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GLOBAL HELICALLY SYMMETRIC SOLUTIONS TO THE STOKES APPROXIMATION EQUATIONS FOR THREE-DIMENSIONAL COMPRESSIBLE VISCOUS FLOWS*

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Abstract. We prove the existence and uniqueness of global strong solutions to the Cauchy problem of the compressible Stokes approximation equations for any (specific heat ratio) $\gamma > 1$ in $\mathbb{R}^3$ when initial data are helically symmetric. Moreover, the large-time behavior of the strong solution and the existence of global weak solutions are obtained simultaneously. The proof is based on a Ladyzhenskaya interpolation type inequality for helically symmetric functions in $\mathbb{R}^3$ and uniform a priori estimtes. The present paper extends Lions' [17] and Lu, Kazhikhov and Ukai's [18] existence theorem in $\mathbb{R}^2$ to the three-dimensional helically symmetric case.

Key words. Stokes approximation equations, Helically symmetric flow, Classical solutions

AMS subject classifications. 35Q30 35Q35

1. Introduction. The Navier-Stokes system of a compressible viscous fluid for the isentropic motion has the form

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \triangle u - (\mu + \lambda)\nabla \mathrm{div} u + \nabla P(\rho) = 0. \end{cases} \tag{1.1}$$

Here $t \geq 0$ is time, $x = (x_1, ..., x_d) \in \mathbb{R}^d$ is the spatial coordinate,

$$\rho(x,t), \quad u(x,t) = (u_1(x,t), ..., u_d(x,t)), \quad P(\rho) = a\rho^\gamma \quad (a > 0, \gamma \geq 1)$$

represent the fluid density, velocity and pressure, respectively; μ, λ are constant viscosity coefficients which satisfy $\mu > 0$, $\lambda d + 2\mu \geq 0$.

In the last decades, the system (1.1) has been investigated by many mathematicians and significant progress has been made on the mathematical topics. Concerning the global existence and the large-time behavior of solutions for sufficiently small data, the system (1.1) (as well as the full compressible Navier-Stokes equations including the conservation law of energy) is well-understood in the sense that if data are small enough, then there exists a (smooth or weak) solution which is time-asymptotically stable.

The situation, however, becomes complex when data are large, and a number of important questions, for example the existence of global solutions in the case of heat-conducting gases and the uniqueness of weak solutions, still remain open. The first general result was obtained by Lions in [17], where he used the method of weak convergence and established delicate techniques to obtain global weak solutions provided the specific heat ratio γ is appropriately large, for example $\gamma \geq 3d/(d+2)$, $d = 2, 3$. Then, by combining Lions' techniques and convex analysis to reduce the integrability

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requirement of the density around the origin, Jiang and Zhang [11] showed the global existence of weak solutions for any $\gamma > 1$ to the Cauchy problem with spherically symmetric data. Recently, Feireisl, Novotný and Petzeltová [5, 6] exploit the curl-div lemma to derive certain compactness, and apply Lions' idea [17] and a technique from [11] to extend Lions' existence result to the case $\gamma > d/2$ ($d = 2, 3$). More recently, the global existence of axisymmetric and helically symmetric weak solutions for any $\gamma \geq 1$ was studied in [12, 13, 26] by combining different ideas from [17, 5, 6, 11, 24] and Lions' concentrated compactness arguments for the stationary isothermal flow, and the result in [13] extends that of Hoff [7]. More other results on the existence for the compressible isentropic Navier-Stokes equations can be found, e.g., in [30] in which strong solutions in two space dimensions are shown to exist globally in time provided that μ varies with ρ in a very specific way, and in [2, 3] where self-gravitating fluids and non-monotone pressure laws are investigated, in [4] where the motion of rigid body in a viscous compressible fluid is studied, and among others.

On the other hand, approaches to the compressible Navier-Stokes problem have been intensively sought with simplified hydrodynamic models, which possess some common features of the compressible Navier-Stokes equations and for which one could discuss the existence of *strong* solutions. One of the best-known simplifications of the Navier-Stokes system is the Stokes approximation:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \bar{\rho} u_t - \mu \triangle u - \xi \nabla \operatorname{div} u + \nabla P(\rho) = 0, \end{cases} \tag{1.2}$$

where $\bar{\rho}$ is a positive constant and $\xi = \mu + \lambda$. This is a good approximation for strongly viscous fluids. For the system (1.2), the global existence of weak solutions is known for the Cauchy problem and (general) initial boundary value problems under the restriction of $\gamma \geq 2d/(d+2)$ ($d \geq 2$) (see [17, 29]).

Recently, Lu, Kazhikhov and Ukai [18] proved the global existence of weak and classical solutions to the Cauchy problem of (1.2) with large initial data satisfying $\rho(x) \to \bar{\rho}$ as $|x| \to \infty$ in $\mathbb{R}^2$. One can easily see from the proof that their method can be applied to the case $\rho(x) \to 0$ as $|x| \to \infty$ and a similar conclusion can be obtained. However, the techniques in [18] fail to apply to the three-dimensional case because they strongly depend on the two-dimensional Sobolev embedding and interpolation inequality. Moreover, the large-time behavior of the solutions is not investigated in [18].

In this paper, we will study the existence, uniqueness and the large-time behavior of global strong solutions to the system (1.2) in $\mathbb{R}^3$ when initial data are helically symmetric.

The existence of *weak* solutions in the helically symmetric class to the Stokes approximation equations for three-dimensional compressible flows can be obtained by a procedure very similar to that in [26]. Moreover, as found out in [19] for incompressible fluids, the helically symmetric case remains essentially two-dimensional, and therefore it can be shown as we will do here, that there exists a global *strong* solution. These special features obtained in two dimensions are systematically exploited in this paper. It should be pointed out that the method used in this paper is different from those in [7, 10, 11, 12, 13]. Here we only use the properties of helical symmetry of the velocity field to derive a Ladyzhenskaya interpolation type inequality (see Lemma 3.2 below) which is the key estimate to obtain the global strong solution. Thus, we can avoid the complicated and tedious calculations in cylindrical coordinates.

This paper is organized as follows: The problem is first formulated and then the main result, Theorem 2.1, of this paper is stated in Section 2. In Section 3, we give some necessary preliminaries and a priori estimates. The global estimate of the strong solution which is the key point in the proof of the main theorem is given in Section 4.

NOTATION. (*used throughout this paper*). Let m be an integer and let $1 \le p \le \infty$. By $W^{m,p}(\mathbb{R}^d)$ $(W_0^{m,p}(\mathbb{R}^d))$ we denote the usual Sobolev space defined over $\mathbb{R}^n$. $W^{m,2}(\mathbb{R}^d) \equiv H^m(\mathbb{R}^d)$ $(W_0^{m,2}(\mathbb{R}^d) \equiv H_0^m(\mathbb{R}^d))$, $W^{0,p} \equiv L^p(\mathbb{R}^d)$ with norm $\|\cdot\|_{L^p(\mathbb{R}^d)}$.

The same letter C (sometimes used as $C(X)$ to emphasize the dependence of C on X) will denote various positive constants which do not depend on T.

