METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 11, No. 1, pp. 033–040, March 2004

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DISCRETE BOUNDARY VALUE PROBLEMS WITH INITIAL AND FINAL CONDITIONS *

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Abstract. In this research, we consider a nonhomogeneous linear difference equation with constant coefficients of order $k \geq 2$ subject to boundary conditions that split into two groups: initial and final conditions. We establish necessary and / or sufficient conditions on the distribution(s) of the zeros of the associated characteristic polynomial that ensure the existence of a unique solution.

1. Introduction. Consider the k^{th} order linear difference equation with constant coefficients:

$$\sum_{j=0}^{k} a_j \; y(t+j) = g(t), \; t = 0, 1, 2, ..., \qquad a_0 a_k \neq 0 \tag{1}$$

subject to boundary conditions (BC) of the form:

$$\text{Initial Condition(s): } y(i) = y_i, \; i = 0, ..., k_1 - 1 \tag{2}$$

$$\text{Final Condition(s): } y(i) = y_i, \; i = N, ..., N + k_2 - 1 \tag{3}$$

where $k, \; k_1, \; k_2$, and N are positive integers such that $k \geq 2$, $k_1, k_2 \geq 1$, $k_1 + k_2 = k$, $N > k_1$, $a_j \in \mathbb{C}$, and $g : \mathbb{N} \cup \{0\} \to \mathbb{C}$.

Our main objective in the present work is to characterize distributions of the characteristic roots, i.e., the zeros of the associated characteristic polynomial of Eq. (1):

$$p(\lambda) = \sum_{j=0}^{k} a_j \lambda^j \tag{4}$$

that ensure the existence of a unique solution of discrete boundary value problem (DBVP) (1)-(3).

Our interest in DBVP was initiated by an open problem in [6] due to Trigiante. In [1], the author et. al. investigated the case when all characteristic roots were assumed to be distinct. The results obtained therein were extended in [2] for more general boundary conditions, namely

$$y(n_i) = y_i, \; i = 1, ..., k, \; 0 = n_1 < n_2 < ... < n_k \; (n_k \geq k). \tag{5}$$

However, the assumption that "the characteristic roots being distinct" was kept. In this paper, we continue our investigation of DBVP (1)-(3) and remove the assumption that all characteristic roots are distinct. Thus, we are dealing with the most general case of repeated characteristic roots. We believe the results that will be established in this paper are extendable in a straight forward manner to boundary conditions of the form (5).

*Received November 28, 2003; accepted for publication April 2, 2004. This research is supported by a grant from the University of Sharjah.
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Observe that if the characteristic polynomial (4) has $r \geq 1$ distinct characteristic roots, then the general solution of Eq. (1) is given by

$$y(t) = y_p(t) + \sum_{j=1}^{r} q_j(t) z_j^t, \quad t \geq 0$$

where $y_p(t)$ is a particular solution of Eq. (1), $q_j(t) = \sum_{i=0}^{m_j-1} c_{ji} t^i$ is a polynomial in t of degree $m_j - 1$, and $m_j \geq 1$ is the multiplicity of characteristic root z_j (see [4, pages 63-78] for proofs and details) such that $\sum_{j=1}^{r} m_j = k$. Applying BC (2)-(3), we obtain:

$$\begin{array}{lcl}
\sum_{j=1}^{r} q_j(0) & = & y_0 - y_p(0) \\
\vdots & & \vdots \\
\sum_{j=1}^{r} q_j(k_1-1) z_j^{k_1-1} & = & y_{k_1-1} - y_p(k_1-1) \\
\sum_{j=1}^{r} q_j(N) z_j^{N} & = & y_N - y_p(N) \\
\vdots & & \vdots \\
\sum_{j=1}^{r} q_j(N+k_2-1) z_j^{N+k_2-1} & = & y_{N+k_2-1} - y_p(N+k_2-1)
\end{array}$$

which is a system of linear equations in which the coefficient matrix M is given by

$$M = \begin{cases} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ z_1^{k_1-1} & \cdots & (k_1-1)^{k-1} z_1^{k_1-1} \\ z_1^{N} & \cdots & N^{k-1} z_1^{N} \\ \vdots & & \vdots \\ z_1^{N+k_2-1} & \cdots & (N+k_2-1)^{k-1} z_1^{N+k_2-1} \end{pmatrix} & \text{if} \quad r = 1 \\ \begin{pmatrix} M_1 & \cdots & M_r \end{pmatrix} & \text{if} \quad r > 1 \end{cases} \tag{6}$$

