

GLOBAL ENTROPY SOLUTIONS IN L^∞ TO THE EULER EQUATIONS AND EULER-POISSON EQUATIONS FOR ISOTHERMAL FLUIDS WITH SPHERICAL SYMMETRY *

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Abstract. We prove the existence of global entropy solutions in L^∞ to the multidimensional Euler equations and Euler-Poisson equations for compressible isothermal fluids with spherically symmetric initial data that allows vacuum and unbounded velocity outside a solid ball. The multidimensional existence problem can be reduced to the existence problem for the one-dimensional Euler equations and Euler-Poisson equations with geometrical source terms. Due to the presence of the geometrical source terms, new variables-weighted density and momentum—are first introduced to transform the nonlinear system into a new nonlinear hyperbolic system to reduce the geometric source effect. We then develop a shock capturing scheme of Lax-Friedrichs type to construct approximate solutions for the weighted density and momentum. Since the velocity may be unbounded, the Courant-Friedrichs-Lewy stability condition may fail for the standard fractional-step Lax-Friedrichs scheme; hence we introduce a cut-off technique to modify the approximate density functions and adjust the ratio of the space and time mesh sizes to construct our approximate solutions. Finally we establish the convergence and consistency of the approximate solutions using the method of compensated compactness and obtain global entropy solutions in L^∞ . The solutions we obtain allow unbounded velocity near vacuum, one of the essential difficulties here, which is different from the isentropic case.

1. Introduction. We are concerned with the convergence and stability of shock capturing schemes and their applications to constructing global entropy solutions to the Euler equations and Euler-Poisson equations for compressible fluids. Shock capturing schemes have played an important role in providing excellent numerical solutions for various physical problems in science and technology; examples of success include the Lax-Friedrichs scheme [27], the Glimm scheme [19], the Godunov scheme [20] and related high order schemes such as ENO and WENO (see Harten-Osher [23], Harten-Engquist-Osher-Chakravarthy [24], Shu-Osher [46], and Shu [45]), van Leer's MUSCL [48], Colella-Wooward's PPM [11], as well as the Lax-Wendroff scheme [29] and its two-step version, the Richtmyer scheme (see [43]) and the MacCormick scheme [37].

In this paper, we present two examples through the Euler equations and the Euler-Poisson equations to show that efficient numerical shock capturing schemes not only provide excellent numerical solutions, but also can yield a mathematical existence theory for global entropy solutions. Other related schemes of success include the Glimm scheme [19] (see [12, 34, 36, 44, 47]), the wave-front tracking algorithm (see [2, 13, 25, 30]), and the Lax-Friedrichs scheme and the Godunov scheme (see [4, 14, 15]).

The Euler equations for compressible isothermal fluids are of the form:

$$(1.1) \quad \begin{aligned} \partial_t \rho + \nabla \cdot \vec{m} &= 0, \\ \partial_t \vec{m} + \nabla \cdot \left(\frac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p &= 0, \end{aligned} \quad \vec{x} \in \mathbf{R}^N, \quad t \in [0, \infty),$$

where ρ , \vec{m} , and p are the density, the momentum, and the pressure of the fluid, respectively, and $p(\rho) = \kappa \rho$ for the isothermal fluid (κ can be chosen as 1 by scaling).

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The Euler-Poisson equations for compressible isothermal fluids are of the form:

$$(1.2) \quad \begin{aligned} \partial_t \rho + \nabla \cdot \vec{m} &= 0, \\ \partial_t \vec{m} + \nabla \cdot \left(\frac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p &= \rho \nabla \phi - \frac{\vec{m}}{\tau}, \\ \Delta \phi &= \rho - D(\vec{x}), \end{aligned} \quad \vec{x} \in \mathbf{R}^N, t \in [0, \infty),$$

where ϕ is the potential function, $D(\vec{x})$ is the doping profile, and $\tau > 0$ is the relaxation time. This system describes the dynamic behavior of many important physical flows including the propagation of electrons in submicron semiconductor devices [1, 17, 18, 40] and the biological transport of ions for channel proteins (see [3] and the references cited therein) in bounded domains.

We are interested in spherically symmetric solutions to system (1.1) with the form:

$$(1.3) \quad (\rho, \vec{m})(\vec{x}, t) = \left(\rho(x, t), m(x, t) \frac{\vec{x}}{x} \right), \quad x = |\vec{x}|,$$

and to system (1.2) for which $D(\vec{x}) = D(|\vec{x}|)$ with the form:

$$(1.4) \quad (\rho, \vec{m}, \phi)(\vec{x}, t) = \left(\rho(x, t), m(x, t) \frac{\vec{x}}{x}, \phi(x, t) \right).$$

Then $(\rho(x, t), m(x, t))$ in (1.3) is governed by the one-dimensional Euler equations with geometric source terms:

$$(1.5) \quad \begin{aligned} \partial_t \rho + \partial_x m &= -\frac{N-1}{x} m, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + \rho \right) &= -\frac{N-1}{x} \frac{m^2}{\rho}, \end{aligned}$$

and $(\rho(x, t), m(x, t), \phi(x, t))$ in (1.4) is governed by the one-dimensional Euler-Poisson equations with geometric source terms:

$$(1.6) \quad \begin{aligned} \partial_t \rho + \partial_x m &= -\frac{N-1}{x} m, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + \rho \right) &= -\frac{N-1}{x} \frac{m^2}{\rho} - \frac{m}{\tau} + \rho \partial_x \phi, \\ \partial_{xx} \phi &= -\frac{N-1}{x} \partial_x \phi + \rho - D(x). \end{aligned}$$

When $N = 1$ and initial data in BV stays away from vacuum, the first existence result in BV for the Euler equations for isothermal fluids was established by Nishida [41] by using the Glimm scheme; and the existence problem for the Euler-Poisson equations was worked out by Poupaud-Rascle-Vila [42]. For large initial data in L^∞ containing vacuum, the existence of entropy solutions in L^∞ containing vacuum for the Euler equations was recently solved by Huang-Wang [26] by combining the method of compensated compactness with analysis of entropy pairs via analytic extension techniques, and the global existence problem for the Euler-Poisson equations was worked out by Li [31].

When $N > 1$ and initial data in BV stays away from vacuum, only available result is due to Makino-Mizohata-Ukai [38] for the existence of global entropy solutions with spherical symmetry by using the Glimm scheme. For the multidimensional case, the geometrical source terms induce the resonance between the characteristic fields and the stationary geometrical sources. Such a nonlinear resonance causes extra difficulties; and more efficient methods have to be developed to solve (1.5) and (1.6) with initial data that allows vacuum.

In this paper, we first focus on the spherically symmetric initial data in L^∞ with vacuum for isothermal fluid flow outside the solid ball $\{x \geq 1\}$. That is, we consider the Euler equations (1.5) for isothermal fluids with the initial-boundary conditions:

$$(1.7) \quad (\rho, m)|_{t=0} = (\rho_0(x), m_0(x)), \quad m|_{x=1} = 0.$$

The natural issues associated with this problem are: (a) whether the solution has the same geometrical structure globally; (b) whether the solution blows up in a finite time, especially the density. These issues are not easily resolved through physical experiments or numerical simulations, especially the second one, due to the limited capacity of available instruments and computers.

To overcome the difficulty of resonance arising from the geometric source terms, one of our ingredients is to introduce new variables

$$\varrho = x^{N-1}\rho, \quad \varpi = x^{N-1}m,$$

the weighted density and momentum so that the resulting hyperbolic system has less geometrical source effect. Indeed, these new variables are then governed by

$$(1.8) \quad \begin{aligned} \partial_t \varrho + \partial_x \varpi &= 0, \\ \partial_t \varpi + \partial_x \left(\frac{\varpi^2}{\varrho} + \varrho \right) &= \frac{N-1}{x} \varrho, \end{aligned}$$

in which the first equation is homogeneous and the second has only the source term of first order in the density. With this special structure, our main strategy is to follow Ding-Chen-Luo [15] to develop a shock capturing scheme of Lax-Friedrichs type to construct approximate solutions so that they are uniformly bounded and the corresponding family of entropy dissipation measures is compact in H^{-1} .

The main obstacle in constructing the approximate solutions and making the required estimates is the unboundedness of velocity near vacuum. Then the characteristic speeds are unbounded near vacuum which yields that the Courant-Friedrichs-Lewy condition fails for the standard fractional-step Lax-Friedrichs scheme as developed in [15]. To overcome this difficulty, we develop a cut-off technique to modify the approximate density functions so that the characteristic speeds can be bounded by $\lambda(h)$ that is unbounded as the space mesh size $h \rightarrow 0$ and then to adjust the ratio of the space and time mesh sizes so that the Courant-Friedrichs-Lewy condition holds for each fixed space and time mesh sizes. Furthermore, the boundedness of the gradients of entropy and entropy flux functions fails since the velocity is unbounded near vacuum; while the boundedness is essential to get the H^{-1} compactness of entropy dissipation measures for the isentropic case. In this paper, we establish the estimates required for the convergence of approximate solutions to an entropy solution in L^∞ .

Consider the following family of entropy pairs (η, q) for the isothermal case:

$$(1.9) \quad \eta = \rho^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} \frac{m}{\rho}}, \quad q = \left(\frac{m}{\rho} + \xi \right) \eta \quad \text{for } \xi \in \mathbf{R}.$$

DEFINITION 1.1. *An L^∞ vector function $(\rho(x, t), m(x, t))$ is called a global entropy solution of the initial-boundary value problem (1.5) and (1.7) in $\Pi := \{(x, t) : x > 1, 0 < t < \infty\}$ provided that*

(i) *for any $\psi \in C_0^\infty(\bar{\Pi})$, $\bar{\Pi} = \{(x, t) : x \geq 1, 0 \leq t < \infty\}$, with $\psi(1, t) = 0$ for $t \geq 0$,*

$$(1.10) \quad \int_0^\infty \int_1^\infty \left(\rho \partial_t \psi + m \partial_x \psi - \frac{N-1}{x} m \psi \right) dx dt + \int_1^\infty \rho_0(x) \psi(x, 0) dx = 0,$$

$$(1.11) \quad \int_0^\infty \int_1^\infty \left(m \partial_t \psi + \left(\frac{m^2}{\rho} + \rho \right) \partial_x \psi - \frac{N-1}{x} \frac{m^2}{\rho} \psi \right) dx dt + \int_1^\infty m_0(x) \psi(x, 0) dx = 0;$$

(ii) when $\epsilon \rightarrow 0$,

$$(1.12) \quad \frac{1}{\epsilon} \int_1^{1+\epsilon} m(x, t) dx \xrightarrow{*} 0 \quad \text{in } L_{loc}^\infty((0, \infty));$$

(iii) for any nonnegative smooth function $\psi \in C_0^\infty(\bar{\Pi})$ with $\psi(1, t) = 0$,

$$(1.13) \quad \int_0^\infty \int_1^\infty \left(\eta(\rho, m) \partial_t \psi + q(\rho, m) \partial_x \psi - \frac{N-1}{x} (\partial_\rho \eta(\rho, m) m + \partial_m \eta(\rho, m) \frac{m^2}{\rho}) \psi \right) dx dt + \int_1^\infty \eta(\rho_0, m_0) \psi(x, 0) dx \geq 0,$$

for any entropy pair (η, q) in (1.9) with $\xi \in (-1, 1)$.

The following theorem is the main result of this paper.

THEOREM 1.1. *Suppose that the initial data $(\rho_0(x), m_0(x))$ satisfies the conditions:*

$$0 \leq \rho_0(x) \leq \frac{C_0}{x^{N-1}}, \quad |m_0(x)| \leq \rho_0(x) (C_0 + |\ln(x^{N-1} \rho_0(x))|)$$

for some constant $C_0 > 0$. Then there exists a global entropy solution $(\rho(x, t), m(x, t))$ of the initial-boundary value problem (1.5) and (1.7) in the sense of Definition 1.1 such that, for any $T > 0$, there exists $M = M(T) > 0$ so that

$$(1.14) \quad 0 \leq \rho(x, t) \leq \frac{M}{x^{N-1}}, \quad |m(x, t)| \leq \rho(x, t) (M + |\ln(x^{N-1} \rho(x, t))|), \quad 0 \leq t \leq T.$$

With this $(\rho(x, t), m(x, t))$ obtained in Theorem 1.1, we set $(\rho, \vec{m})(\vec{x}, t)$ through (1.3). Then $(\rho, \vec{m})(\vec{x}, t)$ is an entropy solution of (1.1) with initial data:

$$(\rho, \vec{m})|_{t=0} = \left(\rho_0(|\vec{x}|), m_0(|\vec{x}|) \frac{\vec{x}}{|\vec{x}|} \right), \quad |\vec{x}| \geq 1.$$

Related results have been obtained for the isentropic case ($p = \kappa \rho^\gamma, \gamma > 1$) with large initial data in L^∞ containing vacuum. For $N = 1$, the global existence for the Euler equations with large initial data in L^∞ was established for $\gamma = 1 + \frac{2}{2n+1}, n \geq 2$, by DiPerna [16] by using the method of compensated compactness, for $1 < \gamma \leq 5/3$ by Ding-Chen-Luo [14] and Chen [4, 5], and for $\gamma > 5/3$ by Lions-Perthame-Tadmor [33] and Lions-Perthame-Souganidis [32]; also recently by Chen-LeFloch [9] for general pressure laws in which the approach further simplifies the proof for the case $\gamma > 1$. The global existence problem for the Euler-Poisson equations with $\gamma > 1$ was worked out by Zhang [49].

For $N > 1$, the local existence of such a weak solution for $1 < \gamma < 5/3$ was constructed by Makino-Mizohata-Ukai [39] with the aid of the compactness theorem

in Ding-Chen-Luo [14] and Chen [4, 5]. A theorem was also established for the general case by Chen [6] to ensure the existence of L^∞ spherically symmetric entropy solutions in the large amplitude, which models outgoing blast and large-time asymptotic solutions. Chen-Glimm [8] obtained the existence of entropy solutions in L^∞ for $\gamma > 1$ by incorporating the Godunov scheme and the steady-state solutions for general initial data in L^∞ . Entropy solutions to the Euler-Poisson equations (1.6) for $\gamma > 1$ were also obtained by Chen-Wang in [10].

