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ORTHOGONAL POLYNOMIALS ASSOCIATED WITH SOME MODIFICATIONS OF A LINEAR FORM*

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Abstract. We show that if v is a regular Laguerre-Hahn (resp. semiclassical) linear form, then the linear form u defined by the relation $\lambda(x-a)u = (x-c)v$ is also regular and a Laguerre-Hahn (resp. semiclassical) linear form for every complex λ except for a discrete set of numbers depending on v, a, and c. We give explicitly the coefficients of the three-term recurrence relation, the structure relation of the orthogonal sequence associated with u, and the class of the linear form u knowing that of v. Finally, we apply the above results to some examples.

Key words. Orthogonal polynomials, Laguerre-Hahn linear forms, integral representations

AMS subject classifications. 42C05, 33C45

Introduction. Let v be a regular linear form. We define a new linear form u by the relation $D(x)u = A(x)v$ where $A(x)$ and $D(x)$ are non-zero polynomials. In terms of the Stieltjes function, u is obtained from v by a linear spectral transform (see [20-21]). The linear form u has been studied by several authors from different points of view. When $D(x) = 1$, v is a positive definite linear form and $A(x)$ is a positive polynomial Christoffel [8] has proved that u is also positive definite linear form. This result has been generalized in [9]. When $A(x) = \lambda \neq 0$ and $D(x) = x-c, x^2$ (resp. $D(x) = x^3$) Maroni [19], [16] (resp. Maroni-Nicolau [14]) found necessary and sufficient conditions for u to be regular. Also, an explicit expression for the orthogonal polynomials (O.P.)with respect to u is given. Finally, it was proved that, if v is a semiclassical linear form (see [2], [5], [18]), then u is also a semiclassical linear form. When $A(x) = D(x)$, u is obtained from v by adding finitely many mass points and their derivatives (see [11]). See also [1] and [13] for some special cases. In particular, in these papers, it was proved that, if v is Laguerre-Hahn (resp. semiclassical) linear form, then u is a Laguerre-Hahn (resp. semiclassical) linear form. When $A(x)$ and $D(x)$ have no non-trivial common factor, J. H. Lee and K. H. Kwon [12] found a necessary and sufficient condition for u to be regular and gave its corresponding O.P. in terms of the O.P. relative to v.

In this paper for a sake of simplicity, we consider the situation when $A(x)$ and $D(x)$ are of degree equal to one. From the point of view of Maroni, we study the linear form u, fulfilling $\lambda(x-a)u = (x-c)v, \quad \lambda \neq 0, a \neq c$.

The first section is devoted to the preliminary results and notations used in the sequel. In the second section, an explicit necessary and sufficient condition for the regularity of the new linear form is given. We obtain the coefficients of the three-term recurrence relation satisfied by the new family of O.P. We also get a characterization of the sequence of O.P. studied by Maroni in [17] (Proposition 2.9). In the third section, we compute the exact class of the Laguerre-Hahn (resp. semiclassical) linear form obtained by the above modification and we give the structure relation of the O.P. sequence relatively to the linear form u. Finally, in the fourth section we apply our results to some examples.

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1. Preliminaries and notations. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For any linear form u, any polynomial h, let $Du = u'$, hu, δ_c, and $(x-c)^{-1}u$ be the linear forms defined by:

$$< u', f >:= - < u, f' >, < hu, f >:=< u, hf >, < \delta_c, f >:= f(c),$$

and $\quad < (x-c)^{-1}u, f >:=< u, \theta_c f >$ where $\big(\theta_c f\big)(x) = \dfrac{f(x)-f(c)}{x-c}$.
It is straightforward to prove that

$$(x-c)(x-c)^{-1}u = u, \tag{1.1}$$

$$(x-c)^{-1}(x-c)u = u - (u)_0\delta_c. \tag{1.2}$$

We also define the right-multiplication of a linear form by a polynomial with

$$(uh)(x) := \Big\langle u, \frac{xh(x)-\xi h(\xi)}{x-\xi}\Big\rangle = \sum_{m=0}^{n}\Big(\sum_{j=m}^{n} a_j (u)_{j-m}\Big)x^m,\ h(x) = \sum_{j=0}^{n} a_j x^j.$$

Next, it is possible to define the product of two linear forms through

$$\langle uv, f\rangle := \langle u, vf\rangle\ , \quad f \in \mathcal{P}.$$

For $f, g \in \mathcal{P}$ and $u \in \mathcal{P}'$ we have the following results [18]

$$\big(u\theta_0 f\big)(x) = \big(\theta_0(uf)\big)(x), \tag{1.3}$$

$$u(fg)(x) = \big((fu)g\big)(x) + xg(x)\big(u\theta_0 f\big)(x), \tag{1.4}$$

$$(fu)' = fu' + f'u. \tag{1.5}$$

Definition 1.1. *A linear form v is called regular (see* [6]*) if we can associate with it a sequence of monic orthogonal polynomials (MOPS) $\{B_n\}_{n\geq 0}$, i.e.:*
(i) The leading coefficient of $B_n(x)$ is equal to 1,
(ii) $\langle v, B_n B_m\rangle = r_n\delta_{nm}, \quad r_n \neq 0, \quad n \geq 0.$

In this case $\{B_n\}_{n\geq 0}$ is said to be orthogonal with respect to v. It is a very well known fact that the sequence $\{B_n\}_{n\geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [6])

$$\begin{aligned} &B_{n+2}(x) = \big(x-\beta_{n+1}\big)B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0, \\ &B_1(x) = x - \beta_0, \qquad B_0(x) = 1 \end{aligned} \tag{1.6}$$

with $(\beta_n, \gamma_{n+1}) \in \mathbb{C}\times\mathbb{C} - \{0\}, n \geq 0$. By convention we set $\gamma_0 = (v)_0 = 1$.

Definition 1.2. [6] *The sequence* $\{B_n^{(1)}\}_{n\geq 0}$ *defined by*

$$B_n^{(1)}(x) := \big(v\theta_0 B_{n+1}\big)(x) = \left\langle v, \frac{B_{n+1}(x)-B_{n+1}(\xi)}{x-\xi}\right\rangle, n\geq 0$$

is said to be the sequence of associated polynomials of first kind for the sequence $\{B_n\}_{n\geq 0}$. *They can also be described by the shifted recurrence relation*

$$\begin{aligned} &B_{n+2}^{(1)}(x) = \big(x-\beta_{n+2}\big)B_{n+1}^{(1)}(x) - \gamma_{n+2}B_n^{(1)}(x), \quad n\geq 0,\\ &B_1^{(1)}(x) = x-\beta_1, \qquad B_0^{(1)}(x) = 1. \end{aligned} \tag{1.7}$$

Definition 1.3. [7] *The sequence* $\{B_n(.,\mu)\}_{n\geq 0}$ *defined by*

$$\begin{aligned} &B_{n+2}(x,\mu) = \big(x-\beta_{n+1}\big)B_{n+1}(x,\mu) - \gamma_{n+1}B_n(x,\mu), \quad n\geq 0,\\ &B_1(x,\mu) = x-\beta_0-\mu, \qquad B_0(x,\mu) = 1, \end{aligned} \tag{1.8}$$

is called the co-recursive polynomials for the sequence $\{B_n\}_{n\geq 0}$.

The polynomials of the sequence $\{B_n(.,\mu)\}_{n\geq 0}$ satisfy the relation [6]

$$B_{n+1}(x,\mu) = B_{n+1}(x) - \mu B_n^{(1)}(x), \quad n\geq 0. \tag{1.9}$$

2. Algebraic properties. Let v be a regular, normalized linear form (i.e. $(v)_0 = 1$) and $\{B_n\}_{n\geq 0}$ be the corresponding MOPS. For fixed a, $c\in\mathbb{C}$ and $\lambda\in\mathbb{C}-\{0\}$, we can define a new normalized linear form $u\in\mathcal{P}'$ by the relation

$$\lambda(x-a)u = (x-c)v. \tag{2.1}$$

Equivalently, from (1.1) and (1.2) we have

$$\lambda u = (x-a)^{-1}(x-c)v + \lambda\delta_a = v + (a-c)(x-a)^{-1}v + (\lambda-1)\delta_a. \tag{2.2}$$

The case $a = c$ is treated in [1] and [13], so henceforth, we assume $a\neq c$.

When u is regular, let $\{\tilde{B}_n\}_{n\geq 0}$ be its corresponding MOPS. It satisfies

$$\begin{aligned} &\tilde{B}_{n+2}(x) = \big(x-\tilde{\beta}_{n+1}\big)\tilde{B}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{B}_n(x), \quad n\geq 0,\\ &\tilde{B}_1(x) = x-\tilde{\beta}_0, \qquad \tilde{B}_0(x) = 1. \end{aligned} \tag{2.3}$$

Lemma 2.1. [15] *Let* $\{P_n\}_{n\geq 0}$ *be orthogonal with respect to* u *and* $\{Q_n\}_{n\geq 0}$ *be orthogonal with respect to* v. *If there exist a number* $\lambda\neq 0$ *and two monic polynomials* Φ *and* B, *respectively, with degrees* t *and* s *such that*

$$\lambda\Phi(x)u = B(x)v,$$

then we have

$$\begin{aligned} \Phi(x)Q_n(x) &= \sum_{\nu=n-s}^{n+t}\lambda_{n,\nu}P_\nu(x), \quad \lambda_{n,n-s}\neq 0, \quad n\geq s,\\ B(x)P_m(x) &= \sum_{\nu=m-t}^{m+s}\tilde{\lambda}_{m,\nu}Q_\nu(x), \quad \tilde{\lambda}_{m,m-t}\neq 0, \quad m\geq t. \end{aligned}$$

From (2.1) and Lemma 2.1, the sequence $\{\tilde{B}_n\}_{n\geq 0}$, when it exists, satisfies the following finite-type relation [15]

$$\begin{aligned}&(x-c)\tilde{B}_{n+1}(x) = B_{n+2}(x) + b_{n+1}B_{n+1}(x) + a_n B_n(x), \quad n \geq 0,\\ &(x-c)\tilde{B}_0(x) = B_1(x) + b_0 B_0(x),\end{aligned} \tag{2.4}$$

with $(a_n, b_n) \in \big(\mathbb{C} - \{0\}\big) \times \mathbb{C}$.

In this condition , the sequence $\{\tilde{B}_n\}_{n\geq 0}$ is orthogonal with respect to u if and only if

$$\langle u, \tilde{B}_{n+1}\rangle = 0, \quad \langle u, \tilde{B}_n^2\rangle \neq 0, \quad n \geq 0.$$

Substituting x by c in (2.4), we get

$$a_n B_n(c) + b_{n+1}B_{n+1}(c) = -B_{n+2}(c), \tag{2.5}$$

$$b_0 = \beta_0 - c. \tag{2.6}$$

Subtracting (2.5) from (2.4), we obtain after dividing by $(x-c)$

$$\tilde{B}_{n+1}(x) = \big(\theta_c B_{n+2}\big)(x) + b_{n+1}\big(\theta_c B_{n+1}\big)(x) + a_n\big(\theta_c B_n\big)(x). \tag{2.7}$$

From (2.2), we have

$$\begin{aligned}\lambda\langle u, \big(\theta_c B_n\big)(x)\rangle &= \langle (x-a)^{-1}(x-c)v + \lambda\delta_a, \big(\theta_c B_n\big)(x)\rangle\\ &= B^{(1)}_{n-1}(a) + (\lambda-1)\big(\theta_c B_n\big)(a), \quad n \geq 0.\end{aligned}$$

From (2.7) the condition $\langle u, \tilde{B}_{n+1}\rangle = 0, n \geq 0$, implies that

$$\begin{aligned}&\Big\{B^{(1)}_{n-1}(a) + (\lambda-1)\big(\theta_a B_n\big)(c)\Big\}\, a_n +\\ &\qquad + \Big\{(B^{(1)}_{n}(a) + (\lambda-1)\big(\theta_a B_{n+1}\big)(c)\Big\}\, b_{n+1}\\ &= -B^{(1)}_{n+1}(a) - (\lambda-1)\big(\theta_a B_{n+2}\big)(c).\end{aligned} \tag{2.8}$$