where M_j is the $k \times m_j$ matrix given by

$$M_j = \begin{cases} \begin{pmatrix} 1 \\ \vdots \\ z_j^{k_1-1} \\ z_j^{N} \\ \vdots \\ z_j^{N+k_2-1} \end{pmatrix} & \text{if} \quad m_j = 1 \\ \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ z_j^{k_1-1} & \cdots & (k_1-1)^{m_j-1} z_j^{k_1-1} \\ z_j^{N} & \cdots & N^{m_j-1} z_j^{N} \\ \vdots & & \vdots \\ z_j^{N+k_2-1} & \cdots & (N+k_2-1)^{m_j-1} z_j^{N+k_2-1} \end{pmatrix} & \text{if} \quad m_j > 1 \end{cases} \tag{7}$$

for $j = 1, ..., r$. Therefore, the existence of a unique solution of DBVP (1)-(3) is equivalent to the non-singularity of the block matrix M which we call, following Trigiante in [6], *Generalized Mosaic Vandermonde Matrix*.

REMARK 1.1. It is worth mentioning that the forcing term $g(t)$ appears in the solution through $y_p(t)$. If one is interested in the existence of solutions only, the presence of $g(t)$ will play a crucial role in the desired condition(s). However, the non-singularity of the matrix M restricts the effect of $g(t)$ to the solution formula.

The present paper is organized as follows. In Section 2, we recall and establish all needed results for the proof of our main result, Theorem 3.1 which will be established in Section 3. In Section 4, we relate the developed conditions to the coefficients of the given difference equation. This connection raises some related questions that are important in their own right.

2. Preliminary Results. To establish our main result in this paper, we need to recall the following two results:

THEOREM 2.1 (Flowe and Harris [5]). *Let $m_1, ..., m_r$ be positive integers, $\vec{v}(z) = (1, z, ..., z^m)^t$, $m = m_1 + ... + m_r - 1$, and $A(z_1, ..., z_r; m_1, ..., m_r) = (A_1 \ \cdots \ A_r)$ where*

$$A_j = \left(\vec{v}(z_j), \cdots, \vec{v}^{(m_j-1)}(z_j)\right).$$

If $D(z_1, ..., z_r; m_1, ..., m_r) = det(A(z_1, ..., z_r; m_1, ..., m_r))$, then

$$D(z_1, ..., z_r; m_1, ..., m_r) = \left(\prod_{i=1}^{r} \prod_{j_i=0}^{m_i-1} j_i!\right) \prod_{j=2}^{r} \prod_{i=1}^{j-1} (z_j - z_i)^{m_i m_j}.$$

REMARK 2.1. In Theorem 2.1, the convention that *empty product, i.e., the upper limit is less than the lower one, is equal 1* is adopted. Thus, if $r = 1$ and so $m_1 = k$, then $D(z_1; m_1) = \prod_{j_1=0}^{m_1-1} j_1!$.

THEOREM 2.2 (Abu-Saris and Al-Dosary [2]). *If*

$$D_k(z_1, ..., z_k; n_2, ..., n_k) = det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1^{n_2} & z_2^{n_2} & \cdots & z_k^{n_2} \\ \vdots & \vdots & & \vdots \\ z_1^{n_k} & z_2^{n_k} & \cdots & z_k^{n_k} \end{pmatrix},$$

then

$$D_k(z_1, ..., z_k; n_2, ..., n_k) = \Phi(z_1, ..., z_k) V_k(z_1, ..., z_k)$$

where

$$\Phi(z_1, ..., z_k) = \sum_{\|\alpha\| = n - k(k-1)/2} a_\alpha \ z^\alpha, \qquad n = n_2 + ... + n_k$$

such that

$$\alpha=(\alpha_1,...,\alpha_k),\ \ \alpha_i\in\{0,...,n_k-k+1\},\ \ i=1,...,k,\ \ \|\alpha\|=\alpha_1+\cdots+\alpha_k,$$

$$z^\alpha=z_1^{\alpha_1}\cdots z_k^{\alpha_k},\ \ \textit{and}\ a_\alpha=a_{\alpha_1\cdots\alpha_k}>0.$$

We, also, need to establish the following two lemmas.