In Section 2, we introduce some basic results which are used in subsequent sections. In Section 3, we develop a shock capturing scheme of Lax-Friedrichs type to construct a family of approximate solutions. The uniform L^∞ estimate of the approximate solutions is achieved in Section 4 by estimating the corresponding Riemann invariants of the approximate solutions and by identifying invariant regions for Riemann solutions at each time step. In Section 5, the difficulty of unbounded velocity is overcome and a compactness embedding technique is used to prove that the family of entropy dissipation measures is compact in H_{loc}^{-1} . Then, in Section 6, we employ the compensated compactness framework to obtain a convergent subsequence, and we then show that the limit of this subsequence is a physical entropy solution. In Section 7, we employ the approaches and ideas developed in Sections 3–6 to construct global entropy solutions to the Euler-Poisson equations for isothermal fluids with spherical symmetry.

2. Preliminaries. In this section, we first introduce some basic facts about the homogeneous Euler system of (1.8):

$$(2.1) \quad \begin{aligned} \partial_t \varrho + \partial_x \varpi &= 0, \\ \partial_t \varpi + \partial_x \left(\frac{\varpi^2}{\varrho} + \varrho \right) &= 0. \end{aligned}$$

For more details, see [26, 4, 14, 28, 49].

First, system (2.1) can be rewritten in the form of vector $v = (\varrho, \varpi)$ as

$$(2.2) \quad \partial_t v + \partial_x f(v) = 0,$$

where $f(v) = (\varpi, \frac{\varpi^2}{\varrho} + \varrho)^\top$. The eigenvalues of (2.1) are

$$(2.3) \quad \lambda_1 = \frac{\varpi}{\varrho} - 1, \quad \lambda_2 = \frac{\varpi}{\varrho} + 1,$$

and the Riemann invariants are

$$(2.4) \quad w = \frac{\varpi}{\varrho} + \ln \varrho, \quad z = \frac{\varpi}{\varrho} - \ln \varrho.$$

Consider the classical Riemann problem for (2.1) with Riemann data:

$$(2.5) \quad (\varrho, \varpi) |_{t=0} = \begin{cases} (\varrho_l, \varpi_l), & x < x_0, \\ (\varrho_r, \varpi_r), & x > x_0, \end{cases} \quad x_0 > 1,$$

and the lateral Riemann problem for (2.1) with lateral Riemann data:

$$(2.6) \quad (\varrho, \varpi) |_{t=0} = (\varrho_r, \varpi_r), \quad x > 1; \quad \varpi |_{x=1} = 0,$$

where $\varrho_l, \varrho_r, \varpi_l$, and ϖ_r are constants satisfying

$$0 \leq \varrho_l, \varrho_r \leq C_0, \quad \left| \frac{\varpi_l}{\varrho_l} \right| \leq C_0 + |\ln \varrho_l|, \quad \left| \frac{\varpi_r}{\varrho_r} \right| \leq C_0 + |\ln \varrho_r|.$$

There are two distinct types of rarefaction waves and shock waves called *elementary waves*, which are labeled 1-rarefaction or 2-rarefaction waves and 1-shock or 2-shock waves, respectively.

LEMMA 2.1. *There exists a global entropy solution of the Riemann problem (2.5), which is a piecewise smooth vector function satisfying*

$$\begin{aligned} w(x, t) &\equiv w(\varrho(x, t), \varpi(x, t)) \leq \max\{w(\varrho_l, \varpi_l), w(\varrho_r, \varpi_r)\}, \\ z(x, t) &\equiv z(\varrho(x, t), \varpi(x, t)) \geq \min\{z(\varrho_l, \varpi_l), z(\varrho_r, \varpi_r)\}. \end{aligned}$$

Also there exists a global entropy solution of the lateral Riemann problem (2.6), which is a piecewise smooth vector function satisfying

$$\begin{aligned} w(x, t) &\equiv w(\varrho(x, t), \varpi(x, t)) \leq \max\{w(\varrho_r, \varpi_r), -z(\varrho_r, \varpi_r)\}, \\ z(x, t) &\equiv z(\varrho(x, t), \varpi(x, t)) \geq \min\{z(\varrho_l, \varpi_l), z(\varrho_r, \varpi_r)\}, \end{aligned}$$

where w and z are the Riemann invariants in (2.4).

It follows that the regions $\Lambda = \{(\varrho, \varpi) : w \leq \max(w_0, -z_0), z \geq z_0\}$ are invariant regions for the Riemann problem (2.5)–(2.6). More precisely, if the Riemann data lies in Λ , then the corresponding Riemann solution of (2.5)–(2.6) also lies in Λ . This implies that there exists $C > 0$ such that the Riemann solution $(\rho(x, t), m(x, t))$ satisfies

$$0 \leq \varrho(x, t) \leq C, \quad \left| \frac{\varpi(x, t)}{\varrho(x, t)} \right| \leq C + |\ln \varrho(x, t)|.$$

LEMMA 2.2. *If $(\varrho, \varpi) : (a, b) \rightarrow \Lambda$, then*

$$\left(\frac{1}{b-a} \int_a^b \varrho(x) dx, \frac{1}{b-a} \int_a^b \varpi(x) dx \right) \in \Lambda.$$

These two properties above for $\gamma = 1$ are the same as the case $\gamma > 1$.

DEFINITION 2.1. *A pair of mapping $(\eta, q) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is called an entropy pair of system (2.1) if (η, q) satisfies the following linear hyperbolic system:*

$$\nabla q = \nabla \eta \nabla f.$$

If $\eta|_{\varrho=0} = 0$, then η is called a weak entropy.

The following entropy pairs (η, q) of (2.1):

$$(2.7) \quad \eta = \varrho^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} \frac{\varpi}{\varrho}}, \quad q = \left(\frac{\varpi}{\varrho} + \xi \right) \varrho^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} \frac{\varpi}{\varrho}}$$

satisfy

$$(2.8) \quad \eta_{\varrho\varrho} \eta_{\varpi\varpi} - \eta_{\varrho\varpi}^2 = \frac{\xi^4}{(1-\xi^2)^3} \varrho^{\frac{2\xi^2}{1-\xi^2}-2} e^{\frac{2\xi}{1-\xi^2} \frac{\varpi}{\varrho}} \quad \text{for } \xi \in \mathbf{R}.$$

Thus, η is a weak and convex entropy for any $\xi \in (-1, 1)$.

The following compactness theorem for $\gamma = 1$ was established in Huang-Wang [26].

THEOREM 2.1. *Assume that the family of functions $(\varrho_h(x, t), \varpi_h(x, t))$, defined on a bounded open set $\Omega \subset \Pi$, satisfies the following conditions:*

(i) *There exists some constant $M = M(\Omega) > 0$ such that*

$$0 \leq \varrho_h(x, t) \leq M, \quad |\varpi_h(x, t)| \leq \varrho_h(x, t) (M + |\ln \varrho_h(x, t)|) \quad a.e.$$

(ii) *The sequence of entropy dissipation measures $\eta_t(\varrho_h, \varpi_h) + q_x(\varrho_h, \varpi_h)$ is compact in $H_{loc}^{-1}(\Omega)$, for any entropy pair (η, q) in (2.7) with $\xi \in (-1, 1)$.*

Then there exist $(\varrho, \varpi) \in L^\infty$ and a subsequence (still denoted) (ϱ_h, ϖ_h) such that

$$(\varrho_h, \varpi_h) \longrightarrow (\varrho, \varpi) \quad a.e. \text{ as } h \rightarrow 0$$

and

$$0 \leq \varrho(x, t) \leq M, \quad |\varpi(x, t)| \leq \varrho(x, t) (M + |\ln \varrho(x, t)|).$$

We will use this compactness theorem in proving the convergence of our approximate solutions.

3. Construction of approximate solutions. We now develop a shock capturing scheme of Lax-Friedrichs type to construct approximate solutions of (1.7)–(1.8).

Let $h > 0$ be the space mesh length. We partition the interval $[1, \infty)$ into cells with the j^{th} cell centered at $x_j = 1 + jh, j = 0, 1, 2, \dots$. To ensure the Courant-Friedrichs-Lewy (CFL) stability condition, we choose the time mesh length

$$(3.1) \quad k = \frac{h}{10(1 + |\ln h|)}.$$

We then set $t_i = ik$.

To ensure the CFL condition, we employ a cut-off technique so that the approximate density functions stay away from vacuum by h^β , with $2 \leq \beta \leq 3$, starting from the initial data:

$$\bar{\varrho}_0 = \max(\varrho_0, h^\beta), \quad \bar{\varpi}_0 = \varpi_0.$$

Let $\bar{v}_0 = (\bar{\varrho}_0, \bar{\varpi}_0)$. We define

$$\underline{v}(x, 0 + 0) = \begin{cases} \bar{v}_j^0 & \text{for } x_{j-1} \leq x \leq x_{j+1}, j \geq 4 \text{ even,} \\ \bar{v}_2^0 & \text{for } 1 \leq x \leq x_3, \end{cases}$$

where \bar{v}_j^0 is the average value of the function \bar{v}_0 in each cell:

$$\bar{v}_j^0 = \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{v}_0 dx, \quad j \geq 4 \text{ even;} \quad \bar{v}_2^0 = \frac{1}{3h} \int_1^{1+3h} \bar{v}_0 dx.$$

Then we solve the Riemann problem (2.5) in the region

$$R_j^1 \equiv \{(x, t) : x_j \leq x \leq x_{j+2}, 0 \leq t < t_1\}, \quad j \geq 2,$$

with Riemann data:

$$\underline{v}_h|_{t=0} = \begin{cases} \bar{v}_j^0 & \text{for } x < x_{j+1}, \\ \bar{v}_{j+2}^0 & \text{for } x > x_{j+1}, \quad j = 2, 4, \dots \end{cases}$$

and the lateral Riemann problem (2.6) in $\{(x, t) : 1 \leq x \leq x_2, 0 \leq t < t_1\}$ with Riemann data:

$$\underline{v}_h|_{t=0} = \bar{v}_2^0, \quad x > 1; \quad \varpi|_{x=1} = 0$$

to obtain $\underline{v}_h(x, t)$ for $0 \leq t < t_1$. Then we set

$$v_h(x, t) = \underline{v}_h(x, t) + V(\underline{v}_h(x, t))t \quad \text{for } 0 < t < t_1,$$

where $V = (V_1, V_2) = (0, \frac{N-1}{x}\varrho)$.

Suppose that we have defined approximate solutions $v_h(x, t)$ for $0 \leq t < t_i$. Then we set

$$(3.2) \quad \bar{\varrho}_h(x, t) = \max(\varrho_h(x, t), h^\beta), \quad \bar{\varpi}_h(x, t) = \varpi_h(x, t) \quad \text{for } t_{i-1} \leq t < t_i.$$

We define $\underline{v}(x, t_i + 0)$ as follows: When $i \geq 1$ is odd,

$$\underline{v}(x, t_i + 0) = \begin{cases} \bar{v}_j^i & \text{for } x_{j-1} \leq x \leq x_{j+1}, j \geq 3 \text{ odd,} \\ \bar{v}_1^i & \text{for } 1 \leq x \leq x_2; \end{cases}$$

and, when $i \geq 2$ is even,

$$\underline{v}(x, t_i + 0) = \begin{cases} \bar{v}_j^i & \text{for } x_{j-1} \leq x \leq x_{j+1}, j \geq 4 \text{ even,} \\ \bar{v}_2^i & \text{for } 1 \leq x \leq x_3, \end{cases}$$

where \bar{v}_j^i is the average value of the function $\bar{v}_h(x, t_i - 0)$ in each cell as follows: When i is odd,

$$(3.3) \quad \bar{v}_j^i = \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{v}_h(x, t_i - 0) dx, \quad j \geq 3 \text{ odd}; \quad \bar{v}_1^i = \frac{1}{2h} \int_1^{1+2h} \bar{v}_h(x, t_i - 0) dx,$$

and, when i is even,

$$(3.4) \quad \bar{v}_j^i = \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{v}_h(x, t_i - 0) dx, \quad j \geq 4 \text{ even}; \quad \bar{v}_2^i = \frac{1}{3h} \int_1^{1+3h} \bar{v}_h(x, t_i - 0) dx.$$

Then we solve the Riemann problem (2.5) in the region

$$R_j^i \equiv \{(x, t) : x_j \leq x \leq x_{j+2}, t_i \leq t < t_{i+1}\}$$

with initial data:

$$(3.5) \quad \underline{v}_h|_{t=t_i} = \begin{cases} \bar{v}_j^i, & x < x_{j+1}, \\ \bar{v}_{j+2}^i, & x > x_{j+1} \end{cases}$$

for $j \geq 2$ even when i is even and for $j \geq 1$ odd when i is odd, and the lateral Riemann problem (2.6) in $\{(x, t) : 1 \leq x \leq x_2, t_i \leq t < t_{i+1}\}$ when i is even with the Riemann data:

$$(3.6) \quad \underline{v}_h|_{t=t_i} = \bar{v}_2^i, \quad 1 < x \leq x_2; \quad \varpi|_{x=1} = 0,$$

and $\{(x, t) : 1 \leq x \leq x_1, t_i \leq t < t_{i+1}\}$ when i is odd with the Riemann data:

$$(3.7) \quad \underline{v}_h|_{t=t_i} = \bar{v}_1^i, \quad 1 < x \leq x_1; \quad \varpi|_{x=1} = 0$$

to obtain $\underline{v}_h(x, t), 0 \leq t_i < t_{i+1}$. We set

$$(3.8) \quad v_h(x, t) = \underline{v}_h(x, t) + V(\underline{v}_h(x, t))(t - t_i) \quad \text{for } t_i \leq t < t_{i+1}.$$

We summarize the above process as follows:

$$\bar{v}^{i+1} = C_h \circ R \circ E_k \circ A_h(\bar{v}^i),$$

or

$$v^{i+1} = R \circ E_k \circ A_h \circ C_h(v^i),$$

where C_h is the cut-off operator (3.2), A_h is the cell-averaging operator (3.4), E_k is the exact evolution operator (3.5), and R is the reconstruction step (3.8).

