(t,s) \to (t,s,X(t,s))$ a "subrelativistic" string, but not a relativistic one.

Proof of Theorem 1.1. The proof is elementary. Notice first that (3) implies that $\mathbf{F}_\kappa$ is made of uniformly Lipschitz functions X. With normalization $X(0,0)$=0, this is enough, according to Ascoli's theorem, to see that $\mathbf{F}_\kappa$ is relatively compact for the uniform convergence on any compact subset of $\mathbf{R}^2$. Next, notice that, just as condition (3), condition (7) can be written

$$|\partial_t X \underline{+} \kappa \partial_s X|^2 \leq 1 - \kappa^2. \tag{8}$$

Thus, both conditions are preserved by the wave equation (4). Let us consider a sequence X_n in $\mathbf{F}_\kappa$. Up to extracting a subsequence, we may assume that $X_n(t,s)$ converges to some limit $X(t,s)$ uniformly on any compact subset of $\mathbf{R}^2$, meanwhile $\partial_t X_n$ and $\partial_s X_n$ respectively converge to $\partial_t X$ and $\partial_s X$ for the weak-* topology of $L^\infty(\mathbf{R}^2;\ \mathbf{R}^d)$. Thus $Y = \partial_s X$ and $W = -\partial_t X$ must take their values in the closed convex hull of

$$S = \{(Y,W) \in \mathbf{R}^{d+d}\ ; \quad |W \underline{+} \kappa Y|^2 = 1 - \kappa^2\}, \tag{9}$$

which exactly is

$$\{(Y,W) \in \mathbf{R}^{d+d}\ ; \quad |W \underline{+} \kappa Y|^2 \leq 1 - \kappa^2\}. \tag{10}$$

Conversely, let us consider a solution X to the wave equation (2) that satisfies $X(0,0) = 0$ and (8), which implies that $(Y_0, W_0) = (\partial_s X, -\partial_t X)(t=0,\cdot)$ is valued in the closed convex hull of S. Then, at time 0, we may find a sequence of (smooth) functions (Y_n^0, W_n^0) valued in S that converges to (Y_0, W_0) for the weak-* topology of $L^\infty(\mathbf{R};\ \mathbf{R}^d)$. (This is a well known and very useful property of weak topologies, see [Ta] for instance.) Let us consider, for each n, the unique solution X_n to (2) such that

$$X_n(0,0) = 0, \quad (\partial_s X_n, -\partial_t X_n)(t=0,\cdot) = (Y_n^0, W_n^0).$$

Then we observe that X_n converges to X uniformly on any compact subset of $\mathbf{R}^2$ and satisfies condition (3). The proof of Theorem 1.1 is now complete.

Comparison with the Euclidean case. There is no result like Theorem 1.1 in the Riemannian case, with the Euclidean metric $(+1,\cdots,+1)$. In that case, the area of a graph

$$(t,s) \in \overline{\Omega} \to (t,s,X(t,s)) \in \mathbf{R}^2 \times \mathbf{R}^d,$$

where Ω is a smooth, bounded, connected open subset of $\mathbf{R}^2$, is given by:

$$\mathbf{A}_\Omega(X) = \int_\Omega \sqrt{(1+\partial_s X^2)(1+\partial_t X^2) - (\partial_t X \cdot \partial_s X)^2}\, dsdt.$$

Then, the minimal surface equation is just:

$$\partial_t(B\partial_t X + C\partial_s X) + \partial_s(C\partial_t X + D\partial_s X) = 0,$$

where

$$B = \frac{1+\partial_s X^2}{A}, \quad A = \sqrt{(1+\partial_s X^2)(1+\partial_t X^2) - (\partial_t X \cdot \partial_s X)^2},$$

$$C = \frac{\partial_t X \cdot \partial_s X}{A}, \quad D = \frac{1+\partial_t X^2}{A}.$$

Let us assume that X is harmonic:

$$\partial_{tt} X + \partial_{ss} X = 0,$$

and $a = \partial_t X \cdot \partial_s X$, $b = \partial_t X^2 - \partial_s X^2$ both vanish along $\partial\Omega$. Since a and b are also harmonic, they must vanish inside Ω. Thus $B = D = 1$, $C = 0$, and X is also a solution to the minimal surface equation. Let us now consider a sequence of such functions X_n and assume that the restriction of X_n to the boundary $\partial\Omega$ converges to some limit $X_{\partial\Omega}$, say in $C^0(\partial\Omega)$. Then $X_{\partial\Omega}$ has a harmonic extension X and, due to elliptic regularity, X_n converges to X in $C^\infty(\Omega)$. Thus, X must satisfy $\partial_t X \cdot \partial_s X = \partial_t X^2 - \partial_s X^2 = 0$, and, therefore, is still a solution to the minimal surface equation. So, in this (over)simplified framework, there is no way to converge to a graph that is not a minimal surface. Of course, this is can be discussed in a much more general framework, as in [Fe] (chapter 5.4), or, also, in terms of weak continuity of determinants and polyconvexity (cf. [Ev], for instance).

2. The augmented relativistic string equations. Theorem 1.1 is just a motivation to study more comprehensively global solutions X to the string equation (1). Do they have some compactness properties? What are their limits?
To achieve this goal, we first embed the string equation in a larger, augmented system.

