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A SECOND ORDER THREE-POINT BOUNDARY VALUE PROBLEM WITH MIXED NONLINEAR BOUNDARY CONDITIONS*

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Abstract. We apply the generalized quasilinearization method to a second order three-point boundary value problem involving mixed nonlinear boundary conditions and obtain a monotone sequence of approximate solutions converging to the unique solution of the problem possessing a convergence of order $k(k \geq 2)$.

Key words. Quasilinearization, Three-point boundary value problem, Rapid convergence

AMS subject classifications. 34A37, 34B15

1. Introduction. The method of quasilinearization developed by Bellman and Kalaba [1] and generalized by Lakshmikantham [2-3] later on, has been studied and extended in several diverse disciplines. In fact, it is generating a rich history and an extensive bibliography can be found in [4-10].

Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [11], have been addressed by many authors, for example, see [12-14]. In particular, Eloe and Gao [15] discussed the quasilinearization method for a three-point boundary value problem. In this paper, we study the generalized quasilinearization method for a second order three-point boundary value problem with mixed nonlinear boundary conditions. In fact, a sequence of approximate solutions converging monotonically to a solution of the nonlinear three-point problem with the order of convergence $k(k \geq 2)$ has been presented.

2. Preliminary results. Consider a three-point boundary value problem with mixed nonlinear boundary conditions

$$x''(t) = f(t, x(t)), \tag{1.1}$$

$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = g(x(\frac{1}{2})), \tag{1.2}$$

where f is continuous with $f_x > 0$ on $[0,1] \times R$, $p, q > 0$ with $p > 1$ and $g : R \longrightarrow R$ is continuous. By Green's function method, the solution, $x(t)$ of (1.1)-(1.2) can be written as

$$x(t) = a(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + g(x(\frac{1}{2}))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] + \int_0^1 G(t,s)f(s,x(s))ds,$$

where the Green's function $G(t,s)$ for the mixed three-point boundary value problem is given by

$$G(t,s) = \frac{1}{(p^2+2pq)} \begin{cases} (pt+q)(p(s-1)-q), & \text{if } 0 \leq t \leq s \leq 1, \\ (p(t-1)-q)(ps+q), & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

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Notice that $G(t,s) < 0$ on $[0,1] \times [0,1]$.

We say that $\alpha \in C^2[0,1]$ is a lower solution of the boundary value problem (1.1)-(1.2) if

$$\alpha''(t) \geq f(t,\alpha),\ t \in [0,1],$$

$$p\alpha(0) - q\alpha'(0) \leq a, \qquad p\alpha(1) + q\alpha'(1) \leq g(\alpha(\frac{1}{2})),$$

and $\beta \in C^2[0,1]$ is an upper solution of the boundary value problem (1.1)-(1.2) if

$$\beta''(t) \leq f(t,\beta),\ t \in [0,1],$$

$$p\beta(0) - q\beta'(0) \geq a, \qquad p\beta(1) + q\beta'(1) \geq g(\beta(\frac{1}{2})).$$

THEOREM 1. Assume that f is continuous with $f_x > 0$ on $[0,1] \times R$ and g is continuous with $0 \leq g' < 1$ on R. Let β and α be the upper and lower solutions of (1.1)-(1.2) respectively. Then $\alpha(t) \leq \beta(t),\ t \in [0,1]$.

Proof. Define $h(t) = \alpha(t) - \beta(t)$. For the sake of contradiction, we suppose that $h(t) > 0$ for some $t \in [0,1]$. First we take $t_0 \in (0,1)$. Then by the definition of lower and upper solutions together with $f_x > 0$, we obtain

$$h''(t_0) = \alpha''(t_0) - \beta''(t_0) \geq f(t_0,\alpha(t_0)) - f(t_0,\beta(t_0)) > 0. \tag{1.3}$$