2. Helically symmetric Stokes approximation. In this section we first formulate the helically symmetric form of (1.2) in $\mathbb{R}^3$ and then give the main result.

For the sake of simplicity of the presentation, let us assume that $\xi = 0$ and $\mu = a = 1$ in (1.2). It is easy to see, from the proof throughout this paper, that the case $\xi > 0$, μ, $a \ne 1$ will not arouse any new difficulties. Therefore, we consider the system:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ u_t - \triangle u + \nabla P(\rho) = 0, \end{cases} \qquad x \in \mathbb{R}^3,\ t > 0 \tag{2.1}$$

with initial conditions

$$\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}^3, \tag{2.2}$$

where $P(\rho) = \rho^\gamma$, $\gamma > 1$.

The following existence of weak solutions to (2.1), (2.2) can be found in [17]:

PROPOSITION 2.1. *Assume that* $\gamma \ge 6/5$, $\rho_0 \in L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$ *and* $u_0 \in L^2(\mathbb{R}^3)$. *Then, there exists a global weak solution* (ρ, u) *of (2.1), (2.2) satisfying*

$$\begin{aligned} &\rho \in C([0,\infty); L^1(\mathbb{R}^3)) \cap C([0,\infty); L^\gamma(\mathbb{R}^3) - w) \cap L^{5\gamma/3}(\mathbb{R}^3 \times (0,T)), \\ &u \in L^2(0,T; H^1(\mathbb{R}^3)) \cap C([0,\infty); L^2(\mathbb{R}^3) - w) \quad \textit{for any } T \in (0,\infty). \end{aligned}$$

We also quote without proof a uniqueness assertion (see [17] for the proof).

PROPOSITION 2.2. *For* $T > 0, q > 3$, *a solution* (ρ, u) *of the problem* (2.1), (2.2) *in the function space:*

$$\rho \in L^2(0,T; W^{1,q}(\mathbb{R}^3)), \quad u \in L^1(0,T; W^{1,\infty}(\mathbb{R}^3))$$

is unique on $\mathbb{R}^3 \times (0,T)$.

In this paper, we will study the helically symmetric *strong* solutions of the problem (2.1), (2.2). For this purpose, we consider solutions of (2.1), (2.2) of $2\pi/\alpha$-period in x_3:

$$\begin{cases} \rho(x_1,x_2,x_3,t) = \rho(x_1,x_2,x_3 + 2\pi/\alpha, t), \\ u(x_1,x_2,x_3,t) = u(x_1,x_2,x_3 + 2\pi/\alpha, t), \end{cases} \qquad x = (x_1,x_2,x_3) \in \mathbb{R}^3, \tag{2.3}$$

$$u(x_1,x_2,x_3) \to 0, \quad \text{as } \sqrt{x_1^2 + x_2^2} \to \infty, \tag{2.4}$$

where α is a positive real number.

For helically symmetric flow, in cylindrical coordinates (r, θ, z) $(0 < r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty)$, the velocity vector u and the density ρ do not depend on θ and z independently, but only on the linear combination $\xi = n\theta + \alpha z$ where n and α take assigned integer and real values, respectively. Namely, for helically symmetric flow,

$$\begin{aligned} &\rho(x,t) = \rho(r,\xi,t), \\ &u(x,t) = (\tfrac{x_1}{r}u_1(r,\xi,t) - \tfrac{x_2}{r}u_2(r,\xi,t), \tfrac{x_2}{r}u_1(r,\xi,t) + \tfrac{x_1}{r}u_2(r,\xi,t), u_3(r,\xi,t)) \end{aligned} \tag{2.5}$$

for some (u_1, u_2, u_3), where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $r = \sqrt{x_1^2 + x_2^2}$, ρ and (u_1, u_2, u_3) are periodic in ξ of period $4n\pi$. Then, the helical symmetry of the initial data (ρ_0, u_0) means that

$$\rho_0(x) \equiv \rho_0(r,\xi),\ \ u_0(x) \equiv (\tfrac{x_1}{r}u_1^0(r,\xi) - \tfrac{x_2}{r}u_2^0(r,\xi), \tfrac{x_2}{r}u_1^0(r,\xi) + \tfrac{x_1}{r}u_2^0(r,\xi), u_3^0(r,\xi)) \tag{2.6}$$

for some (u_1^0, u_2^0, u_3^0) with ρ_0 and (u_1^0, u_2^0, u_3^0) being periodic in ξ of period $4n\pi$.

If we still denote the fluid density and velocity in the helically symmetric case by $\rho(r,\xi,t)$ and $u = (u_1(r,\xi,t), u_2(r,\xi,t), u_3(r,\xi,t))$, respectively, the Stokes approximation equations (2.1) for helically symmetric isentropic flows are:

$$\begin{cases} \partial_t\rho + \dfrac{1}{r}\partial_r(r\rho u_1) + \dfrac{n}{r}\partial_\xi(\rho u_2) + \alpha\partial_\xi(\rho u_3) = 0, \\ \partial_t u_1 - \Big\{\dfrac{1}{r}\partial_r(r\partial_r u_1) + (\alpha^2 + \dfrac{n^2}{\xi^2})\partial_\xi^2 u_1 - \dfrac{u_1}{r^2} - \dfrac{2n}{r^2}\partial_\xi u_2\Big\} + \partial_r P = 0, \\ \partial_t u_2 - \Big\{\dfrac{1}{r}\partial_r(r\partial_r u_2) + (\alpha^2 + \dfrac{n^2}{\xi^2})\partial_\xi^2 u_2 - \dfrac{u_2}{r^2} + \dfrac{2n}{r^2}\partial_\xi u_1\Big\} + \dfrac{n}{r}\partial_\xi P = 0, \\ \partial_t u_3 - \Big\{\dfrac{1}{r}\partial_r(r\partial_r u_3) + (\alpha^2 + \dfrac{n^2}{\xi^2})\partial_\xi^2 u_3\Big\} + \alpha\partial_\xi P = 0, \end{cases} \tag{2.7}$$

together with initial and boundary conditions

$$\rho(r,\xi,0) = \rho_0(r,\xi), \quad (r,\xi) \in \mathbb{R}^+ \times \mathbb{R}, \tag{2.8}$$

$$u_1(0,\xi,t) = u_2(0,\xi,t) = 0, \quad \partial_r u_3(0,\xi,t) = 0,\ \ t \ge 0 \tag{2.9}$$

and the condition that $\rho(r,\xi,t)$, $(u_1,u_2,u_3)(r,\xi,t)$ are periodic in ξ of period $4n\pi$ (provided that $(\rho,u)(x,t)$ satisfies (2.3)). Here, for simplicity, *we have assumed that n is an even interger* (cf. Remark 2.1).

