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BOUNDEDNESS FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS ON TRIEBEL-LIZORKIN SPACES *

LIU LANZHE†

Abstract. In this paper, we prove the boundedness in the context of Triebel-Lizorkin spaces for multilinear Littlewood-Paley operators.

1. Introduction and results. Let $\delta > 0$ and ψ be a function on R^n which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y) - \psi(x)| \leq C|y|(1+|x|)^{-(n+2-\delta)}$ when $2|y| < |x|$.

Fixed $\mu > 1$. Let m be a positive integer and A be a function on R^n. The multilinear Littlewood-Paley operator is defined by

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{R^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leq m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Let $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [18]).

Let H be the Hilbert space $H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty\right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$g_\mu^A(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x,y)\right|\right|,$$

$$g_\mu(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right|\right|.$$

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†College of Mathematics and Computer, Changsha University of Science and Technology, Changsha 410077, P.R. of China (lanzheliu@263.net).

Note that when $m = 0$, g_μ^A is just the commutator of Littlewood-Paley operator (see [1][13-16]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3-6][9][10]). In [11][17], Janson and Paluszynski obtain the boundedness of commutators generated by the Calderón-Zygmund operator or fractional integral operator with Lipschitz functions on Triebel-Lizorkin spaces. The main purpose of this paper is to discuss the boundedness of the multilinear Littlewood-Paley operators in the context of Triebel-Lizorkin spaces. First, let us introduce some notation. We will work on R^n, $n > 1$. Throughout this paper, $M(f)$ will denote the Hardy-littlewood maximal function of f, Q will denote a cube of R^n with side parallel to the axes, and for a cube Q, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space. The Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x,\,h\,\in\,R^n\\ h\neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see[17]).

We shall prove the following theorems in Section 3.

Theorem 1. Let $0 \le \delta < n$, $0 < \beta < 1/2$, $1 < p < n/\delta$, $1/p - 1/q = \delta/n$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Then g_μ^A is bounded from $L^p(R^n)$ to $\dot{F}_q^{\beta,\infty}(R^n)$.

Theorem 2. Let $0 \le \delta < n$, $0 < \beta < 1/2$, $1 < p < n/(\delta+\beta)$, $1/p - 1/q = (\delta+\beta)/n$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Then g_μ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Theorem 3. Let $0 \le \delta < n$, $0 < \beta < 1/2$, $\delta + \beta < n$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Then for any $\lambda > 0$,

$$\left|\{x \in R^n : g_\mu^A(f)(x) > \lambda\}\right| \le C\left(\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} ||f||_{L^1}/\lambda\right)^{n/(n-\delta-\beta)}.$$

2. Some lemmas. We begin with some preliminary lemmas.

Lemma 1 (see [17]). For $0 < \beta < 1$, $1 < p < \infty$, we have

$$||f||_{\dot{F}_p^{\beta,\infty}} \approx \left|\left|\sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x) - f_Q|dx\right|\right|_{L^p}$$
$$\approx \left|\left|\sup_{\cdot\in Q}\inf_c \frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x) - c|dx\right|\right|_{L^p}.$$

Lemma 2 (see [17]). For $0 < \beta < 1, 1 \le p \le \infty$, we have

$$||f||_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x) - f_Q|dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{1/p}.$$

Lemma 3 (see [2]). For $1 \le r < \infty$ and $0 < \delta < n$, let

$$M_{\delta,r}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\delta r/n}}\int_Q |f(y)|^r dy\right)^{1/r}.$$

Suppose that $r < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then $||M_{\delta,r}(f)||_{L^q} \le C||f||_{L^p}$.

LEMMA 4 (see [8]). Let $Q_1 \subset Q_2$. Then

$$|f_{Q_1} - f_{Q_2}| \le C||f||_{\dot{\Lambda}_\beta}|Q_2|^{\beta/n}.$$

LEMMA 5 (see [6]). Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

LEMMA 6. Let $0 \le \delta < n$, $1 < p < \infty$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$, $1 < r \le \infty$, $1/q = 1/p + 1/r - \delta/n$. Then g_μ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is

$$||g_\mu^A(f)||_{L^q} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} ||f||_{L^p}.$$

Proof. By the Minkowski' inequality and the condition of ψ, we have

$$\begin{aligned}
&g_\mu^A(f)(x)\\
&\le \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}}\right)^{1/2} dz\\
&\le C\int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}\\
&\quad \times \left(\int_0^\infty \int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}} \frac{dydt}{t^{1+n}}\right)^{1/2} dz\\
&\le C\int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}\\
&\quad \times \left[\int_0^\infty \left(t^{-n}\int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{dy}{(t+|y-z|)^{2n+2-2\delta}}\right) tdt\right]^{1/2} dz.
\end{aligned}$$

Noting that

$$\begin{aligned}
t^{-n}\int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{dy}{(t+|y-z|)^{2n+2-2\delta}} &\le CM\left(\frac{1}{(t+|x-z|)^{2n+2-2\delta}}\right)\\
&\le \frac{C}{(t+|x-z|)^{2n+2-2\delta}}
\end{aligned}$$

and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$\begin{aligned}
g_\mu^A(f)(x) &\le C\int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2} dz\\
&= C\int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} dz,
\end{aligned}$$

thus, the lemma follows from [3].