By the standard methodology, let $h(t)$ have a local positive maximum at $t_0 \in (0,1)$, then $h'(t_0) = 0$ and $h''(t_0) \leq 0$, which contradicts (1.3). Thus, for $t_0 \in (0,1)$, we have $\alpha(t) \leq \beta(t)$. Now, suppose that $h(t)$ has a local positive maximum at $t_0 = 1$, then $h'(1) = 0$ and $h''(1) < 0$. On the other hand, by definition of lower and upper solutions and in view of the condition $0 \leq g' < 1$, we find that

$$\begin{aligned} ph(1) + qh'(1) &\leq g(\alpha(\frac{1}{2})) - g(\beta(\frac{1}{2})) \\ &= \frac{g(\alpha(\frac{1}{2})) - g(\beta(\frac{1}{2}))}{\alpha(\frac{1}{2}) - \beta(\frac{1}{2})}[\alpha(\frac{1}{2}) - \beta(\frac{1}{2})] \\ &\leq \alpha(\frac{1}{2}) - \beta(\frac{1}{2}) \\ &= h(\frac{1}{2}). \end{aligned}$$

Thus, $ph(1) \leq h(\frac{1}{2})$ or $h(1) < h(\frac{1}{2})$ for $p > 1$, which is a contradiction. Similarly, we get a contradiction for $t_0 = 0$. Hence we conclude that $\alpha(t) \leq \beta(t)$ on $[0,1]$.

THEOREM 2. Assume that f is continuous on $[0,1] \times R$ with $f_x > 0$ and g is continuous on R satisfying $0 \leq g' < 1$. Further, we assume that there exist an upper solution β and a lower solution α of (1.1)-(1.2) such that $\alpha(t) \leq \beta(t),\ t \in [0,1]$. Then there exists a solution $x(t)$ of (1.1)-(1.2) satisfying $\alpha(t) \le x(t) \le \beta(t),\ t \in [0,1]$.

Proof. Define F and G by

$$F(t,x) = \begin{cases} f(t,\beta) + \frac{x-\beta}{1+x-\beta}, & \text{if } x(t) > \beta(t), \\ f(t,x), & \text{if } \alpha(t) \le x(t) \le \beta(t), \\ f(t,\alpha) + \frac{x-\alpha}{1+|x-\alpha|}, & \text{if } x(t) < \alpha(t), \end{cases}$$

$$G(x) = \begin{cases} g(\beta(\frac{1}{2})), & \text{if } x > \beta(\frac{1}{2}), \\ g(x), & \text{if } \alpha(\frac{1}{2}) \le x \le \beta(\frac{1}{2}), \\ g(\alpha(\frac{1}{2})), & \text{if } x < \alpha(\frac{1}{2}). \end{cases}$$

Since $F(t,x)$ and $G(x)$ are continuous and bounded, a standard application of Schauder's fixed point theorem ensures the existence of a solution, x of the problem

$$x''(t) = F(t,x(t)),\ t \in [0,1],$$

$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = G(x(\frac{1}{2})).$$

In order to complete the proof, we need to show that $\alpha(t) \le x(t) \le \beta(t)$ on $[0,1]$ which can be done using the procedure employed in the proof of theorem 1. In this case, G satisfies $0 \le G' \le 1$ on $[\alpha(\frac{1}{2}), \beta(\frac{1}{2})]$.

Remark. In case of the problem $-x''(t) = f(t,x(t))$, we require the condition $f_x < 0$ and the corresponding Green's function $G(t,s)$ is nonnegative, that is,

$$G(t,s) \ge \frac{q^2}{(p^2+2pq)},\ (t,s) \in [0,1]\times[0,1].$$

3. Main result.

Theorem 3. Assume that

(A_1) $\frac{\partial^i}{\partial x^i} f(t,x),\ i = 0,1,2,...,k$, are continuous on $[0,1]\times R$ satisfying $\frac{\partial^i}{\partial x^i} f(t,x) \ge 0,\ i = 0,1,2,...,k-1$, with $\frac{\partial^k}{\partial x^k}(f(t,x)+\phi(t,x)) \le 0$, where $\frac{\partial^i}{\partial x^i}\phi(t,x),\ i = 0,1,2,...,k$ are continuous and $\frac{\partial^k}{\partial x^k}\phi(t,x) \le 0$ for some function $\phi(t,x)$.

(A_2) $\alpha, \beta \in C^2[0,1], R]$ are lower and upper solutions of (1.1)-(1.2) respectively.