Proposition 2.1. *Let us consider a solution X to the relativistic string equation (1). Then, the following quantities*

$$\tau = \frac{-L}{1+Y^2}, \quad v = \frac{Y \cdot W}{1+Y^2}, \quad \eta = \frac{-L}{1+Y^2} Y, \quad \zeta = W - \frac{Y \cdot W}{1+Y^2} Y, \tag{11}$$

where

$$Y = \partial_s X, \quad W = -\partial_t X, \quad L = -\sqrt{(1+Y^2)(1-W^2) + (Y \cdot W)^2},$$

are solutions to the "augmented system":

$$\partial_t \tau + v\partial_s \tau = \tau \partial_s v, \quad \partial_t v + v\partial_s v = \tau\partial_s \tau,$$

$$\partial_t \eta + v\partial_s \eta = -\tau\partial_s \zeta, \quad \partial_t \zeta + v\partial_s \zeta = -\tau\partial_s \eta. \tag{12}$$

In addition, they satisfy the following constraints:

$$\tau > 0, \quad \tau^2 + v^2 + \eta^2 + \zeta^2 = 1, \quad \tau v = \eta \cdot \zeta. \tag{13}$$

Next, we introduce

Definition 2.2. *We call subrelativistic strings all solutions* (τ, v, η, ζ) *of the augmented system (12) that satisfy the following algebraic inequalities:*

$$\tau \geq 0, \quad \tau^2 + v^2 + \eta^2 + \zeta^2 + 2|\tau v - \eta \cdot \zeta| \leq 1. \tag{14}$$

We will say that a subrelativistic string (τ, v, η, ζ) is global whenever it is a global solution to the augmented system, i.e. for $-\infty < t < +\infty$. We will see later that a necessary condition for (τ, v, η, ζ) to be a global subrelativistic string is the existence of a real constant α such that

$$\tau \underline{+} (v - \alpha) > 0$$

holds true at $t = 0$, meanwhile a sufficient condition is the further existence of some constant $\delta > 0$ such that:

$$\delta \leq \tau \underline{+} (v - \alpha) \leq \frac{1}{\delta}.$$

Our main result, which will be precisely stated as Theorem 4.1, asserts, roughly speaking, that global subrelativistic strings form a natural completion for global relativistic strings. The main steps of the analysis will be:
1) an almost explicit resolution of the augmented system for a large class of initial data, thanks to d'Alembert formula,
2) a weak convergence argument, using that (14) defines the closed convex hull of (13).
Let us finally mention, before proving Proposition 2.1, that it has been known for a long time that relativistic string equations can be solved using d'Alembert's formula. (Just like minimal surfaces can be reduced to harmonic functions.) See [Po], for instance. It turns out that this is also true for generalized strings described by the augmented system.

Proof of Proposition 2.1. Let

$$Y = \partial_s X, \quad W = -\partial_t X. \tag{15}$$

Then the relativistic string equation may be equivalently obtained by varying the Action $\int L(Y, W) dt ds$ where Y and W are subject to

$$\partial_t Y + \partial_s W = 0,$$

and the Lagrangian density L is given by

$$L(Y, W) = -\sqrt{(1 + Y^2)(1 - W^2) + (W \cdot Y)^2}.$$

The resulting equations are

$$\partial_t Y + \partial_s W = 0, \quad \partial_t Z + \partial_s V = 0,$$

where Z and V are defined by

$$Z = \frac{\partial L}{\partial W}(Y, W) = \frac{(1 + Y^2)W - (W \cdot Y)Y}{-L}.$$

$$V = -\frac{\partial L}{\partial Y}(Y,W) = \frac{(1-W^2)Y - (W\cdot Y)W}{-L}.$$

In order to write W and V as functions of the evolution variables Y and Z, we introduce the hamiltonian function h defined as the partial Legendre transform

$$h(Y,Z) = \sup_{W\in\mathbb{R}^d} Z\cdot W - L(Y,W) = \sqrt{1+Y^2+Z^2+(Y\cdot Z)^2}.$$

Let us introduce

$$q = Y\cdot Z. \tag{16}$$

Thus

$$h = \sqrt{1+Y^2+Z^2+q^2}, \tag{17}$$

$$V = \frac{\partial h}{\partial Y}(Y,Z) = \frac{Y+qZ}{h}, \quad W = \frac{\partial h}{\partial Z}(Y,Z) = \frac{Z+qY}{h}. \tag{18}$$

The relativistic string equation now reads:

$$\partial_t Y + \partial_s(\frac{Z+qY}{h}) = 0, \quad \partial_t Z + \partial_s(\frac{Y+qZ}{h}) = 0, \tag{19}$$

where q and h are defined by (16,17).