3. Proof of theorems.

Proof of Theorem 1. Fix a cube $Q = Q(x_0, l)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$
\begin{aligned}
F_t^A(f)(x,y) &= \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) f(z) dz \\
&= \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) f_2(z) dz + \int_{R^n} \frac{R_m(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) f_1(z) dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-z)^\alpha \psi_t(y-z)}{|x-z|^m} D^\alpha \tilde{A}(z) f_1(z) dz,
\end{aligned}
$$

then

$$
\begin{aligned}
&\left| g_\mu^A(f)(x) - g_\mu^{\tilde{A}}(f_2)(x_0) \right| \\
&= \left| \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^A(f)(x,y) \right\| - \left\| \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x_0,y) \right\| \right| \\
&\le \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t \left(\frac{R_m(\tilde{A}; x, \cdot)}{|x-\cdot|^m} f_1 \right)(y) \right\| \\
&\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t \left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1 \right)(y) \right\| \\
&\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x,y) - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x_0,y) \right\| \\
&:= A(x) + B(x) + C(x),
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q \left| g_\mu^A(f)(x) - g_\mu^{\tilde{A}}(f_2)(x_0) \right| dx \\
&\le \frac{1}{|Q|^{1+\beta/n}} \int_Q A(x) dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q B(x) dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q C(x) dx \\
&:= I + II + III.
\end{aligned}
$$

Now, let us estimate I, II and III, respectively. First, for $x \in Q$ and $z \in \tilde{Q}$, using Lemma 2 and Lemma 5, we get

$$
\begin{aligned}
|R_m(\tilde{A}; x, z)| &\le C|x-z|^m \sum_{|\alpha|=m} \sup_{x \in \tilde{Q}} |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}| \\
&\le C|x-z|^m |Q|^{\beta/n} \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta},
\end{aligned}
$$

thus, taking r, s such that $1 < r < p$ and $1/s = 1/r - \delta/n$, by the (L^r, L^s)-boundedness of g_μ(see Lemma 6) and the Hölder'inequality, we obtain

$$\begin{aligned} I &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \frac{1}{|Q|} \int_Q |g_\mu(f_1)(x)| dx \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} ||g_\mu(f_1)||_{L^s} |Q|^{-1/s} \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} ||f_1||_{L^r} |Q|^{-1/s} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \left(\frac{1}{|\tilde{Q}|^{1-r\delta/n}} \int_{\tilde{Q}} |f(y)|^r dy \right)^{1/r} \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

Secondly, using the following inequality(see [17]):

$$||(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f \chi_{\tilde{Q}}||_{L^r} \le C|Q|^{1/s+\beta/n} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,r}(f),$$

and similar to the proof of I, we gain

$$\begin{aligned} II &\le \frac{C}{|Q|^{1+\beta/n}} \sum_{|\alpha|=m} ||g_\mu((D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1)||_{L^s} |Q|^{1-1/s} \\ &\le C|Q|^{-\beta/n-1/r} \sum_{|\alpha|=m} ||(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1||_{L^r} \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

To estimate III, we write

$$\begin{aligned} &\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x,y) - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x_0,y) \\ &= \int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \left[\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m}\right] R_m(\tilde{A}; x, z) \psi_t(y-z) f_2(z) dz \\ &\quad + \int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \frac{\psi_t(y-z) f_2(z)}{|x_0-z|^m} [R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)] dz \\ &\quad + \int_{R^n} \left[\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\right] \frac{R_m(\tilde{A}; x_0, z) \psi_t(y-z) f_2(z)}{|x_0-z|^m} dz \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \frac{(x-z)^\alpha}{|x-z|^m} - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} \frac{(x_0-z)^\alpha}{|x_0-z|^m}\right] \\ &\quad \times D^\alpha \tilde{A}(z) \psi_t(y-z) f_2(z) dz \\ &:= III_1 + III_2 + III_3 + III_4. \end{aligned}$$

Note that $|x-z| \sim |x_0-z|$ for $x \in Q$ and $z \in R^n \setminus \tilde{Q}$. By the condition of ψ and similar to the proof of Lemma 6, we obtain