The determinant of the system defined by first (2.5) and (2.8) is

$$\begin{aligned}d_n = {}& B_n(c)B^{(1)}_n(a) - B_{n+1}(c)B^{(1)}_{n-1}(a) +\\ &+(\lambda-1)\frac{B_{n+1}(c)B_n(a) - B_{n+1}(a)B_n(c)}{c-a},\end{aligned} \tag{2.9}$$

$$d_{n+1} = \gamma_{n+1}d_n + \Big\{(a-c)B^{(1)}_n(a) + (\lambda-1)B_{n+1}(a)\Big\}\, B_{n+1}(c). \tag{2.10}$$

When $d_n \neq 0, n \geq 0$, by solving of the above system, we obtain

$$a_n = \frac{d_{n+1}}{d_n}, \quad n \geq 0, \qquad (a_{-1} = d_0 = \lambda) \tag{2.11}$$

$$b_{n+1} = \beta_{n+1} - c + \frac{(1-\lambda)B_{n+1}(a) + (c-a)B_n^{(1)}(a)}{d_n} B_n(c), n \geq 0. \tag{2.12}$$

Consequently, using Theorem3.4 in [12], we easily deduce the following result:

Proposition 2.2. *The linear form u is regular if and only if $d_n \neq 0$, $n \geq 0$.*

Proposition 2.3. *We may write* (Compare with [4])

$$\tilde{\gamma}_{n+1} = \frac{a_n}{a_{n-1}}\gamma_n, \quad n \geq 0, \tag{2.13}$$

$$\tilde{\beta}_n = \beta_{n+1} + b_n - b_{n+1}, \quad n \geq 0, \tag{2.14}$$

$$\tilde{\gamma}_{n+1} = \gamma_{n+2} + b_{n+1}\{\beta_{n+1} - \beta_{n+2} + b_{n+2} - b_{n+1}\} + a_n - a_{n+1}, \quad n \geq 0, \tag{2.15}$$

$$b_n\tilde{\gamma}_{n+1} = b_{n+1}\gamma_{n+1} + a_n\{\beta_n - \beta_{n+2} + b_{n+2} - b_{n+1}\}, \quad n \geq 0. \tag{2.16}$$

Proof. After multiplication of (2.3) by $(x-c)$, we substitute $(x-c)\tilde{B}_{k+1}$ by $B_{k+2} + b_{k+1}B_{k+1} + a_kB_k$ with $k = n+1, n, n-1$ and we apply the recurrence relation (1.6), the comparison of the coefficients of B_{n+1}, B_n, and B_{n-1} (resp. B_{n+2}), yields (2.13), and (2.15)-(2.16) (resp. (2.14) for $n \geq 1$).

From (2.4) with $n = 0$, we easily obtain $\tilde{\beta}_0 = \beta_1 + b_1 - b_0$. □

Proposition 2.4. *For $n \geq 0$, we have*

$$\begin{cases} (x-a)B_n(x) = \dfrac{\rho_n}{a_{n-1}}\tilde{B}_{n+1}(x) + \dfrac{\gamma_n}{a_{n-1}}\sigma_{n+1}(x)\tilde{B}_n(x), \\ (x-a)B_{n+1}(x) = \left(\sigma_n(x) - \rho_n\dfrac{b_n}{a_{n-1}}\right)\tilde{B}_{n+1}(x) - \dfrac{\gamma_n}{a_{n-1}}\rho_{n+1}\tilde{B}_n(x), \end{cases} \tag{2.17}$$

where $\quad \rho_n = a_{n-1} - \gamma_n \quad$ *and* $\quad \sigma_n(x) = x - \beta_n + b_n$.

Proof. We take $\Phi(x) = (x-a)$ and $B(x) = (x-c)$ in Lemma 2.1, we obtain

$$(x-a)B_{n+1}(x) = \lambda_{n+1,n+2}\tilde{B}_{n+2}(x) + \lambda_{n+1,n+1}\tilde{B}_{n+1}(x) + \lambda_{n+1,n}\tilde{B}_n(x), n \geq 0, \tag{2.18}$$

$$(x-a)B_0(x) = \lambda_{0,1}\tilde{B}_1(x) + \lambda_{0,0}\tilde{B}_0(x),$$

by (2.4) and the second formula (2.16) of [15, p.301], we have

$$\lambda_{n,n+1} = 1, \quad \lambda_{n,n} = \frac{b_n\gamma_n}{a_{n-1}}, \quad \lambda_{n+1,n} = \frac{\gamma_{n+1}\gamma_n}{a_{n-1}}, \quad n \geq 0.$$

This yields

$$\begin{cases} (x-a)B_n(x) = \dfrac{\gamma_n}{a_{n-1}}\sigma_{n+1}(x)\tilde{B}_n(x) + \dfrac{\rho_n}{a_{n-1}}\tilde{B}_{n+1}(x), \\ (x-a)B_{n+1}(x) = -\dfrac{\gamma_n}{a_{n-1}}\rho_{n+1}\tilde{B}_n(x) + \\ \qquad + \left(\sigma_{n+2}(x) - \rho_{n+1}\dfrac{b_{n+1}}{a_n}\right)\tilde{B}_{n+1}(x), n \geq 0. \end{cases} \tag{2.19}$$

Next, from (2.4) and (1.6), we have

$$\begin{cases} \left(\frac{a_{n-1}}{\gamma_n}(x-\beta_n)+b_n\right)B_n(x)+(1-\frac{a_{n-1}}{\gamma_n})B_{n+1}(x)=(x-c)\tilde{B}_n(x) \\ (a_n-\gamma_{n+1})B_n(x)+(x-\beta_{n+1}+b_{n+1})B_{n+1}(x)=(x-c)\tilde{B}_{n+1}(x), \end{cases} \tag{2.20}$$

which by inversion gives

$$\begin{cases} H_n(x)B_n(x)=\left(\frac{a_{n-1}}{\gamma_n}-1\right)(x-c)\tilde{B}_{n+1}(x)+ \\ \qquad\qquad +(x-\beta_{n+1}+b_{n+1})(x-c)\tilde{B}_n(x), n\geq 0, \\ H_n(x)B_{n+1}(x)=\left(\frac{a_{n-1}}{\gamma_n}(x-\beta_n)+b_n\right)(x-c)\tilde{B}_{n+1}(x)- \\ \qquad\qquad -(a_n-\gamma_{n+1})(x-c)\tilde{B}_n(x), n\geq 0 \end{cases} \tag{2.21}$$

Comparing with (2.19), we obtain

$$H_n(x)=\frac{a_{n-1}}{\gamma_n}(x-a)(x-c),\ n\geq 0, \tag{2.22}$$

and

$$\sigma_{n+2}(x)-\rho_{n+1}\frac{b_{n+1}}{a_n}=\sigma_n(x)-\left(1-\frac{\gamma_n}{a_{n-1}}\right)b_n, n\geq 0. \tag{2.23}$$

Hence (2.17) follows. □

Remark. (2.23) leads to

$$b_{n+2}+\frac{\gamma_{n+1}}{a_n}b_{n+1}=\frac{\gamma_1}{a_0}b_1+b_2+\beta_{n+2}-\beta_2+\beta_{n+1}-\beta_1,\ \ n\geq 0. \tag{2.24}$$

Particular case: v is symmetric and $c=0$. A linear form v is called symmetric if $(v)_{2n+1}=0$, $n\geq 0$. In (1.6), we have $\beta_n=0, n\geq 0$ [6]. In the sequel, the linear form v will be supposed regular and symmetric, and $c=0$, then we have necessarily $a\neq 0$.

Proposition 2.5. *When the linear form v is symmetric and $c=0\neq a$, then u (defined by (2.1)) is regular if and only if*

$$\triangle_n:=B_{2n}^{(1)}(a)+\frac{(\lambda-1)}{a}B_{2n+1}(a)\neq 0, \quad n\geq 0. \tag{2.25}$$

Proof. Taking into account (1.6), with $\beta_n=0$, we get $B_{n+2}(0)=-\gamma_{n+1}B_n(0)$. Consequently,

$$B_{2n+1}(0)=0,\ \ B_{2n+2}(0)=(-1)^{n+1}\prod_{\nu=0}^{n}\gamma_{2\nu+1}\neq 0, \quad n\geq 0. \tag{2.26}$$

Replacing n by $2n$ in (2.9)-(2.10), we obtain

$$\begin{cases} d_{2n}=B_{2n}(0)(B_{2n}^{(1)}(a)+\frac{(\lambda-1)}{a}B_{2n+1}(a)) \\ d_{2n+1}=\gamma_{2n+1}d_{2n} \end{cases}, \ n\geq 0. \tag{2.27}$$

Using Proposition 2.2, we obtain the desired result. □

In such a condition, from (2.11)-(2.14), and (2.6) we get

$$a_{-1} = \lambda,\ a_{2n} = \gamma_{2n+1},\ a_{2n+1} = -\frac{\triangle_{n+1}}{\triangle_n},\ n \geq 0, \tag{2.28}$$

$$b_{2n+1} = -a, \quad b_{2n} = 0, \quad n \geq 0, \tag{2.29}$$

$$\tilde{\beta}_n = (-1)^n a, \quad \tilde{\gamma}_{2n+1} = \frac{\gamma_{2n+1}\gamma_{2n}}{a_{2n-1}}, \quad \tilde{\gamma}_{2n+2} = a_{2n+1}, \quad n \geq 0. \tag{2.30}$$

Remarks. 1. If v is symmetric, on account of (2.30) the linear form u is not symmetric. 2. From (2.15) where $n \longrightarrow 2n$, we obtain

$$\tilde{\gamma}_{2n+1} + \tilde{\gamma}_{2n+2} = \gamma_{2n+1} + \gamma_{2n+2} - a^2. \tag{2.31}$$

3. Proposition 2.2 and the Proposition 2.5. give necessary and sufficient conditions so that the sequence $\{a_n\}_{n\geq 0}$ satisfies $a_n \neq 0, \quad n \geq 0$. For the applications, we must give a simple sufficient conditions (see below).

Let us recall some general features. Consider the quadratic decomposition of $\{B_n\}_{n\geq 0}$ and $\{B_n^{(1)}\}_{n\geq 0}$ [17]

$$B_{2n}(x) = P_n(x^2) \quad , \quad B_{2n+1}(x) = xR_n(x^2), \quad n \geq 0, \tag{2.32}$$

$$B_{2n}^{(1)}(x) = R_n(x^2, -\gamma_1), \quad B_{2n+1}^{(1)}(x) = xP_n^{(1)}(x^2), \quad n \geq 0. \tag{2.33}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to σv and $x\sigma v$ where σv is the even part of v defined by

$$\langle \sigma v, f\rangle := \langle v, \big(\sigma f\big)(x)\rangle = \langle v, f(x^2)\rangle.$$

Moreover, $\{R_n\}_{n\geq 0}$ satisfies the recurrence relation

$$\begin{cases} R_{n+2}(x) = (x - \beta_{n+1}^R)R_{n+1}(x) - \gamma_{n+1}^R R_n(x) \quad n \geq 0, \\ R_1(x) = x - \beta_0^R, \quad R_0(x) = 1, \end{cases} \tag{2.34}$$

with $\beta_n^R = \gamma_{2n+1} + \gamma_{2n+2}, \quad \gamma_{n+1}^R = \gamma_{2n+2}\gamma_{2n+3}, \quad n \geq 0.$

From (2.32), (2.33), and (1.9) we obtain

$$B_{2n}^{(1)}(a) + \frac{(\lambda - 1)}{a} B_{2n+1}(a) = \lambda R_n\left(a^2, -\frac{\gamma_1}{\lambda}\right), \quad n \geq 0. \tag{2.35}$$

Remark. As an immediate consequence of (2.35), we have: when v is symmetric and positive definite, $\lambda \in \mathbb{R}-\{0\}$, then u is regular for every a such that $a^2 \notin \mathbb{R}$.

