SELF-SIMILAR SOLUTIONS FOR NONLINEAR SCHRODINGER EQUATIONS *

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Dedicated to Professor Yulin Zhou on the occasion of his eightieth birthday

Abstract. In this paper we study self-similar solutions for nonlinear Schrödinger equations using a scaling technique and the partly contractive mapping method. We establish the small global well-posedness of the Cauchy problem for nonlinear Schrödinger equations in some non-reflexive Banach spaces which contain many homogeneous functions. This we do by establishing some a priori nonlinear estimates in Besov spaces, employing the mean difference characterization and multiplication in Besov spaces. These new global solutions to nonlinear Schrödinger equations with small data admit a class of self-similar solutions. Our results improve and extend the well-known results of Planchon [18], Cazenave and Weissler [4, 5] and Ribaud and Youssfi [20].

1. Introduction. This paper is devoted to the study of the Cauchy problem for the nonlinear Schrödinger equation

$$iu_t + \triangle u = \mu |u|^\alpha u, \quad x \in \mathbb{R}^n, \quad t \geq 0 \tag{1.1}$$

$$u(0, x) = \psi(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\mu \in \mathbb{R}$ is a constant, $u = u(t, x)$ is a complex-valued function defined on $\mathbb{R} \times \mathbb{R}^n$ and the initial data ψ is a complex-valued function defined in $\mathbb{R}^n$. The Cauchy problem (1.1)-(1.2) for the Schrödinger equation, which models many physics phenomena, has been studied extensively by many authors, and in particular, the well-posedness and the scattering theory in Sobolev-type spaces have been established using the Strichartz estimates and other analytic methods (see, e.g. [2, 6, 8, 9, 10, 11, 23, 24, 27]).

In this paper we study the global well posedness in $\dot{B}_{2,\infty}^{s_c}$ of the Cauchy problem (1.1)-(1.2) with small data for $\alpha \notin 2\mathbb{N}$, where $s_c = n/2 - 2/\alpha$. These new global solutions to the nonlinear Schrödinger equation with small data admit a class of self-similar solutions. Our results improve and extend the well-known results of Planchon [18], Cazenave and Weissler [4, 5] and Ribaud and Youssfi [20]. From the scaling principle it is easy to see that if $u(t, x)$ is a solution of the Cauchy problem (1.1)-(1.2) then $u_\lambda(t, x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x)$ with $\lambda > 0$ is also a solution of (1.1)-(1.2) with the initial data $\psi(x)$ replaced by $\lambda^{\frac{2}{\alpha}} \psi(\lambda x)$. We thus have the following definition.

DEFINITION 1.1. *$u(t, x)$ is said to be a self-similar solution to the Schrödinger equation* (1.1) *if*

$$u_\lambda(t, x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x) \equiv u(t, x), \quad \forall \lambda > 0. \tag{1.3}$$

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REMARK 1.1.

(i) It is well-known that the self-similar solution to (1.1) is of the form

$$u(t,x) = t^{-\frac{1}{\alpha}} V(\frac{x}{\sqrt{t}}). \tag{1.4}$$

(ii) Self-similar solutions to nonlinear evolution equations can be studied through the study of the associated nonlinear elliptic equations. For example, if $V(x)$ is a solution of the elliptic equation

$$\triangle V - \frac{i}{\alpha} V - \frac{i}{2} x \cdot \nabla V = \mu |V|^{\alpha} V, \tag{1.5}$$

then $u(x,t)$, defined by (1.4), is a self-similar solution to the Schrödinger equation (1.1) (see [11, 12]). However, it is usually very difficult to solve the elliptic equation (1.5).

(iii) Another important way of looking for self-similar solutions for the nonlinear Schrödinger equation (1.1) is to study the small global well-posedness in some suitable function spaces of the associated Cauchy problem (1.1)-(1.2). These new global solutions admit a class of self-similar solutions. In fact, let the initial data ψ satisfy $\psi(x) = \lambda^{\frac{2}{\alpha}} \psi(\lambda x)$ with $\lambda > 0$ (for example, ψ can be taken as

$$\psi(x) = \frac{\Omega(x')}{|x|^{\frac{2}{\alpha}}}, \quad x' = \frac{x}{|x|}, \tag{1.6}$$

where the function $\Omega(x')$ is defined on the unit sphere in $\mathbb{R}^n$). Then, if $u(t,x)$ is the unique solution of the Cauchy problem (1.1)-(1.2) with the initial data ψ given by (1.6) then $u(t,x)$ is a self-similar solution for the Schrödinger equation (1.1).

(iv) It is necessary to point out that the self-similar solution given by (1.4) with V being the solution of the elliptic equation (1.5) is different from that defined by the solution of the Cauchy problem (1.1) with homogeneous data (1.2) (see [14] for details).

Self-similar solutions for nonlinear evolution equations are very important since they describe the large time behavior of general global solutions to the evolution equations under certain conditions. For example, global solutions of the nonlinear heat equation have been shown to be asymptotically close to the self-similar solutions [11, 14]. Also, for the nonlinear Schrödinger equation (1.1) it has been proved that the long time asymptotic behavior of general solutions can sometimes be governed by self-similar solutions (see [4]). On the other hand, if $u(t,x)$ is a self-similar solution to the problem (1.1)-(1.2), then the function $u(0,x) = \psi(x) = \lambda^{\frac{2}{\alpha}} \psi(\lambda x)$ so ψ is homogeneous of degree $-2/\alpha$. Such homogeneous functions, in general, do not belong to the usual spaces where the global well-posedness of the Cauchy problem holds. Thus, in order to construct self-similar solutions of the problem (1.1)-(1.2) we have to choose a suitable Banach space X which includes the homogeneous data of degree $-2/\alpha$ and to show that the problem (1.1)-(1.2) generates a global flow in X.