(A_3) $\frac{d^i}{dx^i} g(x),\ i = 0,1,2,...,k$, are continuous on R satisfying $0 \le \frac{d^i}{dx^i} g(x) < \frac{M}{(\beta-\alpha)^{i-1}}$ with $\frac{d^k}{dx^k} g(x) \ge 0$ and $0 < M < \frac{1}{3}$.

Then there exists a monotone sequence of approximate solutions $\{w_n\}$ converging to the unique solution, x of (1.1)-(1.2) with the order of convergence $k (k \ge 2)$.

Proof. Define $F : [0,1]\times R \longrightarrow R$ by

$$F(t,x) = f(t,x) + \phi(t,x).$$

Using (A_1), (A_3) and the generalized mean value theorem, we obtain

$$f(t,x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t,y)\frac{(x-y)^i}{i!} - \phi(t,x),$$

$$g(x) \geq \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}.$$

Set

$$F^{**}(t,x,y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t,y) \frac{(x-y)^i}{i!} - \phi(t,x), \tag{1.4}$$

and

$$h^*(x,y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}. \tag{1.5}$$

Observe that $F^{**}(t,x,y)$ and $h^*(x,y)$ are continuous and

$$f(t,x) = \min_y F^{**}(t,x,y), \qquad f(t,x) = F^{**}(t,x,x), \tag{1.6}$$

$$g(x) = \max_y h^*(x,y), \qquad g(x) = h^*(x,x). \tag{1.7}$$

Expanding $\phi(t,x)$ by Taylor's theorem, (1.4) takes the form

$$F^{**}(t,x,y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^k}{k!}. \tag{1.8}$$

Differentiating (1.8) and using (A_1), we get

$$F_x^{**}(t,x,y) > \sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^{i-1}}{(i-1)!} \geq 0, \tag{1.9}$$

which implies that $F_x^{**}(t,x,y)$ is increasing in x for each $(t,y) \in [0,1] \times R$. Similarly, differentiation of (1.5) together with (A_3) yields

$$h_x^*(x,y) = \sum_{i=1}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!},$$

which is clearly nonnegative and further

$$\begin{aligned} h_x^*(x,y) &= \sum_{i=1}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!} \\ &\leq \sum_{i=1}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(\beta-\alpha)^{i-1}}{(i-1)!} \\ &\leq \sum_{i=1}^{k-1} \frac{M}{(i-1)!} < M(1 + \sum_{i=1}^{k-2} \frac{1}{2^{i-1}}) = M(3 - \frac{1}{2^{k-3}}) \\ &< 3M < 1, \end{aligned}$$

where $\alpha \leq y \leq x \leq \beta$. Select $\alpha = w_0$ and consider the following mixed problem

$$x'' = F^{**}(t, x(t), w_0(t)), \ t \in [0,1], \tag{1.10}$$

$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = h^*(x(\frac{1}{2}), w_0(\frac{1}{2})). \tag{1.11}$$

Using (A_3), (1.6) and (1.7), we obtain

$$w_0'' \geq f(t, w_0) = F^{**}(t, w_0, w_0), t \in [0, 1],$$

$$pw_0(0) - qw_0'(0) \leq a, \qquad pw_0(1) + qw_0'(1) \leq g(w_0(\frac{1}{2})) = h^*(w_0(\frac{1}{2}), w_0(\frac{1}{2})),$$

and

$$\beta'' \leq f(t, \beta) \leq F^{**}(t, \beta, w_0), t \in [0, 1],$$

$$p\beta(0) - q\beta'(0) \geq a, \qquad p\beta(1) + q\beta'(1) \geq g(\beta(\frac{1}{2})) \geq h^*(\beta(\frac{1}{2}), w_0(\frac{1}{2})),$$

which imply that w_0 and β are lower and upper solutions of (1.10)-(1.11) respectively. It follows by Theorems 1 and 2 that there exists a unique solution, w_1 of (1.10)-(1.11) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), t \in [0, 1].$$

Now, we consider the problem

$$x'' = F^{**}(t, x(t), w_1(t)), t \in [0, 1], \tag{1.12}$$

$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = h^*(x(\frac{1}{2}), w_1(\frac{1}{2})). \tag{1.13}$$