Proposition 2.6. *When the linear form v is symmetric, $c = 0 \neq a$ and $\gamma_{2n+1} + \gamma_{2n+2} = a^2$, then the linear form u is regular for every $\lambda \neq 0$.*

Proof. We have $\beta_n^R = \gamma_{2n+1} + \gamma_{2n+2} = a^2$, by virtue of formula (2.34) we obtain $R_{n+2}(a^2) = -\gamma_{n+1}^R R_n(a^2)$, $n \geq 0$. Thus, we get successively

$$R_{2n+1}(a^2) = 0, \quad R_{2n+2}(a^2) = (-1)^{n+1} \prod_{\nu=0}^{n} \gamma_{2\nu+1}^R \neq 0, \; n \geq 0,$$

$$R_{2n+1}^{(1)}(a^2) = 0, \quad R_{2n+2}^{(1)}(a^2) = (-1)^{n+1} \prod_{\nu=0}^{n} \gamma_{2\nu+2}^R \neq 0, \; n \geq 0.$$

Using (1.9), we obtain $R_n\left(a^2, -\frac{\gamma_1}{\lambda}\right) \neq 0$, $n \geq 0$. Then, from (2.25) and (2.35), the linear form u is regular for any $\lambda \neq 0$. □

Under the condition of Proposition 2.6, we can precise the relations (2.28) and (2.30). From (2.28) and (2.30)-(2.31), we obtain

$$a_{-1} = \lambda \quad , \quad a_{2n+1} = -\frac{\gamma_{2n}\gamma_{2n+1}}{a_{2n-1}}, \quad n \geq 0,$$

Putting $\quad a_{2n-1} = \dfrac{\xi_{n+1}}{\xi_n}, \xi_n \neq 0, n \geq 0 \quad$ in the above equation, we deduce

$$a_{4n+1} = -\frac{\gamma_1}{\lambda} \prod_{\nu=0}^{n-1} \frac{\gamma_{4\nu+5}\gamma_{4\nu+4}}{\gamma_{4\nu+3}\gamma_{4\nu+2}} \,, \quad n \geq 0, \tag{2.36}$$

$$a_{4n+3} = \gamma_{4n+3}\gamma_{4n+2} \frac{\lambda}{\gamma_1} \prod_{\nu=0}^{n-1} \frac{\gamma_{4\nu+3}\gamma_{4\nu+2}}{\gamma_{4\nu+5}\gamma_{4\nu+4}} \,, \quad n \geq 0. \tag{2.37}$$

with $\displaystyle\prod_{\nu=0}^{-1} = 1$. So (2.30) becomes for $n \geq 0$

$$\tilde{\beta}_n = (-1)^n a, \;\; \tilde{\gamma}_{4n+2} = -\tilde{\gamma}_{4n+1} = a_{4n+1}, \;\; \tilde{\gamma}_{4n+4} = -\tilde{\gamma}_{4n+3} = a_{4n+3}. \tag{2.38}$$

PROPOSITION 2.7. *Under the conditions of Proposition 2.5, the MOPS $\{\tilde{B}_n\}_{n\geq 0}$ can be decomposed in the following way*

$$\tilde{B}_{2n}(x) = \tilde{P}_n(x^2), \quad \tilde{B}_{2n+1}(x) = (x-a)\tilde{R}_n(x^2), \quad n \geq 0, \qquad \text{where}$$

$$\tilde{P}_n(x) = R_n(x) + a_{2n-1}R_{n-1}(x), \quad \tilde{R}_n(x) = R_n(x), n \geq 0. \tag{2.39}$$

and $\quad R_{-1}(x) := 0$. *The sequences $\{\tilde{P}_n\}_{n\geq 0}$ and $\{\tilde{R}_n\}_{n\geq 0}$ are respectively orthogonal with respect to* $\lambda^{-1}(x-a^2)^{-1}(x\sigma v) + \delta_{a^2}$ and $x\sigma v$.

Proof. From (1.6), (2.4), (2.28), (2.29), and (2.32) we have

$$\begin{cases} x\tilde{B}_{2n+1}(x) = B_{2n+2}(x) + b_{2n+1}B_{2n+1}(x) + a_{2n}B_{2n}(x) = x(x-a)R_n(x^2) \\ x\tilde{B}_{2n}(x) = B_{2n+1}(x) + b_{2n}B_{2n}(x) + a_{2n-1}B_{2n-1}(x) = x\big(R_n(x^2) + a_{2n-1}R_{n-1}(x^2)\big). \end{cases}$$

Then, $\tilde{B}_{2n}(x) = \tilde{P}_n(x^2)$, $\tilde{B}_{2n+1}(x) = (x-a)\tilde{R}_n(x^2)$, $n \geq 0$, with $\tilde{P}_n(x)$ and $\tilde{R}_n(x)$ are given by (2.39). But, $\tilde{R}_n(x) = R_n(x)$ implies that $\tilde{R}_n(x)$ is orthogonal with respect to $x\sigma v$. Moreover, it was shown in [19] that the linear form $\lambda^{-1}(x-a^2)^{-1}(x\sigma v) + \delta_{a^2}$ is regular if and only if $\frac{\gamma_1}{\lambda} \neq 0$ and $R_n\left(a^2, -\frac{\gamma_1}{\lambda}\right) \neq 0$ which are fulfilled according to (2.25) and (2.35).

In such conditions, the corresponding MOPS $\left\{R_n\left(., -\frac{\gamma_1}{\lambda}, a^2\right)\right\}_{n\geq 0}$ is given by

$$R_n\left(x, -\frac{\gamma_1}{\lambda}, a^2\right) = R_n(x) - \frac{R_n\left(a^2, -\frac{\gamma_1}{\lambda}\right)}{R_{n-1}\left(a^2, -\frac{\gamma_1}{\lambda}\right)} R_{n-1}(x).$$

But, from (2.11),(2.27), and (2.35), we have $a_{2n-1} = -\frac{R_n\left(a^2, -\frac{\gamma_1}{\lambda}\right)}{R_{n-1}\left(a^2, -\frac{\gamma_1}{\lambda}\right)}$.

Therefore $\tilde{P}_n(x) = R_n\left(x, -\frac{\gamma_1}{\lambda}, a^2\right)$.

Hence, $\{\tilde{P}_n\}_{n\geq 0}$ is orthogonal with respect to $\lambda^{-1}(x-a^2)^{-1}(x\sigma v) + \delta_{a^2}$. □

Remark. The sequence $\{\tilde{P}_n\}_{n\geq 0}$ satisfies the recurrence relation (2.34) with

$$\beta_0^{\tilde{P}} = \frac{\gamma_1}{\lambda} + a^2, \quad \beta_{n+1}^{\tilde{P}} = \frac{\gamma_{2n+2}\gamma_{2n+3}}{a_{2n+1}} + a_{2n+1} + a^2, \quad \gamma_{n+1}^{\tilde{P}} = \frac{\gamma_{2n+1}\gamma_{2n}a_{2n+1}}{a_{2n-1}}, n \geq 0.$$

Proposition 2.8. *Under the conditions of Proposition 2.5, if $\tilde{\gamma}_{2n+2} \neq \gamma_{2n+2}$, then the MOPS $\{B_n\}_{n\geq 0}$ and $\{\tilde{B}_n\}_{n\geq 0}$ satisfy the following relation*

$$B_n(x) + s_n B_{n-1}(x) = \tilde{B}_n(x) + t_n \tilde{B}_{n-1}(x), \quad n \geq 1, \tag{2.40}$$

with

$$\begin{cases} (s_1, t_1) \in \mathbb{C}^2\ ,\ s_1 t_1 \neq 0, s_1 \neq t_1, s_{2n+2} = t_{2n+2} = a^{-1}(\tilde{\gamma}_{2n+2} - \gamma_{2n+2}), \\ s_{2n+3} = -\dfrac{a\gamma_{2n+2}}{\tilde{\gamma}_{2n+2} - \gamma_{2n+2}}, t_{2n+3} = -\dfrac{a\tilde{\gamma}_{2n+2}}{\tilde{\gamma}_{2n+2} - \gamma_{2n+2}}, n \geq 0. \end{cases}$$

Proof. From (1.6), (2.3) where $n \to 2n+2$ and Proposition 2.7, we have

$$\begin{cases} B_{2n+2}(x) = R_{n+1}(x^2) + \gamma_{2n+2}R_n(x^2), \quad \tilde{B}_{2n+2}(x) = R_{n+1}(x^2) + \tilde{\gamma}_{2n+2}R_n(x^2), \\ B_{2n+1}(x) = xR_n(x^2), \quad \tilde{B}_{2n+1}(x) = (x-a)R_n(x^2), \quad n \geq 0. \end{cases}$$

Then, $B_n(x) \neq \tilde{B}_n(x)$, $n \geq 1$. From [4, Theorem 2.4], there exist complex sequences $\{s_n\}_{n\geq 1}$, $\{t_n\}_{n\geq 1}$ with $s_1 \neq t_1$ and $s_n t_n \neq 0$ for $n \geq 1$, such that $\{B_n\}_{n\geq 0}$ and $\{\tilde{B}_n\}_{n\geq 0}$ are related to (2.40). By (2.3), (2.18), $n \to 2n+1$ and taking into account (1.6), (2.4), (2.19), and (2.28)-(2.30), we obtain

$$\begin{cases} \tilde{B}_{2n+3}(x) + a\tilde{B}_{2n+2}(x) + \tilde{\gamma}_{2n+2}\tilde{B}_{2n+1}(x) = B_{2n+3} + \tilde{\gamma}_{2n+2}B_{2n+1}(x), \\ B_{2n+3}(x) - aB_{2n+2}(x) + \gamma_{2n+2}B_{2n+1}(x) = \tilde{B}_{2n+3} + \gamma_{2n+2}\tilde{B}_{2n+1}(x), n \geq 0. \end{cases}$$

This leads to (for $n \geq 0$)

$$a\tilde{B}_{2n+2}(x) + (\tilde{\gamma}_{2n+2} - \gamma_{2n+2})\tilde{B}_{2n+1}(x) = aB_{2n+2}(x) + (\tilde{\gamma}_{2n+2} - \gamma_{2n+2})B_{2n+1}(x).$$

Consequently $s_{2n+2} = t_{2n+2} = a^{-1}(\tilde{\gamma}_{2n+2} - \gamma_{2n+2})$, since for $n \geq 2$ s_n and t_n are unique. Next, by formulas (2.7)-(2.8) of [4] , we have

$$s_{2n+3} = -\frac{a\gamma_{2n+2}}{\tilde{\gamma}_{2n+2} - \gamma_{2n+2}} \quad \text{and} \quad t_{2n+3} = -\frac{a\tilde{\gamma}_{2n+2}}{\tilde{\gamma}_{2n+2} - \gamma_{2n+2}}. \qquad \square$$

It is possible to characterize the sequence $\{\tilde{B}_n\}_{n\geq 0}$, studied by Maroni in [17], in the particular case where it satisfies (2.3) with $\tilde{\beta}_n$ and $\tilde{\gamma}_{n+1}$ given by (2.30) or (2.38).

Proposition 2.9. *Let u be a normalized and regular linear form and $\{\tilde{B}_n\}_{n\geq 0}$ be its corresponding MOPS and $a \in \mathbb{C} - \{0\}, a^2 \neq (u)_2$. The following statements are equivalent*

a) *There exists a normalized , regular and symmetric linear form v such that*

$$\lambda(x-a)u = xv, \quad \lambda = \frac{(v)_2}{(u)_2 - a^2}.$$

b) *The sequence $\{\tilde{B}_n\}_{n\geq 0}$ satisfies* (2.3) *with $\tilde{\beta}_n$ and $\tilde{\gamma}_{n+1}$ given by* (2.30).

Proof. $a) \Rightarrow b)$. On account of (2.30) and Proposition 2.5.

$b) \Rightarrow a)$. Suppose b). It was shown in ([17] , p.21 , p.43) that the sequence $\{\tilde{B}_n\}_{n\geq 0}$ has the following quadratic decomposition

$$\tilde{B}_{2n}(x) = \tilde{P}_n(x^2), \quad \tilde{B}_{2n+1}(x) = (x-a)\tilde{R}_n(x^2), \quad n \geq 0.$$

In addition, sequences $\{\tilde{P}_n\}_{n\geq 0}$ and $\{\tilde{R}_n\}_{n\geq 0}$ respectively are orthogonal with respect to σu and w_1 , where $\tilde{\gamma}_1 w_1 = (x - a^2)\sigma u$.