LEMMA 2.1. *If* $\vec{u}(z)=(z^{n_1},z^{n_2},...,z^{n_k})^t$ *and* $\vec{v}_j(z)=\left(n_1^jz^{n_1},n_2^jz^{n_2},...,n_k^jz^{n_k}\right)^t$, *then*

$$\vec{v}_j(z)=\sum_{\ell=0}^{j}\frac{1}{\ell!}\ (\Delta^\ell n^j)|_{n=0}\ z^\ell\ \vec{u}^{(\ell)}(z)$$

Proof. First, by Newton's forward interpolating polynomial [3, page 128]

$$n^j=\sum_{\ell=0}^{j}\begin{pmatrix} n\\ \ell\end{pmatrix}\ (\Delta^\ell n^j)|_{n=0}=\sum_{\ell=0}^{j}\frac{(\Delta^\ell n^j)|_{n=0}}{\ell!}\ \prod_{i=0}^{\ell-1}(n-i).$$

Therefore,

$$n^jz^n=\sum_{\ell=0}^{j}\frac{(\Delta^\ell n^j)|_{n=0}}{\ell!}\ z^\ell\ \frac{d^\ell z^n}{dz^\ell}.$$

Since $(\Delta n_i^j)|_{n_i=0}$ is independent of i, the result follows. □

By Lemma 2.1 and the well-known properties of determinants, we have the following result:

LEMMA 2.2. *Let* $k,\ m_1,\ ...,\ m_r$ *be positive integers such that* $m_1+m_2+...+m_r=k$. *Let* $\vec{u}(z)=(z^{n_1},z^{n_2},...,z^{n_k})^t$ *and* $\vec{v}_i(z)=\left(n_1^{i-1}z^{n_1},n_2^{i-1}z^{n_2},...,n_k^{i-1}z^{n_k}\right)^t$.

If $B=(B_1\ \cdots\ B_r)$ *such that* $B_j=(\vec{v}_1(z_j)\ \cdots\ \vec{v}_{m_j}(z_j))$, *then*

$$det(B)=\left(\prod_{j=1}^{r}z_j^{m_j(m_j-1)/2}\right)\ det(A)$$

where $A=(A_1\ \cdots\ A_r)$ *and* $A_j=\left(\vec{u}(z_j),\cdots,\vec{u}^{(m_j-1)}(z_j)\right)$.

3. An Existence and Uniqueness Theorem. Our main result on the existence of a unique solution of DBVP (1)-(3) is stated in the next theorem.

THEOREM 3.1. *If* $z_j=|z_j|e^{i\theta},\ j=1,...,r$ *where* $|z_i|\neq|z_j|$ *whenever* $i\neq j$, *i.e., the characteristic roots are on the same ray with different moduli, then DBVP (1)-(3) has a unique solution. In particular, if* $r=1$, *i.e., there is one characteristic root only, then there will always be a unique solution.*

Proof. First, by Lemma 2.2, we have

$$\det(M) = \left(\prod_{j=1}^{r} z_j^{m_j(m_j-1)/2}\right) \; E(z_1, ..., z_r; m_1, ..., m_r),$$

where

$$E(z_1, ..., z_r; m_1, ..., m_r) = \det\left(A_1 \;\cdots\; A_r\right),$$

$$A_j = \left(\vec{u}(z_j), \cdots, \vec{u}^{(m_j-1)}(z_j)\right),$$

and

$$\vec{u}(z) = (1, ..., z^{k_1-1}, z^N, ..., z^{N+k_2-1}).$$

Next we introduce the variables $t_1, ..., t_k$ such that $t_1 = z_1$, $t_{m_1+1} = z_2$, $t_{m_1+m_2+1} = z_3, ...,$ $t_{m_1+...+m_{r-1}+1} = z_r$, and define the determinant:

$$\tilde{E}(t_1, ..., t_k) = \det\left(\vec{u}(t_1) \cdots \vec{u}(t_k)\right),$$

which, by Theorem 2.2,

$$\tilde{E}(t_1, ..., t_k) = \Phi(t_1, ..., t_k) \; V_k(t_1, ..., t_k).$$

But (see [7, pages 97-99] for the derivative of determinants and related interesting results),