Again, using (A_3), (1.6) and (1.7), we get

$$w_1'' = F^{**}(t, w_1, w_0) \geq F^{**}(t, w_1, w_1), t \in [0, 1],$$

$$pw_1(0) - qw_1'(0) \leq a, \qquad pw_1(1) + qw_1'(1) = h^*(w_1(\frac{1}{2}), w_0(\frac{1}{2})) \leq h^*(w_1(\frac{1}{2}), w_1(\frac{1}{2})),$$

and

$$\beta'' \leq f(t, \beta) \leq F^{**}(t, \beta, w_1), t \in [0, 1],$$

$$p\beta(0) - q\beta'(0) \geq a, \qquad p\beta(1) + q\beta'(1) \geq g(\beta(\frac{1}{2})) \geq h^*(\beta(\frac{1}{2}), w_1(\frac{1}{2})),$$

implying that w_1 and β are lower and upper solutions of $(1.12) - (1.13)$ respectively. By the earlier arguments, we find a solution, w_2 of $(1.12) - (1.13)$ such that

$$w_0(t) \leq w_2(t) \leq \beta(t), \ t \in [0, 1].$$

Continuing this process successively, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq ... \leq w_n(t) \leq \beta(t), \ t \in [0, 1],$$

where each element w_n of the sequence is a solution of the following problem

$$x'' = F^{**}(t, x(t), w_{n-1}(t)),\ t \in [0,1],$$

$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = h^*(x(\frac{1}{2}), w_{n-1}(\frac{1}{2})),$$

and is given by

$$\begin{aligned} w_n(t) &= a(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + h^*(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2}))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] \\ &+ \int_0^1 G(t,s)F^{**}(s, w_n, w_{n-1})ds. \end{aligned} \tag{1.14}$$

In view of the fact that $[0,1]$ is compact and the monotone convergence is pointwise, it follows that the convergence of the sequence is uniform. If $x(t)$ is the limit point of the sequence, then passing onto the limit $n \to \infty$, (1.14) gives

$$\begin{aligned} x(t) &= a(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + h^*(x(\frac{1}{2}), x(\frac{1}{2}))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] \\ &+ \int_0^1 G(t,s)F^{**}(s, x(s), x(s))ds \\ &= a(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + g(x(\frac{1}{2}))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] \\ &+ \int_0^1 G(t,s)f(s, x(s))ds. \end{aligned}$$

Thus $x(t)$ is the solution of (1.1)-(1.2).

Now, we show that the convergence of the sequence of iterates is of order $k (k \geq 2)$. For that, we set $e_n(t) = x(t) - w_n(t),\ a_n(t) = w_{n+1}(t) - w_n(t),\ t \in [0,1]$ and note that $e_n(t) \geq 0,\ a_n(t) \geq 0,\ e_n(t) - a_n(t) = e_{n+1}(t)$. Also $e_n(t) \geq a_n(t)$ and hence by induction $e_n^k(t) \geq a_n^k(t)$. Further

$$pe_n(0) - qe_n'(0) = 0,\ pe_n(1) + qe_n'(1) = g(x(\frac{1}{2})) - h^*(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})).$$

Using the generalized mean value theorem, we have

$$\begin{aligned} e_{n+1}''(t) &= x'' - w_{n+1}'' \\ &= \textstyle\sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{(x-w_n)^i}{i!} + \frac{\partial^k}{\partial x^k} f(t,\xi) \frac{(x-w_n)^k}{k!} \\ &- \textstyle\sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{(w_{n+1}-w_n)^i}{i!} + \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(w_{n+1}-w_n)^k}{k!} \\ &= \textstyle\sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{(e_n^i - a_n^i)}{i!} + \frac{\partial^k}{\partial x^k} f(t,\xi) \frac{(e_n)^k}{k!} + \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(a_n)^k}{k!} \\ &\geq (\sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{1}{i!} \sum_{j=0}^{k-1} e_n^j a_n^{i-1-j}) e_{n+1} + (\frac{\partial^k}{\partial x^k} f(t,\xi) + \frac{\partial^k}{\partial x^k} \phi(t,\xi)) \frac{(e_n)^k}{k!} \\ &\geq \frac{\partial^k}{\partial x^k} F(t,\xi) \frac{(e_n)^k}{k!} \geq -M \|e_n|^k, \end{aligned} \tag{1.15}$$