Let $\alpha \in \mathbb{C} - \{0\}$ such that $\tilde{R}_n(0, -\alpha) \neq 0$, $n \geq 0$, then $w_2 = \alpha x^{-1} w_1 + \delta_0$ is regular on the basis of [19]. From [17, p.42, Proposition 2.3.], we have $\sigma\big((x-a)u\big) = 0$ and $\tilde{\gamma}_1 w_1 = \sigma\big(x(x-a)u\big)$. Hence $(x-a)u = \mathcal{A}$ where $\mathcal{A}$ is an antisymmetric form and then $\tilde{\gamma}_1 w_1 = \sigma(x\mathcal{A})$. But $w_1 = \alpha^{-1} x\sigma v$ where $v = w(w_2)$ is the symmetrized form of w_2 [see 17, p.28, p.35].

Consequently $\sigma(\lambda^{-1}x^2 v - x\mathcal{A}) = 0$, which implies $\lambda^{-1}x^2 v - x\mathcal{A} = 0$, since this form is both symmetric and antisymmetric. Whence

$$\mathcal{A} = \lambda^{-1}x^1 v, \quad (\lambda^{-1} = \alpha^{-1}\tilde{\gamma}_1).$$

It is clear that $(v)_2 = \alpha$, and the condition $\langle u, \tilde{B}_2\rangle = 0$ gives $\tilde{\gamma}_1 = (u)_2 - a^2$. $\square$

3. The Laguerre-Hahn case.

Definition 3.1. [2] *The regular linear form v is called Laguerre-Hahn if its formal Stieltjes function satisfies the Riccati equation*

$$\Phi(z)S'\big(v\big)(z) = B(z)S^2\big(v\big)(z) + C_0(z)S\big(v\big)(z) + D_0(z), \tag{3.1}$$

where Φ monic , B , C_0, and D_0 are polynomials and $S\big(v\big)(z) = -\sum_{n\geq 0} \frac{(v)_n}{z^{n+1}}$.

It was shown in [10] that equation (3.1) is equivalent to

$$\big(\Phi(x)v\big)' + \Psi v + B\big(x^{-1}v^2\big) = 0, \tag{3.2}$$

with

$$\Psi(x) = -\Phi'(x) - C_0(x). \tag{3.3}$$

We also have the following relation

$$D_0(x) = -\big(v\theta_0\Phi\big)'(x) - \big(v\theta_0\Psi\big)(x) - \big(v^2\theta_0^2 B\big)(x).$$

Proposition 3.2.[2] *We define* $d = \max\big(\deg(\Phi), \deg(B)\big)$ *and* $p = \deg(\Psi)$. *The Laguerre-Hahn linear form* v *satisfying* (3.2) *is of class* $s = \max(d-2, p-1)$ *if and only if*

$$\prod_{b\in\mathcal{Z}}\Big\{|\Phi'(b)+\Psi(b)| + |B(b)| + |\langle v, \theta_b^2\Phi + \theta_b\Psi + v\theta_0\theta_b B\rangle|\Big\} \neq 0,$$

where $\mathcal{Z}$ *denotes the set of zeros of* Φ.

Corollary 3.3. *The Laguerre-Hahn linear form* v *satisfying (3.2) is of class* $s = \max(d-2, p-1)$ *if and only if*

$$\prod_{b\in\mathcal{Z}}\Big\{|C_0(b)| + |B(b)| + |D_0(b)|\Big\} \neq 0. \tag{3.4}$$

Remark. (3.4) is equivalent to the fact that the polynomial coefficients in (3.1) are coprime.

Definition 3.4.(*see* [5],[18]) *A linear form* v *is semiclassical if it is regular and there exist two polynomials* Φ *(monic),* Ψ, $\deg(\Psi) \geq 1$ *such that*

$$\big(\Phi v\big)' + \Psi v = 0.$$

The class of v *is* $s = \max(\deg\Psi - 1, \deg\Phi - 2)$ *if and only if*

$$\prod_{b\in\mathcal{Z}} |\Phi'(b) + \Psi(b)| + |\left\langle u, \theta_b\Psi + \theta_b^2\Phi\right\rangle| \neq 0.$$

Remark.[18] When $B = 0$ in (3.1) or (3.2), the linear form v is semiclassical.

Proposition 3.5. *If* v *is a Laguerre-Hahn linear form and satisfies (3.1), then for every* $a, c \in \mathbb{C}, a \neq c$ *and every* $\lambda \in \mathbb{C} - \{0\}$ *such that* $d_n \neq 0, n \geq 0$, *the linear form* u *defined by (2.1) is regular and Laguerre-Hahn. It satisfies*

$$\tilde{\Phi}(z)S'\big(u\big)(z) = \tilde{B}(z)S^2\big(u\big)(z) + \tilde{C}_0(z)S\big(u\big)(z) + \tilde{D}_0(z), \tag{3.5}$$

where

$$\begin{cases} \tilde{\Phi}(z) = (z-c)(z-a)\Phi(z), \quad \tilde{B}(z) = \lambda(z-a)^2 B(z), \\ \tilde{C}_0(z) = (c-a)\Phi(z) + (z-a)\Big((z-c)C_0(z) - 2(1-\lambda)B(z)\Big), \\ \lambda\tilde{D}_0(z) = (z-c)^2 D_0(z) + (\lambda-1)\Big((z-c)C_0(z) + \Phi(z)\Big) + (\lambda-1)^2 B(z), \end{cases} \tag{3.6}$$

and u is of class $\tilde{s}$ such that $\tilde{s} \leq s+2$.

Proof. We have [10]

$$(x-c)S\big(v\big)(z) = S\big((\xi-c)v\big)(z) - \big(v\theta_0(\xi-c)\big)(z) = S\big((\xi-c)v\big)(z) - 1.$$

Using (2.1), we get

$$(x-c)S(v)(z) = \lambda S\big((\xi-a)u\big)(z) - 1 = \lambda(z-a)S(u)(z) + \lambda - 1. \tag{3.7}$$

Multiplying (3.1) by $(x-c)^2$ and taking into account (3.7) , we obtain (3.5)-(3.6). From (3.3), and (3.5)-(3.6), the linear form u satisfies the distributional equation

$$\big(\tilde{\Phi}(x)u\big)' + \tilde{\Psi}u + \tilde{B}(x^{-1}v^2) = 0, \tag{3.8}$$

where $\tilde{\Phi}$ and $\tilde{B}$ are the polynomials defined in (3.6) and

$$\tilde{\Psi}(x) = -\tilde{\Phi}'(x) - \tilde{C}_0(x) = (x-a)\Big((x-c)\Psi(x) - 2\Phi(x) + 2(1-\lambda)B(x)\Big), \tag{3.9}$$

then $\deg(\tilde{\Phi}) \leq s+4$, $\deg(\tilde{B}) \leq s+4$ and $\deg(\tilde{\Psi}) = \tilde{p} \leq s+3$.

Thus $\tilde{d} = \max\Big(\deg(\tilde{\Phi}), \deg(\tilde{B})\Big) \leq s+4$ and $\tilde{s} = \max\big(\tilde{d}-2, \tilde{p}-1\big) \leq s+2$. □

PROPOSITION 3.6. *Let u be a Laguerre-Hahn linear form satisfying (3.8). For every zero of $\tilde{\Phi}$ different from a and c, the equation (3.5) is irreducible.*

Proof. Since v is a Laguerre-Hahn linear form of class s, $S(v)(z)$ satisfies (3.1), where the polynomials Φ, B, C_0, and D_0 are coprime. Let $\tilde{\Phi}, \tilde{B}, \tilde{C}_0$, and $\tilde{D}_0$ as in the Proposition 3.4. Let b be a zero of $\tilde{\Phi}$ different from a and c, this implies that $\Phi(b) = 0$. We know that $|B(b)| + |C_0(b)| + |D_0(b)| \neq 0$,

i) if $B(b) \neq 0$, then $\tilde{B}(b) \neq 0$,

ii) if $B(b) = 0$ and $C_0(b) \neq 0$, then $\tilde{C}_0(b) \neq 0$,

iii)if $B(b) = C_0(b) = 0$, then $\tilde{D}_0(b) \neq 0$, whence $|\tilde{B}(b)| + |\tilde{C}_0(b)| + |\tilde{D}_0(b)| \neq 0$. □

Concerning the class of u, we have the following result (see Proposition 3.8). But first, let us recall this technical lemma.

LEMMA 3.7. *We have the following properties*

R_1. *The equation* $(3.5)-(3.6)$ *is irreducible in a if and only if*

$$|\Phi(a)| + |(a-c)^2D_0(a) + (\lambda-1)(a-c)C_0(a) + (\lambda-1)^2B(a)| \neq 0.$$

R_2. *The equation* $(3.5)-(3.6)$ *is divisible by* $(x-a)$ *but not by* $(x-a)^2$ *if and only if*

$$\begin{cases} |\Phi(a)| + |(a-c)^2D_0(a) + (\lambda-1)(a-c)C_0(a) + (\lambda-1)^2B(a)| = 0, \\ |(a-c)^2D_0'(a) + (\lambda-1)(a-c)C_0'(a) + (\lambda-1)^2B'(a)| + \\ \qquad\qquad + |(a-c)\big(\Phi'(a) - C_0(a)\big) + 2(1-\lambda)B(a)| \neq 0. \end{cases}$$

R_3. *The equation* $(3.5)-(3.6)$ *is irreducible in c if and only if*

$$\big(\Phi(c), B(c)\big) \neq (0,0).$$

R_4. *The equation* (3.5) − (3.6) *is divisible by* $(x-c)$ *but not by* $(x-c)^2$ *if and only if*

$$\big(\Phi(c), B(c)\big) = (0,0) \quad \text{and} \quad \big(\Psi(c), B'(c)\big) \neq (0,0).$$

Proof. From (3.6), we have $\tilde{B}(a) = 0$ and $\tilde{C}_0(a) = (c-a)\Phi(a)$. If $\Phi(a) \neq 0$, then $\tilde{C}_0(a) \neq 0$. If $\Phi(a) = 0$, then $\lambda\tilde{D}_0(a) = (a-c)^2 D_0(a) + (\lambda-1)(a-c)C_0(a) + (\lambda-1)^2 B(a)$. So, by virtue of (3.4), we obtain R_1.

Now, if $\Phi(a) = 0$ and

$$(a-c)^2 D_0(a) + (\lambda-1)(a-c)C_0(a) + (\lambda-1)^2 B(a) = 0, \tag{3.10}$$

the equation (3.5)-(3.6) is divisible by $(z-a)$ according to (3.4). Thus $S(u)(z)$ satisfies (3.5) with

$$\begin{cases} \tilde{\Phi}(z) = (z-c)\Phi(z), \quad \tilde{B}(z) = \lambda(z-a)B(z), \\ \tilde{C}_0(z) = (c-a)\big(\theta_a\Phi\big)(z) + (z-c)C_0(z) - 2(1-\lambda)B(z), \\ \lambda\tilde{D}_0(z) = (z+a-2c)D_0(z) + (a-c)^2\big(\theta_a D_0\big)(z) + \\ \quad +(\lambda-1)\Big(C_0(z) + (a-c)\big(\theta_a C_0\big)(z) + \big(\theta_a\Phi\big)(z)\Big) + (\lambda-1)^2\big(\theta_a B\big)(z). \end{cases} \tag{3.11}$$

Then, $\tilde{C}_0(a) = (c-a)\Phi'(a) + (a-c)C_0(a) - 2(1-\lambda)B(a)$. If $(c-a)\Phi'(a) + (a-c)C_0(a) - 2(1-\lambda)B(a) \neq 0$, then the equation (3.5)-(3.11) is irreducible in a. If

$$(c-a)\Phi'(a) + (a-c)C_0(a) - 2(1-\lambda)B(a) = 0, \tag{3.12}$$

we have to evaluate $\tilde{D}_0(a)$. From (3.11), we obtain

$$\begin{aligned} \lambda\tilde{D}_0(a) = {} & 2(a-c)D_0(z) + (a-c)^2 D_0'(a) + \\ & +(\lambda-1)\Big(C_0(a) + (a-c)C_0'(a) + \Phi'(a)\Big) + (\lambda-1)^2 B'(a). \end{aligned} \tag{3.13}$$

Multiplying respectively (3.10) and (3.12) by 2 and $\lambda-1$, we get after making the difference between the equations formulated

$$2(a-c)^2 D_0(a) + (\lambda-1)(a-c)C_0(a) + (\lambda-1)(a-c)\Phi'(a) = 0.$$

Thus,

$$2(a-c)D_0(a) + (\lambda-1)C_0(a) + (\lambda-1)\Phi'(a) = 0. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\lambda\tilde{D}_0(a) = (a-c)^2 D_0'(a) + (\lambda-1)(a-c)C_0'(a) + (\lambda-1)^2 B'(a).$$

Then, we deduce R_2.