Next, we follow an idea used in [Br] for the Born-Infeld system (for which we also refer to [BDLL, Gi, Se]), by adding to system (19) two additional conservation laws, for h and q, respectively:

$$\partial_t h + \partial_s q = 0, \tag{20}$$

$$\partial_t q + \partial_s(\frac{q^2-1}{h}) = 0. \tag{21}$$

System (20,21) is known under many different names, such as the Chaplygin gas equation, the (one-dimensional) Born-Infeld equations or the Eulerian version of the linear wave equation. [BDLL], [Se], [Se2]. As we will see in the next section, this system can be easily integrated by using d'Alembert's formula. Let us now establish equations (20,21) from the string equation written in form (16,17,18,19). We first get

$$\partial_t h = \frac{\partial h}{\partial Y}\cdot\partial_t Y + \frac{\partial h}{\partial Z}\cdot\partial_t Z$$

$$= -V\cdot\partial_s W - W\cdot\partial_s V = -\partial_s(W\cdot V)$$

where

$$W\cdot V = \frac{(Z+(Y\cdot Z)Y)(Y+(Y\cdot Z)Z)}{h^2} = Z\cdot Y,$$

which leads to (20). Next, we have

$$-\partial_t q = Z\cdot\partial_s W + Y\cdot\partial_s V = Z\cdot\partial_s(\frac{Z+(Y\cdot Z)Y}{h}) + Y\cdot\partial_s(\frac{Y+(Y\cdot Z)Z}{h}).$$

$$= Z \cdot \partial_s(\frac{(Y \cdot Z)Y}{h}) + Y \cdot \partial_s(\frac{(Y \cdot Z)Z}{h}) + h\partial_s \frac{Z^2 + Y^2}{2h^2}.$$

Observe that

$$\partial_s \frac{q^2}{h} = \partial_s \frac{(Y \cdot Z)^2}{h} = Z \cdot \partial_s \frac{Y(Y \cdot Z)}{h} + \partial_s Z \cdot \frac{Y(Y \cdot Z)}{h}$$

$$= Z \cdot \partial_s \frac{Y(Y \cdot Z)}{h} + Y \cdot \partial_s \frac{Z(Y \cdot Z)}{h} - Y \cdot Z \partial_s \frac{Y \cdot Z}{h}$$

$$= Z \cdot \partial_s \frac{Y(Y \cdot Z)}{h} + Y \cdot \partial_s \frac{Z(Y \cdot Z)}{h} - h\partial_s \frac{(Y \cdot Z)^2}{2h^2}.$$

Thus

$$\partial_t q + \partial_s \frac{q^2}{h} = -h\partial_s \frac{Z^2 + Y^2 + (Y \cdot Z)^2}{2h^2} = -h\partial_s \frac{h^2 - 1}{2h^2} = \partial_s \frac{1}{h},$$

which is just (21).
Let us finally introduce the rescaled variables:

$$\tau = \frac{1}{h}, \quad v = \frac{q}{h}, \quad \eta = \frac{Y}{h}, \quad \zeta = \frac{Z}{h}. \tag{22}$$

Because of (16,17), they must satisfy

$$\tau > 0, \quad \tau^2 + v^2 + \eta^2 + \zeta^2 = 1, \quad \tau v = \eta \cdot \zeta,$$

which exactly is (13).
After straightforward calculations, the "augmented" system (19,20,21) can be written in terms of τ, v, η, ζ:

$$\partial_t \tau + v\partial_s \tau = \tau \partial_s v, \quad \partial_t v + v\partial_s v = \tau \partial_s \tau,$$

$$\partial_t \eta + v\partial_s \eta = -\tau \partial_s \zeta, \quad \partial_t \zeta + v\partial_s \zeta = -\tau \partial_s \eta,$$

which is nothing but (12). Thus, the proof of Proposition 2.1 is now complete.

Comments on the augmented system. The augmented system (12) makes sense for all $U = (\tau, v, \eta, \zeta) \in \mathbf{R}^{1+1+d+d}$, even if (13) is not satisfied. (Notice that τ may even change sign!) As a matter of fact system (12) can be written as

$$\partial_t U + A(U)\partial_s U = 0,$$

where $A(U)$ is a symmetric matrix. Therefore, this system is a symmetric hyperbolic system of first order PDEs. As a consequence, the Cauchy problem, with initial data at time $t = 0$, is solvable in a neighborhood of $t = 0$, for all smooth initial data $s \in \mathbf{R} \rightarrow U_0(s) \in \mathbf{R}^{1+1+d+d}$, with appropriate behaviour near $s = \underline{+}\infty$.
Surprisingly enough, the augmented system (12) is Galilean invariant, under the following transform:

$$(t, s) \rightarrow (t, s + ut), \quad (\tau, v, \eta, \zeta) \rightarrow (\tau, v + u, \eta, \zeta), \tag{23}$$

where $u \in \mathbf{R}$ is a fixed velocity. Observe that this transform, which is certainly ruled out by the relativistic constraint (13), is compatible with the "subrelativistic" condition (14), provided $|u|$ is not too large.

Comment on relativistic and non-relativistic strings. As shown in Proposition 2.1, we can attach a solution $U = (\tau, v, \eta, \zeta)$ of the augmented system (12) to each graph $(t, s) \to (t, s, X(t, s))$ corresponding to a relativistic string, through (11). Of course, by construction of the augmented system, such solutions automatically satisfy constraint (13). Conversely, given a smooth solution $U = (\tau, v, \eta, \zeta)$ to the augmented system (12), such that $\tau > 0$, we may define $X(t, s)$ (up to a normalization such as $X(0, 0) = 0$) by

$$\partial_s X = \frac{\eta}{\tau}, \quad \partial_t X = -\zeta - v\frac{\eta}{\tau}.$$

Then, if U satisfies (13), we can check from (12) that, indeed, X solves the relativistic string equation (1).