Remark 2.1. *i) The choice of the boundary conditions (2.9) follows from the fact that a smooth helically symmetric solution to (2.1)–(2.4) satisfies (2.9) at $r = 0$ automatically. Moreover, test functions of $2\pi/\alpha$-period in-x_3 in the weak form of (2.1) satisfy (2.9) automatically when (ρ, u) is helically symmetric (cf. [26] for details).*
ii) When n is odd, we have to impose the following boundary conditions, instead of (2.9),

$$(u_1 + n\partial_\xi u_2)(0,\xi,t) = (u_2 - n\partial_\xi u_1)(0,\xi,t) = \partial_\xi u_3(0,\xi,t) = 0$$

because of the same reason as in i). In this case, we have to modify (3) in Definition 2.1 appropriately (cf. Remark 2.2 in [26]). Global existence similar to Theorem 2.1 and Proposition 2.3 can be obtained without essential changes in the arguments for n being even.

To state our result more precisely, we give the definition of weak and classical solutions in the helically symmetric class:

DEFINITION 2.1. *We call (ρ, u) a weak solution in the helically symmetric class to the problem* (2.1), (2.2), *if*

(1) (ρ, u) is helically symmetric, i.e., (2.5) holds;

(2) $\rho \geq 0, \rho, P(\rho), u, \nabla u \in L^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^3)$, and the boundary conditions (2.3) *hold;*

(3) For any test function $\varphi, \psi \in C^1_0(\mathbb{R} \times \mathbb{R}^3)$, the problem (2.1), (2.2) *is satisfied in the following sense:*

$$\int_{\mathbb{R}^3} \rho_0 \varphi(\cdot, 0) dx + \int_0^\infty \int_{\mathbb{R}^3} (\rho \varphi_t + \rho u \cdot \nabla \varphi) dx dt = 0, \tag{2.10}$$

$$\int_{\mathbb{R}^3} u_0 \psi(\cdot, 0) dx + \int_0^\infty \int_{\mathbb{R}^3} (u \psi_t + P(\rho) div \psi + \nabla u \cdot \nabla \psi) dx dt = 0. \tag{2.11}$$

DEFINITION 2.2. *We call (ρ, u) a classical solution in the helically symmetric class to the problem* (2.1), (2.2), *if (ρ, u) is helically symmetric, i.e., (2.5) holds, and $\rho \in C^1([0,\infty) \times \mathbb{R}^3)$, $u \in C^2([0,\infty) \times \mathbb{R}^3)$ satisfy the boundary conditions* (2.3), *and* (2.1), (2.2) *hold everywhere in $(0,\infty) \times \mathbb{R}^3$.*

Obviously, due to periodicity in x_3, we can pay our attention to the problem (2.1), (2.2) in the following domain

$$\Omega := \mathbb{R}^2 \times \Big(0, \frac{2\pi}{\alpha}\Big).$$

The following existence of a weak solution in the helically symmetric class to the Stokes approximation equations for three-dimensional compressible flows can be obtained by a procedure very similar to that in the proof of Theorem 1.1 in [26], and therefore, we omit its proof here.

PROPOSITION 2.3. *Let $\gamma > 1$, $0 \leq \rho_0 \in L^1(\Omega) \cap L^\gamma(\Omega)$ and $u_0 \in L^2(\Omega)$ be helically symmetric and periodic in x_3 of period $2\pi/\alpha$. Then, there exists a global helically symmetric weak solution (ρ, u) of* (2.1)–(2.4), *such that for any $T, L > 0$ and $\beta \in (0,1)$,*

$$\int_0^T \int_0^1 \int_{-L}^L \rho^\gamma(r, \xi, t) r^\beta dr d\xi dt \leq C.$$

Now, the main theorem of this paper reads:

THEOREM 2.1. *Let $\gamma > 1$. Assume that*

$$\rho_0 \geq 0, \quad \rho_0 \in W^{l,q}(\Omega) \cap L^1(\Omega), \quad u_0 \in W^{l+1,q}(\Omega) \cap W^{1,2}(\Omega)$$

for some $q > 2$, $l \geq 1$, and ρ_0, u_0 are helically symmetric and periodic in x_3 of period $2\pi/\alpha$. Then problem (2.1)–(2.4) *has a unique solution (ρ, u) with $\rho \geq 0$ in the helically symmetric class, such that for any $T > 0$,*

$$\partial_t^k \rho \in L^\infty(0, T; W^{l-k,q}(\Omega)), \quad \partial_t^k u \in L^\infty(0, T; W^{l-k+1,q}(\Omega)), \tag{2.12}$$

for any $0 \le k \le l$. Furthermore, we have

$$\sup_{0\le t\le T} \|\rho(\cdot,t)\|_{L^\infty(\Omega)} \le C, \tag{2.13}$$

where C is a positive constant independent of T, and

$$\lim_{t\to\infty} (\|\rho(\cdot,t)\|_{L^\alpha(\Omega)} + \|u(\cdot,t)\|_{L^\beta(\Omega)}) = 0 \tag{2.14}$$

for $1 < \alpha < \infty$, $2 < \beta < \infty$. If $l \ge 2$, then the unique solution (ρ, u) is also a classical one.

Remark 2.2. *If $\rho \to \bar\rho$, as $|x| \to \infty$, we can define $P = \rho^\gamma - \bar\rho^\gamma$, and $B = \operatorname{div} u - \rho^\gamma + \bar\rho^\gamma$, arguing similarly to that in [18], we can obtain a similar conclusion without essential changes in arguments. Moreover, we have $\rho(x,t) > 0$ if $\rho_0(x) > 0$, and*

$$\lim_{t\to\infty} (\|\rho(\cdot,t) - \bar\rho\|_{L^\alpha(\Omega)} + \|u(\cdot,t)\|_{L^\beta(\Omega)}) = 0,$$

for $1 < \alpha < \infty, 2 < \beta < \infty$.

Remark 2.3. *Without essential changes in arguments, we can obtain a similar conclusion for viscous flow in an infinite circular pipe driven by a constant pressure gradient along the Z-axis with superimposed solid body rotation in the azimuthal direction, the so-called rotating Hagen-Poeiseuille flow or rotating pipe flow from the physical viewpoint (see [19]). In this case, the domain reads*

$$\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} < 1, 0 < x_3 < \frac{2\pi}{\alpha}\},$$

and supplemented with the following mixed boundary conditions:

$$\begin{cases} \rho(x_1,x_2,x_3,t) = \rho(x_1,x_2,x_3+2\pi/\alpha,t), \\ u(x_1,x_2,x_3,t) = u(x_1,x_2,x_3+2\pi/\alpha,t), \end{cases} \quad x \in \bar\Omega,$$
$$u(x_1,x_2,x_3) = 0, \quad \text{on} \quad \bar\Gamma,$$

where

$$\Gamma := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} = 1, 0 < x_3 < \frac{2\pi}{\alpha}\}.$$

Remark 2.4. *We should point out that the discussion in this paper depends strongly on a Ladyzhenskaya interpolation type inequality (see Lemma 3.2 below) which needs $n \neq 0$ consequently. This is to say, our conclusion cannot cover the axisymmetric case (i.e., $n = 0$).*

3. A priori estimates. In this section, we derive L^p estimates for a strong solution (ρ, u) of the problem (2.1) in the function class given in Theorem 2.1. These estimates will be used in the derivation of uniform a priori estimates in the next section.