From (3.6), we get

$$\begin{cases} \tilde{B}(z) = \lambda(c-a)^2 B(c), \\ \tilde{C}_0(c) = (c-a)\Phi(c) + 2(c-a)(1-\lambda)B(c), \\ \lambda\tilde{D}_0(c) = (\lambda-1)\Phi(c) + (\lambda-1)^2 B(c). \end{cases}$$

We can deduce that $|\tilde{B}(c)|+|\tilde{C}_0(c)|+|\tilde{D}_0(c)|\neq 0$ if and only if $\big(\Phi(c),B(c)\big)\neq(0,0)$. Thus R_3 is proved.

If $\big(\Phi(c),B(c)\big)=(0,0)$, then the equation (3.5)-(3.6) can be divided by $(z-c)$ according to (3.4). In this case, $S(u)(z)$ satisfies (3.5) with

$$\begin{cases} \tilde{\Phi}(z)=(z-a)\Phi(z), \quad \tilde{B}(z)=\lambda(z-a)^2\big(\theta_c B\big)(z), \\ \tilde{C}_0(z)=(c-a)\big(\theta_c\Phi\big)(z)+(z-a)\Big(C_0(z)-2(1-\lambda)\big(\theta_c B\big)(z)\Big), \\ \lambda\tilde{D}_0(z)=(z-c)D_0(z)+(\lambda-1)\Big(C_0(z)+\big(\theta_c\Phi\big)(z)\Big)+(\lambda-1)^2\big(\theta_c B\big)(z). \end{cases} \tag{3.15}$$

Substituting z by c in (3.15) and using (3.3), we obtain

$$\begin{cases} \tilde{B}(c)=\lambda(c-a)^2B'(c), \\ \tilde{C}_0(z)=-(c-a)\Psi(c)-2(1-\lambda)B'(c), \\ \lambda\tilde{D}_0(z)=-(\lambda-1)\Psi(c)+(\lambda-1)^2B'(c). \end{cases}$$

Then (3.5)-(3.15) is irreducible in c if and only if $\big(\Psi(c),B'(c)\big)\neq(0,0)$. Hence R_4. □

Proposition 3.8. *Under the conditions of Proposition 3.5, for the class of u, we have the two different cases:*

1) $|\Phi(a)|+|(a-c)^2D_0(a)+(\lambda-1)(a-c)C_0(a)+(\lambda-1)^2B(a)|\neq 0.$
   i) $\tilde{s}=s+2$ *if* $\big(\Phi(c),B(c)\big)\neq(0,0)$.
   ii) $\tilde{s}=s+1$ *if* $\big(\Phi(c),B(c)\big)=(0,0)$ *and* $\big(\Psi(c),B'(c)\big)\neq(0,0)$.

2) $$\begin{cases} |\Phi(a)|+\big|(a-c)^2D_0(a)+(\lambda-1)(a-c)C_0(a)+(\lambda-1)^2B(a)\big|=0, \\ \big|(a-c)^2D_0'(a)+(\lambda-1)(a-c)C_0'(a)+(\lambda-1)^2B'(a)\big|+ \\ \qquad +\big|(a-c)\big(\Phi'(a)-C_0(a)\big)+2(1-\lambda)B(a)\big|\neq 0. \end{cases}$$
   i) $\tilde{s}=s+1$ *if* $\big(\Phi(c),B(c)\big)\neq(0,0)$.
   ii) $\tilde{s}=s$ *if* $\big(\Phi(c),B(c)\big)=(0,0)$ *and* $\big(\Psi(c),B'(c)\big)\neq(0,0)$.

Proof. From Proposition 3.6, the class of u depends only on the zeros c and a. For the zero a we consider the following situations:

A) $|\Phi(a)|+|(a-c)^2D_0(a)+(\lambda-1)(a-c)C_0(a)+(\lambda-1)^2B(a)|\neq 0$. In this case the equation (3.5)-(3.6) is irreducible in a according to R_1. But what about the zero c? We will analyze the following cases:

i) $\big(\Phi(c),B(c)\big)\neq(0,0)$, the equation (3.5)-(3.6) is irreducible in c according to R_3. Then (3.5)-(3.6) is irreducible and $\tilde{s}=s+2$. Thus we proved 1) i)

ii) $\big(\Phi(c),B(c)\big)=(0,0)$ and $\big(\Psi(c),B'(c)\big)\neq(0,0)$.
From R_4., (3.5)-(3.6) is divisible by $(x-c)$ but not by $(x-c)^2$ and thus the order of the class of u decreases in one unit. In fact, $S\big(u\big)(z)$ satisfies the irreducible equation (3.5)-(3.15) and then $\tilde{s}=s+1$. Thus we proved 1) ii).

B)
$$\begin{cases} |\Phi(a)|+|(a-c)^2D_0(a)+(\lambda-1)(a-c)C_0(a)+(\lambda-1)^2B(a)|=0 \\ |(a-c)^2D_0'(a)+(\lambda-1)(a-c)C_0'(a)+(\lambda-1)^2B'(a)|+ \\ \qquad +|(a-c)\big(\Phi'(a)-C_0(a)\big)+2(1-\lambda)B(a)|\neq 0. \end{cases}$$

In this condition, (3.5)-(3.6) is divisible by $(x-a)$ but not by $(x-a)^2$ according to R_2. But what about the zero c? We have the two following cases:

i) $\big(\Phi(c),B(c)\big)\neq(0,0)$, the equation (3.5)-(3.6) is irreducible in c according to R_3. Then, $S(u)(z)$ satisfies the irreducible equation (3.5)-(3.11) and then $\tilde{s}=s+1$. Thus 2) i) is proved.

ii) $\big(\Phi(c), B(c)\big) = (0,0)$ and $\big(\Psi(c), B'(c)\big) \neq (0,0)$. From R_4, (3.5)-(3.6) is divisible by $(x-c)$ but not by $(x-c)^2$. Therefore, $S(u)(z)$ satisfies the irreducible equation (3.5) with

$$\begin{cases} \tilde{\Phi}(z) = \Phi(z), \quad \tilde{B}(z) = \lambda(z-a)\big(\theta_c B\big)(z), \\ \tilde{C}(z) = \big(\theta_c \Phi\big)(z) - \Big(\big(\theta_c \Phi\big)(z) + C_0(z) - 2(1-\lambda)\big(\theta_c B\big)(z)\Big), \\ \lambda \tilde{D}_0(z) = D_0(z) + (a-c)\big(\theta_a D_0\big)(z) + (\lambda - 1)\Big(\big(\theta_a C_0\big)(z) + \\ \qquad\qquad + \big(\theta_a \theta_c \Phi\big)(z)\Big) + (\lambda-1)^2\big(\theta_a\theta_c B\big)(z). \end{cases} \tag{3.16}$$

Then $\tilde{s} = s$ and 2) ii) is also proved. □

Corollary 3.9. *Let v be a semi-classical linear form of class s, satisfying (3.1) with $B = 0$. For every $a, c \in \mathbb{C}, a \neq c$, and every $\lambda \in \mathbb{C}^*$ such that $d_n \neq 0, n \geq 0$, the linear form u defined by (2.1) is regular and semi-classical of class $\tilde{s} \leq s+2$. In fact, we have the two different cases:*

1) $|\Phi(a)| + |(a-c)^2 D_0(a) + (\lambda-1)(a-c)C_0(a)| \neq 0$,
   i) $\tilde{s} = s+2$ *if* $\Phi(c) \neq 0$.
   ii) $\tilde{s} = s+1$ *if* $\Phi(c) = 0$ *and* $\Psi(c) \neq 0$.
2) $|\Phi(a)| + |(a-c)^2 D_0(a) + (\lambda-1)(a-c)C_0(a)| = 0$ *and*
   $|(a-c)^2 D_0'(a) + (\lambda-1)\big((a-c)C_0'(a) + \Phi'(a)\big)| + |(a-c)\big(\Phi'(a) - C_0(a)\big)| \neq 0$,
   i) $\tilde{s} = s+1$ *if* $\Phi(c) \neq 0$.
   ii) $\tilde{s} = s$ *if* $\Phi(c) = 0$ *and* $\Psi(c) \neq 0$.

Proof. It follows from Proposition 3.7, with $B = 0$. □

The structure relation. Note that the MOPS relatively to a Laguerre-Hahn linear form satisfies a structure relation [2]. Then, if we consider that the linear form v is Laguerre-Hahn, its MOPS $\{B_n\}_{n\geq 0}$ fulfils the following structure relation

$$\Phi(x)B'_{n+1}(x) - B(x)B_n^{(1)}(x) = \\ \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)B_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)B_n(x)\ ,\ n \geq 0, \tag{3.17}$$

with

$$\begin{array}{l} C_{n+1}(x) = -C_n(x) + 2(x-\beta_n)D_n(x) \\ \gamma_{n+1}D_{n+1}(x) = -\Phi(x) + \gamma_n D_{n-1}(x) - \\ \qquad\qquad -(x-\beta_n)C_n(x) + (x-\beta_n)^2 D_n(x) \end{array} \quad , n \geq 0, \tag{3.18}$$

where $C_0(x)$ and $D_0(x)$ are given by (3.1) and $\gamma_0 D_{-1}(x) = B(x)$.

Replacing n by $n-1$ in (3.20) and using (1.6), we obtain

$$\Phi(x)B'_n(x) - B(x)B_{n-1}^{(1)}(x) = \\ D_n(x)B_{n+1}(x) + \left(\frac{1}{2}\big(C_n(x) - C_0(x)\big) - (x-\beta_n)D_n(x)\right)B_n(x). \tag{3.19}$$

Remarks. 1. When $B = 0$ in (3.17)-(3.18), we meet the structure relation in the semiclassical case $\big($i.e. v is semiclassical linear form$\big)$ (see [18]).

2. In the Laguerre-Hahn $\big($resp. semiclassical $\big)$ case, the polynomials C_n and D_n of (3.17) enable to obtain the coefficients of the fourth-order $\big($resp. second-order$\big)$ differential equation satisfied by each B_n , $n \geq 0$. See, for instance [10, p.90] $\big($ resp. [19, p.236] $\big)$.

From Proposition 3.5, the linear form u is also Laguerre-Hahn and its MOPS $\{\tilde{B}_n\}_{n\geq 0}$ satisfies a structure relation. In general, $\{\tilde{B}_n\}_{n\geq 0}$ fulfils

$$\begin{aligned}\tilde{\Phi}(x)\tilde{B}'_{n+1}(x) - \tilde{B}(x)\tilde{B}^{(1)}_n(x) = &\\ \frac{1}{2}\big(\tilde{C}_{n+1}(x) - \tilde{C}_0(x)\big)\tilde{B}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{D}_{n+1}(x)\tilde{B}_n(x), n \geq 0,&\end{aligned} \tag{3.20}$$

with

$$\begin{array}{l}\tilde{C}_{n+1}(x) = -\tilde{C}_n(x) + 2(x - \tilde{\beta}_n)\tilde{D}_n(x),\\ \tilde{\gamma}_{n+1}\tilde{D}_{n+1}(x) = -\tilde{\Phi}(x) + \tilde{\gamma}_n\tilde{D}_{n-1}(x)-\\ \qquad\qquad -(x - \tilde{\beta}_n)\tilde{C}_n(x) + (x - \tilde{\beta}_n)^2\tilde{D}_n(x)\end{array}, n \geq 0,$$

where $\tilde{C}_0(x)$, $\tilde{D}_0(x)$ are given by (3.6) and $\tilde{\gamma}_0\tilde{D}_{-1}(x) = \tilde{B}(x)$.

We are going to establish the expression of $\tilde{C}_n$ and $\tilde{D}_n$, $n \geq 0$ in terms of those of the sequence $\{B_n\}_{n\geq 0}$.