$$E = \left(\frac{\partial^{m_r-1}}{\partial^{m_r-1} t_k} \cdots \frac{\partial}{\partial t_{m_1+...+m_{r-1}+2}}\right) \cdots\cdots \left(\frac{\partial^{m_1-1}}{\partial^{m_1-1} t_{m_1}} \cdots \frac{\partial}{\partial t_2}\right) \tilde{E}$$

when $t_1 = \cdots = t_{m_1} = z_1, ..., t_{m_1+...+m_{r-1}+1} = \cdots = t_k = z_r$, and by the well-known properties of determinants

$$\begin{aligned}
V_k\big|_{t_1=\cdots=t_{m_1}=z_1} &= 0 \\
\frac{\partial^{j_3}}{\partial t_3^{j_3}} \frac{\partial^{j_2}}{\partial t_2^{j_2}} V_k\Big|_{t_1=\cdots=t_{m_1}=z_1} &= 0 \text{ if } j_3 < 2, \text{ and for all } j_2 \le 1 \\
\frac{\partial^{j_4}}{\partial t_4^{j_4}} \frac{\partial^{j_3}}{\partial t_3^{j_3}} \frac{\partial^{j_2}}{\partial t_2^{j_2}} V_k\Big|_{t_1=\cdots=t_{m_1}=z_1} &= 0 \text{ if } j_4 < 3, \text{ and for all } j_3 \le 2, \; j_2 \le 1 \\
\vdots \quad & \vdots
\end{aligned}$$

Therefore, differentiating $\tilde{E}$ appropriately, and substituting $t_1 = \cdots = t_{m_1} = z_1, ...,$ $t_{m_1+...+m_{r-1}+1} = \cdots = t_k = z_r$, one can see that all terms obtained from differentiation vanish, except

$$\Phi \; \underbrace{\left(\frac{\partial^{m_r-1}}{\partial^{m_r-1} t_k} \cdots \frac{\partial}{\partial t_{m_1+...+m_{r-1}+2}}\right) \cdots\cdots \left(\frac{\partial^{m_1-1}}{\partial^{m_1-1} t_{m_1}} \cdots \frac{\partial}{\partial t_2}\right) V_k}_{D(z_1,...,z_r;m_1,...,m_r)} \; .$$

Now, suppose that $z_j = |z_j|\ e^{i\theta},\ j = 1, ..., r$ where $|z_i| \neq |z_j|$ whenever $i \neq j$. Then, by Theorem 2.1,

$$E_k(z_1, ..., z_r) = \underbrace{e^{i[n+\|\alpha\|]\theta}}_{\neq 0}\ \underbrace{\Phi(|z_1|, ..., |z_r|)}_{>0}\ \underbrace{D(|z_1|, ..., |z_r|)}_{\neq 0} \neq 0.$$

where $n = \sum_{j=2}^{r}\sum_{i=1}^{j-1} m_i m_j$. This completes the proof. □

We illustrate the applicability and limitation of Theorem 3.1 by the following examples. In the first two examples the conditions of Theorem 3.1 are applicable whereas in the last one they are not.

Example 3.1. The difference equation

$$y(t+4) - 5y(t+3) + 9y(t+2) - 7y(t+1) + 2y(t) = 0$$

has the characteristic polynomial

$$p(\lambda) = (\lambda - 1)^3(\lambda - 2).$$

Since $z_1 = 1$ and $z_2 = 2$ are both positive real numbers, a unique solution can always be constructed given any set of boundary conditions as described in (2)-(3).

Example 3.2. The difference equation

$$y(t+5) - 7i\ y(t+4) - 19\ y(t+3) + 25i\ y(t+2) + 16\ y(t+1) - 4i\ y(t) = 0,$$

where $i = \sqrt{-1}$ has the characteristic polynomial

$$p(\lambda) = (\lambda - i)^3(\lambda - 2i)^2.$$

Once more, since $z_1 = i$ and $z_2 = 2i$ are both on the positive imaginary axis, a unique solution can always be constructed given any set of boundary conditions as described in (2)-(3).

Example 3.3. Consider the DBVP

$$y(t+2) - 4y(t) = 0, \quad y(0) = y_0, \quad y(N) = y_N.$$

Since the characteristic polynomial is

$$p(\lambda) = \lambda^2 - 4,$$

the characteristic roots are $z_1 = 2$ and $z_2 = -2$. Thus Theorem 3.1 is not applicable. However,

$$y(t) = c_1 2^t + c_2(-2)^t,$$

and so a solution exists if and only if the system

$$\begin{pmatrix} 1 & 1 \\ 2^N & (-2)^N \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_N \end{pmatrix}$$

is consistent. This implies that, if N is odd, then a unique solution is guaranteed to exist. On the other hand, if N is even, then a solution exists if and only if $y_N = 2^N y_0$, in which case the solution is not unique.