Other solutions to the augmented system (12) may describe graphs $(t, s) \to (t, s, X(t, s))$ that are not necessarily relativistic strings. For instance, consider a non relativistic string, for which X solves the wave equation (2) but not necessarily equation (1). Then, assuming $v = 0$, $\tau = \kappa$, in system (12), we get

$$\partial_t \eta + \kappa \partial_s \zeta = \partial_t \zeta + \kappa \partial_s \eta = 0,$$

and, by setting:

$$\eta = \kappa \partial_s X, \quad \zeta = -\partial_t X,$$

we recover the wave equation (2). Such a string is relativistic only if (13) is satisfied, which means

$$\kappa^2 + \eta^2 + \zeta^2 = 1, \quad \eta \cdot \zeta = 0,$$

or, in other words,

$$\kappa^2 |\partial_s X|^2 + |\partial_t X|^2 = 1 - \kappa^2, \quad \kappa \partial_s X \cdot \partial_t X = 0,$$

which exactly is condition (3).

3. Integration of the augmented system. The augmented system (12) can also be written in "diagonal" form:

$$D_t^\epsilon (v - \epsilon\tau) = 0, \quad D_t^\epsilon (\eta + \epsilon\zeta) = 0, \tag{24}$$

where $\epsilon \in \{-1, +1\}$ and

$$D_t^\epsilon = \partial_t + (v + \epsilon\tau)\partial_s. \tag{25}$$

It follows that, ϵ being fixed in $\{-1, +1\}$, for any real function f and any constant r, the level set

$$\{U = (\tau, b, \eta, \zeta); \quad f(v - \epsilon\tau, \eta + \epsilon\zeta) = r\},$$

is an invariant set for system (24). As a consequence, the following sets are also invariant:

$$G_{\alpha,\delta} = \{U = (\tau, v, \eta, \zeta) \in \mathbf{R}^{1+1+d+d}; \quad \delta \le \tau \underline{+} (v - \alpha) \le \frac{1}{\delta} \}, \tag{26}$$

for any constants $\alpha \in \mathbf{R}$ and $0 < \delta < 1$. Observe that $G_{\alpha,\delta}$ is included in

$$\{U = (\tau, v, \eta, \zeta); \quad \delta \le \tau \le \frac{1}{\delta}\ \}.$$

Other invariant sets are:

$$M_\epsilon = \{(\tau, v, \eta, \zeta) \in \mathbf{R}^{1+1+d+d}\ ; \quad (v + \epsilon\tau)^2 + |\eta - \epsilon\zeta|^2 = 1\}$$

for $\epsilon = -1, +1$, as well as their intersection:

$$M = \{(\tau, v, \eta, \zeta) \in \mathbf{R}^{1+1+d+d}; \quad \tau v = \eta \cdot \zeta, \quad \tau^2 + v^2 + \eta^2 + \zeta^2 = 1\}, \tag{27}$$

which precisely corresponds to the relativistic string constraint (13).
It is now easy to integrate system (24) for solutions valued in $G_{\alpha,\delta}$.

Proposition 3.1. *Let $\alpha \in \mathbf{R}$, $\delta > 0$ be fixed constants. All solutions $U = (\tau, v, \eta, \zeta)$ to the augmented system (12) valued in the invariant set $G_{\alpha,\delta}$ (26) are global and implicitly defined by:*

$$(v \underline{+} \tau)(t, \xi(t, y)) = (v \underline{+} \tau)(0, \xi(0, y \overline{+} t))), \quad \forall (t, y) \in \mathbf{R}^2$$

$$(\eta \overline{+} \zeta)(t, \xi(t, y)) = (\eta \overline{+} \zeta)(0, \xi(0, y \overline{+} t)), \tag{28}$$

where, for each t, $y \to \xi(t, y)$ is a bi-Lipschitz homeomorphism of the real line, with

$$\partial_y \xi(t, y) = \tau(t, \xi(t, y)), \tag{29}$$

valued in $[\delta, \frac{1}{\delta}]$. In addition, ξ is completely determined by $\xi(0, 0) = 0$ and:

$$(\partial_t \xi \underline{+} \partial_s \xi)(t, y) = v(0, \xi(0, y \underline{+} t)) \underline{+} \tau(0, \xi(0, y \underline{+} t)). \tag{30}$$

Proof. Since $0 < \delta \le \tau \le \delta^{-1}$, we can define $\xi(t, y)$ for all $(t, y) \in \mathbf{R}^2$ by (29) in such a way that, in addition,

$$\partial_t \xi(t, y) = v(t, \xi(t, y)), \quad \partial_{tt} \xi = \partial_{yy} \xi,$$

hold true. This is possible, due to the two first equations of the augmented system (12). Of course, we can normalize $\xi(0, 0) = 0$. Next,

$$(\partial_t \overline{+} \partial_s)(\partial_t \xi \underline{+} \partial_s \xi)$$

follows, and we deduce (30) from d'Alembert's formula. Notice that (30) and $\xi(0, 0) = 0$ entirely determine ξ, given τ and v at time $t = 0$.
Then, using (24), we get:

$$\partial_t[(v \underline{+} \tau)(t, \xi(t, y \underline{+} t))] = 0, \quad \partial_t[(\eta \overline{+} \zeta)(t, \xi(t, y \underline{+} t))] = 0,$$

which leads to formula (28) and completes the proof.