PROPOSITION 3.10. *We have for $n \geq 0$*

$$\rho_{n+1}B^{(1)}_{n-1}(x) + \sigma_{n+1}(x)B^{(1)}_n(x) = \lambda(x - a)\tilde{B}^{(1)}_n(x) + (1 - \lambda)\tilde{B}_{n+1}(x), \tag{3.21}$$

$$\left\{\begin{array}{l}\frac{1}{2}\big(\tilde{C}_{n+1}(x) - \tilde{C}_0(x)\big) = \left(\sigma_n(x) - \rho_n\dfrac{b_n}{a_{n-1}}\right)U_n(x) - \dfrac{\rho_n}{a_{n-1}}V_n(x)-\\ \qquad\qquad -(x - a)\big(\Phi(x) + (\lambda - 1)B(x)\big),\\ \tilde{\gamma}_{n+1}\tilde{D}_{n+1} = \dfrac{\gamma_n}{a_{n-1}}\rho_{n+1}U_n(x) + \dfrac{\gamma_n}{a_{n-1}}\sigma_{n+1}(x)V_n(x),\end{array}\right. \tag{3.22}$$

$$\left\{\begin{array}{l}U_n(x) = \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)\sigma_{n+1}(x) + \rho_{n+1}D_n(x) + \Phi(x),\\ V_n(x) = \frac{1}{2}\big(C_n(x) + C_0(x)\big)\rho_{n+1} + \gamma_{n+1}\sigma_{n+1}(x)D_{n+1}(x),\end{array}\right. \tag{3.23}$$

where $\tilde{C}_0(x)$ and $\tilde{D}_0(x)$ are given by (3.6).

Proof. From (2.4), we can write

$$(x - c)\tilde{B}_{n+1}(x) = \rho_{n+1}B_n(x) + \sigma_{n+1}(x)B_{n+1}(x), n \geq 0. \tag{3.24}$$

On the one hand using (1.3), and (1.4) we obtain

$$\big(u\theta_0\big((\xi - c)\tilde{B}_{n+1}(\xi)\big)\big)(x) =< \tilde{B}_{n+1}(\xi)u, 1 > +(x - c)\big(u\theta_0\tilde{B}_{n+1}\big)(x).$$

From Definition 1.2 and the fact that $\{\tilde{B}_n\}_{n\geq 0}$ is orthogonal with respect to u, we get

$$\big(u\theta_0\big((\xi - c)\tilde{B}_{n+1}(\xi)\big)\big)(x) = (x - c)\tilde{B}^{(1)}_n(x), n \geq 0. \tag{3.25}$$

On the other hand, from (3.24) we obtain

$$\big(u\theta_0((\xi-c)\tilde{B}_{n+1}(\xi))\big)(x) = \rho_{n+1}\big(u\theta_0 B_n\big)(x) + \\ +\sigma_{n+1}(x)\big(u\theta_0 B_{n+1}\big)(x) + < B_{n+1}(\xi)u, 1 > . \tag{3.26}$$

The functional equation (2.1), leads to

$$\begin{aligned}\lambda\big(u\theta_0 B_n\big)(x) &= \left\langle (\xi-a)^{-1}(\xi-c)v + \lambda\delta_a, \frac{B_n(\xi)-B_n(x)}{\xi-x}\right\rangle \\ &= \left\langle (\xi-c)v, \frac{(a-\xi)\big(B_n(\xi)-B_n(x)\big)+(x-\xi)\big(B_n(a)-B_n(\xi)\big)}{(\xi-x)(a-x)(\xi-a)}\right\rangle \\ &\quad +\lambda\big(\theta_a B_n\big)(x).\end{aligned}$$

Then

$$\lambda\big(u\theta_0 B_n\big)(x) = \\ \frac{1}{a-x}\left((c-x)B^{(1)}_{n-1}(x) + (a-c)B^{(1)}_{n-1}(a)\right) + (\lambda-1)\big(\theta_a B_n\big)(x). \tag{3.27}$$

From (2.2), we have

$$< \lambda u, B_{n+1}(x) >= (a-c)B^{(1)}_{n}(a) + (\lambda-1)B_{n+1}(a). \tag{3.28}$$

By substituting (3.27)-(3.28) in (3.26), we obtain

$$\begin{aligned}&\lambda\big(u\theta_0((\xi-c)\tilde{B}_{n+1}(\xi))\big)(x) = (a-c)B^{(1)}_{n}(a) + (\lambda-1)B_{n+1}(a) + \\ &+\sigma_{n+1}(x)\left(\frac{1}{a-x}\left((c-x)B^{(1)}_{n}(x) + (a-c)B^{(1)}_{n}(a)\right) + (\lambda-1)\big(\theta_a B_{n+1}\big)(x)\right) + \\ &+\rho_{n+1}\left(\frac{1}{a-x}\left((c-x)B^{(1)}_{n-1}(x) + (a-c)B^{(1)}_{n-1}(a)\right) + (\lambda-1)\big(\theta_a B_n\big)(x)\right).\end{aligned} \tag{3.29}$$

Substituting x by c in (3.24), we get

$$\rho_{n+1}B_n(c) + \sigma_{n+1}(c)B_{n+1}(c) = 0, n \geq 0. \tag{3.30}$$

Now, substituting x by c in (3.29) and using (3.25), and (3.30), we obtain

$$\rho_{n+1}\Big(\big(a-c\big)B^{(1)}_{n-1}(a) + \big(\lambda-1\big)B_n(a)\Big) + \\ +\sigma_{n+1}(a)\Big(\big(a-c\big)B^{(1)}_{n}(a) + \big(\lambda-1\big)B_{n+1}(a)\Big) = 0. \tag{3.31}$$

From (3.24), (3.29), and (3.31), we get

$$\lambda\big(u\theta_0\big((\xi-c)\tilde{B}_{n+1}(\xi)\big)\big)(x) = \\ \frac{c-x}{a-x}\left(\rho_{n+1}B^{(1)}_{n-1}(x) + \sigma_{n+1}(x)B^{(1)}_{n}(x) - (1-\lambda)\tilde{B}_{n+1}(x)\right),\ n \geq 0. \tag{3.32}$$

Then the relation (3.21) is a consequence of (3.25) and (3.32).

Now, we are going to prove (3.22). After derivation, we multiply (3.24) by $(x-a)\Phi(x)$, we obtain

$$(x-c)(x-a)\Phi(x)\tilde{B}'_{n+1}(x)+(x-a)\Phi(x)\tilde{B}_{n+1}(x)=\rho_{n+1}(x-a)\Phi(x)B'_n(x)+ \\ +\sigma_{n+1}(x)(x-a)\Phi(x)B'_{n+1}(x)+(x-a)\Phi(x)B_{n+1}(x). \quad (3.33)$$

Multiplying (3.21) by $(x-a)B(x)$, subtracting from (3.33) the equation we have found and taking into account (3.17)-(3.19), and (2.17), we get

$$(x-a)(x-c)\Phi(x)\tilde{B}'_{n+1}(x)-\lambda(x-a)^2B(x)\tilde{B}^{(1)}_n(x)= \\ \left(\left(\sigma_n(x)-\rho_n\frac{b_n}{a_{n-1}}\right)U_n(x)-\frac{\rho_n}{a_{n-1}}V_n(x)-(x-a)\big(\Phi(x)+(\lambda-1)B(x)\big)\right)\tilde{B}_{n+1}(x)- \\ -\left(\frac{\gamma_n}{a_{n-1}}\rho_{n+1}U_n(x)+\frac{\gamma_n}{a_{n-1}}\sigma_{n+1}(x)V_n(x)\right)\tilde{B}_n(x).$$

where $U_n(x)$ and $V_n(x)$ are defined by (3.23).

Comparing with (3.20), we get (3.22). □

4. Examples.

4.1. We study the problem (2.1), with $v=GG$ where GG is the Generalized Gegenbauer linear form. In this case, the linear form v is symmetric semiclassical of class $s=1$. Thus, we have [2,6]

$$\gamma_{2n+1}=\frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},\ n\geq 0,$$

$$\gamma_{2n+2}=\frac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)},\ n\geq 0. \quad (4.1)$$

The regularity conditions are $\alpha\neq -n\ ,\ \beta\neq -n\ ,\ \alpha+\beta\neq -(n+1)\ ,\ n\geq 1.$

We also have

$$\Phi(x)=x(x^2-1) \quad , \quad \Psi(x)=-2(\alpha+\beta+2)x^2+2(\beta+1). \quad (4.2)$$

$$\begin{array}{l} C_n(x)=(2n+2\alpha+2\beta+1)x^2+(-1)^{n+1}(2\beta+1) \\ D_n(x)=2(n+\alpha+\beta+1)x \end{array} ,\ n\geq 0. \quad (4.3)$$

In addition, the MOPS $\{B_n\}_{n\geq 0}$ satisfies (see [6]).

$$B_{2n}(x)=P_n(x^2), \quad B_{2n+1}(x)=xR_n(x^2), \quad n\geq 0,$$

with $P_n(x)=\frac{1}{2^n}P_n^{\alpha,\beta}(2x-1)$ and $R_n(x)=\frac{1}{2^n}P_n^{\alpha,\beta+1}(2x-1)$, $n\geq 0$, where $P_n^{\alpha,\beta}(x)$ denotes the classical Jacobi's polynomials which are orthogonal with respect to $\mathcal{J}(\alpha,\beta)$.

For greater convenience we take $c=0$, $a=1$, and $\alpha\neq 0$.

Using (2.26), we obtain

$$B_{2n}(0)=\frac{1}{2^n}P_n^{\alpha,\beta}(-1)=(-1)^n\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)}, \quad n\geq 0. \quad (4.4)$$

We have [18]

$$B_{2n+1}(1) = \frac{1}{2^n} P_n^{\alpha,\beta+1}(1) = \frac{(-1)^n}{2^n} P_n^{\beta+1,\alpha}(-1)$$

since $P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x)$. So, if we replace (α,β) by $(\beta+1,\alpha)$ in the previous equation, then we get

$$B_{2n+1}(1) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+2)}, \quad n \geq 0. \tag{4.5}$$

Taking into account (2.33) and (1.7), with $\beta_n = 0$, we have

$$B_{2n}^{(1)}(0) = \frac{1}{2^n} P_n^{\alpha,\beta+1}(-1, -2\gamma_1) = (-1)^n \prod_{\nu=0}^{n} \gamma_{2\nu}, \quad n \geq 0.$$

Therefore, using (4.1) we get for $n \geq 0$

$$\frac{1}{2^n} P_n^{\alpha,\beta+1}(-1, \frac{-2(\beta+1)}{\alpha+\beta+2}) = (-1)^n \frac{\Gamma(\alpha+\beta+2)\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+2)}. \tag{4.6}$$

Using (2.33) and (1.9), we obtain

$$B_{2n}^{(1)}(1) = (1+\frac{\beta+1}{\alpha})B_{2n+1}(1) - \frac{\beta+1}{\alpha}\frac{(-1)^n}{2^n} P_n^{\beta+1,\alpha}(-1, \frac{-2\alpha}{\alpha+\beta+2})\ ,\ n \geq 0.$$

From (2.25), (4.5) and (4.6), we get

$$\begin{aligned} \triangle_n = & -\frac{\Gamma(\alpha+\beta+2)\Gamma(n+1)\Gamma(n+\beta+2)}{\Gamma(\beta+2)\Gamma(2n+\alpha+\beta+2)} + \\ & +(\lambda+\frac{\beta+1}{\alpha})\frac{\Gamma(n+1+\alpha)\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+2)}. \end{aligned} \tag{4.7}$$

Then, the linear form u is regular for every λ such that

$$(\lambda+\frac{\beta+1}{\alpha}) \neq \frac{\Gamma(\alpha)\Gamma(\alpha+\beta+2)\Gamma(n+1)\Gamma(n+\beta+2)}{\Gamma(\beta+1)\Gamma(n+1+\alpha)\Gamma(n+\alpha+\beta+2)}.$$

Since v is semiclassical, then according to Proposition 3.5 (with B(x) =0) the linear form u is also semiclassical. It satisfies (3.5) and (3.8) with

$$\begin{cases} \tilde{\Phi}(x) = x(x-1)^2(x+1), \tilde{B}(x) = 0 \\ \tilde{\Psi}(x) = (x-1)\Big(-(2\alpha+2\beta+5)x^2+2\beta+3\Big), \\ \tilde{C}_0(x) = (x-1)\Big((2\alpha+2\beta+1)x^2 - x - 2(\beta+1)\Big), \\ \lambda\tilde{D}_0(x) = 2\lambda(\alpha+\beta+1)x^2 - 2(\lambda-1)(\beta+1). \end{cases} \tag{4.8}$$

According to Corollary 3.9, we have the following results:

$\star$ If $\lambda \neq -\dfrac{\beta+1}{\alpha}$, then the class of u is $\tilde{s} = 2$.