4. A uniform bound for the approximate solutions. To obtain a uniform bound for the approximate solutions, we first estimate the Riemann invariants $w_h(x, t)$ and $z_h(x, t)$. For $t_i \leq t < t_{i+1}$, we have

$$(4.1) \quad w_h(x, t) = \underline{w}_h(x, t) + \frac{N-1}{x}(t - t_i), \quad z_h(x, t) = \underline{z}_h(x, t) + \frac{N-1}{x}(t - t_i),$$

where \underline{w}_h and \underline{z}_h are Riemann invariants corresponding to the Riemann solutions \underline{v}_h .

THEOREM 4.1. *Suppose that $(\varrho_0(x), \varpi_0(x))$ satisfies the following conditions:*

$$(4.2) \quad 0 \leq \varrho_0(x) \leq C_0, \quad |\varpi_0(x)| \leq \varrho_0(x)(C_0 + |\ln \varrho_0|).$$

Then, for any given $T > 0$, $(\bar{\varrho}_h(x, t), \bar{\varpi}_h(x, t))$ are uniformly bounded in the region $\bar{\Pi}_T := \{(x, t) : x \geq 1, 0 \leq t \leq T\}$; that is, there exists $M = M(T) > 0$ such that

$$0 < h^\beta \leq \bar{\varrho}_h(x, t) \leq M, \quad |\bar{\varpi}_h(x, t)| \leq \bar{\varrho}_h(x, t)(M + |\ln \bar{\varrho}_h(x, t)|).$$

Proof. Let

$$(4.3) \quad \begin{aligned} w_h(x, t) &= w(v_h(x, t)), & \underline{w}_h(x, t) &= w(\underline{v}_h(x, t)), & \bar{w}_h(x, t) &= w(\bar{v}_h(x, t)), \\ z_h(x, t) &= z(v_h(x, t)), & \underline{z}_h(x, t) &= z(\underline{v}_h(x, t)), & \bar{z}_h(x, t) &= z(\bar{v}_h(x, t)). \end{aligned}$$

For $t_i \leq t < t_{i+1}$, (4.1) and Lemma 2.1 imply

$$(4.4) \quad w_h(x, t) = \underline{w}_h(x, t) + \frac{N-1}{x}(t - t_i) \leq \sup_x \underline{w}_h(x, t_i + 0) + (N-1)k,$$

$$(4.5) \quad z_h(x, t) = \underline{z}_h(x, t) + \frac{N-1}{x}(t - t_i) \geq \inf_x \underline{z}_h(x, t_i + 0).$$

By the assumption that $0 \leq \varrho_0(x) \leq C_0$ and $|\varpi_0(x)| \leq \varrho_0(x)(C_0 + |\ln \varrho_0(x)|)$, there exists $\alpha_0 > 0$ such that

$$(4.6) \quad \sup_x w(\varrho_0(x), \varpi_0(x)) \leq \alpha_0, \quad \inf_x z(\varrho_0(x), \varpi_0(x)) \geq -\alpha_0.$$

It is easy to check that, for the Riemann invariants corresponding to the modified functions by the cut-off technique, we have

$$(4.7) \quad w(\bar{\varrho}_0(x), \bar{\varpi}_0(x)) \leq \alpha_0, \quad z(\bar{\varrho}_0(x), \bar{\varpi}_0(x)) \geq -\alpha_0.$$

For $0 \leq t < t_1$, the properties of Riemann invariants in Lemmas 2.1 and 2.2 yield

$$(4.8) \quad w(\underline{\varrho}_h(x, t), \underline{\varpi}_h(x, t)) \leq \alpha_0, \quad z(\underline{\varrho}_h(x, t), \underline{\varpi}_h(x, t)) \geq -\alpha_0.$$

By (4.4) and (4.5), we obtain

$$w_h(x, t) = w(\varrho_h, \varpi_h) \leq \alpha_0 + (N - 1)k, \quad z_h(x, t) = z(\varrho_h, \varpi_h) \geq -\alpha_0 - (N - 1)k.$$

Performing the same procedure, we conclude that, for $0 \leq t < T$,

$$\bar{w}_h = w(\bar{\varrho}_h, \bar{\varpi}_h) \leq \alpha_0 + (N - 1)T, \quad \bar{z}_h = z(\bar{\varrho}_h, \bar{\varpi}_h) \geq -\alpha_0 - (N - 1)T,$$

which implies that there exists $M = M(T) > 0$ such that

$$(4.9) \quad 0 \leq \bar{\varrho}_h(x, t) \leq M, \quad |\bar{\varpi}_h(x, t)| \leq \bar{\varrho}_h(x, t)(M + |\ln \bar{\varrho}_h(x, t)|).$$

On the other hands, by the definition of $\bar{\varrho}_h$, we have $\bar{\varrho}_h \geq h^\beta$. This completes the proof.

With the uniform bound for the approximate solutions in $\bar{\Pi}_T$, there exists $h_0 = h_0(T) > 0$ such that, when $h \leq h_0$, the CFL condition holds, which implies that the approximate solutions are well-defined in Π_T .

5. H^{-1} compactness of entropy dissipation measures. In this section, we estimate the H^{-1} compactness of entropy dissipation measures

$$\partial_t \eta(\bar{v}_h) + \partial_x q(\bar{v}_h)$$

associated with any weak entropy pair (η, q) in (2.7) for the approximate solutions \bar{v}_h , constructed in Section 3. For simplicity, we drop the subscript h of the approximate solutions \bar{v}_h , and, from now on, we denote C a generic constant independent of h , which may be different at each occurrence.

LEMMA 5.1. *Let $\bar{v}(x, t)$ be the approximate solutions. Then, for any given $L \geq 1$ and $T > 0$, there is a constant $C = C(L, T) > 0$ independent of h such that*

$$(5.1) \quad \sum_{ik \leq T} \sum_{1+(j+1)h \leq L} \int_{x_{j-1}}^{x_{j+1}} (\bar{v}(x, t_i - 0) - \bar{v}_j^i)^2 dx \leq C,$$

where $\bar{v}_j^i = \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{v}_h(x, t_i - 0) dx$ for $j \geq 4$ even with $\bar{v}_2^i = \frac{1}{3h} \int_1^{1+3h} \bar{v}_h(x, t_i - 0) dx$ when i is even, and $j \geq 3$ odd with $\bar{v}_1^i = \frac{1}{2h} \int_1^{1+2h} \bar{v}_h(x, t_i - 0) dx$ when i is odd.

Proof. Fix a strictly convex entropy η in (2.7) with $\xi \in (-\frac{1}{2}, \frac{1}{2})$. For the Riemann solutions \underline{v} for (2.5) and (2.6) in the time strip $t_i \leq t < t_{i+1}$, the Green formula implies

$$(5.2) \quad \sum_{1+(j+1)h \leq L} \int_{x_{j-1}}^{x_{j+1}} (\eta(\underline{v}^{i+1}) - \eta(\bar{v}_j^i)) dx + \int_{t_i}^{t_{i+1}} \sum (\sigma[\eta] - [q]) dt \leq 0,$$

where $\underline{v}^{i+1} = \underline{v}(x, t_{i+1} - 0)$, the summation \sum is taken over all the shock waves in \underline{v} at a fixed t between t_i and t_{i+1} , σ is the propagating speed of the shock wave, and $[\eta]$ and $[q]$ denote the jumps of η and q across the shock wave from the left to the right, respectively. That is, if $S = (x(t), t)$ denotes a shock wave in \underline{v} , then

$$[\eta] = \eta(\underline{v}(x(t) + 0, t)) - \eta(\underline{v}(x(t) - 0, t)), \quad [q] = q(\underline{v}(x(t) + 0, t)) - q(\underline{v}(x(t) - 0, t)).$$

Summing over all i in (5.2) yields

$$(5.3) \quad I_h + II_h + III_h + IV_h \leq C,$$

where

$$\begin{aligned} I_h &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\underline{v}^i) - \eta(v^i)) dx, & II_h &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(v^i) - \eta(\bar{v}^i)) dx, \\ III_h &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\bar{v}^i) - \eta(\bar{v}_j^i)) dx, & IV_h &= \int_0^T \sum (\sigma[\eta] - [q]) dt, \end{aligned}$$

with $1 \leq 1 + (j + 1)h \leq L$, $0 \leq ik \leq T$, $i + j$ even, and $v^i = v(x, t_i - 0)$.

By the reconstruction step (3.8),

$$\varrho^i = \underline{\varrho}^i, \quad \varpi^i - \underline{\varpi}^i = \underline{\varrho}^i \frac{N-1}{x} k.$$

Noting that $\eta_{\varpi} = \frac{\xi}{1-\xi^2} \varrho^{\frac{1-\xi^2}{1-\xi^2}-1} e^{\frac{\xi}{1-\xi^2} \frac{\varpi}{\varrho}} = \frac{\xi}{1-\xi^2} \frac{1}{\varrho} \eta$, we have

$$\begin{aligned} \left| \int_{x_{j-1}}^{x_{j+1}} (\eta(\underline{v}^i) - \eta(v^i)) dx \right| &\leq \int_{x_{j-1}}^{x_{j+1}} |\eta(\underline{v}^i) - \eta(v^i)| dx \\ &= \int_{x_{j-1}}^{x_{j+1}} \left| \left(\int_0^1 \nabla \eta(\underline{v}^i + \theta(v^i - \underline{v}^i)) d\theta \right) \cdot (v^i - \underline{v}^i) \right| dx \\ &\leq k \int_{x_{j-1}}^{x_{j+1}} \left(\int_0^1 \frac{|\xi|}{1-\xi^2} \eta(\varrho^i, \tilde{\varpi}_\theta^i) d\theta \right) \frac{N-1}{x} dx, \end{aligned}$$

where $\tilde{\varpi}_\theta^i = \theta \varpi^i + (1-\theta) \underline{\varpi}^i$. Notice that, using Theorem 4.1, $|\frac{\tilde{\varpi}_\theta^i}{\varrho^i}| \leq M + |\ln \varrho^i|$ and, hence

$$(5.4) \quad \eta(\varrho^i, \tilde{\varpi}_\theta^i) = (\varrho^i)^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} \frac{\tilde{\varpi}_\theta^i}{\varrho^i}} \leq C(\varrho^i)^{\left(\frac{1}{1-\xi^2} - \frac{|\xi|}{1-\xi^2}\right)} = C(\varrho^i)^{\frac{1-|\xi|}{1-\xi^2}}.$$

Since $\frac{1-|\xi|}{1-\xi^2} > \frac{1}{2}$ when $\xi \in (-1, 1)$, both functions $\eta(\varrho^i, \tilde{\varpi}_\theta^i)$ and $|\ln \varrho^i| \eta(\varrho^i, \tilde{\varpi}_\theta^i)$ are bounded. Then we conclude

$$\left| \int_{x_{j-1}}^{x_{j+1}} (\eta(\underline{v}^i) - \eta(v^i)) dx \right| \leq Chk.$$

Summing over all cells, we have

$$(5.5) \quad |I_h| \leq C,$$

where C depends only on L and T .

When $\varrho < h^\beta$,

$$\eta = \varrho^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} \frac{\varpi}{\varrho}} \leq e^{\frac{\xi}{1-\xi^2} M} \varrho^{\frac{1-|\xi|}{1-\xi^2}} \leq e^{\frac{\xi}{1-\xi^2} M} h^{\beta \frac{1-|\xi|}{1-\xi^2}}.$$

If $\xi \in (-1, 1)$, $\frac{1-|\xi|}{1-\xi^2} \in (\frac{1}{2}, 1]$. Since $\beta \geq 2 > 1 + |\xi|$, then $\beta \frac{1-|\xi|}{1-\xi^2} > 1$. By (3.1), we

have

$$\begin{aligned}
 |II_h| &= \left| \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(v^i) - \eta(\bar{v}^i)) dx \right| \\
 &= \left| \sum_{i,j} \int_{[x_{j-1}, x_{j+1}] \cap \{\varrho^i > h^\beta\}} (\eta(v^i) - \eta(\bar{v}^i)) dx \right. \\
 &\quad \left. + \sum_{i,j} \int_{[x_{j-1}, x_{j+1}] \cap \{\varrho^i \leq h^\beta\}} (\eta(v^i) - \eta(\bar{v}^i)) dx \right| \\
 &\leq 0 + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} 2e^{\frac{\xi}{1-\xi^2}} M h^{\beta \frac{1-|\xi|}{1-\xi^2}} \leq C(1 + |\ln h|) h^{\beta \frac{1-|\xi|}{1-\xi^2} - 1}.
 \end{aligned}$$

Thus we have

$$(5.6) \quad II_h \longrightarrow 0 \quad \text{as } h \rightarrow 0.$$

Therefore,

$$III_h + IV_h \leq C.$$

Since η is convex, the entropy inequality $\sigma[\eta] - [q] \geq 0$ holds across the shock waves [14] which implies $IV_h \geq 0$. Also, since \bar{v}_j^i is the average of \bar{v}^i and η is convex, we have $III_h \geq 0$. Then we obtain

$$(5.7) \quad 0 \leq III_h = \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\bar{v}^i) - \eta(\bar{v}_j^i)) dx \leq C, \quad 0 \leq IV_h = \int_0^T \sum (\sigma[\eta] - [q]) dt \leq C.$$

Moreover, since η is strictly convex for $\xi \in (-\frac{1}{2}, \frac{1}{2})$ so that

$$\begin{aligned}
 C &\geq III_h = \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\bar{v}^i) - \eta(\bar{v}_j^i)) dx \\
 &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\nabla \eta(\bar{v}_j^i) (\bar{v}^i - \bar{v}_j^i) + (\bar{v}^i - \bar{v}_j^i)^\top \nabla^2 \eta(\theta_j^i) (\bar{v}^i - \bar{v}_j^i)) dx \\
 &\geq 0 + \bar{\alpha} \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\bar{v}^i - \bar{v}_j^i)^2 dx \quad \text{for some } \bar{\alpha} > 0.
 \end{aligned}$$

Therefore, we conclude (5.1).