Generalized solutions. Just as d'Alembert's formula does for the linear wave equation, formulae (30,28) provide a natural notion of (global) generalized solutions for the augmented system (12), globally and uniquely defined for *each* Lebesgue measurable initial condition valued in $G_{\alpha,\delta}$, which means

$$\delta \leq \tau(0,\cdot)\underline{+}(v(0,\cdot)-\alpha) \leq \frac{1}{\delta}\,, \tag{31}$$

for some constants $\alpha \in \mathbf{R}$, $0<\delta<1$. Of course, for each smooth initial condition, the corresponding generalized solution automatically is a classical, global, smooth solution to the augmented system (12). As a matter of fact, condition (31) is nearly a necessary condition to define a global solution (τ, v, η, ζ) to system (12) with $\tau > 0$. Indeed, because of (29), $\partial_y\xi(t,y)$ must stay positive for all $(t,y) \in \mathbf{R}^2$. Because of d'Alembert formula (30), this is possible only if

$$v(0,s)-\tau(0,s) < v(0,s')+\tau(0,s'), \quad \forall s,s' \in \mathbf{R},$$

which exactly means $\tau(0,\cdot)\underline{+}(v(0,\cdot)-\alpha) > 0$, for some constant $\alpha \in \mathbf{R}$. In the rest of the paper, we will consider only generalized solutions valued in one of the $G_{\alpha,\delta}$.

Weak form. For each generalized solution valued in $G_{\alpha,\delta}$, $y \to \xi(t,y)$ is a bi-Lipschitz homeomorphism of $\mathbf{R}$, for each $t \in \mathbf{R}$, since $0 < \delta \leq \partial_y\xi \leq \delta^{-1} < +\infty$. The inverse of $\xi(t,\cdot)$ is denoted by $\xi^{-1}(t,\cdot)$. This allows us to write (29) in the following "weak" form:

$$\int_{-\infty}^{+\infty} \frac{g(s)}{\tau(t,s)}ds = \int_{-\infty}^{+\infty} g(\xi(t,y))dy, \tag{32}$$

for all functions $g \in L^1(\mathbf{R})$. Similarly, (30,28) reads:

$$\int_{-\infty}^{+\infty} (\partial_t\xi\underline{+}\partial_s\xi)(t,y)g(y)dy = \int_{-\infty}^{+\infty} \frac{v\underline{+}\tau}{\tau}(0,s)g(\xi^{-1}(0,s)\overline{+}t)ds, \tag{33}$$

$$\int_{-\infty}^{+\infty} \frac{v\underline{+}\tau}{\tau}(t,s)g(s)ds = \int_{-\infty}^{+\infty} \frac{v\underline{+}\tau}{\tau}(0,s)g(\xi(t,\xi^{-1}(0,s)\overline{+}t))ds,$$

$$\int_{-\infty}^{+\infty} \frac{\eta\overline{+}\zeta}{\tau}(t,s)g(s)ds = \int_{-\infty}^{+\infty} \frac{\eta\overline{+}\zeta}{\tau}(0,s)g(\xi(t,\xi^{-1}(0,s)\overline{+}t))ds, \tag{34}$$

for all functions $g \in L^1(\mathbf{R})$. Using the original variables (h,q,Y,Z) instead of (τ,v,η,ζ), we respectively get:

$$\int_{-\infty}^{+\infty} g(s)h(t,s)ds = \int_{-\infty}^{+\infty} g(\xi(t,y))dy, \tag{35}$$

$$\int_{-\infty}^{+\infty} (\partial_t\xi\underline{+}\partial_s\xi)(t,y)g(y)dy = \int_{-\infty}^{+\infty} (q(0,s)\underline{+}1)g(\xi^{-1}(0,s)\overline{+}t)ds, \tag{36}$$

$$\int_{-\infty}^{+\infty} (q(t,s)\underline{+}1)g(s)ds = \int_{-\infty}^{+\infty} (q(0,s)\underline{+}1)g(\xi(t,\xi^{-1}(0,s)\overline{+}t))ds,$$

$$\int_{-\infty}^{+\infty} (Y\overline{+}Z)(t,s)g(s)ds = \int_{-\infty}^{+\infty} (Y\overline{+}Z)(0,s)g(\xi(t,\xi^{-1}(0,s)\overline{+}t))ds, \tag{37}$$

for all functions $g \in L^1(\mathbf{R})$.

4. Weak completion of global relativistic strings. In this last section, we study the subset of all global generalized solutions to the augmented system (12) valued in the invariant subset $G_{\alpha,\delta}$ (defined by (26) for some fixed constants $\alpha \in \mathbf{R}$, $0 < \delta < 1$, which, in addition, satisfy the relativistic constraint (13), or, in other words, are valued in the invariant region M defined by (27), and, therefore, correspond to global relativistic strings.