$\star$ If $\lambda = -\dfrac{\beta+1}{\alpha}$, then the class of u is $\tilde{s} = 1$.

Now, we give the coefficients of the recurrence relation satisfied by $\{\tilde{B}_n\}_{n\geq 0}$. For this, first we calculate the coefficients a_n and b_n, $n \geq 0$, given by (2.28)-(2.29).

$$a_{-1} = \lambda, a_{2n} = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad a_{2n+1} = -\frac{\triangle_{n+1}}{\triangle_n}, \quad n \geq 0,$$

$$b_{2n} = 0, \quad b_{2n+1} = -1, \quad n \geq 0.$$

Using the above results and (2.30), we obtain

$$\left.\begin{array}{l} \tilde{\beta}_n = (-1)^n, \quad \tilde{\gamma}_1 = \dfrac{\beta+1}{\lambda(\alpha+\beta+2)} \\ \tilde{\gamma}_{2n+2} = -\frac{\triangle_{n+1}}{\triangle_n}, \quad \tilde{\gamma}_{2n+3} = -\dfrac{\gamma_{2n+2}\gamma_{2n+3}\triangle_n}{\triangle_{n+1}} \end{array}\right. , n \geq 0.$$

Then according to Proposition 3.10, we give the elements of the structure relation of the sequence $\{\tilde{B}_n\}_{n\geq 0}$ (for $n \geq 0$)

$$\begin{cases} \dfrac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2} = (x-1)\times \\ \left((2n+1)x^2 + x - \dfrac{2(a_{2n-1}-\gamma_{2n})(n+\beta+1)(n+\alpha+\beta+1)}{a_{2n-1}(2n+\alpha+\beta+1)} + 2(\beta+1)\right), \\ \dfrac{\tilde{C}_{2n+2}(x) - \tilde{C}_0(x)}{2} = (x-1)\times \\ \quad \left(2(n+1)x^2 + 2(a_{2n+1}-\gamma_{2n+2})(2n+\alpha+\beta+2)\right), \\ \tilde{\gamma}_{2n+1}\tilde{D}_{2n+1}(x) = \dfrac{2\gamma_{2n}(n+\beta+1)(n+\alpha+\beta+1)}{a_{2n-1}(2n+\alpha+\beta+1)}(x-1)^2, \\ \tilde{\gamma}_{2n+2}\tilde{D}_{2n+2}(x) = 2a_{2n+1}(2n+2\alpha+2\beta+3)x^2 + \\ \qquad +2(a_{2n+1}-\gamma_{2n+2})\Big((a_{2n+1}-\gamma_{2n+2})(2n+\alpha+\beta+2) - \beta - 1\Big). \end{cases} \tag{4.9}$$

The linear form v has the following integral representation [6 p.156], for $\Re\alpha > -1, \Re\beta > -1, f \in \mathcal{P}$,

$$\langle v, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} |x|^{2\beta+1}(1-x^2)^\alpha f(x)dx. \tag{4.10}$$

From (2.2), we obtain

$$\langle \lambda u, f\rangle = (\lambda-1)f(1) + \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} |x|^{2\beta+1}(1-x^2)^\alpha f(x)dx +$$
$$+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} |x|^{2\beta+1}(1-x^2)^\alpha \frac{f(x)-f(1)}{x-1}dx.$$

But, when $\Re\alpha > 0$ we have

$$\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} |x|^{2\beta+1}(1-x^2)^\alpha \frac{1}{x-1}dx = -\frac{\alpha+\beta+1}{\alpha}.$$

Therefore for $\Re\beta > -1, \Re\alpha > 0, f \in \mathcal{P}$,

$$\langle u, f\rangle = \left(1 + \frac{(\beta+1)}{\lambda\alpha}\right) f(1) - \tag{4.11}$$
$$-\frac{\Gamma(\alpha+\beta+2)}{\lambda\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} x|x|^{2\beta+1}(1+x)^\alpha(1-x)^{\alpha-1} f(x)dx.$$

It is interesting to give, When $\lambda = -\dfrac{\beta+1}{\alpha}$, the elements of the sequence $\{\tilde{B}_n\}_{n\geq 0}$. In this case, the linear form u is non-symmetric semiclassical of class $s = 1$, we have

$$\begin{cases} \tilde{\beta}_n = (-1)^n \\ \tilde{\gamma}_{2n+1} = -\dfrac{(n+\alpha)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \\ \tilde{\gamma}_{2n+2} = -\dfrac{(n+1)(n+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}, \ n \geq 0. \end{cases} \tag{4.12}$$

From Proposition 2.8, we have

$$\begin{cases} B_n(x) + s_n B_{n-1}(x) = \tilde{B}_n(x) + t_n \tilde{B}_{n-1}(x), \quad n \geq 1, \\ s_{2n+2} = t_{2n+2} = -\dfrac{n+1}{2n+\alpha+\beta+2}, \\ s_{2n+1} = -\dfrac{n+\alpha}{2n+\alpha+\beta+1}, \quad t_{2n+1} = \dfrac{n+\beta+1}{2n+\alpha+\beta+1}, \quad n \geq 0. \end{cases} \tag{4.13}$$

The linear form u satisfies (3.5) , (3.8), and (3.20) with

$$\begin{cases} \tilde{\Phi}(x) = x(x^2-1), \ \tilde{\Psi}(x) = -(2\alpha+2\beta+2)x^2 + x + 2\beta + 3, \\ \tilde{C}_0(x) = (2\alpha+2\beta+7)x^2 + x - 2\beta - 4, \ \tilde{D}_0(x) = 2(\alpha+\beta+1)(x+1), \\ \dfrac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2} = (2n+1)x^2 + x - 2(n+\alpha), \\ \dfrac{\tilde{C}_{2n+2}(x) - \tilde{C}_0(x)}{2} = (2n+1)(x^2-1), \\ \tilde{D}_{2n+1}(x) = 2(2n+\alpha+\beta+2)(x-1), \\ \tilde{D}_{2n+2}(x) = 2(2n+\alpha+\beta+3)(x+1), \quad n \geq 0. \end{cases}$$

The linear form u has the following integral representation for $\Re\beta > -1, \Re\alpha > 0, f \in \mathcal{P}$,

$$\langle u, f \rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha)\Gamma(\beta+2)} \int_{-1}^{1} x|x|^{2\beta+1}(1+x)^{\alpha}(1-x)^{\alpha-1} f(x)dx. \tag{4.14}$$

Remarks. 1. (4.13) is an example which illustrates the results of Theorem 2.4 in [4] , when $(x-c)v$ is not regular linear form.

2. The integral representation (4.14) doesn't exist in the list given in [5].

4.2. We study the problem (2.1), with $v = \mathcal{L}(\alpha)$ where $\mathcal{L}(\alpha)$ is the Laguerre linear form. In this case, the linear form v is not symmetric. This linear form is classical (semiclassical of class $s = 0$ [18]).

We have [19]

$$\beta_n = 2n + \alpha + 1, \quad \gamma_{n+1} = (n+1)(n+\alpha+1), \quad n \geq 0,$$

the regularity condition is $\alpha \neq -n, \ n \geq 1$

$$\Phi(x) = x, \quad \Psi(x) = x - \alpha - 1,$$

$$C_n(x) = -x + (2n+\alpha), \quad D_n(x) = -1, \quad n \geq 0.$$

To simplify, we take $a = 0$, $c = -1$ and $\alpha \neq 0$.

We have [19][6]

$$B_n(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad n \geq 0, \tag{4.15}$$

$$B_n(-1) = (-1)^n n! \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)k!}, \quad n \geq 0. \tag{4.16}$$

Using the three-term recurrence relation satisfied by $\{B_n^{(1)}\}_{n\geq 0}$, we deduce by induction

$$B_n^{(1)}(0) = \frac{(-1)^{n+1}}{\alpha} \left(\Gamma(n+2) - \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)} \right), \quad n \geq 0. \tag{4.17}$$

Taking into account (2.9) and the above results, we get

$$\begin{aligned} d_n = d_n(\lambda) = & \\ = (-1)^{n+1} B_n(-1) & \left\{ \alpha^{-1}(n+1)! + (\lambda - \alpha^{-1} - 1) \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)} \right\} + \\ +(-1)^{n+1} B_{n+1}(-1) & \left\{ \alpha^{-1} n! + (\lambda - \alpha^{-1} - 1) \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \right\}, n \geq 0. \end{aligned} \tag{4.18}$$

In particular, we have

If $\alpha > 0$ then, $d_n(\alpha^{-1}+1) = \alpha^{-1} \sum_{k=0}^{n+1} \frac{n!(n+1)!\Gamma(n+\alpha+1)}{(n-k+1)!k!\Gamma(\alpha+k)} \neq 0, n \geq 0.$

Then, from (2.9)-(2.12), and (2.6) we obtain for every λ such that $d_n(\lambda) \neq 0$

$$\begin{aligned} & a_n = (-1)^{n+1} B_{n+1}(-1) d_n^{-1} \left\{ \alpha^{-1}(n+1)! + (\lambda - \alpha^{-1} - 1) \tfrac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)} \right\} + \\ & \qquad +(n+1)(n+\alpha+1), \\ & b_{n+1} = 2n + \alpha + 3 + (-1)^n B_n(-1) d_n^{-1} \left\{ \alpha^{-1}(n+1)! + (1 + \alpha^{-1} - \lambda) \tfrac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)} \right\}, \\ & b_0 = \alpha + 2, \\ & \tilde{\gamma}_1 = \lambda^{-1}(\alpha+1)(\alpha+3) - \left(\lambda^{-1}(\alpha+2) \right)^2, \quad \tilde{\gamma}_{n+2} = (n+1)(n+\alpha+1) \tfrac{d_{n+2} d_n}{d_{n+1}^2}, \\ & \tilde{\beta}_n = 2n + \alpha + 3 + b_n - b_{n+1}, \quad n \geq 0. \end{aligned}$$

Since v is semiclassical, then according to Proposition 3.5 (when B(x) = 0), the linear form u is also semiclassical. It satisfies (3.5) and (3.8) with

$$\begin{cases} \tilde{\Phi}(x) = (x+1)x^2 \ , \ \tilde{\Psi}(x) = -2x^3 - (\alpha+2)x^2 - \alpha x \ , \\ \tilde{C}_0(x) = -x^3 + \alpha x^2 + \alpha x \ , \ \lambda \tilde{D}_0(x) = -\lambda x^2 + (\lambda\alpha - \alpha - 2)x + (\lambda - 1)\alpha - 1. \end{cases}$$

According to Corollary 3.9, we have the following results:
⋆ If $\lambda \neq \alpha^{-1} + 1$, then the class of u is $\tilde{s} = 2$.
⋆ If $\lambda = \alpha^{-1} + 1$, then the class of u is $\tilde{s} = 1$.