The following energy estimate can be easily obtained by multiplying $(2.1)_2$ by u in $L^2(\Omega)$, and using $(2.1)_1$, (2.3) and (2.4):

$$\frac{d}{dt}\int_\Omega \Big[\frac{1}{2}|u|^2 + \frac{1}{\gamma-1}\rho^\gamma\Big](x,t)dx + \int_\Omega |\nabla u(x,t)|^2 dx \le 0, \quad t \ge 0. \tag{3.1}$$

From the proof of [18], it is clear that in order to obtain the global existence of strong solutions to the two-dimensional Stokes approximation equations, one needs to control the L^4-norm of the velocity u by the L^2-norm of u and ∇u in the following form

$$\|u\|_{L^4(\mathbb{R}^2)} \le C\|u\|^{1/2}_{L^2(\mathbb{R}^2)}\|\nabla u\|^{1/2}_{L^2(\mathbb{R}^2)}. \tag{3.2}$$

We remark that a similar inequality is also available in the three-dimensional case, but with different interpolation powers, that is

$$\|u\|_{L^4(\mathbb{R}^3)} \le C\|u\|^{1/4}_{L^2(\mathbb{R}^3)}\|\nabla u\|^{3/4}_{L^2(\mathbb{R}^3)}.$$

However, the power $3/4$ is too large to be utilized in obtaining the global existence of strong solutions to the three-dimensional Stokes approximation equations. To this end, for our case, we will show that when $u(x_1, x_2, x_3)$ is helically symmetric and $2\pi/\alpha$-period in x_3, then an inequality of the type (3.2) is still valid in three dimensions. In fact, we can obtain a more general inequality than (3.2), which provides the basis for our proof of the global existence.

At first, we will extend the inequality in [14] to the case $p \ge 3$:

Lemma 3.1. *For any $u \in W_0^{1,2}(\mathcal{O})$, $p \ge 3$, one has*

$$\|u\|^p_{L^p(\mathcal{O})} \le \frac{p^2}{4}\|u\|^{p-2}_{L^{p-2}(\mathcal{O})}\|u_{x_1}\|_{L^2(\mathcal{O})}\|u_{x_2}\|_{L^2(\mathcal{G})} \le \frac{p^2}{8}\|u\|^{p-2}_{L^{p-2}(\mathcal{O})}\|\nabla u\|^2_{L^2(\mathcal{O})}. \tag{3.3}$$

where $u_{x_j} := \partial_{x_j} u$, $\mathcal{O} \subset \mathbb{R}^2$ is a domain.

In particular,

$$\|u\|^4_{L^4(\mathcal{O})} \le 4\|u\|^2_{L^2(\mathcal{O})}\|u_{x_1}\|_{L^2(\mathcal{O})}\|u_{x_2}\|_{L^2(\mathcal{O})} \le 2\|u\|^2_{L^2(\mathcal{O})}\|\nabla u\|^2_{L^2(\mathcal{O})}.$$

Proof. It suffices to prove (3.3) for $u(x) \in C_0^\infty(\mathbb{R}^2)$ by virtue of the density argument.

By the inequality

$$|u(x)|^{p/2} \le \frac{p}{2}\int_{-\infty}^{x_k} |u|^{p/2-1}|u_{x_k}|\, dx_k, \quad k = 1, 2,$$

we have

$$\max_{x_k}|u(x)|^{p/2} \le \frac{p}{2}\int_{-\infty}^{\infty} |u|^{p/2-1}|u_{x_k}|\, dx_k, \quad k = 1, 2, \ \ p \ge 3,$$

and by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\int_{\mathbb{R}^2} |u|^p dx &\le \int_{\mathbb{R}^1} \max_{x_2}|u|^{p/2}dx_1 \int_{\mathbb{R}^1}\max_{x_1}|u|^{p/2}dx_2 \\
&\le \frac{p^2}{4}\int_{\mathbb{R}^2}|u|^{p/2-1}|u_{x_2}|\, dx \int_{\mathbb{R}^2}|u|^{p/2-1}|u_{x_1}|\, dx \\
&\le \frac{p^2}{4}\|u\|^{p-2}_{L^{p-2}(\mathbb{R}^2)}\|u_{x_1}\|_{L^2(\mathbb{R}^2)}\|u_{x_2}\|_{L^2(\mathbb{R}^2)} \\
&\le \frac{p^2}{8}\|u\|^{p-2}_{L^{p-2}(\mathbb{R}^2)}\|\nabla u\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}$$

which proves the lemma. ☐

LEMMA 3.2. *Let $u \in H^1(\Omega)$ be helically symmetric, periodic in x_3 of period $2\pi/\alpha$, $\alpha > 0$, and $u(x_1, x_2, x_3) \to 0$ as $r = \sqrt{x_1^2 + x_2^2} \to \infty$. Then for any integer $n \geq 1$,*

$$\|u\|_{L^4(\Omega)} \leq \Big(\frac{C}{n\alpha}\Big)^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)}^{1/2}. \tag{3.4}$$

Moreover, if instead of $u \in H^1(\Omega)$ with $u \in H_0^1(\Omega)$, then

$$C = \frac{2\alpha^2}{\pi}.$$

Proof. Define

$$C^\infty_{\rm per}(\Omega) = \{f \in C^\infty(\tilde\Omega) : f(x_1, x_2, x_3) = f(x_1, x_2, x_3 + \frac{2\pi}{\alpha}) \text{ for every } x \in \tilde\Omega\},$$
$$C^\infty_{0,\rm per}(\Omega) = \{f \in C^\infty_{\rm per}(\tilde\Omega) \text{ and supp} f \text{ is compact in } \Omega\},$$

where $\tilde\Omega$ denotes the closure of Ω. Obviously,

$$H^m_{per}(\Omega) = \text{ the closure of } C^\infty_{\rm per}(\Omega) \text{ in } H^m(\Omega);$$
$$H^m_{0,per}(\Omega) = \text{ the closure of } C^\infty_{0,\rm per}(\Omega) \text{ in } H^m(\Omega).$$

Let $u(x_1, x_2, x_3) \in C^\infty_{\rm per}(\Omega)$ be a helically symmetric function. Then by change of variables to the cylindrical coordinates, we get

$$\|u\|^4_{L^4(\Omega)} = \int_0^{\frac{2\pi}{\alpha}} \int_0^{2\pi} \int_0^\infty |\tilde u(r, \theta, x_3)|^4 r dr d\theta dx_3,$$

where $\tilde u(r, \theta, x_3) = u(x_1, x_2, x_3)$. Now, we make another change of variables

$$r = r, \ \ \xi = n\theta + \alpha x_3, \ \ z = x_3,$$

and due to helical symmetry of u, we set $v(r, \xi) \equiv \tilde u(r, \theta, x_3)$. Because $v(r, \xi)$ is 2π-periodic in ξ, we obtain

$$\|u\|^4_{L^4(\Omega)} = \frac{1}{n} \int_0^{\frac{2\pi}{\alpha}} \int_0^{2\pi} \int_0^\infty |v(r, \xi)|^4 r dr d\xi dz = \frac{2\pi}{n\alpha} \int_0^{2\pi} \int_0^\infty |v(r, \xi)|^4 r dr d\xi. \tag{3.5}$$