We call $\Sigma_{\alpha,\delta}$ the set of all such solutions. We also denote:

$$M_{\alpha,\delta} = M \ \cap \ G_{\alpha,\delta},$$

i.e.

$$M_{\alpha,\delta} = \{U = (\tau, v, \eta, \zeta); \ \ \delta \leq \tau \underline{+} (v - \alpha) \leq \frac{1}{\delta} \ ;$$

$$\tau v = \eta \cdot \zeta, \ \ \tau^2 + v^2 + \eta^2 + \zeta^2 = 1\}. \tag{38}$$

An equivalent definition is:

$$M_{\alpha,\delta} = \{\delta \leq \tau \underline{+} (v - \alpha) \leq \frac{1}{\delta} \ , \quad (v \underline{+} \tau)^2 + |\eta \overline{+} \zeta|^2 = 1 \ \} \ . \tag{39}$$

From the topological point of view, we confer to $\Sigma_{\alpha,\delta}$ the topology induced by the space $C^0(\mathbf{R}; L^\infty_{weak^*}(\mathbf{R}; \mathbf{R}^{1+1+d+d}))$ through the one-to-one transform

$$T \ : U = (\tau, v, \eta, \zeta) \to u = (h, q, Y, Z) = \frac{1}{\tau}(1, v, \eta, \zeta), \tag{40}$$

defined on $\mathbf{R}_+ \times \mathbf{R}^{1+d+d}$. More precisely, we say that $U_n = (\tau_n, v_n, \eta_n, \zeta_n)$ converges to $U = (\tau, v, \eta, \zeta)$ if and only if TU_n converges to TU in

$$C^0(\mathbf{R}; L^\infty_{weak^*}(\mathbf{R}; \mathbf{R}^{1+1+d+d})),$$

i.e.

$$\int_{-\infty}^{+\infty} (h_n - h, q_n - q, Y_n - Y, Z_n - Z)(s) g(s) ds \to 0 \tag{41}$$

uniformly in t on any compact subset of $\mathbf{R}$, for all functions $g \in L^1(\mathbf{R})$, or, equivalently

$$\int_{-\infty}^{+\infty} \{\frac{(1, v_n, \eta_n, \zeta_n)}{\tau_n} - \frac{(1, v, \eta, \zeta)}{\tau}\}(s) g(s) ds \to 0. \tag{42}$$

Notice that T and its inverse

$$T^{-1} \ : u = (h, q, Y, Z) \to U = (\tau, v, \eta, \zeta) = \frac{1}{h}(h, q, Y, Z)$$

(which was already used for definition (22)), both preserve straight lines and convexity on $\mathbf{R}_+ \times \mathbf{R}^{1+d+d}$.

Theorem 4.1. *The set $\Sigma_{\alpha,\delta}$ is relatively compact for the toplogy of*

$$C^0(\mathbf{R}; L^\infty_{weak^*}(\mathbf{R}; \mathbf{R}^{1+1+d+d})),$$

induced by T (defined by (40,41)). Its closure is the set of all generalized solutions to the augmented system (12), in the sense of (30,28), valued in $CM \cap G_{\alpha,\delta}$, where

$$CM = \{U = (\tau, v, \eta, \zeta) \in \mathbf{R}^{1+1+d+d}; \ \ \tau^2 + v^2 + \eta^2 + \zeta^2 + 2|\tau v - \eta \cdot \zeta| \leq 1\}. \tag{43}$$

Comment on Theorem 1.1. Theorem (4.1) has Theorem (1.1) as a corollary. Indeed, let us consider a solution X to the wave equation (2), with $0<\kappa<1$. Assume that X satisfies (7) and define $\tau=\kappa$, $v=0$,

$$\eta=\kappa\partial_s X,\quad \zeta=-\partial_t X.$$

Then $U=(\tau,v,\eta,\zeta)$ is valued in $CM\cap G_{\alpha,\delta}$, for $\alpha=0$ and $\delta=\kappa>0$. Thus, U can be approximated by a sequence U_n valued in $M\cap G_{\alpha,\delta}$, which means that there is a sequence of relativistic strings $(t,s)\to X_n(t,s)$, such that

$$\frac{(1,v_n,\eta_n,\zeta_n)}{\tau_n}\to\frac{(1,v,\eta,\zeta)}{\tau}$$

in $C^0(\mathbf{R};L^\infty_{weak^*}(\mathbf{R};\mathbf{R}^{1+1+d+d}))$, which, in particular, implies that $X_n(t,s)$ converges to $X(t,s)$ uniformly on all compact subset of $\mathbf{R^2}$.

