

FINITE ELEMENT ANALYSIS OF TRANSIENT ELECTROMAGNETIC SCATTERING FROM 2D CAVITIES*

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Abstract. We present a finite element method for the electromagnetic scattering from a 2-D cavity embedded in the infinite ground plane. The problem is first discretized in time by the β , γ Newmark time-marching scheme. The resulting semi-discrete problem is well-posed. Error analysis of the fully discrete finite element formulation is performed. Stability criteria of the time-stepping scheme are also established. Numerical experiments demonstrate the accuracy and stability of the method.

Key words. Helmholtz equations, finite element methods, Newmark scheme, nonreflecting boundary condition, scattering

AMS subject classifications. 35L05, 65M60, 78M10, 74J20

1. Introduction. It is well-known that one of the main difficulties in numerically approximating solutions involving cavities is the appearance of spurious modes caused by interior resonances. A variety of techniques have been developed to simulate the scattering by cavities. They include high and low frequency methods [6, 11, 14], the method of moments [21, 13, 22], and the time domain finite difference/element methods [4, 10, 15]. These methods are limited to certain range of frequencies and/or small/simple cavities. Recently, the hybrid finite element-boundary integral methods have gained increasing popularity for their ability to model large and complex cavities [8, 9, 12]. It is observed that many of the findings reported in the engineering literature regarding scattering from cavities are experimental in nature and hence often give rise to disputes (for example, explanation of interior resonances). Partial mathematical research in this area is found in [3, 1, 2, 19] (frequency domain), and [17, 20, 18] (time-domain).

This paper aims to provide a thorough treatment of two-dimensional cavity-backed transient electromagnetic scattering problems. We show the method is mathematically rigorous and numerically accurate and stable. In [18] we showed the semi-discrete problem in time and its corresponding variational formulation are well-posed. Experiments for homogeneous cavity media were performed that reflect the accuracy and stability of the scheme. Here, we further provide finite element error analysis for the fully discrete problem. Stability criteria for the resulting hybrid formulation are also obtained. In addition, numerical experiments for layered cavities are performed and they again demonstrate the accuracy and stability of the method.

Let the cavity embedded in an infinite ground plane be denoted Ω that is a bounded Lipschitz continuous region in \mathbb{R}^2 :

$$\Omega \subset \{\mathbf{r} = (x, y) \in \mathbb{R}^2 : y < 0\}, \quad \bar{\Omega} \cap \{\mathbf{r} = (x, y) \in \mathbb{R}^2 : y = 0\} \neq \emptyset.$$

Let S be the cavity walls, Γ the cavity aperture, $\Gamma^c = \{\mathbf{r} = (x, y) \in \mathbb{R}^2 : y = 0\} \setminus \Gamma$, and $\mathcal{U} = \{\mathbf{r} = (x, y) : y > 0\}$ the upper half plane. The ground plane is perfectly

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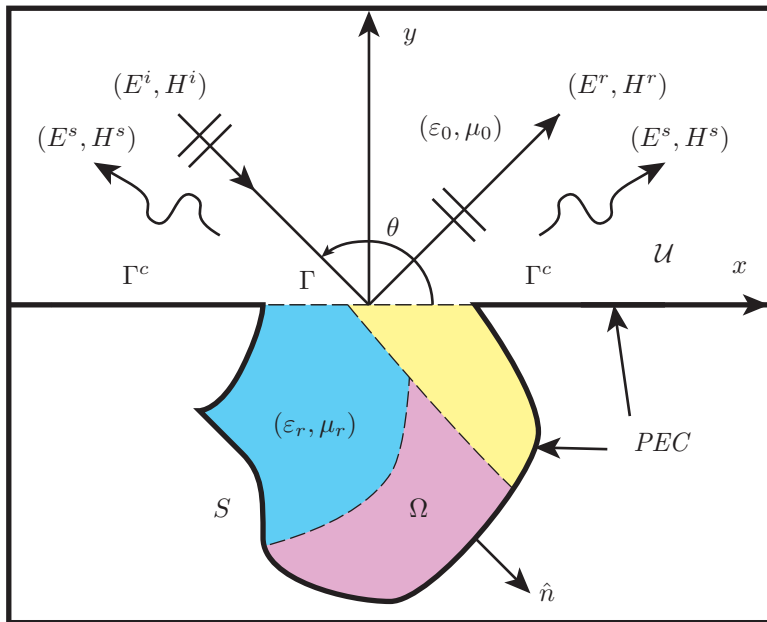


FIG. 1. Cavity setting

electric conducting (PEC). Ω is either empty with $\epsilon_r = \epsilon_0 = 1$ or filled with material whose relative permittivity is $\epsilon_r > 1$, see Figure 1. We assume all media are non-magnetic, hence $\mu_r = \mu_0 = 1$. Given an electromagnetic field $(\mathbf{E}^i, \mathbf{H}^i)$ incident on the cavity Ω , we wish to determine the total fields $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i + \mathbf{E}^r$, $\mathbf{H} = \mathbf{H}^s + \mathbf{H}^i + \mathbf{H}^r$, where $(\mathbf{E}^s, \mathbf{H}^s)$ and $(\mathbf{E}^r, \mathbf{H}^r)$ are the scattered and reflected fields, respectively.

In this paper we shall present the analysis for the transverse magnetic (TM) polarization only for brevity, but numerical results for both TM and TE will be provided.