From Proposition 3.10, we give the elements of the structure relation of the sequence $\{\tilde{B}_n\}_{n\geq 0}$.

$$\frac{\tilde{C}_{n+1}(x)-\tilde{C}_0(x)}{2} = (n+1)x^2 + \Big\{-\alpha+1-(n+1)(5n+3\alpha+7-b_{n+1})+ \\ +\frac{\gamma_n}{a_{n-1}}\big((n+2)b_n+2\gamma_{n+1}-a_n\big)\Big\}x + \frac{\gamma_n}{a_{n-1}}\Big\{-(n+1)(n+2-b_{n+1})(n+1+\alpha+ \\ +b_n)+(n+1+\alpha)(a_n-2\gamma_{n+1})-b_na_n\Big\} + (n+1)\big((3n+2\alpha+2)(n+2-b_{n+1})+ \\ +2(n+\alpha+1)^2\big)+na_n,$$

$$\tilde{\gamma}_{n+1}\tilde{D}_{n+1}(x) = \frac{\gamma_n}{a_{n-1}}\big(2\gamma_{n+1}-a_n\big)x^2 + \frac{\gamma_n}{a_{n-1}}\Big\{(4n+2\alpha+6-b_{n+1})a_n-(8n+ \\ +4\alpha+12-3b_{n+1})\gamma_{n+1}\Big\}x + \frac{\gamma_n}{a_{n-1}}\big(\gamma_{n+1}-a_n\big)\big((n+1)(n+2-b_{n+1})-a_n\big)+ \\ +\frac{\gamma_n}{a_{n-1}}\big(b_{n+1}-2n-\alpha-3\big)\big((n+\alpha+1)a_n-(3n+2\alpha+4-b_{n+1})\gamma_{n+1}\big), n\geq 0.$$

The linear form v has the following integral representation[18]

$$\langle v, f\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty} x^\alpha e^{-x}f(x)dx, \ \ \Re(\alpha)>-1, \ \ f\in\mathcal{P}. \tag{4.19}$$

From (2.2), we have

$$<\lambda u, f> = \frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty} x^\alpha e^{-x}f(x)dx + \\ +\frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty} x^\alpha e^{-x}\frac{f(x)-f(0)}{x}dx + (\lambda-1)f(0).$$

But, when $\Re\alpha>0$ we have $\quad \dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_0^{+\infty} x^\alpha e^{-x}\frac{1}{x}dx = \alpha^{-1}.$

Therefore for λ such that $d_n\neq 0$, $\Re\alpha>0, f\in\mathcal{P}$,

$$<\lambda u, f>= \frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty}(x+1)x^{\alpha-1}e^{-x}f(x)dx + (\lambda-\alpha^{-1}-1)\delta_0. \tag{4.20}$$

4.3. Let $v=\mathcal{J}^{(1)}(\alpha,\beta)$ be the associated linear form of the first kind of Jacobi. So $B_n(x)=P_n^{(1)}(x), n\geq 0$ where $P_n(x)$ denotes the classical Jacobi polynomials. We have [17 , 10]

$$\beta_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}, n\geq 0,$$
$$\gamma_{n+1} = \frac{4(n+2)(n+\alpha+\beta+2)(n+\alpha+2)(n+\beta+2)}{(2n+\alpha+\beta+5)(2n+\alpha+\beta+4)^2(2n+\alpha+\beta+3)}, \quad n\geq 0.$$

The regularity conditions are $\alpha\neq -n$, $\beta\neq -n$, $\alpha+\beta\neq -n$, $n\geq 2$.

$$\Phi(x) = x^2-1, \quad \Psi(x) = -(\alpha+\beta+4)x - \frac{\alpha^2-\beta^2}{\alpha+\beta+2},$$

$$B(x) = \frac{4(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2},$$
$$C_n(x) = (2n+\alpha+\beta+2)x + \frac{\alpha^2-\beta^2}{2n+\alpha+\beta+2}, \quad D_n(x) = 2n+\alpha+\beta+3, \quad n \geq 0.$$

We assume $(\alpha+\beta+1)(\alpha+1)(\beta+1) \neq 0$, then v is a Laguerre-Hahn linear form of class $s = 0$. To simplify, we take $c = 1,\ a = -1$, and $\alpha\beta \neq 0$.

Using the three-term recurrence relation satisfied by $\{P_n^{(1)} = B_n\}_{n\geq 0}$, we deduce by induction

$$P_n^{(1)}(-1) = P_n^{(1)}(-1,\alpha,\beta) = \frac{2^n(-1)^{n+1}(\alpha+\beta+1)}{\beta\Gamma(2n+\alpha+\beta+3)} \times \tag{4.21}$$
$$\times\left(\frac{\Gamma(\alpha+\beta+1)\Gamma(n+2)\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)} - \frac{\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(\beta+1)}\right),$$

and $\quad P_n^{(1)}(1) = (-1)^n P_n^{(1)}(-1,\beta,\alpha),\ n \geq 0.$

From [10, p.80 Theorem 2.2] , we have for $n \geq 0$

$$B_{n-1}^{(1)}(x) = P_{n-1}^{(2)}(x)$$
$$= -\frac{(\alpha+\beta+2)(\alpha+\beta+3)}{2(\alpha+1)}P_n^{(1)}(x) - \frac{(\alpha+\beta+2)^2(\alpha+\beta+3)}{4(\alpha+1)(\beta+1)}P_{n+1}(x),$$

(where $\{P_n^{(2)}\}_{n\geq 0}$ denotes the sequence of associated polynomials of first kind for the sequence $\{P_n^{(1)}\}_{n\geq 0}$.)

From (2.9), (4.4), and the above results, we obtain

$$d_n = d_n(\lambda) = \frac{(-1)^{n+1}2^{2n+1}(\alpha+\beta+1)}{\alpha\Gamma(2n+\alpha+\beta+3)\Gamma(2n+\alpha+\beta+4)} \times \tag{4.22}$$
$$\times\left\{\frac{\alpha+\beta+1}{2\beta(\alpha+1)}\big((\alpha+1)(1-\lambda)-(\alpha+\beta+2)(\alpha+\beta+3)\big)\big(X_n + Y_n(\alpha,\beta) + \right.$$
$$\left. + Y_n(\beta,\alpha) + Z_n\big) + \frac{(\alpha+\beta+3)(\alpha+\beta+2)^2}{2(\alpha+1)(\beta+1)}\big(Y_n(\beta,\alpha)+Z_n\big)\right\}$$

where

$$\begin{cases} X_n = \frac{\Gamma^2(\alpha+\beta+1)\Gamma(n+2)\Gamma(n+3)\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}, \\ Y_n(\alpha,\beta) = -\frac{\Gamma(\alpha+\beta+1)\Gamma(n+2)\Gamma(n+\alpha+2)\Gamma(n+\alpha+3)\Gamma(n+\alpha+\beta+2)}{\Gamma^2(\alpha+1)}, \\ Z_n = \frac{\Gamma(n+\beta+2)\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+\beta+3)}{\Gamma(\alpha+1)\Gamma(\beta+1)}. \end{cases} \tag{4.23}$$

In particular, if $\alpha > 0$ and $\beta > 0$, we have for $\lambda_1 = 1 - \dfrac{(\alpha+\beta+3)(\alpha+\beta+2)}{\alpha+1}$ and $\lambda_2 = 1 - \dfrac{(\alpha+\beta+3)(\alpha+\beta+2)}{(\alpha+\beta+1)(\beta+1)}$

$d_n(\lambda_1) = \frac{(-1)^{n+1}2^{2n}(\alpha+\beta+1)(\alpha+\beta+2)^2(\alpha+\beta+3)}{\alpha(\alpha+1)(\beta+1)\Gamma(2n+\alpha+\beta+3)\Gamma(2n+\alpha+\beta+4)}\big(Y_n(\beta,\alpha)+Z_n\big) \neq 0, n \geq 0,$

$d_n(\lambda_2) = \frac{(-1)^n 2^{2n}(\alpha+\beta+1)(\alpha+\beta+2)^2(\alpha+\beta+3)}{\alpha(\alpha+1)(\beta+1)\Gamma(2n+\alpha+\beta+3)\Gamma(2n+\alpha+\beta+4)}\big(X_n + Y_n(\alpha,\beta)\big) \neq 0, n \geq 0.$

Then, from (2.9), (2.12), we obtain for every λ such that $d_n(\lambda) \neq 0$ ($n \geq 0$)

$$a_n = d_{n+1}d_n^{-1},$$

$$\begin{aligned} b_n = \beta_{n+1} - 1 + &\frac{d_n^{-1}(-1)^{n+1}2^{2n+1}(\alpha+\beta+1)}{\alpha\beta\Gamma(2n+\alpha+\beta+3)\Gamma(2n+\alpha+\beta+5)} \times \\ &\Big\{\frac{(\alpha+\beta+1)}{\alpha+1}\big[(\alpha+1)(1-\lambda) - (\alpha+\beta+2)(\alpha+\beta+3)\big]\big[(n+\alpha+2)X_n + \\ &+(n+2)Y_n(\alpha,\beta)\big] + (\beta+1)^{-1}\big[(\beta+1)(\alpha+\beta+1)(1-\lambda) - \\ &-(\alpha+\beta+2)(\alpha+\beta+2)\big]\big[(n+\alpha+\beta+2)Y_n(\beta,\alpha) + (n+\beta+2)Z_n\big]\Big\}. \end{aligned}$$

Taking into account that the linear form v is Laguerre-Hahn , and by virtue of Proposition 3.5, the linear form u is also Laguerre-Hahn. It satisfies (3.5) and (3.8) with

$$\tilde{\Phi}(x) = (x^2-1)^2, \quad \tilde{B}(x) = \frac{4\lambda(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}(x+1)^2,$$

$$\begin{aligned} \tilde{\Psi}(x) = (x+1)\Big\{&-(x-1)\Big((\alpha+\beta+4)x + \frac{(\alpha+2)^2-\beta^2+2\beta}{\alpha+\beta+2}\Big) + \\ &+\frac{8(1-\lambda)(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\Big\}, \end{aligned}$$

$$\begin{aligned} \tilde{C}_0(x) = (x+1)\Big\{&(x-1)\Big((\alpha+\beta+2)x + \frac{(\alpha+2)^2-\beta^2+2\beta}{\alpha+\beta+2}\Big) - \\ &-\frac{8(1-\lambda)(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\Big\}, \end{aligned}$$

$$\begin{aligned} \lambda\tilde{D}_0(x) = (x-1)\Big(&(2\alpha+2\beta+5)x + \frac{\alpha^2-\beta^2}{\alpha+\beta+2} - (\alpha+\beta+2)\Big) + \\ &+\frac{4(\lambda-1)(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}. \end{aligned}$$

According to Proposition 3.8, we have the following results:

$\star$ If $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$, then the class of u is $\tilde{s} = 2$.

$\star$ $\lambda = \lambda_1$ or $\lambda = \lambda_2$, then the class of u is $\tilde{s} = 1$.

Finally, from Proposition 3.10, we give the elements of the structure relation of the sequence $\{\tilde{B}_n\}_{n\geq 0}$ for $n \geq 0$

$$\begin{aligned} U_n(x) = (n+2)x^2 + (n+1)&\left(\frac{\beta_{n+1}}{(\alpha+\beta+2)} + b_{n+1}\right)x + \\ &+\frac{(n+1)(\beta^2-\alpha^2)(b_{n+1}-\beta_{n+1})}{(\alpha+\beta+2)(2n+\alpha+\beta+4)} + (2n+\alpha+\beta+3)(a_n - \gamma_{n+1}) - 1, \end{aligned}$$

$$\begin{aligned} V_n(x) = &-\Big((n+\alpha+\beta+3)a_n + (n+2)\gamma_{n+1}\Big)x + \\ &+\frac{(n+\alpha+\beta+3)(\alpha^2-\beta^2)(a_n-\gamma_{n+1})}{(\alpha+\beta+2)(2n+\alpha+\beta+4)} + \\ &+(2n+\alpha+\beta+5)(b_{n+1}-\beta_{n+1})\gamma_{n+1}. \end{aligned}$$

$$\begin{cases} \dfrac{\tilde{C}_{n+1}(x)-\tilde{C}_0(x)}{2} = \left(x-\beta_n+\dfrac{\gamma_n b_n}{a_{n-1}}\right)U_n(x) - \left(1-\dfrac{\gamma_n}{a_{n-1}}\right)V_n(x) - \\ \qquad -(x+1)\left(x^2-1+(\lambda-1)\dfrac{4(\lambda-1)(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\right), \\ \tilde{\gamma}_{n+1}\tilde{D}_{n+1}(x) = \dfrac{\gamma_n}{a_{n-1}}\Big((a_n-\gamma_{n+1})U_n(x) + (x-\beta_{n+1}+b_{n+1})V_n(x)\Big). \end{cases}$$

Remark. Unfortunately, we are not able to give an integral representation of u in this case, especially because we still don't know an integral representation of v.

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