4. Conditions on the Coefficients. In applying Theorem 3.1, one may face a difficulty in determining the characteristic roots when the degree of the characteristic polynomial is higher than or equal to 5. Therefore, it is worth formulating the existence and uniqueness conditions in terms of the coefficients $a_j,\ j = 0, ..., k-1$.

The following two lemmas furnish a prelude for one possible approach in this direction. The first lemma asserts that it is enough to investigate polynomials with positive roots. The second one establishes necessary and / or sufficient conditions for a polynomial to have positive roots.

Lemma 4.1. Suppose that $b_j = a_j\ e^{i(k-j)\theta_0}$. *Then All zeros of the polynomial* $q(\lambda) = \lambda^k + \sum_{j=0}^{k-1} b_j \lambda^j$ *lie on the ray* $Arg(z) = \theta_0,\ |z| > 0$ *if and only if the zeros of* $p(\lambda) = \lambda^k + \sum_{j=0}^{k-1} a_j \lambda^j$ *are positive.*

Proof. Suppose that the zeros of the polynomial q are given by $z_\ell = |z_\ell|\ e^{i\theta_0},\ \ell = 1, ..., k$. Then

$$0 = q(|z_\ell|\ e^{i\theta_0}) = |z_\ell|^k\ e^{ik\theta_0} + \sum_{j=0}^{k-1} b_j\ |z_\ell|^j\ e^{i(j\theta_0)} = e^{i\ k\theta_0}\left(|z_\ell|^k + \sum_{j=0}^{k-1} b_j\ |z_\ell|^j e^{-i(k-j)\theta_0}\right)$$

$$= e^{i\ k\theta_0}\left(|z_\ell|^k + \sum_{j=0}^{k-1} a_j |z_\ell|^j\right) = p(|z_\ell|).$$

Hence, the result follows. ◻

Lemma 4.2.

(a) *Suppose that all zeros of the polynomial* $p(\lambda) = \lambda^k + \sum_{j=0}^{k-1} a_j \lambda^j$ *are positive. Then* $(-1)^j a_{k-j} > 0$ *for* $j = 0, ..., k-1$, *i.e., the coefficients have to alternate in sign.*

(b) *Suppose that all the zeros of the polynomial* $p(\lambda) = \lambda^k + \sum_{j=0}^{k-1} a_j \lambda^j$ *are nonzero and real. Then all of them are positive if and only if the coefficients alternate in sign.*

Proof. Part (a) follows immediately from the fact that

$$a_j = (-1)^{k-j} \sum_{\|\alpha\| = k-j} z^\alpha$$

where $z_1, ..., z_k$ are the zeros of $p(\lambda)$, and Part (b) follows from the fact that polynomials with alternating coefficients can't have negative real roots. ◻

Using Theorem 3.1, Lemma 4.1, and the discriminant formulas for quadratic and cubic polynomials [8, pp. 72-82], we have the following corollary.

Corollary 4.1.

(i) *If* $a < 0,\ b > 0$ *and* $a^2 - 4b \geq 0$, *then the DBVP*

$$y(t+2) + a\ y(t+1) + b\ y(t) = g(t), \quad y(0) = y_0, \quad y(N) = y_N$$

has a unique solution.

(ii) *If* $a < 0,\ b > 0,\ c < 0$ *and* $a^2\ b^2 - 4\ b^3 - 4\ a^3\ c - 27\ c^2 + 18\ a\ b\ c \geq 0$*, then the DBVPs*

$$y(t+3) + a\ y(t+2) + b\ y(t+1) + c\ y(t) = g(t),$$
$$y(0) = y_0, \quad y(1) = y_1, \quad y(N) = y_N \qquad (N > 2)$$

and

$$y(t+3) + a\ y(t+2) + b\ y(t+1) + c\ y(t) = g(t),$$
$$y(0) = y_0, \quad y(N) = y_N, \quad y(N+1) = y_{N+1} \qquad (N > 1)$$

have unique solutions.

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