In the TM case, the fields are of the form

$$(1.1) \quad \mathbf{E} = (0, 0, E_z) = (0, 0, u) \quad \text{and} \quad \mathbf{H} = (H_x, H_y, 0).$$

For demonstration purposes, we consider the following two types of incident fields $\mathbf{E}^i(t, \mathbf{r})$: time-harmonic plane wave and Gaussian plane wave polarizing in the z -direction and propagating in the direction $\mathbf{k} = (-\cos \theta_i, -\sin \theta_i)$, $0 \leq \theta_i \leq \pi$. Specifically, we consider $\mathbf{E}^i = (0, 0, u^i)$, where

$$(1.2) \quad u^i(t, \mathbf{r}) = \begin{cases} E_0 \text{Re}\{e^{i\omega_0 \mathbf{r} \cdot \mathbf{k}} e^{i\omega_0 t}\} & \text{(time-harmonic),} \\ E_0 \frac{4}{\sigma\sqrt{\pi}} \exp\left\{-\left[\frac{4(t-t_0 - \mathbf{r} \cdot \mathbf{k})}{\sigma}\right]^2\right\} & \text{(Gaussian).} \end{cases}$$

The associated reflected field E^r is

$$(1.3) \quad u^r(t, \mathbf{r}) = \begin{cases} -E_0 \text{Re}\{e^{i\omega_0 \mathbf{r} \cdot \mathbf{k}^*} e^{i\omega_0 t}\} & \text{(time-harmonic),} \\ -E_0 \frac{4}{\sigma\sqrt{\pi}} \exp\left\{-\left[\frac{4(t-t_0 - \mathbf{r} \cdot \mathbf{k}^*)}{\sigma}\right]^2\right\} & \text{(Gaussian),} \end{cases}$$

where $\mathbf{k}^* = (-\cos\theta_i, \sin\theta_i)$, σ is the width of the Gaussian pulse measuring its temporal duration, and E_0 is the initial field amplitude. The speed of light c is normalized to 1. By Maxwell's equations, the total field $\mathbf{E} = (0, 0, u)$ satisfies the following problem:

$$(1.4) \quad \begin{cases} -\Delta u + \varepsilon_r \frac{\partial^2 u}{\partial t^2} = 0 & \text{in } \Omega \times (0, T), \\ u = u^i + u^r + u^s & \text{on } \Gamma \times (0, T), \\ u = 0 & \text{on } S \cup \Gamma^c \times (0, T), \end{cases}$$

with the initial conditions

$$u(0, \mathbf{r}) = u_0(\mathbf{r}), \quad \frac{\partial u}{\partial t}(0, \mathbf{r}) = u_1(\mathbf{r}).$$

The scattered field is defined in the upper half plane and satisfies

$$(1.5) \quad \begin{cases} -\Delta u^s + \frac{\partial^2 u^s}{\partial t^2} = 0 & \text{in } \mathcal{U} \times (0, T), \\ u^s = 0 & \text{on } \Gamma^c \times (0, T), \end{cases}$$

and the radiation condition

$$(1.6) \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t} \right) u^s(t, \mathbf{r}) = 0, \quad r = |\mathbf{r}|.$$

The magnetic field \mathbf{H} is found in terms of the electric field \mathbf{E} by Maxwell's equations.

The paper is organized as follows. In Section 2, we discretize the equations in time by the Newmark time-stepping scheme to obtain semi-discrete problems defined in an infinite domain. At each time step we construct an exact nonlocal boundary operator on the cavity aperture to couple the fields in the exterior of the cavity to those inside. This coupling enables the semi-discrete problem to be reduced to the minimal region: the cavity itself, where finite elements are applied to approximate the solutions. We prove that at each time step the weak formulation has a unique solution. The problem is fully discretized and its finite element error analysis presented in Section 3. In Section 4, stability criteria for the time-stepping scheme are derived. Numerical experiments that show the accuracy and temporal stability of the Newmark-finite element scheme are performed in Section 5.

2. Semi-discrete problem. Let $\mathcal{N} > 0$ be a positive integer, and $\Delta t = T/\mathcal{N}$ be the constant time step. For each $n = 0, 1, \dots, \mathcal{N}$, $u^n(x)$ and $\dot{u}^n(x)$ denote the temporal approximations of $u(\mathbf{r}, t_n)$ and $\frac{\partial u}{\partial t}(\mathbf{r}, t_n)$ where $t_n = n\Delta t$, and $\mathbf{r} = (x, y) \in \mathbb{R}^2$. We express the Newmark scheme in the following predictor-corrector form:

Prediction:

$$(2.1) \quad \tilde{u}^{n+1} = u^n + \Delta t \dot{u}^n + \frac{(\Delta t)^2}{2} (1 - 2\beta) \ddot{u}^n,$$

$$(2.2) \quad \tilde{\dot{u}}^{n+1} = \dot{u}^n + \Delta t (1 - \gamma) \ddot{u}^n.$$

Solution:

$$(2.3) \quad -\Delta u^{n+1} + \alpha^2 \varepsilon_r u^{n+1} = \alpha^2 \varepsilon_r \tilde{u}^{n+1} \quad \text{in } \Omega,$$

$$u^{n+1} = 0 \quad \text{on } S,$$

$$u^{n+1} = u^{s,n+1} \quad \text{on } \Gamma,$$

since $u^i + u^r = 0$ on Γ .

Correction:

$$(2.4) \quad \ddot{u}^{n+1} = \alpha^2(u^{n+1} - \tilde{u}^{n+1}),$$

$$(2.5) \quad \dot{u}^{n+1} = \tilde{u}^{n+1} + \Delta t \gamma \ddot{u}^{n+1},$$

where $\alpha^2 = \frac{1}{\Delta t^2 \beta}$.

The scattered field $u^{s,n+1}$ satisfies

$$(2.6) \quad \begin{cases} -\Delta u^{s,n+1} + \alpha^2 u^{s,n+1} &= \alpha^2 \tilde{u}^{s,n+1} & \text{in } \mathcal{U}, \\ u^{s,n+1} &= g & \text{on } \Gamma, \\ u^{s,n+1} &= 0 & \text{on } \Gamma^c, \end{cases}$$

where $g := u^{n+1}$ on Γ . Note that $\varepsilon_r = 1$ in \mathcal{U} .

We refer (2.6) as the *exterior* (to the cavity) problem, which we show next can be solved analytically.