Proof. Our proof is elementary and based on closed formulae (35,37). Alternative proofs, based on the Murat-Tartar "div-curl" lemma [Ta], are possible, following Serre's analysis of the one-dimensional Born-Infeld equation [Se].
Let us consider a sequence $U_n=(\tau_n,v_n,\eta_n,\zeta_n)$ in $\Sigma_{\alpha,\delta}$ and the corresponding variables $TU_n=(h_n,q_n,Y_n,Z_n)$. Using definitions (26,27) and formulae (28), we have:

$$\delta\le\tau_n\underline{+}(v_n-\alpha)\le\frac{1}{\delta}\ ;$$

$$\tau_n^2+v_n^2+\eta_n^2+\zeta_n^2=1,\quad \delta\le\tau_n\le 1,$$

$$(v_nn\underline{+}\tau_n)(t,\xi_n(t,y\underline{+}t))=(v_n\underline{+}\tau_n)(0,\xi_n(0,y))),$$

$$(\eta\overline{+}\zeta_n)(t,\xi_n(t,y\underline{+}t))=(\eta\underline{+}\zeta_n)(0,\xi_n(0,y)),$$

where

$$\partial_s\xi_n(t,y)=\tau_n(t,\xi_n(t,y)),\quad \partial_t\xi_n(t,y)=v_n(t,\xi_n(t,y)),$$

with normalization $\xi_n(0,0)=0$. So, we immediately get:

$$(\partial_t\xi_n)^2+(\partial_s\xi_n)^2\le 1,\quad \partial_s\xi_n\ge\delta.$$

and deduce, using Ascoli's theorem, that ξ_n is relatively compact in $C^0(\mathbf{R}^2)$. Thus, up to the extraction of a subsequence, $\xi_n(t,y)$ uniformly converges to a limit $\xi(t,y)$ on any compact subset of $\mathbf{R}^2$. Since $\delta\le\partial_y\xi_n\le\delta^{-1}$, we also have $\xi_n^{-1}(t,s)\to\xi^{-1}(t,s)$ uniformly in (t,s), on any compact subset of $\mathbf{R}^2$.
Let us now consider the initial values $U_n(0,\cdot)$. Because of constraint (13), these functions are bounded in sup norm. Thus, the sequence $TU_n(0,\cdot)$, where T is defined by (40), is bounded in the space $L^\infty(\mathbf{R};\mathbf{R}^{1+1+d+d})$. So, up to the extraction of a further subsequence, we may assume that they converge (in the weak-* sense) to some limit $u_{in}=(h_{in},q_{in},Y_{in},Z_{in})$. Since T and T^{-1} preserve convexity, $U_{in}=T^{-1}u_{in}$ is valued in the closed convex hull of $M_{\alpha,\delta}$, that we denote by $CM_{\alpha,\delta}$. Since $G_{\alpha,\delta}$ is a closed, convex subset of $\mathbf{R}^{1+1+d+d}$ and contains $M_{\alpha,\delta}$, according to definitions

(26,38), it certainly contains $CM_{\alpha,\delta}$. Thus U_{in} is valued in $CM_{\alpha,\delta}$ and $G_{\alpha,\delta}$.
Let us go back to $TU_n = (h_n, q_n, Y_n, Z_n)$. Because of (36), we have

$$\int_{-\infty}^{+\infty} (\partial_t \xi_n \pm \partial_s \xi_n)(t,y) g(y) dy = \int_{-\infty}^{+\infty} (q_n(0,s) \pm 1) g(\xi_n^{-1}(0,s) \mp t) ds,$$

for all functions $g \in L^1(\mathbf{R})$ and $t \in \mathbf{R}$. We deduce, after letting $n \to \infty$,

$$\int_{-\infty}^{+\infty} (\partial_t \xi \pm \partial_s \xi)(t,y) g(y) dy = \int_{-\infty}^{+\infty} (q_{in}(s) \pm 1) g(\xi^{-1}(0,s) \mp t) ds.$$

Next, we use that TU_n satisfies (35,37):

$$\int_{-\infty}^{+\infty} g(s) h_n(t,s) ds = \int_{-\infty}^{+\infty} g(\xi_n(t,y)) dy,$$

$$\int_{-\infty}^{+\infty} (q_n \pm 1)(t,s) g(s) ds = \int_{-\infty}^{+\infty} (q_n \pm 1)(0,s) g(\xi_n(t, \xi_n^{-1}(0,s) \mp t)) ds,$$

$$\int_{-\infty}^{+\infty} (Y_n \mp Z_n)(t,s) g(s) ds = \int_{-\infty}^{+\infty} (Y_n \mp Z_n)(0,s) g(\xi_n(t, \xi_n^{-1}(0,s) \mp t)) ds,$$

for all functions $g \in L^1(\mathbf{R})$ and $t \in \mathbf{R}$. As $n \to +\infty$, each right-hand side of these equations has a well defined limit in terms of ξ and $(h_{in}, q_{in}, Y_{in}, Z_{in})$. This implies that each left-hand side is convergent, uniformly in t on any compact subset of $\mathbf{R}$. Thus, (h_n, q_n, Y_n, Z_n) has a limit (h, q, Y, Z) in the space $C^0(\mathbf{R}; L^\infty_{weak^*}(\mathbf{R}; \mathbf{R}^{1+1+d+d}))$. This limit satisfies

$$\int_{-\infty}^{+\infty} g(s) h(t,s) ds = \int_{-\infty}^{+\infty} g(\xi(t,y)) dy,$$

$$\int_{-\infty}^{+\infty} (q \pm 1)(t,s) g(s) ds = \int_{-\infty}^{+\infty} (q_{in} \pm 1)(s) g(\xi(t, \xi^{-1}(0,s) \mp t)) ds,$$