We now use the duality and the Sobolev interpolation inequality to obtain the H^{-1} compactness by the following lemma, whose proof can be found in [14, 4].

LEMMA 5.2. *Let $\Omega \subset \mathbf{R}^m$ be a bounded open set. Then*

$$(Compact \ set \ of \ W^{-1,p}(\Omega)) \cap (Bounded \ set \ of \ W^{-1,r}(\Omega)) \subset (Compact \ set \ of \ H_{loc}^{-1}(\Omega))$$

for some constants p and r satisfying $1 < p \leq 2 < r < \infty$.

THEOREM 5.1. *Assume that the conditions of Theorem 4.1 are satisfied. Then the sequence of entropy dissipation measures $\partial_t \eta(\bar{v}) + \partial_x q(\bar{v})$ is compact in $H_{loc}^{-1}(\Pi)$ for any (η, q) in (2.7) with $\xi \in (-1, 1)$.*

Proof. For any $\psi \in C_0^\infty(\Pi)$, there exist $L > 0$, $T > 0$, and an open set Ω such that $\text{supp}\psi \subset \Omega \subset (1, L) \times (0, T)$. We consider

$$(5.8) \quad \int_0^\infty \int_1^\infty (\eta(\bar{v})\partial_t\psi + q(\bar{v})\partial_x\psi) dx dt = A(\psi) + R(\psi) + \Sigma(\psi) + S(\psi),$$

where

$$\begin{aligned} A(\psi) &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\bar{v}^i) - \eta(\bar{v}_j^i)) \psi(x, t^i) dx, \\ R(\psi) &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\underline{v}^i) - \eta(v^i)) \psi(x, t^i) dx + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(v^i) - \eta(\bar{v}^i)) \psi(x, t^i) dx, \\ \Sigma(\psi) &= \int_0^\infty \sum (\sigma[\eta] - [q]) \psi(x(t), t) dt, \\ S(\psi) &= S_1(\psi) + S_2(\psi) \end{aligned}$$

with

$$\begin{aligned} S_1(\psi) &= \int_0^\infty \int_1^\infty ((\eta(v) - \eta(\underline{v}))\partial_t\psi + (q(v) - q(\underline{v}))\partial_x\psi) dx dt, \\ S_2(\psi) &= \int_0^\infty \int_1^\infty ((\eta(\bar{v}) - \eta(v))\partial_t\psi + (q(\bar{v}) - q(v))\partial_x\psi) dx dt. \end{aligned}$$

First we have

$$\begin{aligned} A(\psi) &= \sum_{i,j} \psi_j^i \int_{x_{j-1}}^{x_{j+1}} (\eta(\bar{v}^i) - \eta(\bar{v}_j^i)) dx + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\bar{v}^i) - \eta(\bar{v}_j^i)) (\psi^i - \psi_j^i) dx \\ &= A_1(\psi) + A_2(\psi), \end{aligned}$$

where $\psi^i = \psi(x, t_i)$ and $\psi_j^i = \psi(x_j, t_i)$. By (5.7),

$$(5.9) \quad |A_1(\psi)| \leq \|\psi\|_\infty \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\eta(\bar{v}^i) - \eta(\bar{v}_j^i)) dx \leq C\|\psi\|_\infty,$$

and

$$\begin{aligned} |A_2(\psi)| &\leq \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\eta(\bar{v}^i) - \eta(\bar{v}_j^i)| |\psi^i - \psi_j^i| dx \\ &\leq C\|\psi\|_{C_0^\alpha} h^\alpha \\ &\quad \times \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(|\bar{v}^i - \bar{v}_j^i| \int_0^1 \frac{1}{\varrho\theta} \left| 1 - \xi \frac{\varpi\theta}{\varrho\theta} \right| \eta_\theta |d\theta + |\bar{v}^i - \bar{v}_j^i| \int_0^1 \left| \frac{1}{\varrho\theta} \eta_\theta \right| d\theta \right) dx, \end{aligned}$$

where $0 < \alpha < 1$, $v_\theta = \bar{v}_j^i + \theta(\bar{v}^i - \bar{v}_j^i)$, $\eta_\theta = \varrho_\theta^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} \frac{\varpi\theta}{e\theta}}$, and

$$(5.10) \quad \varrho_\theta = \bar{v}_j^i + \theta(\bar{v}^i - \bar{v}_j^i) = \bar{v}^i + (1-\theta)(\bar{v}_j^i - \bar{v}^i).$$

By Theorem 4.1 and Lemma 5.1, we have

$$\begin{aligned}
& \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(|\bar{\varrho}^i - \bar{\varrho}_j^i| \int_0^1 \frac{1}{\varrho_\theta} |1 - \xi \frac{\overline{\omega}_\theta}{\varrho_\theta}| \eta_\theta |d\theta \right) dx \\
&= \sum_{i,j} \left(\int_{[x_{j-1}, x_{j+1}] \cap \{\varrho_\theta > 1\}} + \int_{[x_{j-1}, x_{j+1}] \cap \{\varrho_\theta \leq 1\}} \right) \left(|\bar{\varrho}^i - \bar{\varrho}_j^i| \int_0^1 \frac{1}{\varrho_\theta} |1 - \xi \frac{\overline{\omega}_\theta}{\varrho_\theta}| \eta_\theta |d\theta \right) dx \\
&\leq C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\bar{\varrho}^i - \bar{\varrho}_j^i| dx + C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(|\bar{\varrho}^i - \bar{\varrho}_j^i| \int_0^1 \varrho_\theta^{-\frac{1}{2}} (1 + |\ln \varrho_\theta|) d\theta \right) dx \\
&\leq C \left(\sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\bar{\varrho}^i - \bar{\varrho}_j^i)^2 dx \right)^{\frac{1}{2}} \left(\sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} k dx \right)^{\frac{1}{2}} k^{-\frac{1}{2}} \\
&\quad + C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(|\bar{\varrho}^i - \bar{\varrho}_j^i| \int_0^1 \varrho_\theta^{-\frac{3}{4}} d\theta \right) dx \\
&\leq C k^{-\frac{1}{2}} + C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\bar{\varrho}^i - \bar{\varrho}_j^i|^{\frac{1}{4}} dx \leq C k^{-\frac{7}{8}}
\end{aligned}$$

with C depending only on L , T , and η , where we used an estimate similar to (5.4) and the definition of ϱ_θ in (5.10).

By Theorem 4.1, we have

$$|\overline{\omega}^i - \overline{\omega}_j^i| \leq \bar{\varrho}^i (M + |\ln \bar{\varrho}^i|) + \bar{\varrho}_j^i (M + |\ln \bar{\varrho}_j^i|) \leq C \left((\bar{\varrho}^i)^{\frac{3}{4}} + (\bar{\varrho}_j^i)^{\frac{3}{4}} \right).$$

We now divide four cases.

If $0 \leq \bar{\varrho}^i - \bar{\varrho}_j^i \leq \bar{\varrho}_j^i$, then $|\overline{\omega}^i - \overline{\omega}_j^i| \leq C(\bar{\varrho}_j^i)^{\frac{3}{4}}$, that is,

$$(5.11) \quad |\overline{\omega}^i - \overline{\omega}_j^i|^{-1} \geq C^{-1} (\bar{\varrho}_j^i)^{-\frac{3}{4}}.$$

Thus, by (5.10) and (5.11), we have

$$\begin{aligned}
& \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(|\overline{\omega}^i - \overline{\omega}_j^i| \int_0^1 \frac{1}{\varrho_\theta} \eta_\theta |d\theta \right) dx \\
&\leq C \sum_{i,j} \left(\int_{[x_{j-1}, x_{j+1}] \cap \{\varrho_\theta \leq 1\}} + \int_{[x_{j-1}, x_{j+1}] \cap \{\varrho_\theta > 1\}} \right) \left(|\overline{\omega}^i - \overline{\omega}_j^i| \int_0^1 \varrho_\theta^{-\frac{1}{2}} d\theta \right) dx \\
&\leq C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\overline{\omega}^i - \overline{\omega}_j^i| (\bar{\varrho}_j^i)^{-\frac{1}{2}} dx + C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\overline{\omega}^i - \overline{\omega}_j^i| dx \\
&\leq C \left(\sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\overline{\omega}^i - \overline{\omega}_j^i|^{\frac{1}{3}} dx + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\overline{\omega}^i - \overline{\omega}_j^i| dx \right) \\
&\leq C k^{-\frac{5}{6}} + C k^{-\frac{1}{2}} \leq C k^{-\frac{5}{6}}.
\end{aligned}$$

If $\bar{\varrho}_j^i < \bar{\varrho}^i - \bar{\varrho}_j^i$, then $\bar{\varrho}^i \leq 2(\bar{\varrho}^i - \bar{\varrho}_j^i)$ and hence $|\overline{\omega}^i - \overline{\omega}_j^i| \leq C(\bar{\varrho}^i - \bar{\varrho}_j^i)^{\frac{3}{4}}$, that is,

$$(5.12) \quad |\overline{\omega}^i - \overline{\omega}_j^i|^{-1} \geq C^{-1} (\bar{\varrho}^i - \bar{\varrho}_j^i)^{-\frac{3}{4}}.$$

Thus, we have

$$\begin{aligned}
 & \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(|\bar{\varpi}^i - \bar{\varpi}_j^i| \int_0^1 \left| \frac{1}{\varrho_\theta} \eta_\theta \right| d\theta \right) dx \\
 & \leq C \sum_{i,j} \left(\int_{[x_{j-1}, x_{j+1}] \times \{\varrho_\theta \leq 1\}} + \int_{[x_{j-1}, x_{j+1}] \times \{\varrho_\theta > 1\}} \right) \left(|\bar{\varpi}^i - \bar{\varpi}_j^i| \int_0^1 \varrho_\theta^{-\frac{1}{2}} d\theta \right) dx \\
 & \leq C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\bar{\varpi}^i - \bar{\varpi}_j^i| (\bar{\varrho}^i - \bar{\varrho}_j^i)^{-\frac{1}{2}} dx + C \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\bar{\varpi}^i - \bar{\varpi}_j^i| dx \\
 & \leq C \left(\sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\bar{\varpi}^i - \bar{\varpi}_j^i|^{\frac{1}{3}} dx + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} |\bar{\varpi}^i - \bar{\varpi}_j^i| dx \right) \leq Ck^{-\frac{5}{6}},
 \end{aligned}$$

since $\varrho_\theta = \bar{\varrho}_j^i + \theta(\bar{\varrho}^i - \bar{\varrho}_j^i) \geq \theta(\bar{\varrho}^i - \bar{\varrho}_j^i)$, where we used (5.12).

The cases $0 \leq \bar{\varrho}_j^i - \bar{\varrho}^i \leq \bar{\varrho}^i$ and $\bar{\varrho}^i < \bar{\varrho}_j^i - \bar{\varrho}^i$ can be estimated similarly.

Therefore, by (3.1), we obtain

$$\begin{aligned}
 (5.13) \quad |A_2(\psi)| & \leq C \|\psi\|_{C_0^\alpha} h^\alpha k^{-\frac{7}{8}} \leq C \|\psi\|_{C_0^\alpha} h^{\alpha - \frac{7}{8}} (1 + |\ln h|)^{\frac{7}{8}} \\
 & \leq Ch^{\alpha - \frac{8}{9}} \|\psi\|_{C_0^\alpha} \quad \text{for } \frac{8}{9} < \alpha < 1.
 \end{aligned}$$

Furthermore, by (5.5) and (5.6),

$$(5.14) \quad |R(\psi)| \leq C \|\psi\|_\infty.$$

Using (5.7) yields

$$(5.15) \quad |\Sigma(\psi)| \leq C \|\psi\|_\infty.$$

Set $\tilde{\varpi}_\theta = \underline{\varpi} + \theta(\varpi - \underline{\varpi})$. Then

$$\begin{aligned}
 |S_1(\psi)| & \leq \left| \int_0^\infty \int_1^\infty \left(\int_0^1 \eta_\varpi(\underline{v} + \theta(v - \underline{v})) d\theta \right) (\varpi - \underline{\varpi}) \partial_t \psi dx dt \right| \\
 & \quad + \left| \int_0^\infty \int_1^\infty \left(\int_0^1 \left(\frac{1}{\varrho} \eta(\varrho, \tilde{\varpi}_\theta) + \left(\frac{\tilde{\varpi}_\theta}{\varrho} + \xi \right) \eta_\varpi(\varrho, \tilde{\varpi}_\theta) \right) d\theta \right) (\varpi - \underline{\varpi}) \partial_x \psi dx dt \right| \\
 & \leq Ck \left| \int_0^\infty \int_1^\infty \left(\int_0^1 \frac{1}{\varrho} \eta(\varrho, \tilde{\varpi}_\theta) d\theta \right) \varrho \frac{N-1}{x} \partial_t \psi dx dt \right| \\
 & \quad + Ck \left| \int_0^\infty \int_1^\infty \left(\int_0^1 \left(1 + \left(\frac{\tilde{\varpi}_\theta}{\varrho} + \xi \right) \right) \eta(\varrho, \tilde{\varpi}_\theta) d\theta \right) \frac{N-1}{x} \partial_x \psi dx dt \right| \\
 & \leq Ck \int_0^\infty \int_1^\infty (|\partial_t \psi| + |\partial_x \psi|) dx dt \leq Ch \|\psi\|_{H_0^1(\Omega)},
 \end{aligned}$$

where we used the fact that $\eta(\rho, \tilde{\varpi}_\theta) |\ln \varrho|^s$ is bounded for any $s \geq 0$ by (5.4). Also,

$$\begin{aligned}
 |S_2(\psi)| & = \left| \int_0^\infty \int_1^\infty ((\eta(\bar{v}) - \eta(v)) \partial_t \psi + (q(\bar{v}) - q(v)) \partial_x \psi) dx dt \right| \\
 & \leq Ch^{\beta \frac{1-|\xi|}{1-\xi^2}} \int_0^\infty \int_1^\infty |\partial_t \psi| dx dt + Ch^{\frac{\beta}{2}} \int_0^\infty \int_1^\infty |\partial_x \psi| dx dt \\
 & \leq Ch \|\psi\|_{H_0^1(\Omega)}.
 \end{aligned}$$

Therefore,

$$(5.16) \quad |S(\psi)| \leq Ch\|\psi\|_{H_0^1}.$$

Thus, S is compact in $H^{-1}(\Omega)$. Using the above estimates, we can apply Lemma 5.2 to get the H^{-1} compactness. By (5.9), (5.14), and (5.15), we have

$$\|A_1 + R + \Sigma\|_{(C_0)^*} \leq C,$$

where C depends only on Ω and η .