LEMMA 2.1. *Given $g \in H^{1/2}(\Gamma)$, Problem (2.6) has a unique solution*

$$(2.7) \quad \begin{aligned} u^{s,n+1}(\mathbf{r}) &= \alpha^2 \int_0^\infty \int_{-\infty}^\infty G_\alpha(\mathbf{r}, \mathbf{r}') \tilde{u}^{s,n+1}(\mathbf{r}') \, d\mathbf{r}' \\ &\quad + \frac{1}{\pi} \int_\Gamma \frac{\partial}{\partial y} K_0(\alpha|\mathbf{r} - x'\hat{x}|) g(x') dx', \end{aligned}$$

where K_0 is the modified Bessel function of the second kind of order 0, and

$$G_\alpha(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \left\{ K_0(\alpha|\mathbf{r} - \mathbf{r}'|) - K_0(\alpha|\mathbf{r} - \mathbf{r}'_i|) \right\},$$

where $|\mathbf{r} - \mathbf{r}'| = \sqrt{(x' - x)^2 + (y' - y)^2}$, and $|\mathbf{r} - \mathbf{r}'_i| = \sqrt{(x' - x)^2 + (y' + y)^2}$.

Proof. We observe that the modified Green function G_α satisfies the Dirichlet problem [7],

$$\begin{cases} -\Delta G_\alpha(\mathbf{r}, \mathbf{r}') + \alpha^2 G_\alpha(\mathbf{r}, \mathbf{r}') &= \delta(\mathbf{r} - \mathbf{r}') & \text{in } \mathcal{U}, \\ G_\alpha &= 0 & \text{on } \{y' = 0\} \text{ or } \{y = 0\}. \end{cases}$$

Hence, the solution $u^{s,n+1}$ to (2.6) can be expressed as

$$\begin{aligned} u^{s,n+1}(\mathbf{r}) &= \alpha^2 \int_{\mathcal{U}} G_\alpha(\mathbf{r}, \mathbf{r}') \tilde{u}^{s,n+1}(\mathbf{r}') \, d\mathbf{r}' - \int_\Gamma \frac{\partial G_\alpha}{\partial n'}(\mathbf{r}, x'\hat{x}) g(x') dx' \\ &= \alpha^2 \int_0^\infty \int_{-\infty}^\infty G_\alpha(\mathbf{r}, \mathbf{r}') \tilde{u}^{s,n+1}(\mathbf{r}') \, d\mathbf{r}' - \int_\Gamma \frac{\partial G_\alpha}{\partial y'}(\mathbf{r}; x'\hat{x}) g(x') dx', \end{aligned}$$

for $\mathbf{r} \in \mathcal{U}$. Direct computations yield

$$(2.8) \quad \begin{aligned} \frac{\partial G_\alpha}{\partial y'}(\mathbf{r}, \hat{x}') &= \frac{1}{2\pi} \left[\alpha K'_0(\alpha|\mathbf{r} - x'\hat{x}|) \frac{-y}{|\mathbf{r} - x\hat{x}|} - \alpha K'_0(\alpha|\mathbf{r} - x'\hat{x}|) \frac{y}{|\mathbf{r} - x\hat{x}|} \right] \\ &= -\frac{\alpha}{\pi} K'_0(\alpha|\mathbf{r} - x'\hat{x}|) \frac{y}{|\mathbf{r} - x\hat{x}|} \\ &= -\frac{1}{\pi} \frac{\partial}{\partial y} K_0(\alpha|\mathbf{r} - x'\hat{x}|). \quad \square \end{aligned}$$

Taking the partial derivative of $u^{s,n+1}$ with respect to y gives

$$\frac{\partial u^{s,n+1}(\mathbf{r})}{\partial y} = \alpha^2 \int_0^\infty \int_{-\infty}^\infty \frac{\partial G_\alpha}{\partial y}(\mathbf{r}, \mathbf{r}') \tilde{u}^{s,n+1}(\mathbf{r}') d\mathbf{r}' + \frac{1}{\pi} \int_\Gamma \frac{\partial^2}{\partial y^2} K_0(\alpha|\mathbf{r} - x'\hat{x}|) g(x') dx'.$$

Noting that $\frac{\partial^2}{\partial y^2} K_0(\alpha|\mathbf{r} - x'\hat{x}|) = \frac{\partial^2}{\partial y^2} K_0(\alpha|x - x'|)$ if $y = 0$, we get, as $y \rightarrow 0$,

$$\begin{aligned} \frac{\partial u^{s,n+1}}{\partial y} \Big|_{y=0} &= \alpha^2 \int_0^\infty \int_{-\infty}^\infty \frac{\partial G_\alpha}{\partial y}(x, 0; \mathbf{r}') \tilde{u}^{s,n+1}(\mathbf{r}') d\mathbf{r}' + \frac{1}{\pi} \int_\Gamma \frac{\partial^2}{\partial y^2} K_0(\alpha|x - x'|) g(x') dx' \\ &\equiv \tilde{H}^{n+1}(x) + Tg(x), \end{aligned}$$

where

$$(2.9) \quad \tilde{H}^{n+1}(x) := \alpha^2 \int_0^\infty \int_{-\infty}^\infty \frac{\partial G_\alpha}{\partial y}(x, 0; \mathbf{r}') \tilde{u}^{s,n+1}(\mathbf{r}') d\mathbf{r}',$$

and $T : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ is defined as

$$(2.10) \quad Tg(x) = \frac{1}{\pi} \int_\Gamma \frac{\partial^2}{\partial y^2} K_0(\alpha|x - x'|) g(x') dx'.$$

Since

$$\frac{\partial^2}{\partial y^2} K_0(\alpha|x - x'|) = [K_0'' \frac{y^2}{|\mathbf{r}|^2} + K_0'(\frac{1}{|\mathbf{r}|} - \frac{y^2}{|\mathbf{r}|^3})] \Big|_{y=0} = K_0' \frac{1}{|x|} = -K_1 \frac{1}{|x|} \leq 0,$$

it is clear that $\langle Tu, u \rangle \leq 0, \forall u$.