where M is a bound on $\frac{1}{k!}\frac{\partial^k}{\partial x^k}F(t,\xi)$ for $t \in [0,1]$. Thus, in view of (1.15), we have

$$\begin{aligned} e_{n+1}(t) &= (g(x(\frac{1}{2})) - h^*(w_{n+1}(\frac{1}{2}), w_n(\frac{1}{2})))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] + \int_0^1 G(t,s)e''_{n+1}(t)ds \\ &\le (g(x(\frac{1}{2})) - h^*(w_{n+1}(\frac{1}{2}), w_n(\frac{1}{2})))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] \\ &+ M\|e_n\|^k \int_0^1 |G(t,s)|ds \\ &= [\sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{(x(\frac{1}{2}) - w_n(\frac{1}{2}))^i}{i!} + \frac{d^k}{dx^k}g(\xi(\frac{1}{2}))\frac{(x(\frac{1}{2}) - w_n(\frac{1}{2}))^k}{k!} \\ &- \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))^i}{i!}][\frac{t}{p+2q} + \frac{q}{p^2+2pq}] \\ &+ M_1\|e_n\|^k \\ &= [\sum_{i=1}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{(e_n^i(\frac{1}{2}) - a_n^i(\frac{1}{2}))}{i!} + \frac{d^k}{dx^k}g(\xi(\frac{1}{2}))\frac{(e_n(\frac{1}{2}))^k}{k!}][\frac{t}{p+2q} \\ &+ \frac{q}{p^2+2pq}] + M_1\|e_n\|^k \\ &= [\sum_{i=1}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2}))\frac{1}{i!}\sum_{j=0}^{k-1} e_n^j(\frac{1}{2})a_n^{i-1-j}(\frac{1}{2})e_{n+1}(\frac{1}{2}) \\ &+ \frac{d^k}{dx^k}g(\xi(\frac{1}{2}))\frac{(e_n(\frac{1}{2}))^k}{k!}][\frac{t}{p+2q} + \frac{q}{p^2+2pq}] + M_1\|e_n\|^k \\ &\le [\sum_{i=0}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{i!}\sum_{j=0}^{i-1} e_n^{i-1-j}(\frac{1}{2})a_n^j(\frac{1}{2})]M_3 e_{n+1}(\frac{1}{2}) + M_2M_3\|e_n\|^k + M_1\|e_n\|^k. \end{aligned} \tag{1.16}$$

where M_1 provides a bound for $M\int_0^1 |G(t,s)|ds$, M_2 provides a bound for $\frac{d^k}{dx^k}g(\xi(\frac{1}{2}))\frac{1}{k!}$, and $M_3 = \frac{1}{p+2q} + \frac{q}{p^2+2pq}$. Letting

$$P_n(t) = \sum_{i=0}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{i!}\sum_{j=0}^{i-1} e_n^{i-1-j}(\frac{1}{2})a_n^j(\frac{1}{2}),$$

we find that

$$\lim_{n\to\infty} P_n(t) = \lim_{n\to\infty} \sum_{i=0}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{i!}\sum_{j=0}^{i-1} e_n^{i-1-j}(\frac{1}{2})a_n^j(\frac{1}{2}) = M < \frac{1}{3}.$$

Therefore, we can choose $\lambda < \frac{1}{3}$ and $n_0 \in N$ such that for $n \ge n_0$, we have $P_n(t) < \lambda$ and consequently (1.16) becomes

$$\|e_{n+1}\| < \lambda_1\|e_{n+1}\| + M_4\|e_n\|^k. \tag{1.17}$$

Solving (1.17) algebraically yields

$$\|e_{n+1}\| \le \frac{M_4}{1-\lambda_1}\|e_n\|^k,$$

where $M_4 = M_1 + M_2M_3$, $\lambda_1 = \lambda M_3$ and $\|e_n\| = \max\{|e_n(t)| : t \in [0,1]\}$ is the usual uniform norm. This completes the proof.

Example. As an example, we can take $f(t,x) = e^x$ and $g(x) = x^p$ (for instance, $p = k$) in (1.1)-(1.2) which clearly satisfy the hypotheses of the main result.

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