By the embedding theorem, $(C_0(\Omega))^* \hookrightarrow W^{-1,p_0}(\Omega)$ is compact for $1 < p_0 < 2$. Thus

$$A_1 + R + \Sigma \quad \text{is compact in } W^{-1,p_0}(\Omega).$$

By the Sobolev theorem, $W_0^{1,p_1}(\Omega) \subset C_0^\alpha(\Omega)$ for $0 < \alpha < 1 - \frac{2}{p_1}$, which implies

$$|A_2(\psi)| \leq Ch^{\alpha - \frac{8}{9}} \|\psi\|_{W_0^{1,p_1}(\Omega)} \quad \text{for } p_1 > \frac{2}{1 - \alpha}.$$

It follows from the duality that

$$\|A_2\|_{W^{-1,p_2}(\Omega)} \leq Ch^{\alpha - \frac{8}{9}} \rightarrow 0 \quad \text{as } h \rightarrow 0, \quad \text{for } \alpha > \frac{8}{9} \text{ and } 1 < p_2 < \frac{2}{1 + \alpha}.$$

Then A_2 is compact in $W^{-1,p_2}(\Omega)$. Thus, $A + R + \Sigma = A_1 + A_2 + R + \Sigma$ is compact in $W^{-1,p}(\Omega)$, where $1 < p \leq \min(p_0, p_2)$.

Next, from the uniform bound of \bar{v} , and the continuity of η and q , we have $\partial_t \eta(\bar{v}) + \partial_x q(\bar{v}) - S$ is bounded in $W^{-1,\infty}(\Omega)$. Since Ω is bounded, $\partial_t \eta(\bar{v}) + \partial_x q(\bar{v}) - S$ is bounded in $W^{-1,r}(\Omega)$ for $r > 1$. That is, $A + R + \Sigma$ is bounded in $W^{-1,r}(\Omega)$. By Lemma 5.2,

$$A + R + \Sigma \quad \text{is compact in } H_{loc}^{-1}(\Pi).$$

That is, $\partial_t \eta(\bar{v}) + \partial_x q(\bar{v}) - S$ is compact in $H_{loc}^{-1}(\Pi)$. Therefore, $\partial_t \eta(\bar{v}) + \partial_x q(\bar{v})$ is compact in $H_{loc}^{-1}(\Pi)$ for $\xi \in (-1, 1)$.

Combining Theorem 4.1 with Theorem 5.1, we have the following framework for the approximate solutions $\bar{v}_h(x, t)$ defined in Section 3.

THEOREM 5.2. *Suppose that $(\varrho_0(x), \varpi_0(x))$ satisfies the following conditions:*

$$0 \leq \varrho_0 \leq C_0, \quad |\varpi_0(x)| \leq \varrho_0(x)(C_0 + |\ln \varrho_0|)$$

for some $C_0 > 0$. Then the approximate solutions $\bar{v}_h(x, t)$ satisfy the following:

(1) For any given $T > 0$, there exists $M = M(T) > 0$ such that

$$0 \leq \bar{\varrho}_h(x, t) \leq M, \quad |\bar{\varpi}_h(x, t)| \leq \bar{\varrho}_h(x, t)(M + |\ln \bar{\varrho}_h(x, t)|), \quad \text{for } 0 \leq t \leq T.$$

(2) The sequence of entropy dissipation measures $\partial_t \eta(\bar{v}) + \partial_x q(\bar{v})$ is compact in $H_{loc}^{-1}(\Pi)$ for any (η, q) in (1.9) with $\xi \in (-1, 1)$.

6. Convergence and existence. In this section, we prove the convergence of the approximate solutions $\bar{v}_h(x, t)$ to obtain a global entropy solution to the Euler equations with spherically symmetric initial data for isothermal fluids.

THEOREM 6.1. *Suppose that the conditions of Theorem 5.2 are satisfied. Then*

(1) *There exist a bounded measurable vector function $(\varrho(x, t), \varpi(x, t))$ and a subsequence (still denoted) $\bar{v}_h(x, t) = (\bar{\varrho}_h(x, t), \bar{\varpi}_h(x, t))$ such that*

$$(\bar{\varrho}_h(x, t), \bar{\varpi}_h(x, t)) \longrightarrow (\varrho(x, t), \varpi(x, t)) \quad a.e.$$

and, for any given $T > 0$, there exists $M = M(T) > 0$,

$$0 \leq \varrho(x, t) \leq M, \quad |\varpi(x, t)| \leq \varrho(x, t)(M + |\ln \varrho(x, t)|), \quad 0 \leq t \leq T.$$

(2) *The bounded measurable vector function $(\varrho(x, t), \varpi(x, t))$ is an entropy solution of (1.8); i.e., $(\varrho(x, t), m(x, t)) = (\frac{\varrho(x, t)}{x^{N-1}}, \frac{\varpi(x, t)}{x^{N-1}})$ is an entropy solution of (1.5) and (1.7). Furthermore, $(\rho(\vec{x}, t), \vec{m}(\vec{x}, t)) = (\rho(|\vec{x}|, t), m(|\vec{x}|, t) \frac{\vec{x}}{|\vec{x}|})$ is a spherically symmetric entropy solution to the multidimensional Euler equations (1.1) for isothermal fluids.*

To prove this theorem, we need the following lemma, which can be obtained by a straightforward calculation.

LEMMA 6.1. *If $g(x)$ has constant left state g_l with length h_1 , intermediate state g_m with length h_2 , and right state g_r with length h_3 in the interval $[a, a+h]$, $h_1 + h_2 + h_3 \leq h$, then*

$$\int_a^{a+h} |g(x) - \bar{g}|^2 dx \geq \frac{1}{3} h \left(\frac{h_1}{h} \frac{h_3}{h} (g_r - g_l)^2 + \frac{h_2}{h} \frac{h_3}{h} (g_r - g_m)^2 + \frac{h_1}{h} \frac{h_2}{h} (g_m - g_l)^2 \right)$$

Proof of Theorem 6.1. We divide the proof into five steps.

1. From Theorems 2.1 and 5.2, we obtain a convergent subsequence (still labeled) \bar{v}_h such that

$$(6.1) \quad (\bar{\varrho}_h, \bar{\varpi}_h) \longrightarrow (\varrho, \varpi) \quad a.e.$$

and, when $0 \leq t \leq T$,

$$0 \leq \varrho(x, t) \leq M, \quad |\varpi(x, t)| \leq \varrho(x, t)(M + |\ln \varrho(x, t)|)$$

for some $M = M(T) > 0$.

2. For every $\psi \in C_0^\infty(\bar{\Pi})$, there exist $L > 0$ and $T > 0$ such that $\text{supp } \psi \subset [1, L) \times [0, T)$. Then

$$\begin{aligned} & \int_0^\infty \int_1^\infty (\bar{\varrho}_h \partial_t \psi + \bar{\varpi}_h \partial_x \psi) dx dt + \int_1^\infty \bar{\varrho}_h(x, 0) \psi(x, 0) dx \\ &= \sum_{ik \leq T} \sum_{1+(j+1)h \leq L} \int_{x_{j-1}}^{x_{j+1}} (\underline{\varrho}_h^i - \bar{\varrho}_j^i) \psi^i dx + \int_0^\infty \int_1^\infty (\bar{\varrho}_h - \underline{\varrho}_h) \partial_t \psi dx dt \\ & \quad + \int_0^\infty \int_1^\infty (\bar{\varpi}_h - \underline{\varpi}_h) \partial_x \psi dx dt + \int_1^\infty (\bar{\varrho}_h(x, 0) - \underline{\varrho}_h(x, 0)) \psi(x, 0) dx \\ &= I_h + II_h + III_h + IV_h, \end{aligned}$$

where $\psi^i(x) = \psi(x, t_i)$. Notice that

$$\begin{aligned} |I_h| &= \left| \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\varrho_h^i - \bar{\varrho}_j^i) \psi^i dx \right| \\ &= \left| \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\varrho_h^i - \bar{\varrho}_h^i) \psi^i dx + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\bar{\varrho}_h^i - \bar{\varrho}_j^i) \psi^i dx \right| \\ &\leq \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} h^\beta \|\psi\|_\infty dx + \left| \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\bar{\varrho}_h^i - \bar{\varrho}_j^i) \psi_j^i dx \right| \\ &\quad + \left| \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\bar{\varrho}_h^i - \bar{\varrho}_j^i) (\psi^i - \psi_j^i) dx \right| \\ &\leq \|\psi\|_\infty \frac{C(1 + |\ln h|)}{h} h^\beta + 0 + \|\psi\|_{C_0^\alpha} h^\alpha C k^{-\frac{1}{2}} \\ &\leq C \left((1 + |\ln h|) h^{\beta-1} + (1 + |\ln h|)^{\frac{1}{2}} h^{\alpha-\frac{1}{2}} \right) \rightarrow 0 \quad \text{as } h \rightarrow 0, \end{aligned}$$

for $\frac{8}{9} < \alpha < 1$ and $2 \leq \beta \leq 3$, where $\psi_j^i = \psi(x_j, t_i)$. Also,

$$|II_h| = \left| \int_0^\infty \int_1^\infty (\bar{\varrho}_h - \underline{\varrho}_h) \partial_t \psi dx dt \right| \leq Ch^\beta \rightarrow 0,$$

and

$$\begin{aligned} |III_h| &= \left| \int_0^\infty \int_1^\infty (\bar{\varpi}_h - \underline{\varpi}_h) \partial_x \psi dx dt \right| = \left| \sum_{i,j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_{j+1}} V_2(\underline{\varrho}_h)(t - t_i) \partial_x \psi dx dt \right| \\ &\leq k \|V_2(\underline{\varrho}_h)\|_\infty \int_0^\infty \int_1^\infty |\partial_x \psi| dx dt \leq Ch \rightarrow 0 \quad \text{as } h \rightarrow 0. \end{aligned}$$

Since $\underline{\varrho}_h(x, 0) = \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{\varrho}_h(x, 0) dx$ for $x_{j-1} \leq x < x_j$, then

$$\begin{aligned} |IV_h| &= \left| \sum_j \int_{x_{j-1}}^{x_{j+1}} (\bar{\varrho}_h(x, 0) - \underline{\varrho}_h(x, 0)) \psi(x_j, 0) dx \right. \\ &\quad \left. + \sum_j \int_{x_{j-1}}^{x_{j+1}} (\bar{\varrho}_h(x, 0) - \underline{\varrho}_h(x, 0)) (\psi(x, 0) - \psi(x_j, 0)) dx \right| \\ &\leq 0 + h \|\psi\|_{C^1} \sum_{1+(j+1)h \leq L} \int_{x_{j-1}}^{x_{j+1}} |\bar{\varrho}_h(x, 0) - \underline{\varrho}_h(x, 0)| dx \leq Ch \rightarrow 0. \end{aligned}$$

Thus,

$$(6.2) \quad \lim_{h \rightarrow 0} \left(\int_0^\infty \int_1^\infty (\bar{\varrho}_h \partial_t \psi + \bar{\varpi}_h \partial_x \psi) dx dt + \int \bar{\varrho}_h(x, 0) \psi(x, 0) dx \right) = 0.$$

Notice that

$$\left| \int_1^\infty \bar{\varrho}_h(x, 0) \psi(x, 0) dx - \int_1^\infty \varrho_0(x) \psi(x, 0) dx \right| \leq h^\beta \int_1^\infty |\psi(x, 0)| dx \rightarrow 0,$$

that is,

$$(6.3) \quad \lim_{h \rightarrow 0} \int_1^\infty \bar{\varrho}_h(x, 0) \psi(x, 0) dx = \int_1^\infty \varrho_0(x) \psi(x, 0) dx.$$

By (6.1)–(6.3) and the dominated convergence theorem, we get

$$(6.4) \quad \int_0^\infty \int_1^\infty (\varrho \partial_t \psi + \varpi \partial_x \psi) dx dt + \int_1^\infty \varrho_0(x) \psi(x, 0) dx = 0.$$

3. From (6.4), we conclude

$$(6.5) \quad \varpi(x, \cdot) \xrightarrow{*} 0 \quad \text{as } x \rightarrow 1.$$

On the other hand, using the trace theorem in the theory of divergence-measure fields (Theorem 2.2 in Chen-Frid [7]), the function ϖ has a well-defined trace in L^∞ . Combining this with (6.5), we obtain

$$(6.6) \quad \varpi|_{x=1} = 0$$

in the sense of traces introduced by Chen-Frid in [7], which especially implies (1.12).