LEMMA 2.2. *The operator $T : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ defined in (2.10) is bounded and non-positive.*

REMARK 2.3. *The operator T is actually a pseudodifferential operator of order one and hence is bounded. The proof of the lemma is rather technical and is omitted here for brevity. Interested readers are referred to [16].*

The boundary operator T will be used to couple the total field in the infinite homogeneous upper half plane \mathcal{U} to the total field in the bounded domain Ω . Indeed, since

$$\begin{aligned} \frac{\partial u^{n+1}}{\partial y} &= \frac{\partial u^{inc,n+1}}{\partial y} + \frac{\partial u^{ref,n+1}}{\partial y} + \frac{\partial u^{s,n+1}}{\partial y} \\ &= \frac{\partial u^{inc,n+1}}{\partial y} + \frac{\partial u^{ref,n+1}}{\partial y} + \tilde{H}^{n+1} + Tu^{s,n+1} \\ &= 2 \frac{\partial u^{inc,n+1}}{\partial y} + \tilde{H}^{n+1} + Tu^{n+1} \quad \text{on } \Gamma, \end{aligned}$$

the boundary value problem (2.3) can be rewritten as

$$(2.11) \quad \begin{aligned} -\Delta u^{n+1} + \alpha^2 \varepsilon_r u^{n+1} &= \alpha^2 \varepsilon_r \tilde{u}^{n+1} \quad \text{in } \Omega, \\ u^{n+1} &= 0 \quad \text{on } S, \\ \frac{\partial u^{n+1}}{\partial y} &= 2 \frac{\partial u^{inc,n+1}}{\partial y} + \tilde{H}^{n+1} + Tu^{n+1} \quad \text{on } \Gamma. \end{aligned}$$

Let $V = \{v \in H^1(\Omega) : u = 0 \text{ on } S\}$. The variational problem associated with (2.11) is then

$$(2.12) \quad b(u^{n+1}, v) = F^{n+1}(v), \quad \forall v \in V,$$

where

$$\begin{aligned} b(u, v) &= (\nabla u, \nabla v) + \alpha^2(\varepsilon_r u, v) - \langle Tu, v \rangle_\Gamma, \\ F^{n+1}(v) &= \alpha^2(\varepsilon_r \tilde{u}^{n+1}, v) + \langle \tilde{H}^{n+1}, v \rangle_\Gamma + 2 \left\langle \frac{\partial u^{inc, n+1}}{\partial y}, v \right\rangle_\Gamma. \end{aligned}$$

THEOREM 2.4. *The variational problem (2.12) has a unique solution in V .*

Proof. Since T is bounded, $|b(u, v)| \leq C \|u\|_1 \|v\|_1$ for some $C > 0$. Since T is non-positive, $b(u, u) \geq C^* \|u\|_1^2$ for some $C^* > 0$. Hence a unique solution is guaranteed by Lax-Milgram theorem. \square

3. Finite element error analysis. Assume that Ω is covered by a family of quasi-uniform triangular mesh τ_h where h is the mesh size, that is,

$$h = \max_{K \in \tau_h} h_K,$$

where h_K is the diameter of the element $K \in \tau_h$.

We consider the finite dimensional subspace

$$V_h = \{v_h \in H^1(\Omega) : v_h|_K \text{ is linear}, K \in \tau_h\}.$$

We note that V_h is closed in V and $V_h \rightarrow V$ as $h \rightarrow 0$. The fully-discrete problem is to find $u_h^n \in V_h$, $n = 1, 2, \dots, \mathcal{N}$, such that

$$(3.1) \quad b(u_h^n, v_h) = F^n(v_h), \quad \forall v_h \in V_h,$$

where $b(u_h^n, v_h)$ and $F^n(v_h)$ are as defined in (2.12). We recall that the bilinear form b is coercive and continuous. Hence by Céa's lemma [5], the fully-discrete problem (3.1) has a unique solution $u_h^n \in V_h$ and

$$(3.2) \quad \|u^n - u_h^n\|_V \leq C \inf_{v_h \in V_h} \|u^n - v_h\|_V.$$

Since ε_r is discontinuous in Ω , the solution $u^n \notin H^2(\Omega)$. Hence the inequality (3.2) does not yield a convergence rate in terms of h . In fact, since $V_h \rightarrow V$, for any $\epsilon > 0$, there is an $h_0 = h_0(\epsilon, u^n)$, such that for $0 < h < h_0$, there exists $v_h \in V_h$ satisfying

$$\|u^n - v_h\|_V \leq \epsilon.$$

By (3.2), we have

$$\|u^n - u_h^n\|_V \leq C\epsilon \quad \forall h < h_0(\epsilon, u^n).$$

Thus, the finite element solution u_h^n converges to u^n in V but not necessarily uniformly. We have the following result.

THEOREM 3.1. *Let $u^n \in V$ and $u_h^n \in V_h$ be the solutions to (2.12) and (3.1), respectively, for $F^n \in V'$. Then given $\epsilon > 0$, there is an $h_0 = h_0(\epsilon)$, such that for all $0 < h < h_0$ we have*

$$(3.3) \quad \|u^n - u_h^n\|_{L^2(\Omega)} \leq \epsilon \|u^n - u_h^n\|_V.$$

Furthermore, if $\varepsilon_r \in L^\infty(\Omega)$, hence $\varepsilon_r \tilde{u}^n \in L^2(\Omega)$, then there exists an $h_1 = h_1(\epsilon) > 0$, such that for all $0 < h < h_1$ we have

$$(3.4) \quad \|u^n - u_h^n\|_V \leq C\epsilon \|F^n\|_{L^2(\Omega)},$$

where C is a positive constant independent of h . Consequently, we have

$$\|u^n - u_h^n\|_{L^2(\Omega)} \leq C\epsilon^2 \|F^n\|_{L^2(\Omega)}.$$

We first consider the following lemma.