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q ||III_1||dx \\
&\le \frac{C}{|Q|^{1+\beta/n}}\int_Q\left(\int_{R^n\setminus\tilde{Q}}\frac{|x-x_0|}{|x_0-z|^{m+n+1-\delta}}|R_m(\tilde{A};x,z)||f(z)|dz\right)dx \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^{k+1}\tilde{Q}}\frac{|x-x_0|}{|x_0-z|^{n+1-\delta}}|f(z)|dz \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty 2^{-k}\frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f(z)|dz \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty 2^{-k}M_{\delta,1}(f)(\tilde{x}) \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,1}(f)(\tilde{x}).
\end{aligned}$$

For III_2, by the formula (see [6]):

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)=\sum_{|\eta|<m}\frac{1}{\eta!}R_{m-|\eta|}(D^\eta\tilde{A};x,x_0)(x-z)^\eta$$

and Lemma 5, we get

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}|x-x_0||x_0-z|^{m-1},$$

thus

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q ||III_2||dx \\
&\le C\frac{1}{|Q|^{1+\beta/n}}\int_Q\int_{R^n\setminus\tilde{Q}}\frac{|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n-\delta}}|f(z)|dzdx \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(z)|dz \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(\tilde{x}).
\end{aligned}$$

For III_3, by the inequality: $a^{1/2}-b^{1/2}\le(a-b)^{1/2}$ for $a\ge b>0$, we gain, similar to the proof of Lemma 6,

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q||III_3||dx\\
&\le\frac{C}{|Q|^{1+\beta/n}}\int_Q\int_{R^n}\\
&\left(\int_{R_+^{n+1}}\left[\frac{t^{n\mu/2}|x-x_0|^{1/2}|\psi_t(y-z)||R_m(\tilde{A};x_0,z)||f_2(z)|}{|x-z|^m(t+|x-y|)^{(n\mu+1)/2}}\right]^2\frac{dydt}{t^{n+1}}\right)^{1/2}dzdx\\
&\le\frac{C}{|Q|^{1+\beta/n}}\int_Q\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)||x-x_0|^{1/2}}{|x-z|^m}\\
&\times\left(\int_0^\infty\frac{dt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2}dzdx\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\int_{R^n}\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}}|f_2(z)|dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty 2^{-k/2}M_{\delta,1}(f)(\tilde{x})\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,1}(f)(\tilde{x}).
\end{aligned}
$$

For III_4, by Lemma 4, we get

$$|D^\alpha A(z)-(D^\alpha A)_{\tilde{Q}}|\le||D^\alpha A||_{\dot{\Lambda}_\beta}|x_0-z|^\beta,$$

thus, similar to the proof of III_1 and III_3, we obtain

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q||III_4||dx\\
&\le\frac{C}{|Q|^{1+\beta/n}}\int_Q\sum_{|\alpha|=m}\int_{R^n}\left(\frac{|x-x_0|}{|x_0-z|^{n+1-\delta}}+\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}}\right)|f_2(z)||D^\alpha\tilde{A}(z)|dzdx\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty(2^{k(\beta-1)}+2^{k(\beta-1/2)})M_{\delta,1}(f)(\tilde{x})\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,1}(f)(\tilde{x}).
\end{aligned}
$$

Thus,

$$III\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,1}(f)(\tilde{x}).$$

We now put these estimates together, and taking the supremum over all Q such that $\tilde{x}\in Q$, and using Lemma 1 with Lemma 3, we obtain

$$||g_\mu^A(f)||_{\dot{F}_q^{\beta,\infty}}\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By the same argument as in the proof of Theorem 1, we have, for $1 \le s < p$ and $1/r = 1/s - \delta/n$,

$$\frac{1}{|Q|}\int_Q |g_\mu^A(f)(x) - g_\mu^{\tilde{A}}(f_2)(x_0)|dx \\ \le C\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta}(M_{\beta+\delta,r}(f) + M_{\beta+\delta,1}(f)),$$

thus, the sharp estimate of g_μ^A is obtained as following

$$(g_\mu^A(f))^\# \le C\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta}(M_{\beta+\delta,r}(f) + M_{\beta+\delta,1}(f)).$$

Now, using Lemma 3, we gain

$$||g_\mu^A(f)||_{L^q} \le C||(g_\mu^A(f))^\#||_{L^q} \\ \le C\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta}(||M_{\beta+\delta,r}(f)||_{L^q} + ||M_{\beta+\delta,1}(f)||_{L^q}) \le C||f||_p.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. We first prove the following estimate:

$$|g_\mu^A(f)(x)| \le C\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta}\left(\lambda_1^{\delta+\beta}Mf(x) + \lambda_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r'}(Mf(x))^{1/r}\right)$$

for any $\lambda_1 > 0$ and $n/(n-\delta-\beta) < r$. In fact, fix the cube $Q = Q(x,\lambda_1)$, similar to the proof of Lemma 6, we have

$$|g_\mu^A(f)(x)| \le C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}}dz \\ = C\left(\int_Q + \int_{Q^c}\right)\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}}dz = I_1 + I_2.$$