4. For every $\psi \in C_0^\infty(\bar{\Pi})$ with $\psi(1, t) = 0$, there exist $L > 0$ and $T > 0$ such that $\text{supp } \psi \subset [1, L) \times [0, T)$. By the Green formula,

$$\begin{aligned} & \int_0^\infty \int_1^\infty (\underline{\varpi}_h \partial_t \psi + f_2(\underline{v}_h) \partial_x \psi) dx dt + \int_1^\infty \underline{\varpi}_h(x, 0) \psi(x, 0) dx \\ &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\underline{\varpi}_h^i - \underline{\varpi}_j^i) \psi^i dx, \end{aligned}$$

where $f_2 = \frac{\varpi^2}{\varrho} + \varrho$ and $V_2 = \frac{N-1}{x} \varrho$. Then, since $\overline{\varpi}_h = \varpi_h$, we have

$$\begin{aligned} & \int_0^\infty \int_1^\infty (\overline{\varpi}_h \partial_t \psi + f_2(\overline{v}_h) \partial_x \psi + V_2(\overline{v}_h) \psi) dx dt + \int_1^\infty \overline{\varpi}_h(x, 0) \psi(x, 0) dx \\ &= E(\psi) + R(\psi) + I(\psi) + A(\psi), \end{aligned}$$

where

$$\begin{aligned} E(\psi) &= \int_0^\infty \int_1^\infty ((f_2(\overline{v}_h) - f_2(v_h)) \partial_x \psi + (V_2(\overline{v}_h) - V_2(v_h)) \psi) dx dt, \\ R(\psi) &= \int_0^\infty \int_1^\infty ((f_2(v_h) - f_2(\underline{v}_h)) \partial_x \psi + (V_2(v_h) - V_2(\underline{v}_h)) \psi + (\varpi_h - \underline{\varpi}_h) \partial_t \psi) dx dt, \\ I(\psi) &= \int_1^\infty (\overline{\varpi}_h(x, 0) - \underline{\varpi}_h(x, 0)) \psi(x, 0) dx, \\ A(\psi) &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\underline{\varpi}_h^i - \underline{\varpi}_j^i) \psi^i dx + \int_0^\infty \int_1^\infty V_2(\underline{v}_h) \psi dx dt. \end{aligned}$$

Then

$$\begin{aligned} |E(\psi)| &\leq \iiint_{\Omega \cap \{\varrho_h \leq h^\beta\}} \left(\left(\left| \frac{\varpi_h^2}{\varrho_h} - \frac{\varpi_h^2}{\varrho_h} \right| + |\overline{\varrho}_h - \varrho_h| \right) |\partial_x \psi| + \frac{N-1}{x} |\overline{\varrho}_h - \varrho_h| |\psi| \right) dx dt \\ &\leq C \iint_{\Omega \cap \{\varrho_h \leq h^\beta\}} (\overline{\varrho}_h (1 + |\ln \overline{\varrho}_h|)^2 + h^\beta) dx dt \\ &\leq Ch^\beta (1 + |\ln h|)^2 \rightarrow 0 \quad \text{as } h \rightarrow 0, \end{aligned}$$

$$\begin{aligned}
|R(\psi)| &\leq \sum_{i,j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_{j+1}} \left(k|V_2(\underline{v}_h)| |\partial_t \psi| + \left| \frac{\overline{\varpi}_h + \underline{\varpi}_h}{\varrho_h} (\overline{\varpi}_h - \underline{\varpi}_h) \partial_x \psi \right| \right) dx dt \\
&\quad + \sum_{i,j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_{j+1}} \left(\frac{N-1}{x} |(\varrho_h - \underline{\varrho}_h) \psi| \right) dx dt \\
&\leq C \sum_{ik \leq T} \sum_{1+(j+1)h \leq L} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_{j+1}} \left(k|V_2(\underline{v}_h)| + k|\overline{\varpi}_h + \underline{\varpi}_h| \frac{N-1}{x} \right) dx dt \\
&\leq Ch \longrightarrow 0,
\end{aligned}$$

$$\begin{aligned}
|I(\psi)| &= \left| \sum_j \int_{x_{j-1}}^{x_{j+1}} (\overline{\varpi}_h - \underline{\varpi}_h)(x, 0) \psi(x_j, 0) dx \right. \\
&\quad \left. + \sum_j \int_{x_{j-1}}^{x_{j+1}} (\overline{\varpi}_h - \underline{\varpi}_h)(x, 0) (\psi(x, 0) - \psi(x_j, 0)) dx \right| \\
&\leq 0 + Ch \longrightarrow 0 \quad \text{as } h \rightarrow 0,
\end{aligned}$$

$$\begin{aligned}
A(\psi) &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\overline{\varpi}_h^i - \underline{\varpi}_j^i) (\psi^i - \psi_j^i) dx + \sum_{i,j} \int_{t_{i-1}}^{t_i} \int_{x_{j-1}}^{x_{j+1}} V_2(\underline{v}_h) (\psi - \psi^i) dx dt \\
&\quad + \sum_{i,j} \int_{t_{i-1}}^{t_i} \int_{x_{j-1}}^{x_{j+1}} (V_2(\underline{v}_h) - V_2(\underline{v}_h^i)) \psi^i dx dt \\
&= A_1(\psi) + A_2(\psi) + A_3(\psi).
\end{aligned}$$

By Lemma 5.1, we have

$$\begin{aligned}
|A_1(\psi)| &\leq Ch \sum_{ik \leq T} \sum_{1+(j+1)h \leq L} \int_{x_{j-1}}^{x_{j+1}} |\overline{\varpi}_h^i - \underline{\varpi}_j^i| dx \leq Ch k^{-\frac{1}{2}} \\
&\leq Ch^{\frac{1}{2}} (1 + |\ln h|)^{\frac{1}{2}} \longrightarrow 0 \quad \text{as } h \rightarrow 0, \\
|A_2(\psi)| &\leq \sum_{i,j} \int_{t_{i-1}}^{t_i} \int_{x_{j-1}}^{x_{j+1}} V_2(\underline{v}_h) \frac{|\psi(x, t) - \psi^i|}{|t - t_i|} k dx dt \leq Ch \longrightarrow 0 \quad \text{as } h \rightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
|A_3(\psi)| &\leq C \|\psi\|_\infty \sum_{i,j} \int_{t_{i-1}}^{t_i} \int_{x_{j-1}}^{x_{j+1}} |\varrho_h - \varrho_h^i| dx dt \\
&\leq C \left(\sum_{i,j} \int_{t_{i-1}}^{t_i} \int_{x_{j-1}}^{x_{j+1}} |\varrho_h - \varrho_h^i|^2 dx dt \right)^{1/2}.
\end{aligned}$$

In the rectangle $[x_{j-1}, x_{j+1}] \times [t_{i-1}, t_i]$, we solve the Riemann problem, $i + j$ even, with ϱ_l , ϱ_m , and ϱ_r as the left state, intermediate state, and right state of $\varrho(x, t)$ in the Riemann solution, respectively. Since we choose $k = \frac{h}{10(1+|\ln h|)}$ and $\lambda = \frac{\overline{\varpi}}{\varrho} \pm 1 \leq M(T) + \beta |\ln h| + 1$, then, when $h \leq h_0(T)$, the ratio of the interval of left state and $l = 2h$, and the interval of right state and $l = 2h$ are both bigger than $\frac{1}{3}$. For any given $\delta > 0$, there are two cases.

Case 1. The ratio of length of the interval of intermediate constant state and $l = 2h$ is smaller than δ and

(i) $\sigma_1 < 0, \sigma_2 > 0$, where σ_1 and σ_2 are the speeds of 1-shock and 2-shock waves, respectively. We have

$$\int_{x_{j-1}}^{x_{j+1}} |\varrho_h - \varrho_h^i|^2 dx \leq C\delta h;$$

(ii) Otherwise, by Lemma 6.1,

$$\begin{aligned} \int_{x_{j-1}}^{x_{j+1}} |\varrho_h - \varrho_h^i|^2 dx &\leq 2h (\delta(\varrho_l - \varrho_m)^2 + \delta(\varrho_r - \varrho_m)^2 + (\varrho_l - \varrho_r)^2) \\ &\leq C \left(\delta h + \int_{x_{j-1}}^{x_{j+1}} (\varrho_h^i - \varrho_j^i)^2 dx \right). \end{aligned}$$

Therefore, we have

$$\sum_{i,j} \int_{t_{i-1}}^{t_i} \int_{x_{j-1}}^{x_{j+1}} |\varrho_h - \varrho_h^i|^2 dx dt \leq C\delta + Ck \leq C\delta + Ch.$$

Case 2. The ratio of length of the interval of intermediate constant state and $l = 2h$ is bigger than δ . By Lemma 6.1,

$$\begin{aligned} \sum_{i,j} \int_{t_{i-1}}^{t_i} \int_{x_{j-1}}^{x_{j+1}} |\varrho_h - \varrho_h^i|^2 dx dt &\leq C \sum_{i,j} k \int_{x_{j-1}}^{x_{j+1}} \sum |\epsilon(\varrho_h^i)|^2 dx \\ &\leq C \sum_{i,j} h \int_{x_{j-1}}^{x_{j+1}} \sum |\epsilon(\varrho_h^i)|^2 dx \\ &\leq C\delta^{-1}h \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} (\varrho_h^i - \varrho_j^i)^2 dx \leq C\delta^{-1}h, \end{aligned}$$

where $\sum |\epsilon(\varrho_h^i)|$ denotes the total jump strengths of $\varrho_h(x, t)$ across shock waves in $[x_{j-1}, x_{j+1}] \times [t_{i-1}, t_i]$. Hence,

$$|A_3(\psi)| \leq C(\delta^{-1}h + \delta)^{1/2} \leq C\delta^{1/2} \quad \text{as } h \rightarrow 0.$$

Thus, for any small δ , $A(\psi) \leq C\delta^{1/2}$ as h tends to zero.

Furthermore, using the dominated convergence theorem and

$$\int_1^\infty (\overline{\varpi}_h(x, 0) - \varpi_0(x))\psi(x, 0)dx = 0,$$

we conclude

$$(6.7) \quad \int_0^\infty \int_1^\infty \left(\overline{\varpi} \partial_t \psi + \left(\frac{\overline{\varpi}^2}{\varrho} + \varrho \right) \partial_x \psi + \frac{N-1}{x} \varrho \psi \right) dx dt + \int_1^\infty \varpi_0(x)\psi(x, 0)dx = 0.$$

5. Now we show the entropy inequality. For every nonnegative function $\psi \in C_0^\infty(\overline{\Pi})$ with $\psi(1, t) = 0$, there exist $L > 0$ and $T > 0$ such that $\text{supp } \psi \subset [1, L] \times [0, T]$. For entropy pair (η, q) in (2.7) with $\xi \in (-1, 1)$, we consider the identity:

$$\int_0^\infty \int_1^\infty (\eta(\overline{v}_h) \partial_t \psi + q(\overline{v}_h) \partial_x \psi) dx dt = A(\psi) + B(\psi) + R(\psi) + \Sigma(\psi) + S(\psi),$$

where $A(\psi)$, $R(\psi)$, $\Sigma(\psi)$, and $S(\psi)$ are similar to $A(\psi)$, $R(\psi)$, $\Sigma(\psi)$, and $S(\psi)$ in Step 4, respectively, and

$$B(\psi) = - \int_1^\infty \eta(\underline{v}_h(x, 0))\psi(x, 0)dx.$$

Since η is a convex entropy for $\xi \in (-1, 1)$ and $\psi \geq 0$, then $\Sigma(\psi) \geq 0$ and

$$\begin{aligned} A(\psi) &= \sum_{i,j} \psi_j^i \int_{x_{j-1}}^{x_{j+1}} \left(\eta(\bar{v}_h^i) - \eta(\bar{v}_j^i) \right) dx + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(\eta(\bar{v}_h^i) - \eta(\bar{v}_j^i) \right) (\psi^i - \psi_j^i) dx \\ &\geq \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(\eta(\bar{v}_h^i) - \eta(\bar{v}_j^i) \right) (\psi^i - \psi_j^i) dx \\ &\geq -Ch^{\alpha-\frac{8}{9}} \quad \text{for } \frac{8}{9} < \alpha < 1. \end{aligned}$$

As the argument in Section 5, we have

$$\begin{aligned} S(\psi) &\geq -Ch, \\ R(\psi) &= \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(\eta(v_h^i) - \eta(\bar{v}_h^i) \right) \psi(x, t^i) dx + \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(\eta(\underline{v}_h^i) - \eta(v_h^i) \right) \psi(x, t^i) dx \\ &\geq -C(1 + |\ln h|)h^{\beta\frac{1-|\xi|}{1-\xi^2}-1} \\ &\quad -k \sum_{i,j} \int_{x_{j-1}}^{x_{j+1}} \left(\int_0^1 \partial_\varpi \eta(v_h^i + \theta(\underline{v}_h^i - v_h^i)) d\theta \right) V_2(\underline{v}_h^i) \psi^i dx. \end{aligned}$$

Therefore,

$$\begin{aligned} (6.8) \quad &\int_0^\infty \int_1^\infty \left(\eta(\bar{v}_h) \partial_t \psi + q(\bar{v}_h) \partial_x \psi \right) dx dt \\ &\quad + \sum_{i,j} k \int_{x_{j-1}}^{x_{j+1}} \left(\int_0^1 \partial_\varpi \eta(v_h^i + \theta(\underline{v}_h^i - v_h^i)) d\theta \right) V_2(v_h^i) \psi^i dx \\ &\quad + \int_1^\infty \eta(\underline{v}_h(x, 0)) \psi(x, 0) dx \\ &\geq -Ch^{\alpha-\frac{8}{9}} - Ch - C(1 + |\ln h|)h^{\beta\frac{1-|\xi|}{1-\xi^2}-1} \quad \text{for } \beta\frac{1-|\xi|}{1-\xi^2} > 1. \end{aligned}$$

Note that

$$\underline{v}_h(x, 0) \rightarrow v_0(x), \quad \bar{v}_h(x, t) \rightarrow v(x, t), \quad v_h(x, t) \rightarrow v(x, t) \quad \text{a.e. as } h \rightarrow 0.$$

Then we let $h \rightarrow 0$ in (6.8) to conclude the following entropy inequality:

$$\begin{aligned} (6.9) \quad &\int_0^\infty \int_1^\infty \left(\eta(v) \partial_t \psi + q(v) \partial_x \psi + \partial_\varpi \eta(v) \frac{N-1}{x} \varrho \psi \right) dx dt \\ &\quad + \int_1^\infty \eta(v_0(x)) \psi(x, 0) dx \geq 0. \end{aligned}$$

It is easy to check that (6.4), (6.7), and (6.9) imply (1.10), (1.11), and (1.13). This completes the proof.