Let us first consider the particular case when $u(x_1, x_2, x_3) \in C^\infty_{0,\rm per}(\Omega)$ and prove (3.4). Since $v(\infty, \xi) = 0$, one can apply (3.3) directly to v and gets from (3.5) that

$$\begin{aligned}\|u\|^4_{L^4(\Omega)} \leq \frac{4\pi}{n\alpha} &\Big\{ \int_0^{2\pi} \int_0^\infty |v(r, \xi)|^2 r dr d\xi \Big\} \\ &\times \Big\{ \int_0^{2\pi} \int_0^\infty \Big[|\partial_r v(r, \xi)|^2 + \frac{1}{r^2} |\partial_\xi v(r, \xi)|^2 \Big] r dr d\xi \Big\}.\end{aligned}$$

Hence,

$$\begin{aligned}\|u\|^4_{L^4(\Omega)} \leq \frac{\alpha}{n\pi} &\Big\{ \int_0^{\frac{2\pi}{\alpha}} \int_0^{2\pi} \int_0^\infty |v(r, \xi)|^2 r dr d\xi dz \Big\} \\ &\times \Big\{ \int_0^{\frac{2\pi}{\alpha}} \int_0^{2\pi} \int_0^\infty \Big[|\partial_r v(r, \xi)|^2 + \frac{1}{r^2} |\partial_\xi v(r, \xi)|^2 \Big] r dr d\xi dz \Big\}.\end{aligned} \tag{3.6}$$

Now, we change variables back to (x_1, x_2, x_3) to get, recalling $n \geq 1$, that

$$\begin{aligned} &|\partial_r v(r,\xi)|^2 + \frac{1}{r^2}|\partial_\xi v(r,\xi)|^2 \\ &\leq 2(|\partial_{x_1} u(x_1,x_2,x_3)|^2 + |\partial_{x_2} u(x_1,x_2,x_3)|^2 + |\partial_{x_3} u(x_1,x_2,x_3)|^2). \end{aligned} \tag{3.7}$$

If we substitute (3.8) into (3.6) and use the density of $C^\infty_{0,\text{per}}(\Omega)$ in $H^1_{0,\text{per}}(\Omega)$ to obtain (3.4) in this case.

Now we treat the general case and prove (3.4). At this stage, let us recall the interpolation inequality (see e.g., [28])

$$\|\phi\|^4_{L^4(\mathbb{R}^2)} \leq C\|\phi\|^2_{L^2(\mathbb{R}^2)}\|\nabla\phi\|^2_{L^2(\mathbb{R}^2)},$$

so

$$\begin{aligned} &\int_0^{2\pi}\int_0^\infty v^4(r,\xi) r dr d\xi \leq C\Big\{\int_0^{2\pi}\int_0^\infty |v(r,\xi)|^2 r dr d\xi\Big\} \\ &\times\Big\{\int_0^{2\pi}\int_0^\infty \Big[|\partial_r v(r,\xi)|^2 + \frac{1}{r^2}|\partial_\xi v(r,\xi)|^2\Big] r dr d\xi\Big\}. \end{aligned}$$

We substitute the above inequality in (3.5), and switch back to Cartesian coordinates x_1, x_2, x_3 to get (3.4). □

Next, to bound the pressure P from above uniformly in t, one needs the following elementary lemma:

LEMMA 3.3. *Let* $y(t) \in W^{1,1}[0,T]$ *satisfy*

$$y'(t) \leq g(y) + b'(t) \ \text{on } [0,T], \quad y(0) = y^0$$

with $g \in C(\mathbb{R})$ *and* $b \in W^{1,1}(0,T)$. *If* $g(\infty) = -\infty$ *and* $b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)$ *for all* $0 \leq t_1 < t_2 \leq T$ *with some* $N_0 \geq 0$ *and* $N_1 \geq 0$, *then*

$$y(t) \leq \max\{y^0, \bar{\zeta} + N_0\} < \infty \ \text{on } [0,T] \ . \tag{3.8}$$

where $\bar{\zeta}$ *satisfies* $g(\zeta) \leq -N_1$ *for* $\zeta \geq \bar{\zeta}$.

Proof. Obviously, it is sufficient to verify the inequality (3.8) for the point $t = t_2$, such that $y(t_2) > \lambda^0 = \max\{y^0, \bar{\zeta}\}$. By virtue of the continuity of y on $[0,T]$ (respectively on $[0,\infty)$) and the initial condition $y(0) = y^0 \leq \lambda^0$, for each of such points (if any), there exists a point $t_1 \in [0, t_2)$ such that $y(t) > \lambda^0$ for $t_1 < t \leq t_2$ and $y(t_1) = \lambda^0$. Integrating the differential equation for y over (t_1, t_2) and taking into account the choice of the points t_1 and t_2 and the assumption on b, we obtain

$$y(t_2) \leq y(t_1) + \int_{t_1}^{t_2} g(y(t))dt + b(t_2) - b(t_1) \leq \lambda^0 + N_0,$$

since $g(y(t)) \leq -N_1$ on $[t_1, t_2]$. Thus, the rest of the proof follows in the same way as in [31] (cf. [1]). □

4. Uniform estimates and proof of the main theorem. In this section, we will prove Theorem 2.1. For this purpose, we first derive necessary uniform a priori estimates which can be used to continue a local strong solution globally in time.

Let (ρ, u) be a helically symmetric (strong) solution of (2.1)–(2.4) in the function class given in Theorem 2.1. For simplicity, we use the abbreviation throughout this section: $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}$.

We start with the observation that by virtue of the equation $(2.1)_1$ and the method of characteristics, we easily obtain $\rho(x,t) \geq 0$. The following lemma is the key uniform estimates.

LEMMA 4.1. *Let $\gamma > 1$. If $\rho_0 \geq 0$, $\rho_0 \in L^1(\Omega) \cap L^{3\gamma}(\Omega)$, $u_0 \in W^{1,q}(\Omega)$ for some $2 \leq q \leq 4$, and are periodic in x_3 of period $2\pi/\alpha$, then there is a positive constant C independent of t, such that*

$$\sup_{0\leq s\leq t}(\|B\|_{L^2}^2 + \|u\|_{L^4}^4)(s) + \int_0^t \|\nabla B\|_{L^2}^2 ds \leq C, \tag{4.1}$$

$$\sup_{0\leq s\leq t}\|P(\cdot,s)\|_{L^3}^3 + \int_0^t (\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4) ds \leq C, \tag{4.2}$$

$$\sup_{0\leq s\leq t}\|P(\cdot,s)\|_{L^\infty} \leq C, \tag{4.3}$$

$$\lim_{t\to\infty}(\|\rho(\cdot,t)\|_{L^\alpha} + \|u(\cdot,t)\|_{L^\beta}) = 0, \tag{4.4}$$

for all $1 < \alpha < \infty$, $2 < \beta < \infty$, where $B = \operatorname{div} u - P$ is so-called the effective viscous flux (see, for example, [17]) and $P = \rho^\gamma$.