LEMMA 3.2. *Let Λ be the set of solutions $w \in V$ to*

$$(3.5) \quad b(w, v) = (\psi, v) \quad \text{for all } v \in V,$$

where $\|\psi\|_{L^2(\Omega)} = 1$. Then Λ is compact in V .

Proof. Since $w \in V$ is the solution to (3.5), it satisfies

$$\|w\|_V \leq C\|\psi\|_{L^2(\Omega)}.$$

Thus, the solution mapping $G : \psi \rightarrow G\psi = w$ is continuous from the dual space V' to $V \subset H^1(\Omega)$. Furthermore, the embedding, $I : L^2(\Omega) \subset V'$, is compact. This implies that $\Lambda \subset G \circ I(\{\psi \in L^2(\Omega) : \|\psi\|_{L^2(\Omega)} = 1\})$ is compact in V . \square

We now prove the theorem.

Proof. By viewing $u^n - u_h^n$ as a linear functional in $L^2(\Omega)$, we have

$$\|u^n - u_h^n\|_{L^2(\Omega)} = \sup_{\|\psi\|_{L^2(\Omega)}=1} (u^n - u_h^n, \psi).$$

Let $w \in V$ be the solution to

$$b(v, \eta) = (\psi, \eta) \quad \text{for all } \eta \in V.$$

Then

$$\|w\|_V \leq C\|\psi\|_{L^2(\Omega)}.$$

Thus, for $v_h \in V_h$, by the boundedness of the bilinear form $b(\cdot, \cdot)$ we have

$$\begin{aligned} |(u^n - u_h^n, \psi)| &= |b(u^n - u_h^n, w)| = |b(u^n - u_h^n, w - v_h)| \\ &\leq C\|u^n - u_h^n\|_V \|w - v_h\|_V. \end{aligned}$$

By the density property of V_h in V , we can choose v_h such that $\|w - v_h\|_V \leq \epsilon \|w\|_V$. We then obtain

$$|(u^n - u_h^n, \psi)| \leq C\epsilon \|u^n - u_h^n\|_V \|w\|_V \leq C\epsilon \|u^n - u_h^n\|_V \|\psi\|_{L^2(\Omega)}.$$

Thus,

$$\|u^n - u_h^n\|_{L^2(\Omega)} \leq C\epsilon \|u^n - u_h^n\|_V.$$

This proves the estimate (3.3).

Next, we set

$$\hat{F}^n = \frac{F^n}{\|F^n\|_{L^2(\Omega)}}, \quad \hat{u}^n = \frac{u^n}{\|F^n\|_{L^2(\Omega)}}, \quad \hat{u}_h^n = \frac{u_h^n}{\|F^n\|_{L^2(\Omega)}}.$$

Then, we have

$$\begin{aligned} b(\hat{u}^n, v) &= \hat{F}^n(v), \quad \forall v \in V, \\ b(\hat{u}_h^n, v_h) &= \hat{F}^n(v_h), \quad \forall v_h \in V. \end{aligned}$$

By Céa's Lemma,

$$\|\hat{u}^n - \hat{u}_h^n\|_V \leq C \inf_{v_h \in V_h} \|\hat{u}^n - v_h\|_V.$$

Since the set $\hat{\Lambda} = \{\hat{u}^n : b(\hat{u}^n, \phi) = \hat{F}^n(\phi), \|\hat{F}^n\|_{L^2(\Omega)} = 1\}$ is compact in V , we have, for $0 < h < h_0(\epsilon)$,

$$\inf_{v_h \in V_h} \|\hat{u}^n - v_h\|_V \leq \epsilon.$$

Thus,

$$\|\hat{u}^n - \hat{u}_h^n\|_V \leq C\epsilon,$$

which implies that

$$\|u^n - u_h^n\|_V \leq C\epsilon \|F^n\|_{L^2(\Omega)}.$$

This completes the proof. \square

4. Stability analysis. For stability analysis, we express the Newmark scheme in a three-step formulation. We start with

$$\begin{aligned} -\Delta u^{n+2} + \alpha^2 \varepsilon_r u^{n+2} &= \alpha^2 \varepsilon_r \tilde{u}^{n+2} \\ &= \alpha^2 \varepsilon_r \left[u^{n+1} + \Delta t \dot{u}^{n+1} + \frac{\Delta t^2}{2} (1 - 2\beta) \ddot{u}^{n+1} \right]. \end{aligned}$$

By using (2.1)-(2.5) recursively, we obtain

$$\begin{aligned} -\beta \Delta u^{n+2} - \left(\frac{1}{2} - 2\beta + \gamma\right) \Delta u^{n+1} - \left(\frac{1}{2} + \beta - \gamma\right) \Delta u^n \\ + \beta \alpha^2 \varepsilon_r (u^{n+2} - 2u^{n+1} + u^n) = 0. \end{aligned}$$

Adapting u_h^n for $u^n \in V_h$, we have the following variational form of the above equation

$$\begin{aligned} (4.1) \quad & \frac{1}{\Delta t^2} (\varepsilon_r (u_h^{n+2} - 2u_h^{n+1} + u_h^n), v_h) + a(\beta u_h^{n+2} + \left(\frac{1}{2} - 2\beta + \gamma\right) u_h^{n+1} \\ & \quad + \left(\frac{1}{2} + \beta - \gamma\right) u_h^n, v_h) \\ & = \beta G^{n+2}(v_h) + \left(\frac{1}{2} - 2\beta + \gamma\right) G^{n+1}(v_h) + \left(\frac{1}{2} + \beta - \gamma\right) G^n(v_h) \end{aligned}$$