7. Global entropy solutions to the Euler-Poisson equations. In this section, we develop the shock capturing scheme in Sections 3–6 to construct spherically symmetric entropy solutions to the Euler-Poisson equations. For concreteness and from physical motivation, we focus on the domain $\{(\vec{x}, t) \in \mathbf{R}^N \times \mathbf{R}_+ : 1 \leq |\vec{x}| \leq 2\}$ in this section; the analysis extends to any domain $\{(\vec{x}, t) \in \mathbf{R}^N \times \mathbf{R}_+ : 0 < \epsilon \leq |\vec{x}| \leq L \leq \infty\}$.

First, the solution of the Poisson equation is given by

$$(7.1) \quad \phi(x, t) = \int_1^x \frac{1}{s^{N-1}} \left(\int_1^s y^{N-1} (\rho(y, t) - D(y)) dy + \Phi(t) \right) ds + \phi(1, t),$$

where

$$(7.2) \quad \Phi(t) = \frac{\phi(2, t) - \phi(1, t) - \int_1^2 \frac{1}{s^{N-1}} \left(\int_1^s y^{N-1} (\rho(y, t) - D(y)) dy \right) ds}{\int_1^2 \frac{1}{s^{N-1}} ds},$$

$\phi(1, t)$ and $\phi(2, t)$ stand for the applied bias at $x = 1$ and $x = 2$. Then system (1.6) for isothermal fluids becomes

$$(7.3) \quad \begin{aligned} \partial_t \rho + \partial_x m &= -\frac{N-1}{x} m, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + \rho \right) &= -\frac{N-1}{x} \frac{m^2}{\rho} - \frac{m}{x} \\ &\quad + \frac{1}{x^{N-1}} \left(\int_1^x y^{N-1} (\rho(y, t) - D(y)) dy + \Phi(t) \right) \rho, \end{aligned}$$

with the following initial-boundary conditions:

$$(7.4) \quad \begin{cases} (\rho, m)|_{t=0} = (\rho_0(x), m_0(x)), & 1 \leq x \leq 2, \\ m|_{x=\ell} = 0, & \ell = 1, 2. \end{cases}$$

As for the Euler problem, we set

$$\varrho = x^{N-1} \rho, \quad \varpi = x^{N-1} m.$$

Then (7.3) becomes

$$(7.5) \quad \begin{aligned} \partial_t \varrho + \partial_x \varpi &= 0, \\ \partial_t \varpi + \partial_x \left(\frac{\varpi^2}{\varrho} + \varrho \right) &= \frac{N-1}{x} \varrho - \frac{\varpi}{x} + \frac{1}{x^{N-1}} \left(\int_1^x (\varrho(y, t) - y^{N-1} D(y)) dy + \Phi(t) \right) \varrho. \end{aligned}$$

THEOREM 7.1. *Suppose that the initial-boundary data satisfies the conditions:*

$$0 \leq \rho_0(x) \leq \frac{C_0}{x^{N-1}}, \quad |m_0(x)| \leq \rho_0(x) (C_0 + |\ln(x^{N-1} \rho_0(x))|),$$

for some positive C_0 , and $\phi(1, t), \phi(2, t) \in L^\infty([0, \infty))$. Let $D(x) \in L^1([1, 2])$. Then there exists a global entropy solution $(\rho(x, t), m(x, t))$ of the initial-boundary value problem (7.3)–(7.4) satisfying that, for any $T \in (0, \infty)$,

$$(7.6) \quad 0 \leq \rho(x, t) \leq \frac{\tilde{M}}{x^{N-1}}, \quad |m(x, t)| \leq \rho(x, t) (\tilde{M} + |\ln(x^{N-1} \rho(x, t))|), \quad 0 \leq t \leq T,$$

for some $\tilde{M} = \tilde{M}(T) > 0$.

Proof. 1. We first develop the shock capturing scheme in Sections 3–6 to construct the approximate solutions of (7.3)–(7.4).

Let $h > 0$ be the space mesh length so that there exists an integer J such that $Jh = 1$. We partition the interval $[1, 2]$ into cells with the j^{th} cell centered at $x_j = 1 + jh, j = 0, 1, 2, \dots, J$. To ensure the Courant-Friedrichs-Lewy (CFL) stability condition, we choose the time mesh length

$$(7.7) \quad k = \frac{h}{10(1 + |\ln h|)}.$$

We then set $t_i = ik$.

As for the Euler equations, to ensure the CFL condition, we employ the cut-off technique so that the approximate density functions stay away from vacuum by h^β , starting from the initial data:

$$\bar{\varrho}_0 = \max(\varrho_0, h^\beta), \quad \bar{\varpi}_0 = \varpi_0.$$

Let $\bar{v}_0 = (\bar{\varrho}_0, \bar{\varpi}_0)$. We define

$$\underline{v}(x, 0 + 0) = \begin{cases} \bar{v}_j^0 & \text{for } x_{j-1} \leq x \leq x_{j+1}, j \geq 4 \text{ even,} \\ \bar{v}_2^0 & \text{for } 1 \leq x \leq x_3, \\ \bar{v}_{H_0-1}^0 & \text{for } x_{H_0-2} \leq x \leq 2, \end{cases}$$

where \bar{v}_j^0 is the average value of the function \bar{v}_0 in each cell:

$$\begin{aligned} \bar{v}_j^0 &= \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{v}_0 dx, \quad j \text{ is even and } j \geq 4, \\ \bar{v}_2^0 &= \frac{1}{3h} \int_1^{1+3h} \bar{v}_0 dx, \quad \bar{v}_{H_0-1}^0 = \frac{1}{1 - (H_0 - 2)h} \int_{1+(H_0-2)h}^2 \bar{v}_0 dx, \end{aligned}$$

where

$$H_i = \begin{cases} J & \text{if } J + i \text{ is odd,} \\ J - 1 & \text{if } J + i \text{ is even.} \end{cases}$$

Then we solve the Riemann problem (2.5) in the region $R_j^1 \equiv \{(x, t) : x_j \leq x \leq x_{j+2}, 0 \leq t < t_1\}, j \geq 2$, with Riemann data:

$$\underline{v}_h|_{t=0} = \begin{cases} \bar{v}_j^0, & x < x_{j+1}, \\ \bar{v}_{j+2}^0, & x > x_{j+1}, \quad j = 2, 4, \dots \end{cases}$$

and the lateral Riemann problem (2.6) in $\{(x, t) : 1 \leq x \leq x_2, 0 \leq t < t_1\}$ with Riemann data:

$$\underline{v}_h|_{t=0} = \bar{v}_2^0, \quad 1 < x \leq x_2; \quad \varpi|_{x=1} = 0,$$

and the lateral Riemann problem in $\{(x, t) : x_{H_0-1} \leq x \leq 2, 0 \leq t < t_1\}$ with Riemann data:

$$\underline{v}_h|_{t=0} = \bar{v}_{H_0-1}^0, \quad x_{H_0-1} \leq x < 2; \quad \varpi|_{x=2} = 0,$$

to obtain $\underline{v}_h(x, t), 0 \leq t < t_1$.

We set

$$v_h(x, t) = \underline{v}_h(x, t) + \tilde{V}(\underline{v}_h(x, t))t, \quad 0 < t < t_1,$$

where $\tilde{V} = (V_1, \tilde{V}_2) = (0, \frac{N-1}{x}\varrho + \frac{1}{x^{N-1}}(\int_1^x(\varrho(y,t) - y^{N-1}D(y))dy + \Phi(t))\varrho - \frac{\varpi}{\tau})$.

Suppose that we have defined approximate solutions $v_h(x, t)$ for $0 \leq t < t_i$. Then we set

$$(7.8) \quad \bar{\varrho}_h(x, t) = \max(\varrho_h(x, t), h^\beta), \quad \bar{\varpi}_h(x, t) = \varpi_h(x, t), \quad \text{for } t_{i-1} \leq t < t_i.$$

We define $\underline{v}(t_i + 0)$ as follows: When $i \geq 1$ is odd,

$$\underline{v}(t_i + 0) = \begin{cases} \bar{v}_j^i & \text{for } x_{j-1} \leq x \leq x_{j+1}, j \geq 3 \text{ odd,} \\ \bar{v}_1^i & \text{for } 1 \leq x \leq x_2, \\ \bar{v}_{H_i-1}^i & \text{for } x_{H_i-2} \leq x \leq 2, \end{cases}$$

and, when $i \geq 2$ is even,

$$\underline{v}(t_i + 0) = \begin{cases} \bar{v}_j^i & \text{for } x_{j-1} \leq x \leq x_{j+1}, j \geq 4 \text{ even,} \\ \bar{v}_2^i & \text{for } 1 \leq x \leq x_3, \\ \bar{v}_{H_i-1}^i & \text{for } x_{H_i-2} \leq x \leq 2, \end{cases}$$

where \bar{v}_j^i is the average value of the function $\bar{v}_h(x, t_i - 0)$ in each cell as follows:

When i is odd,

$$\bar{v}_j^i = \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{v}_h(x, t_i - 0) dx, \quad j \geq 3 \text{ odd,}$$

$$\bar{v}_1^i = \frac{1}{2h} \int_1^{1+2h} \bar{v}_h(x, t_i - 0) dx, \quad \bar{v}_{H_i-1}^i = \frac{1}{1 - (H_i - 2)h} \int_{1+(H_i-2)h}^2 \bar{v}_h(x, t_i - 0) dx,$$

and, when i is even,

$$\bar{v}_j^i = \frac{1}{2h} \int_{x_{j-1}}^{x_{j+1}} \bar{v}_h(x, t_i - 0) dx, \quad j \geq 4 \text{ even,}$$

$$\bar{v}_2^i = \frac{1}{3h} \int_1^{1+3h} \bar{v}_h(x, t_i - 0) dx, \quad \bar{v}_{H_i-1}^i = \frac{1}{1 - (H_i - 2)h} \int_{1+(H_i-2)h}^2 \bar{v}_h(x, t_i - 0) dx.$$

Then we solve the Riemann problem (2.5) in the region $R_j^i \equiv \{(x, t) : x_j \leq x \leq x_{j+2}, t_i \leq t < t_{i+1}\}$ with initial data:

$$(7.9) \quad \underline{v}_h |_{t=t_i} = \begin{cases} \bar{v}_j^i, & x < x_{j+1}, \\ \bar{v}_{j+2}^i, & x > x_{j+1}, \end{cases}$$

for $j \geq 2$ even when i is even and for $j \geq 1$ odd when i is odd, and the lateral Riemann problem (2.6) in $\{(x, t) : 1 \leq x \leq x_2, t_i \leq t < t_{i+1}\}$ when i is even with the Riemann data:

$$(7.10) \quad \underline{v}_h |_{t=t_i} = \bar{v}_2^i, \quad 1 < x \leq x_2; \quad \varpi|_{x=1} = 0,$$

and in $\{(x, t) : 1 \leq x \leq x_1, t_i \leq t < t_{i+1}\}$ when i is odd with the Riemann data:

$$(7.11) \quad \underline{v}_h |_{t=t_i} = \bar{v}_1^i, \quad 1 < x \leq x_1; \quad \varpi|_{x=1} = 0,$$

and for the lateral Riemann problem in $\{(x, t) : x_{H_i-1} \leq x \leq 2, t_i \leq t < t_{i+1}\}$ with the Riemann data:

$$(7.12) \quad \underline{v}_h |_{t=t_i} = \bar{v}_{H_i-1}^i, \quad x_{H_i-1} \leq x < 2; \quad \varpi|_{x=2} = 0,$$

to obtain $\underline{v}_h(x, t), t_i \leq t < t_{i+1}$.

Then we set

$$(7.13) \quad v_h(x, t) = \underline{v}_h(x, t) + \tilde{V}(\underline{v}_h(x, t))(t - t_i), \quad t_i \leq t < t_{i+1}.$$

Then $\bar{v}_h = (\bar{\varrho}_h(x, t), \bar{\varpi}_h(x, t))$ with

$$\bar{\varrho}_h(x, t) = \max(\varrho_h(x, t), h^\beta), \quad \bar{\varpi}_h(x, t) = \varpi_h(x, t)$$

are the approximate solutions.