Proof. We infer from (3.1) that

$$\frac{1}{2}\sup_{0\leq s\leq t}\|u(\cdot,s)\|_{L^2}^2 + \frac{1}{\gamma-1}\sup_{0\leq s\leq t}\|P(\cdot,s)\|_{L^1} + \int_0^t \|\nabla u\|_{L^2}^2 ds \leq C. \tag{4.5}$$

Since u is helically symmetric, we deduce from Lemma 3.2 that

$$\int_0^t \|u\|_{L^4}^4 ds \leq C. \tag{4.6}$$

In view of $(1.1)_1$, P satisfies the following equation

$$P_t + \operatorname{div}(Pu) + (\gamma-1)P\operatorname{div} u = 0. \tag{4.7}$$

So, it is easy to see that B satisfies

$$\begin{cases} B_t - \triangle B = -P_t = \operatorname{div}(Pu) + (\gamma-1)P\operatorname{div} u, \\ B(x,0) = B_0(x) \equiv \operatorname{div} u_0 - P_0. \end{cases} \tag{4.8}$$

Due to helical symmetry of (ρ, u), Lemma 3.2 leads to

$$\|B\|_{L^4}^4 \leq C\|B\|_{L^2}^2\|\nabla B\|_{L^2}^2. \tag{4.9}$$

Multiplying (4.7) by P^2 and integrating it in both space and time, we derive that

$$\|P(t)\|_{L^3}^3 + \int_0^t \|P\|_{L^4}^4 ds \leq C\|P_0\|_{L^3}^3 + C\int_0^t \|B\|_{L^4}^4 ds. \tag{4.10}$$

Now, decompose u and u_0 into $u = v+w$ and $u_0 = v_0+w_0$ with $\operatorname{div} w = \operatorname{div} w_0 = 0$ and $\operatorname{curl} v = \operatorname{curl} v_0 = 0$. One then deduces by a straightforward calculation based on (2.1) that

$$\begin{cases} v_t - \triangle v + \nabla P = 0, \\ v(x,0) = v_0(x), \quad v_t = \nabla B, \end{cases} \tag{4.11}$$

and

$$\begin{cases} w_t - \triangle w = 0, \\ w(x,0) = w_0(x). \end{cases} \tag{4.12}$$

Multiplying (4.12) by $-\triangle w$ and integrating the resulting equation in both space and time, one finds that

$$\|\nabla w(\cdot,t)\|_{L^2}^2 + \int_0^t \|\triangle w\|_{L^2}^2 ds \le \|\nabla w_0\|_{L^2}^2,$$

whence,

$$\int_0^t \|w_t\|_{L^2}^2 ds = \int_0^t \|\triangle w\|_{L^2}^2 ds \le C. \tag{4.13}$$

It follows from (4.11) and (4.8) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 &= -\int_\Omega Pu\cdot\nabla B dx + (\gamma-1)\int_\Omega PB \operatorname{div} u dx \\ &\le -\frac{1}{2}\int_\Omega P\partial_t |u|^2 dx + C\int_\Omega (|Pu|\,|w_t| + |\nabla u|^2|B| + |\nabla u|B^2)dx. \end{aligned} \tag{4.14}$$

Notice that

$$\begin{aligned} &-\frac{1}{2}\int_\Omega P\partial_t|u|^2 dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega P|u|^2 dx + \frac{1}{2}\int_\Omega |u|^2 P_t dx \\ &\le -\frac{1}{2}\frac{d}{dt}\int_\Omega P|u|^2 dx + C\int_\Omega |u|^2|\nabla u| P dx \\ &\le -\frac{1}{2}\frac{d}{dt}\int_\Omega P|u|^2 dx + C\int_\Omega (|u|^2|\nabla u|^2 + |u|^2|\nabla u|\,|B|)dx \end{aligned}$$

and

$$\frac{d}{dt}\|u\|_{L^4}^4 \le C\int_\Omega (|u|^2|\nabla u|^2 + |u|^2|\nabla u||B|)dx. \tag{4.15}$$

Using these estimates, one infers from (4.14) that

$$\begin{aligned} &\int_\Omega (P|u|^2 + |B|^2 + |u|^4)dx + \int_0^t \|\nabla B\|_{L^2}^2 ds \\ &\quad\le C\int_0^t\int_\Omega (|\nabla u||B|^2 + |\nabla u|^2|B| + |u|^2|\nabla u||B| + |u|^2|\nabla u|^2)dxds \\ &\qquad + C\int_0^t \|Pu\|_{L^2}\|w_t\|_{L^2} ds + C \equiv I_1 + I_2 + I_3 + I_4 + I_5 + C. \end{aligned} \tag{4.16}$$

Now, we estimate every term on the right-hand side of (4.16). First, the inequality (4.9) implies that

$$\begin{aligned} I_1 &\le \int_0^t \|\nabla u\|_{L^2}\|B\|_{L^4}^2 ds \le C\int_0^t \|\nabla u\|_{L^2}\|B\|_{L^2}\|\nabla B\|_{L^2} ds \\ &\le \varepsilon\int_0^t \|\nabla B\|_{L^2}^2 ds + C_\varepsilon \int_0^t \|\nabla u\|_{L^2}^2\|B\|_{L^2}^2 ds. \end{aligned}$$

Then, we infer from (4.5), (4.9) and Lemma 3.2 that

$$
\begin{aligned}
I_2+I_3 &\leq \int_0^t (\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}\|B\|_{L^4}+\|u\|_{L^4}^2\|\nabla u\|_{L^4}\|B\|_{L^4})ds\\
&\leq C\int_0^t (\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}\|B\|_{L^4}+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}\|B\|_{L^4})ds\\
&\leq C\int_0^t \|\nabla u\|_{L^2}\|\nabla u\|_{L^4}\|B\|_{L^4}ds\\
&\leq C(\int_0^t \|\nabla u\|_{L^4}^4 ds)^{1/4}(\int_0^t \|\nabla u\|_{L^2}^{4/3}\|B\|_{L^2}^{2/3}\|\nabla B\|_{L^2}^{2/3}ds)^{3/4}\\
&\leq C(\int_0^t \|\nabla u\|_{L^4}^4 ds)^{1/4}(\int_0^t \|\nabla B\|_{L^2}^2 ds)^{1/4}(\int_0^t \|\nabla u\|_{L^2}^2\|B\|_{L^2}ds)^{1/2}\\
&\leq \varepsilon(\int_0^t \|\nabla u\|_{L^4}^4 ds)^{1/2}+\varepsilon\int_0^t \|\nabla B\|_{L^2}^2 ds+C_\varepsilon\int_0^t \|\nabla u\|_{L^2}^2\|B\|_{L^2}^2 ds
\end{aligned}
$$

for any small $\varepsilon > 0$.