$\forall v_h \in V_h$, where

$$a(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h - \int_{\Gamma} T(u_h) v_h,$$

and

$$G^n(v_h) = \int_{\Gamma} 2 \frac{\partial u^i}{\partial y} v_h.$$

Consider the eigenvalue problem

$$(4.2) \quad a(w_h, v_h) = \lambda_h(w_h, v_h), \quad \forall v_h \in V_h.$$

Since $a(u, v)$ is linear, symmetric, and bounded, (4.2) has positive eigenvalues and corresponding orthonormal eigenvectors:

$$0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \dots \leq \lambda_{h,M}, \\ w_{h,1}, w_{h,2}, \dots, w_{h,M},$$

where $\dim V_h = M$. Without confusion, we write $w_i = w_{h,i}$ and $\lambda_i = \lambda_{h,i}$. Substituting w_i for v_h in (4.1), noting that

$$a(u_h^n, w_i) = a(w_i, u_h^n) = \lambda_i(w_i, u_h^n) = \lambda_i(u_h^n, w_i),$$

yields

$$(4.3) \quad \begin{aligned} & \frac{1}{\Delta t^2} (\varepsilon_r (u^{n+2} - 2u^{n+1} + u_h^n), w_i) \\ & + \lambda_i (\beta u_h^{n+1} + (\frac{1}{2} - 2\beta + \gamma) u_h^{n+1} + (\frac{1}{2} + \beta - \gamma) u_h^n, w_i) \\ & = \beta G^{n+2}(w_i) + (\frac{1}{2} - 2\beta + \gamma) G^{n+1}(w_i) + (\frac{1}{2} + \beta - \gamma) G^n(w_i) \\ & \equiv \Psi^n, \end{aligned}$$

where Ψ^n is independent of u -terms.

For stability analysis, we need only consider the corresponding homogeneous equation, where $\Psi^n = 0$. It is also true that ε_r can be considered a constant and hence without loss of generality, let $\varepsilon_r = 1$. Indeed, for $1 < \varepsilon_r \in L^\infty(\Omega)$, we may consider the space $L^2(\Omega, \varepsilon_r)$ with the weighted inner product

$$(u, v)_{\varepsilon_r} = (\varepsilon_r u, v) = (u, \varepsilon_r v).$$

Then the corresponding eigenvalue problem

$$(4.4) \quad a(u_h, v_h) = \lambda_h(u_h, v_h)_{\varepsilon_r} \quad \forall v_h \in V_h,$$

has similar properties as that of (4.2), namely (4.4) has positive eigenvalues λ_i and corresponding orthonormal eigenvectors w_i , $i = 1, 2, \dots, M$, such that

$$(w_i, w_j)_{\varepsilon_r} = \delta_{ij}.$$

Hence we may consider (4.3) with $\varepsilon_r = 1$ and $\Psi^n = 0$. By substituting $u_h^n = \sum_{i=1}^M u_i^n w_{h,i}$ into (4.3), we obtain, for $i = 1, 2, \dots, M$,

$$(4.5) \quad \frac{1}{\Delta t^2} (u_i^{n+2} - 2u_i^{n+1} + u_i^n) + \lambda_i (\beta u_i^{n+2} + (\frac{1}{2} - 2\beta + \gamma) u_i^{n+1} + (\frac{1}{2} + \beta - \gamma) u_i^n) = 0,$$

that is,

$$\begin{aligned}
 u_i^{n+2} &= \frac{\frac{2}{\Delta t^2} - \lambda_i(\frac{1}{2} - 2\beta + \gamma)}{\frac{1}{\Delta t^2} + \lambda_i\beta} u_i^{n+1} - \frac{\frac{1}{\Delta t^2} + \lambda_i(\frac{1}{2} + \beta - \gamma)}{\frac{1}{\Delta t^2} + \lambda_i\beta} u_i^n \\
 &\equiv \eta u_i^{n+1} - \kappa u_i^n.
 \end{aligned}$$

Thus, (4.5) can be written in a matrix form as

$$\begin{pmatrix} u_i^{n+2} \\ u_i^{n+1} \end{pmatrix} = \begin{pmatrix} \eta & -\kappa \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_i^{n+1} \\ u_i^n \end{pmatrix} \equiv B \begin{pmatrix} u_i^{n+1} \\ u_i^n \end{pmatrix}.$$

By denoting $X_i^n = \begin{pmatrix} u_i^{n+1} \\ u_i^n \end{pmatrix}$, for $n = 1, 2, \dots, N$, $i = 1, 2, \dots, M$, we obtain the recursive relation

$$X_i^{n+1} = B(\lambda_i) X_i^n,$$

or equivalently,

$$(4.6) \quad X_i^{n+1} = B^n X_i^0.$$

For stability analysis we wish to establish conditions on β , γ , and Δt such that $|X_i^n| = (|u_i^{n+1}|^2 + |u_i^n|^2)^{1/2}$ for all i , and hence $|u^n| = (\sum_{i=1}^M |u_i^n|^2)^{1/2}$, is bounded independent of n .

We observe that, if B is diagonalizable with the spectral radius $\rho(B) \leq 1$, then

$$|X^n| = |B^n X^0| \leq \|G^{-1}\| \rho^n(B) \|G\| |X^0| \leq C,$$

for some matrix G . For simplicity, we seek conditions on β , γ , and Δt such that B has distinct eigenvalues (hence, diagonalizable) of lengths less than or equal to 1.

We shall assume that $\beta \geq 0$. We consider the characteristic equation of B

$$\det(\mu I - B) = \mu^2 - \mu\eta + \kappa = 0.$$

The solutions μ_1, μ_2 are

$$\mu_{1,2} = \frac{\eta \pm \sqrt{\eta^2 - 4\kappa}}{2}.$$

We consider the following two cases.