$$\int_{-\infty}^{+\infty} (Y \mp Z)(t,s) g(s) ds = \int_{-\infty}^{+\infty} (Y_{in} \mp Z_{in})(s) g(\xi(t, \xi^{-1}(0,s) \mp t)) ds,$$

for all functions $g \in L^1(\mathbf{R})$.
Since $u = (h, q, Y, Z)$ belongs to $C^0(\mathbf{R}; L^\infty_{weak^*}(\mathbf{R}; \mathbf{R}^{1+1+d+d}))$, we deduce from the previous equations, taken at $t = 0$, that the initial value $u(t = 0, \cdot)$ must be equal to $u_{in} = (h_{in}, q_{in}, Y_{in}, Z_{in})$. Thus, $u = (h, q, Y, Z)$ is a generalized solution to the augmented system (12) in the sense of (35,36,37).
We have proven so far that, up to a sequence, any sequence U_n in $\Sigma_{\alpha,\delta}$ converges (in the sense of (41)) to a generalized solution $U = T^{-1}u$. This solution is valued in $CM_{\alpha,\delta}$, the closed convex hull of $M_{\alpha,\delta}$. This shows that $\Sigma_{\alpha,\delta}$ is relatively compact and its closure is contained in the set of generalized solutions valued in $CM_{\alpha,\delta}$.
Conversely, let us show that all generalized solutions U valued in the closed convex hull of $M_{\alpha,\delta}$ belong to the closure of $\Sigma_{\alpha,\delta}$. Because T preserves convexity, $u = TU$ is

valued in the closed convex hull of $T(M_{\alpha,\delta})$. Thus, according to a well known property of weak convergence (see [Ta], for instance), the initial value $u_{in} = u(t=0,\cdot)$ can be approached, in the L^∞ weak-* sense, by a sequence $u_{in,n}$ valued in the manifold $T(M_{\alpha,\delta})$. Then, we see that the unique generalized solution U_n with initial condition $T^{-1}u_{in,n}$ must converge to U (in the sense of (41)).
At this point, we have shown that the closure of $\Sigma_{\alpha,\delta}$ is exactly equal to the set of all generalized solutions valued in the closed convex hull $CM_{\alpha,\delta}$ of $M_{\alpha,\delta}$.
So, the proof of Theorem 4.1 will be complete when we are able to show that $CM_{\alpha,\delta} = CM \cap G_{\alpha,\delta}$. More concretely, we have to prove that

$$\{U;\ \ v-\tau \le \alpha-\delta < \alpha+\delta \le v+\tau, \quad \tau^2+v^2+\eta^2+\zeta^2+2|\tau v - \eta\cdot\zeta| \le 1\}$$

indeed is the closed convex hull of

$$\{U;\ \ \delta \le \tau \underline{+} (v-\alpha) \le \frac{1}{\delta}\ ; \quad \tau v = \eta\cdot\zeta, \ \ \tau^2+v^2+\eta^2+\zeta^2=1\}.$$

We first observe that these sets are equivalently defined by

$$\{U;\ \ \delta \le \tau \underline{+} (v-\alpha) \le \frac{1}{\delta}\ ; \quad (v\underline{+}\tau)^2 + |\eta \overline{+} \zeta|^2 \le 1\}$$

and

$$\{U;\ \ \delta \le \tau \underline{+} (v-\alpha) \le \frac{1}{\delta}\ ; \quad (v\underline{+}\tau)^2 + |\eta \overline{+} \zeta|^2 = 1\},$$

respectively. So, the first set, which is compact and convex, certainly contains the closed convex hull of the second one. Thus, it is now enough to show that any extremal point U of the first subset is indeed a point of the second one. For such a point, for either $\epsilon = 1$ or $\epsilon = -1$, we must have

$$(v+\epsilon\tau)^2 + |\eta - \epsilon\zeta|^2 = 1.$$

Assume $\epsilon = 1$ for simplicity, so that

$$(v+\tau)^2 + |\eta-\zeta|^2 = 1.$$

If $(v-\tau)^2 + |\eta+\zeta|^2 = 1$, then U belongs to the second set, as expected. Otherwise, we have $(v-\tau)^2 + |\eta+\zeta|^2 < 1$.
Let us introduce

$$U' = (\tau', v', \eta', \zeta') = (0, 0, e, -e),$$

where $e \in \mathbf{R}^d$, different from zero, is fixed. we see that for $\lambda \in \mathbf{R}$ near zero, the first set still contains $U + \lambda U'$, which contradicts the assumption that U is one of its extremal point. Indeed, for small λ, we keep

$$(v+\lambda v' - \tau - \lambda\tau')^2 + |(\eta+\lambda\eta' + (\zeta+\lambda\zeta')|^2 < 1,$$

while we conserve

$$(v+\lambda v' + \tau + \lambda\tau')^2 + |(\eta+\lambda\eta' - (\zeta-\lambda\zeta')|^2 = 1,$$

as well as

$$\delta \le (\tau+\lambda\tau')\underline{+}(v+\lambda v' - \alpha) \le \frac{1}{\delta}\ .$$

The proof of Theorem 4.1 is now complete.

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