If we use (4.5), Lemma 3.2 and Hölder's inequality, we deduce that

$$
\begin{aligned}
I_4 &\leq C\int_0^t \|u\|_{L^6}^2\|\nabla u\|_{L^3}^2 ds\\
&\leq C\int_0^t \|u\|_{L^4}^{4/3}\|\nabla u\|_{L^4}^{4/3}\|\nabla u\|_{L^2}^{4/3}ds\\
&\leq \varepsilon(\int_0^t \|\nabla u\|_{L^4}^4 ds)^{1/2}+C_\varepsilon\int_0^t \|u\|_{L^4}^4\|\nabla u\|_{L^2}^2 ds.
\end{aligned}
$$

Finally, from (4.6) and (4.13), it follows that

$$
I_5 \leq \varepsilon(\int_0^t \|P\|_{L^4}^4 ds)^{1/2}+C_\varepsilon.
$$

Next, we show the following estimate:

$$
(\int_0^t \|\nabla u\|_{L^4}^4 ds)^{1/2} \leq C(\int_0^t \|P\|_{L^4}^4 ds)^{1/2}+C. \tag{4.17}
$$

In fact, if we define φ and $\bar{u}$ by solving the following equations

$$
\begin{aligned}
&\varphi_t-\triangle\varphi = P, \quad \varphi(x,0)=0,\\
&\bar{u}_t-\triangle\bar{u}=0, \quad \bar{u}(x,0)=u_0(x),
\end{aligned}
$$

we then obtain

$$
u=-\nabla\varphi+\bar{u},
$$

and by the well-known parabolic estimates ([15, 22]), we see that

$$
\|\varphi_t\|_{L^4((0,T)\times\Omega)}+\|\nabla^2\varphi\|_{L^4((0,T)\times\Omega)} \leq C\|P\|_{L^4((0,T)\times\Omega)}
$$

and

$$
\|\nabla\bar{u}\|_{L^4((0,T)\times\mathbb{R}^3)} \leq C\|u_0\|_{W^{1,4}(\Omega)}.
$$

Thus, we have proved (4.17).

So, denoting

$$A_1(t)=\sup_{0\le s\le t}\int_\Omega(|B|^2+|u|^4+P|u|^2)(x,s)dx,\quad A_2(t)=\int_0^t\|\nabla B\|_{L^2}^2ds,$$

and inserting $I_1, I_2, ..., I_5$ into (4.16), we get from (4.17) that

$$\begin{aligned}&A_1(t)+A_2(t)\\&\le C_\varepsilon+C\varepsilon(\int_0^t\|B\|_{L^4}^4ds)^{1/2}+C\varepsilon A_2(t)+C\varepsilon\int_0^t\|\nabla u\|_{L^2}^2(\|B\|_{L^2}^2+\|u\|_{L^4}^4)ds\\&\le C_\varepsilon+C\varepsilon(A_1(t)+A_2(t))+C_\varepsilon\int_0^t\|\nabla u\|_{L^2}^2(\|B\|_{L^2}^2+\|u\|_{L^4}^4)ds,\end{aligned}$$

which, by choosing ε appropriately small and recalling the definition of $A_1(t)$ and $A_2(t)$, yields

$$\sup_{0\le s\le t}(\|B\|_{L^2}^2+\|u\|_{L^4}^4)+\int_0^t\|\nabla B\|_{L^2}^2ds\le C+C\int_0^t\|\nabla u\|_{L^2}^2(\|B\|_{L^2}^2+\|u\|_{L^4}^4)ds.$$

Hence, from (4.5) and Gronwall's inequality, it follows that

$$\sup_{0\le s\le t}(\|B\|_{L^2}^2+\|u\|_{L^4}^4)+\int_0^t\|\nabla B\|_{L^2}^2ds\le C. \tag{4.18}$$

We use (4.9), (4.10), (4.17) and (4.18) to conclude

$$\sup_{0\le t\le T}\|P(\cdot,s)\|_{L^3}^3+\int_0^t(\|P\|_{L^4}^4+\|\nabla u\|_{L^4}^4)ds\le C. \tag{4.19}$$

The Galiardo-Nirenberg inequality and (4.18)-(4.19) give that

$$\begin{aligned}\int_0^T\|u\|_{L^\infty}^{16/3}ds&\le C\int_0^T\|\nabla u\|_{L^4}^4\|u\|_{L^4}^{4/3}ds\\&\le C\int_0^T\|\nabla u\|_{L^4}^4ds\le C.\end{aligned}$$

From this and (4.19), we find that

$$\int_0^T\|Pu\|_{L^3}^{16/3}ds\le\sup_{0\le s\le T}\|P\|_{L^3}^{16/3}\int_0^T\|u\|_{L^\infty}^{16/3}ds\le C.$$

and

$$\begin{aligned}&\int_0^T\|Pu\|_{L^4}^{16/7}ds\le\int_0^T\|u\|_{L^\infty}^{16/7}\|P\|_{L^4}^{16/7}ds\\&\le C(\int_0^T\|u\|_{L^\infty}^{16/3})^{3/7}(\int_0^T\|P\|_{L^4}^4)^{4/7}\le C.\end{aligned}$$

Thus,

$$\int_0^T\|Pu\|_{L^{64/19}}^{64/19}ds\le C(\int_0^T\|Pu\|_{L^3}^{16/3}ds)^{27/76}(\int_0^T\|Pu\|_{L^4}^{16/7}ds)^{49/76}\le C. \tag{4.20}$$

From (4.19), it follows that

$$(\int_0^T \|P\mathrm{div}u\|_{L^2}^2 ds)^{1/2} \le C(\int_0^T \|P\|_{L^4}^4 ds)^{1/4}(\int_0^T \|\nabla u\|_{L^4}^4 ds)^{1/4} \le C.$$

In view of (4.9) and (4.19), we find that

$$\sup_{0\le s\le T} \|P\|_{L^k} \le C, \qquad \forall\, 1 \le k \le 3.$$

Thus

$$\int_0^T \|P\mathrm{div}u\|_{L^{156/103}}^4 ds \le C \sup_{0\le s\le T} \|P\|_{L^{39/16}}^4 ds \int_0^T \|\nabla u\|_{L^4}^4 ds \le C,$$

and

$$\int_0^T \|P\mathrm{div}u\|_{L^{192/121}}^{64/19} ds \le (\int_0^T \|P\mathrm{div}u\|_{L^2}^2 ds)^{6/19}(\int_0^T \|P\mathrm{div}u\|_{L^{156/103}}^4 ds)^{13/19} \le C.$$

We have by the classical estimates for the parabolic equation (4.8) that

$$\begin{aligned}\int_0^T \|\nabla B\|_{L^{64/19}}^{64/19} ds &\le C + C\int_0^T (\|\nabla^{-1}\mathrm{div}\,(Pu)\|_{L^{64/19}}^{64/19} + \|\nabla^{-1}(P\mathrm{div}\,u)\|_{L^{64/19}}^{64/19}) ds \\ &\le C + C\int_0^T \|Pu\|_{L^{64/19}}^{64/19} ds + \int_0^T \|P\mathrm{div}\,u\|_{L^{192/121}}^{64/19} ds \le C.\end{aligned}$$