2. Notice that \tilde{V} involves the nonlocal term as $\int_1^x \varrho(y, t) dy$.

First, the conservation law of mass in (7.5) and $\varpi|_{x=\ell} = 0, \ell = 1, 2$, implies

$$(7.14) \quad \int_1^2 \underline{\varrho}_h(x, t_i + 0) dx = \int_1^2 \underline{\varrho}_h(x, t_{i+1} - 0) dx = \int_1^2 \varrho_h(x, t_{i+1} - 0) dx.$$

Since $\bar{\varrho}_h(x, t)$ are the cut-off functions of $\varrho_h(x, t)$, we have

$$\int_1^2 \bar{\varrho}_h(x, t_{i+1} - 0) dx \leq \int_1^2 \varrho_h(x, t_{i+1} - 0) dx + h^\beta.$$

For the initial data, we have

$$\int_1^2 \underline{\varrho}_0 dx = \int_1^2 \bar{\varrho}_0 dx \leq \int_1^2 \varrho_0 dx + h^\beta.$$

Therefore,

$$(7.15) \quad \int_1^2 \varrho_h(x, t_{i+1} + 0) dx = \int_1^2 \bar{\varrho}_h(x, t_{i+1} - 0) dx \leq \int_1^2 \varrho_0 dx + \frac{T}{k} h^\beta \leq \int_1^2 \varrho_0 dx + C.$$

LEMMA 7.1. *Suppose that $(\varrho_0(x), \varpi_0(x))$ satisfies the following conditions:*

$$(7.16) \quad 0 \leq \varrho_0 \leq C_0, \quad |\varpi_0(x)| \leq \varrho_0(x)(C_0 + |\ln \varrho_0|),$$

then, for any given $T > 0$, $(\bar{\varrho}_h, \bar{\varpi}_h)$ are uniformly bounded in the region $\{(x, t) : 1 \leq x \leq 2, 0 \leq t \leq T\}$; that is, there exists $\tilde{M} = \tilde{M}(T) > 0$ such that

$$0 \leq \bar{\varrho}_h(x, t) \leq \tilde{M}, \quad |\bar{\varpi}_h(x, t)| \leq \bar{\varrho}_h(x, t)(\tilde{M} + |\ln \bar{\varrho}_h(x, t)|).$$

This can be seen as follows. Let $w(x, t), \underline{w}(x, t), \bar{w}(x, t), z(x, t), \underline{z}(x, t)$, and $\bar{z}(x, t)$ be defined as in (4.3). For $t_i \leq t < t_{i+1}$, (7.13), (7.15), and Lemma 2.1 imply that

$$(7.17) \quad \begin{aligned} w_h(x, t) &= \underline{w}_h(x, t) \left(1 - \frac{t - t_i}{2\tau}\right) - \underline{z}_h(x, t) \frac{(t - t_i)}{2\tau} \\ &\quad + \left(\frac{N-1}{x} + \frac{1}{x^{N-1}} \int_1^x \varrho(y, t) dy - \frac{1}{x^{N-1}} \int_1^x y^{N-1} D(y) dy + \frac{1}{x^{N-1}} \Phi(t) \right) \\ &\quad \times (t - t_i) \\ &\leq \sup_x \underline{w}_h(x, t_i + 0) \left(1 - \frac{t - t_i}{2\tau}\right) - \inf_x \underline{z}_h(x, t_i + 0) \frac{t - t_i}{2\tau} + Ck, \\ z_h(x, t) &\geq \inf_x \underline{z}_h(x, t_i + 0) \left(1 - \frac{t - t_i}{2\tau}\right) - \sup_x \underline{w}_h(x, t_i + 0) \frac{t - t_i}{2\tau} - Ck. \end{aligned}$$

The initial assumption (7.16) implies that there exists $\alpha_0 > 0$ such that

$$\sup_x w(\varrho_0(x), \varpi_0(x)) \leq \alpha_0, \quad \inf_x z(\varrho_0(x), \varpi_0(x)) \geq -\alpha_0.$$

It is easy to check that, for the Riemann invariants corresponding to the cut off functions, we also have

$$w(\bar{\varrho}_0(x), \bar{\varpi}_0(x)) \leq \alpha_0, \quad z(\bar{\varrho}_0(x), \bar{\varpi}_0(x)) \geq -\alpha_0.$$

For $0 \leq t < t_1$, by the properties of Riemann invariants in Lemmas 2.1 and 2.2, we have

$$w(\underline{\varrho}_h(x, t), \underline{\varpi}_h(x, t)) \leq \alpha_0, \quad z(\underline{\varrho}_h(x, t), \underline{\varpi}_h(x, t)) \geq -\alpha_0.$$

By (7.17), we get

$$w_h(x, t) = w(\varrho_h, \varpi_h) \leq \alpha_0 + Ck, \quad z_h(x, t) = z(\varrho_h, \varpi_h) \geq -\alpha_0 - Ck.$$

Then the same procedure yields that, for $0 \leq t < T$,

$$\bar{w}_h = w(\bar{\varrho}_h, \bar{\varpi}_h) \leq \alpha_0 + CT, \quad \bar{z}_h = z(\bar{\varrho}_h, \bar{\varpi}_h) \geq -\alpha_0 - CT,$$

which implies that there exists $\tilde{M} = \tilde{M}(T) > 0$ such that

$$(7.18) \quad 0 \leq \bar{\varrho}_h(x, t) \leq \tilde{M}, \quad |\bar{\varpi}_h(x, t)| \leq \bar{\varrho}_h(x, t)(\tilde{M} + |\ln \bar{\varrho}_h(x, t)|), \quad 0 \leq t \leq T.$$

Again, with the uniform bounds for the approximate solutions in $\{(x, t) : 1 \leq x \leq 2, 0 \leq t \leq T\}$, there exists $h_0 = h_0(T) > 0$ such that, when $h \leq h_0$, the CFL condition holds, which implies that the approximate solutions are well-defined in $\{(x, t) : 1 \leq x \leq 2, 0 \leq t \leq T\}$.

The rest part of the proof is similar to the proof of Theorem 1.1 so that we omit it here.

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REFERENCES

- [1] BLOTEKJAER, K., *Transport equations for electrons in two-valley semiconductors*, IEEE Transactions on Electron Devices, ED-17, 1 (1970), pp. 38–47.
- [2] BRESSAN, A., *Hyperbolic Systems of Conservation Laws: The One Dimensional Cauchy Problem*, Oxford University Press: Oxford, 2000.
- [3] CHEN, D. P., EISENBERG, R. S., JEROME, J. W., AND C. W. SHU, *A hydrodynamic model of temperature change in open ionic channels*, Biophys. J., 69:6 (1995), pp. 2304–2322.
- [4] CHEN, G.-Q., *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics III*, Acta Math. Sci., 6 (1986), pp. 75–120 (in English); 8 (1988), pp. 243–276 (in Chinese).
- [5] CHEN, G.-Q., *Compactness methods and nonlinear hyperbolic conservation laws*, AMS/IP Studies in Advanced Mathematics, 15 (2000), pp. 34–75.
- [6] CHEN, G.-Q., *Remarks on spherically symmetric solutions of the compressible Euler equations*, Proc. Royal Soc. Edinburgh, 127A (1997), pp. 243–259.
- [7] CHEN, G.-Q. AND H. FRID, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Rational Mech. Anal., 147 (1999), pp. 89–118.

- [8] CHEN, G.-Q. AND J. GLIMM, *Global solutions to the compressible Euler equations with geometrical structure*, Commun. Math. Phys., 180 (1996), pp. 153–193.
- [9] CHEN, G.-Q. AND P. LEFLOCH, *Compressible Euler equations with general pressure law*, Arch. Ration. Mech. Anal., 153 (2000), pp. 221–259; *Existence theory for the isentropic Euler equations*, Arch. Rational Mech. Anal., 166 (2003), pp. 81–98..
- [10] CHEN, G.-Q. AND D. WANG, *Convergence of shock capturing schemes for the compressible Euler-Poisson equations*, Comm. Math. Phys., 179 (1996), pp. 333–364.
- [11] COLELLA, P. AND P. WOODWARD, *The piecewise parabolic method (PPM) for gas dynamical simulations*, J. Comp. Phys., 54 (1984), pp. 174–201.
- [12] DAFERMOS, C. M., *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag: Berlin, 2000.
- [13] DAFERMOS, C. M., *Polygonal approximations of solutions of the initial value problem for a conservation laws*, J. Math. Anal. Appl., 38 (1972), pp. 33–41.
- [14] DING, X., CHEN, G.-Q., AND P. LUO, *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics*: I. Acta Math. Sci., 5 (1985), pp. 415–432 (in English); 7 (1987), pp. 467–480 (in Chinese); II. Acta Math. Sci., 5 (1985), pp. 433–472 (in English); 8 (1988), pp. 61–94 (in Chinese).
- [15] DING, X., CHEN, G.-Q., AND P. LUO, *Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics*, Comm. Math. Phys., 121 (1989), pp. 63–84.
- [16] DIPERNA, R. J., *Convergence of viscosity method for isentropic gas dynamics*, Commun. Math. Phys., 91 (1983), pp. 1–30.
- [17] FATEMI, E., JEROME, J., AND S. OSHER, *Solution of the hydrodynamic device model using high-order nonoscillatory shock capturing algorithms*, IEEE Transactions on Computer-Aided Design, 10 (1991), pp. 232–243.
- [18] GARDNER, C. L., *Numerical simulation of a steady-state electron shock wave in a submicron semiconductor device*, IEEE Transactions on Electron Devices, 38 (1991), pp. 392–398.
- [19] GLIMM, J., *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., 18 (1965), pp. 697–715.
- [20] GODUNOV, S. K., *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*, Mat. Sb., 47:89 (1959), pp. 271–360.
- [21] HARTEN, A., *On a class of high resolution total-variation-stable finite-difference schemes*, SIAM J. Numer. Anal., 21 (1984), pp. 1–23.
- [22] HARTEN, A., LAX, P. D., AND B. VAN LEER, *On upstream differencing and Godunov-type schemes for hyperbolic conservation laws*, SIAM Rev., 25 (1983), pp. 35–61.
- [23] HARTEN, A. AND S. OSHER, *Uniformly high order accurate non-oscillatory schemes I*, SIAM J. Numer. Anal., 24 (1987), pp. 279–309.
- [24] HARTEN, A., ENGQUIST, B., OSHER, S., AND S. CHAKRAVARTHY, *Uniformly high order accurate non-oscillatory schemes III*, J. Comp. Phys., 71 (1987), pp. 231–303.
- [25] HOLDEN, H. AND N. RISEBRO, *Front Tracking for Hyperbolic Conservation Laws*, Springer-Verlag: New York, 2002.
- [26] HUANG, F.-M. AND Z. WANG, *Convergence of viscosity for isothermal gas dynamics*, SIAM J. Math. Anal., 34 (2002), pp. 595–610.
- [27] LAX, P. D., *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure Appl. Math., 7 (1954), pp. 159–193.
- [28] LAX, P. D., *Hyperbolic Systems of Conservation Laws and Mathematical Theory of Shock Waves*, Regional Conf. Series in App. Math., 11, SIAM: Philadelphia, 1973.
- [29] LAX, P. D. AND B. WENDORFF, *Systems of conservation laws*, Comm. Pure Appl. Math., 13 (1960), pp. 217–237.
- [30] LEFLOCH, P., *Hyperbolic Systems of Conservation Laws: The Theory of Classical and Non-classical Shock Waves*, Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag: Basel, 2002.
- [31] LI, T.-H., *Convergence of the Lax-Friedrichs scheme for isothermal gas dynamics with semiconductor devices*, Preprint, September 2003.
- [32] LIONS, P.-L., PERTHAME, B., AND P. SOUGANIDIS, *Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates*, Comm. Pure Appl. Math., 49 (1996), pp. 599–638.
- [33] LIONS, P., PERTHAME, B., AND E. TADMOR, *Kinetic formulation of the isentropic gas dynamics and p -system*, Commun. Math. Phys., 163 (1994), pp. 415–431.
- [34] LIU, T.-P., *The deterministic version of the Glimm scheme*, Commun. Math. Phys., 57 (1977), pp. 135–148.
- [35] LIU, T.-P., *Quasilinear hyperbolic systems*, Commun. Math. Phys., 68 (1979), pp. 141–172.

- [36] LIU, T.-P. AND T. YANG, *Well-posedness theory for hyperbolic conservation laws*, Comm. Pure Appl. Math., 52 (1999), pp. 1553–1586.
- [37] MACCORMACK, R. W., *The effect of viscosity in hyperbolicity impact cratering*, AIAA Paper, (1969), pp. 69–354.
- [38] MAKINO, T., MIZOHATA, K. AND S. UKAI, *The global weak solutions of compressible Euler equation with spherical symmetry*, Japan J. Indust. Appl. Math., 9 (1992), pp. 431–449; 11 (1994), pp. 417–426.
- [39] MAKINO, T. AND S. TAKENO, *Initial-boundary value problem for the spherically symmetric motion of isentropic gas*, Japan J. Indust. Appl. Math., 11 (1994), pp. 171–183.
- [40] MARKOWICH, P. A., RINGHOFER, C., AND C. SCHMEISER, *Semiconductor Equations*, Springer-Verlag: Berlin, Heidelberg, New York, 1990.
- [41] NISHIDA, T., *Global solution for an initial boundary value problem of a quasilinear hyperbolic system*, Proc. Japan Acad., 44 (1968), pp. 642–646.
- [42] POUPAUD, F., RASCLE, M., AND J.-P. VILA, *Global solutions to the isothermal Euler-Poisson system with arbitrarily large data*, J. Diff. Eqs., 123 (1995), pp. 93–121.
- [43] RICHTMYER, R. D. AND K. W. MORTON, *Difference Methods for Initial Value Problems*, 2nd Ed. Wiley-Interscience: New York, 1967.
- [44] SERRE, D., *Systems of Conservation Laws*, I,II, Cambridge University Press: Cambridge, 1999.
- [45] SHU, C.-W., *High order ENO and WENO schemes for computational fluid dynamics*, In: High-Order Methods for Computational Physics, pp. 439–582, Lect. Notes Comput. Sci. Eng., 9, Springer: Berlin, 1999.
- [46] SHU, C.-W. AND S. OSHER, *Efficient implementation of essentially non-oscillatory shock capturing schemes*, J. Comp. Phys., 83 (1989), pp. 32–51.
- [47] SMOLLER, J., *Shock Waves and Reaction-Diffusion Equations*, 2nd edition, Springer-Verlag: New York, 1994.
- [48] VAN LEER, B., *Towards the ultimate conservative difference schemes, V: A second order sequel to Godunov's method*, J. Comp. Phys., 43 (1981), pp. 357–372.
- [49] ZHANG, B., *Convergence of the Godunov scheme for a simplified one-dimensional hydrodynamic model for semiconductor devices*, Commun. Math. Phys., 157 (1993), pp. 1–22.