Case 1: Suppose $\Delta = \eta^2 - 4\kappa < 0$. Then $\mu_1 = \bar{\mu}_2$ and $|\mu_1| = |\mu_2| = \sqrt{\kappa}$. Thus, we require $\kappa \leq 1$ which implies that $\gamma \geq \frac{1}{2}$. We have

$$\Delta = [2 - \lambda_i \Delta t^2 (\frac{1}{2} - 2\beta + \gamma)]^2 - 4(1 + \lambda_i \Delta t^2 \beta) [1 + \lambda_i \Delta t^2 (\frac{1}{2} + \beta - \gamma)] < 0,$$

which is

$$-4\lambda_i \Delta t^2 + (\lambda_i \Delta t^2)^2 [(1 + \gamma)^2 - 4\beta] < 0,$$

or equivalently,

$$\frac{1}{4} (\frac{1}{2} + \gamma)^2 - \beta < \frac{1}{\lambda_i \Delta t^2}, \quad \forall i = 1, 2, \dots, M.$$

Case 2: Suppose $\Delta > 0$, that is,

$$(4.7) \quad \frac{1}{4}\left(\frac{1}{2} + \gamma\right)^2 - \beta > \frac{1}{\lambda_i \Delta t^2}, \quad \forall i = 1, 2, \dots, M.$$

Without loss of generality, $\mu_1 < \mu_2$. Let

$$-1 \leq \mu_1 = \frac{\eta}{2} - \frac{\sqrt{\Delta}}{2} < \frac{\eta}{2} + \frac{\sqrt{\Delta}}{2} = \mu_2 \leq 1.$$

The inequality $\mu_1 \geq -1$ implies $1 + \eta + \kappa \geq 0$, or

$$(4.8) \quad \frac{\gamma}{2} - \beta \leq \frac{1}{\lambda_i \Delta t^2}.$$

$\mu_2 \leq 1$ implies $1 - \eta + \kappa \geq 0$. So we require $\kappa \geq -1$, which implies

$$(4.9) \quad \frac{1}{2}\left(\gamma - \frac{1}{2} - 2\beta\right) \leq \frac{1}{\lambda_i \Delta t^2}.$$

By combining (4.8) and (4.9), we have

$$(4.10) \quad \frac{\gamma}{2} - \beta - \frac{1}{4} \leq \frac{1}{\lambda_i \Delta t^2}.$$

However, the inequalities (4.7) and (4.10) are inconsistent, so we ignore Case 2.

Thus, $X^n = B^n X^0$ is stable if

$$(4.11) \quad \gamma \geq \frac{1}{2} \quad \text{and} \quad \frac{1}{4}\left(\frac{1}{2} + \gamma\right)^2 - \beta < \frac{1}{\lambda_i \Delta t^2}, \quad i = 1, 2, \dots, M.$$

We summarize the above analysis in the following theorem.

THEOREM 4.1. *The Newmark scheme for the TM variational problem is stable if $\gamma \geq \frac{1}{2}$, $\beta \geq 0$, and*

$$(4.12) \quad \frac{1}{4}\left(\frac{1}{2} + \gamma\right)^2 - \beta < \frac{1}{\lambda_i \Delta t^2}, \quad i = 1, 2, \dots, M,$$

where $\lambda_{h,i}$ are the eigenvalues of $a(w, v_h) = \lambda_h(w, v_h)$, $\forall v_h \in V_h$.

REMARK 4.2. *In Theorem 4.1, the time-marching scheme is unconditionally stable if*

$$\gamma \geq \frac{1}{2}, \quad \text{and} \quad \beta \geq \frac{1}{4}\left(\frac{1}{2} + \gamma\right)^2,$$

in which case (4.12) holds for all $\Delta t > 0$.

5. Numerical results. For numerical experiments, let Ω be the rectangular cavity of dimension $1 m \times 0.25 m$ as in Figure 3. We consider two types of excitations: continuous wave and Gaussian pulse. The cavity is covered by a uniform mesh of triangles so that there are 20 nodes on the longer sides and 5 nodes on the shorter ones. We set

$$\Delta t = 1/20, \quad \gamma = .9, \quad \beta = .25(.5 + \gamma)^2.$$

We start the time-marching procedure at $t = 0$.

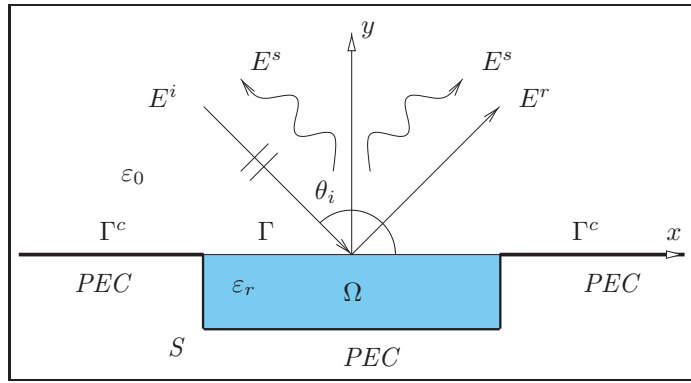


FIG. 2. Rectangular cavity Ω of dimension $1\text{ m} \times .25\text{ m}$

5.1. Incident continuous wave. In this example, the cavity is empty and the incident field is of the form

$$u_i = \text{Re} \left\{ e^{ik_0(x \cos \theta_i + y \sin \theta_i)} e^{ik_0 t} \right\},$$

where $k_0 = 2\pi/\lambda$ is the wave number and $\theta_i = \pi/2$. For a given λ , the steady-state solution follows a basic pattern of the time-harmonic excitation after a number of cycles. Figure 3 shows the RCS obtained by the time-domain finite element method (TDFEM) compared to that by the frequency-domain finite element method (FDFEM), [19], for $\lambda = 1$ meter (300 MHz), and $\lambda = 1.5625$ meter (468.75 MHz), respectively. The results agree well.