Hence, the above estimate, together with the Galiardo-Nirenberg inequality and (4.18), results in

$$\int_0^T \|B\|_{L^\infty}^{206/57} ds \le C\int_0^T \|B\|_{L^2}^{14/57}\|\nabla B\|_{L^{64/19}}^{64/19} ds \le C. \tag{4.21}$$

Setting $D_t w = w_t + u\cdot\nabla w$, we conclude from (2.1) and (4.7) that

$$D_t \log(P+\varepsilon) = -\gamma P - \gamma\frac{P}{P+\varepsilon}B + \gamma\frac{P\varepsilon}{P+\varepsilon}, \qquad \varepsilon > 0 \text{ small.} \tag{4.22}$$

Now, we transfer (4.22) to the Lagrangian coordinates. Taking $y = \log(P+\varepsilon)$, $g(y) = -\gamma e^y$ and

$$b(t) = \int_0^t \gamma\{\varepsilon - \frac{P}{P+\varepsilon}B + \frac{P\varepsilon}{P+\varepsilon}\}(x(s), s) ds.$$

Observing that

$$\frac{P}{P+\varepsilon} \le 1,\ \ \frac{P\varepsilon}{P+\varepsilon} \le \varepsilon,$$

we utilize (4.21) and Hölder's inequality to deduce that

$$|b(t_2) - b(t_1)| \le \int_0^T \|B(\cdot,s)\|_{L^\infty}^{206/57} ds + C(t_2 - t_1) \le C + C(t_2 - t_1)$$

for $0 \le t_1 < t_2 \le T$, which combined with Lemma 3.3 proves (4.3).

To complete the proof, it remains to verify (4.4). First, we claim that

$$\lim_{t\to\infty}(\|P(\cdot,t)\|_{L^4} + \|u(\cdot,t)\|_{L^4}) = 0. \tag{4.23}$$

In fact, setting

$$h(t) = \|P\|_{L^4}^4 + \|u\|_{L^4}^4,$$

and from (4.6) and (4.19), one gets

$$\int_0^\infty h(t)dt \le C.$$

Therefore, we derive from (4.7) and (4.15) that

$$\begin{aligned}\int_0^\infty |h'(t)|dt &\le C\int_0^\infty \{\|\rho\|_{L^\infty}^{2\gamma}\|P\|_{L^4}^2\|\nabla u\|_{L^2} + \|\nabla u\|_{L^4}^2\|u\|_{L^4}^2 + \|B\|_{L^4}\|\nabla u\|_{L^4}\|u\|_{L^4}^2\}dt\\ &\le C\int_0^\infty \{\|\rho\|_{L^\infty}^{4\gamma}\|P\|_{L^4}^4 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|u\|_{L^4}^4 + \|B\|_{L^4}^4\}dt \le C.\end{aligned}$$

Consequently,

$$\lim_{t\to\infty} h(t) = 0.$$

This shows that (4.23) remains valid. Moreover, (4.23) and (4.3) lead to

$$\lim_{t\to\infty}\|\rho(\cdot,t)\|_{L^\alpha} = 0, \tag{4.24}$$

for any $\alpha \in [1,\infty)$.

(4.3) and (4.19) imply that P satisfies

$$\int_0^\infty \|P(\cdot,t)\|_{L^p}^p dt \le C, \qquad \forall\, 2 \le p < \infty.$$

Hence,

$$\sup_{0\le t<\infty}\|u(t)\|_{L^q} \le C, \tag{4.25}$$

for all $2 \le p < \infty$.

Finally, the limit (4.4) follows easily from (4.5), (4.24), (4.25) and (4.23). This completes the proof. □

Remark 4.1. *Recently, Li and Xin [16] proved the same conclusion for initial boundary value problems with space-periodicity condition or no-stick boundary condition, and for the Cauchy problem in two dimensions.*

Proof of Theorem 2.1. With the help of Lemma 4.1, we are able to derive the estimates for higher derivatives. Indeed, from (2.1) we know

$$u(x,t) = S(t)u_0(x) - \int_0^t S(t-\tau)\nabla P(\tau,x)d\tau, \tag{4.26}$$

where $S(t)$ denotes the heat operator. It follows from the classical estimates of the heat semigroup (see, e.g., Appendix F of [24] and Lemma 4.1 of [18]) that for $T>0$,

$$\|\nabla u\|_{L^\infty(\Omega\times(0,T))} \le C(T)\|P\|_{L^\infty(\Omega\times(0,T))}\log(1+\|\nabla P\|_{L^\infty(0,T;L^q(\Omega))}), \quad q>2. \tag{4.27}$$

Differentiation of the equation (4.7) with respect to x gives

$$\nabla P_t + (u\cdot\nabla)\nabla P + (\nabla P\cdot\nabla)u + \gamma \operatorname{div} u\nabla P + \gamma P\nabla \operatorname{div} u = 0.$$

Multiplying this by $q|\nabla P|^{q-2}\nabla P$ for $q>2$, using Lemma 4.1 and integrating over $\mathbb{R}^3$, one concludes

$$\frac{d}{dt}\|\nabla P\|_{L^q}^q \le C(T)\Big(\|\nabla^2 u\|_{L^q}^q + (\|\nabla u\|_{L^\infty(\Omega\times(0,T))}+1)\|\nabla P\|_{L^q}^q\Big). \tag{4.28}$$

Then, for the non-negative function $y_q(t)=\sup_{0<\tau<t}\|\nabla P\|_{L^q}^q$, we obtain by integrating (4.28) over $(0,t)$ and using (4.27) that

$$y_q(t) \le y_q(0) + C\int_0^t y_q(\tau)[1+\log(1+y_q(\tau))^{1/q}]d\tau. \tag{4.29}$$

Here we have used the following classical parabolic estimate for $(2.1)_2$

$$\|u_t\|_{L^q(\Omega\times(0,T))} + \|\nabla^2 u\|_{L^q(\Omega\times(0,T))} \le C + \|\nabla P\|_{L^q(\Omega\times(0,T))}. \tag{4.30}$$

The inequality (4.29) gives immediately a bound for $y_q(t)$, and consequently, (4.30) implies

$$\|u_t\|_{L^q(\Omega\times(0,T))} + \|\nabla^2 u\|_{L^q(\Omega\times(0,T))} \le C,$$

while the equation (4.7) leads to

$$\sup_{0\le t\le T}\|\frac{\partial P}{\partial t}\|_{L^q} \le C(T), \qquad \forall\, q>2.$$

If we differentiate the equation (4.7) with respect to x several times, we can obtain bounds of the higher derivatives. Thus, in view of (3.1) and Lemma 4.1, we have established the necessary a priori estimates needed in Theorem 2.1. Once the a priori estimates have been established, the existence of global strong or classical solutions can be obtained by a standard procedure. One first establishes the existence and uniqueness of a local strong or classical solution by means of a straightforward contraction argument and then uses the derived a priori estimates to continue this local solution globally in time. The proof of Theorem 2.1 is complete. □

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