For I_1, we let that, for $k > 0$,

$$\tilde{A}_k(y) = A(y) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{2^{-k}Q}y^\alpha,$$

then , by Lemma 5, for $z \in 2^{-k}Q$,

$$|R_{m+1}(\tilde{A}_k;x,z)| \le C\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta}(2^{-k}\lambda_1)^\beta|x-z|^m,$$

thus, by Lemma 4 and Lemma 5,

$$\begin{aligned}
I_1 &\le C\sum_{k=0}^{\infty}\int_{2^{-k}Q\setminus 2^{-k-1}Q}\frac{|f(z)||R_{m+1}(\tilde{A}_k;x,z)|}{|x-z|^{m+n-\delta}}dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\sum_{k=0}^{\infty}(2^{-k}\lambda_1)^\beta\int_{2^{-k}Q\setminus 2^{-k-1}Q}\frac{|f(z)|}{|x-z|^{n-\delta}}dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\sum_{k=0}^{\infty}(2^{-k}\lambda_1)^{\beta+\delta-n}\int_{2^{-k}Q\setminus 2^{-k-1}Q}|f(z)|dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\lambda_1^{\beta+\delta}Mf(x);
\end{aligned}$$

For I_2, taking $\varepsilon > 0$ such that $(n+\varepsilon)/(n-\delta-\beta) < r$, we write $n-\delta = (n+\varepsilon)/r + n/r' - \varepsilon/r - \delta$, then, by the Hölder's inequality,

$$\begin{aligned}
I_2 &\le C\left(\int_{Q^c}\frac{|f(z)|dz}{|x-z|^{n+\varepsilon}}\right)^{1/r}\left[\int_{Q^c}\frac{|f(z)|}{|x-z|^{n-(\delta+\varepsilon/r)r'}}\left(\frac{|R_{m+1}(A;x,z)|}{|x-z|^m}\right)^{r'}dz\right]^{1/r'}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\left(\sum_{k=0}^{\infty}(2^k\lambda_1)^{-\varepsilon-n}\int_{|x-z|<2^k\lambda_1}|f(z)|dz\right)^{1/r}\\
&\quad\times\left(\sum_{k=0}^{\infty}(2^k\lambda_1)^{\beta r'}\int_{2^{-k}Q\setminus 2^{-k-1}Q}\frac{|f(z)|dz}{|x-z|^{n-(\delta+\varepsilon/r)r'}}\right)^{1/r'}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\left(\sum_{k=0}^{\infty}2^{-k\varepsilon}\lambda_1^{-\varepsilon}Mf(x)\right)^{1/r}\lambda_1^{\delta+\beta-n/r'+\varepsilon/r}\\
&\quad\times\left(\sum_{k=0}^{\infty}2^{k(\delta+\beta-n/r'+\varepsilon/r)r'}\int_{2^{-k}Q\setminus 2^{-k-1}Q}|f(z)|dz\right)^{1/r'}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\lambda_1^{-\varepsilon/r}(Mf(x))^{1/r}\lambda_1^{\delta+\beta-n/r'+\varepsilon/r}||f||_{L^1}^{1/r'}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\lambda_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r'}(Mf(x))^{1/r}.
\end{aligned}$$

Thus, our claim holds. Now we can prove Theorem 3. For any $\lambda > 0$ and $f \in L^1(R^n)$, taking $\lambda_1 = (\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}||f||_{L^1}\lambda^{-1})^{1/(n-\delta-\beta)}$ in above estimate, we gain, by the weak type boundedness of M,

$$
\begin{aligned}
&\left|\left\{x \in R^n : g_\mu^A(f)(x) > \lambda\right\}\right| \\
\le &\left|\left\{x \in R^n : Mf(x) > \frac{\lambda}{2C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\lambda_1^{\delta+\beta}}\right\}\right| \\
&+\left|\left\{x \in R^n : Mf(x) > \left(\frac{\lambda}{2C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\lambda_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r}}\right)^r\right\}\right| \\
\le &\, C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\lambda_1^{\delta+\beta}||f||_{L^1}/\lambda + C\left(\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\lambda_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r}/\lambda\right)^r \\
\le &\, C\left(\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}||f||_{L^1}/\lambda\right)^{n/(n-\delta-\beta)}.
\end{aligned}
$$

This completes the proof of Theorem 3.

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