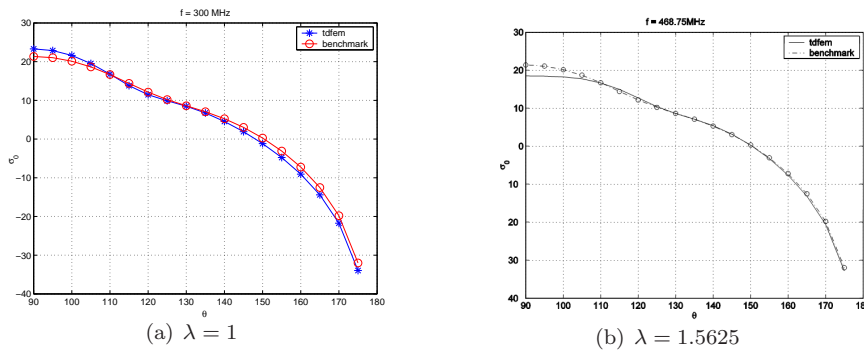


FIG. 3. RCS for $\epsilon_r = 1$.

5.2. Incident Gaussian pulse. We consider the Gaussian pulse represented by

$$(5.1) \quad u_i(x, y, t) = \frac{4}{T\sqrt{\pi}} e^{-\tau^2},$$

where

$$\tau = \frac{4(t - t_0 + x \cos \theta_i + y \sin \theta_i)}{T}, \quad \theta_i \in [\pi/2, \pi].$$

In what follows, we set $\theta_i = \pi/2$, $T = 2$, and $t_0 = 3$. This means that the Gaussian pulse will reach its maximum at the origin $(0, 0)$ at $t_0 = 3$. Figure 4 are contour plots of the TM and TE solutions for an empty cavity. In particular, Figure 4 (a) is a snap shot of the TM fields at $t = 4.025$ and Figure 4(b) is that of the TE fields at $t = 2.025$. No spurious modes are present. Both manifest the expected stability.

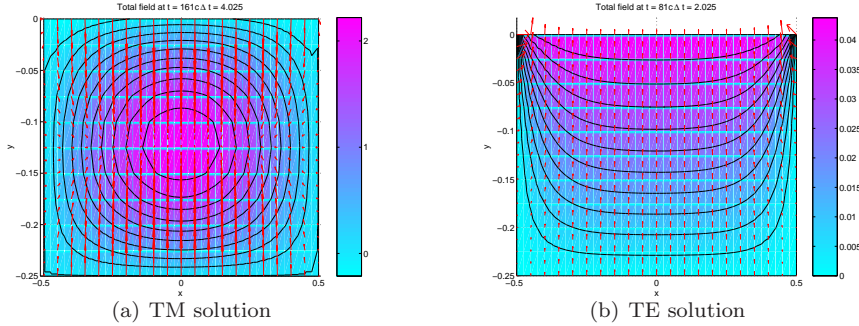


FIG. 4. Solutions on Ω for $\epsilon_r = 1$.

We next consider layered cavities. TM solutions are plotted in Figure 5 for a Left-Right filled cavity, and Figure 6 shows the TE solutions of a Top-Bottom filled cavity. In each case, we plot TM and TE solutions at the center of the cavity opening $(0, 0)$ and at an interior point. Solutions oscillate in the early time and then exponentially decay, clearly showing the expected stability. LM in the plots denotes light-meter, i.e., the amount of time for light to travel $1m$ in free space.

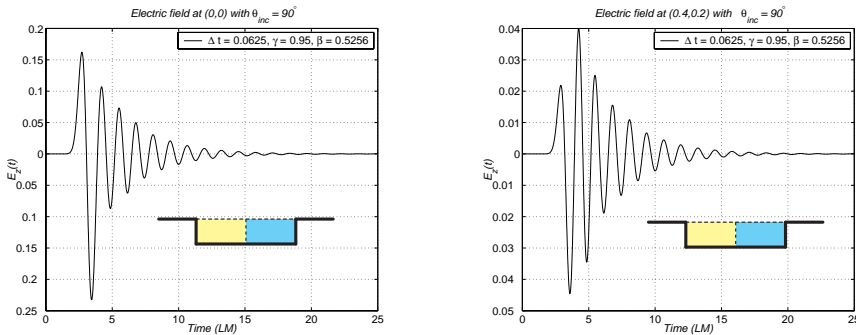


FIG. 5. TM Solution: Left-Right filled with $\epsilon_{lf} = 1$, $\epsilon_{rt} = 4$

6. Conclusion. We have presented a two-dimensional hybrid FETD/TDIE method for analyzing transient electromagnetic scattering from inhomogeneous cavities embedded in the infinite ground plane. The method is shown to lead to a well-posed discrete problem. Finite element error estimates in both the H^1 and L^2 norms are obtained. Stability criteria for the time-marching scheme are also established. The method is fully implemented and numerical results for both filled and unfilled cavities show the accuracy and stability of the scheme.

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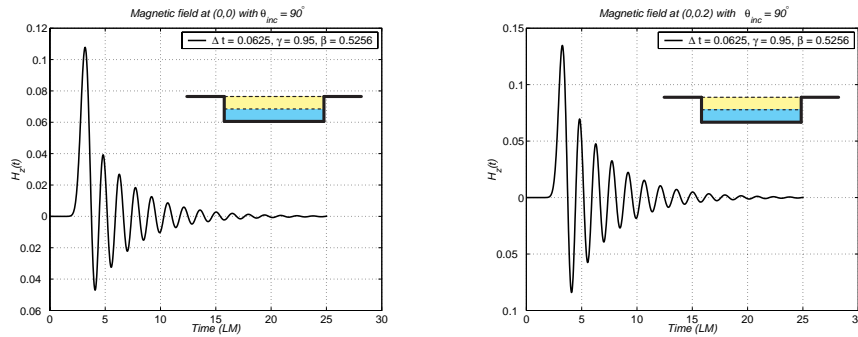


FIG. 6. *TE Solution: Top-Bottom filled with $\epsilon_{top} = 1$, $\epsilon_{bm} = 4$*

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