Before discussing self-similar solutions and stating our main result, we need to introduce several definitions and notations. Denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the Schwartz space and the space of Schwartz distributions respectively. For integer m, $C^m(\mathbb{R}^n)$ denotes the usual space of m-times continuously differentiable functions on $\mathbb{R}^n$, and

for $1 \leq r \leq \infty$, $L^r(\mathbb{R}^n)$ denotes the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|\cdot\|_r$. For $s \in \mathbb{R}$ and $1 < r < \infty$, let $H^{s,r}(\mathbb{R}^n) = (1 - \triangle)^{-\frac{s}{2}} L^r(\mathbb{R}^n)$, the inhomogeneous Sobolev space in terms of Bessel potentials, let $\dot{H}^{s,r}(\mathbb{R}^n) = (-\triangle)^{-\frac{s}{2}} L^r(\mathbb{R}^n)$, the homogeneous Sobolev space in terms of Riesz potentials, and write $H^s(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n) = \dot{H}^{s,2}(\mathbb{R}^n)$. For $s \in \mathbb{R}$ and $1 \leq r, m \leq \infty$, denote by $B^s_{r,m}(\mathbb{R}^n)$ the Besov space defined as the space of distributions u such that $\{2^{js}\|\varphi_j * u\|_r\}_{j=0}^{\infty} \in \ell^m$, where $*$ stands for the convolution and $\{\varphi_j\}$ is a dyadic decomposition on $\mathbb{R}^n$, and by $\dot{B}^s_{r,m}(\mathbb{R}^n)$ the homogeneous Besov space defined as the space of distributions u modulo polynomials such that $\{2^{js}\|\psi_j * u\|_r\}_{j=-\infty}^{\infty} \in \ell^m$, where $\{\psi_j\}$ is a dyadic decomposition on $\mathbb{R}^n \backslash \{0\}$. For the detailed definitions of the above function spaces see, e.g. [1, 15, 22, 25, 26]. We shall omit $\mathbb{R}^n$ from spaces and norms. For any interval $I \subset \mathbb{R}$ and any Banach space X we denote by $C(I;X)$ the space of strongly continuous functions from I to X, by $L^q(I;X)$ the space of strongly measurable functions from I to X with $\|u(\cdot);X\| \in L^q(I)$, and by $C_*(I;X)$ the space of functions in $L^\infty(I;X)$ that are continuous in the distributional sense. For a given function space X we denote by $\dot{X}$ its homogeneous space. Finally, for any $q > 0$, q' stands for the dual to q, i.e., $1/q + 1/q' = 1$.

By using the Strichartz and nonlinear estimates in Besov spaces, Cazenave and Weissler in [6] established the local well-posedness in H^s with $s \geq \dfrac{n}{2} - \dfrac{2}{\alpha} \geq 0$ for the problem (1.1)-(1.2) and in particular, the global well-posedness in the critical space H^{s_c} with $s_c = n/2 - 2/\alpha$ provided that $\|\psi(x)\|_{H^{s_c}} \ll 1$. However, homogeneous functions such as ψ given in (1.6) do not belong to the usual Sobolev space H^s with $s \geq 0$, so the results in [6] do not apply for constructing self-similar solutions for the problem (1.1)-(1.2).

Recently in his Ph.D. dissertation [3], Cannone constructed self-similar solutions in some Besov spaces to the Navier-Stokes equations employing the Littlewood-Paley theory. Planchon then shown that these self-similar solutions provide the large time asymptotic behavior of the global solutions [17]. It was proved in [14] that these results are also valid for the heat equation:

$$u_t - \triangle u = \mu |u|^\alpha u, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad u(0,x) = \psi(x), \quad x \in \mathbb{R}^n, \qquad (1.7)$$

observing that, though ψ given by (1.6) is not in the Sobolev space $\dot{H}^{s_p,p}$, it is in the Besov space $\dot{B}^{s_p}_{p,\infty}$ with $s_p = n/p - 2/\alpha$. In establishing the above results in [3, 14, 17] an important role was played by the fact that

$$\sup_{t \geq 0} t^{-\frac{s}{2}} \|H(t)\psi\|_p = \|\psi\|_{\dot{B}^s_{p,\infty}}, \quad s \geq 0, \qquad (1.8)$$

where $H(t) = e^{t\triangle}$. However, unlike the heat semigroup $H(t)$, the Schrödinger semigroup $S(t) = e^{i\triangle t}$ does not provide an equivalent norm for $\dot{B}^s_{p,\infty}$ since it does not satisfy (1.8). Thus, to study self-similar solutions for the Schrödinger equation, Cazenave and Weissler [5] (also Ribaud and Youssfi [20]) introduced the new function space $E_{s,p} = E_{s,p}(\mathbb{R} \times \mathbb{R}^n)$ which consists of all Bochner measurable functions $u : (0,\infty) \to \dot{H}^{s,p}(\mathbb{R}^n)$ such that $\|u\|_{E_{s,p}} = \sup_{t>0} t^\sigma \|u(t,x)\|_{\dot{H}^{s,p}} < \infty$, where $2 \leq p < \infty$, $0 \leq s < n/p$ and

$$\sigma = \sigma(s,p) := \frac{1}{2}\left(\frac{2}{\alpha} - \frac{n}{p} + s\right).$$

They then established existence of global self-similar solutions in $E_{s,p}$ for the problem (1.1)-(1.2) under the condition that $\|\varphi; E_{s,p}(\mathbb{R}^n)\| < \varepsilon$, where

$$E_{s,p}(\mathbb{R}^n) = \{\varphi(x) \in \dot{H}^{s,p}(\mathbb{R}^n) \mid \|\varphi\|_{E_{s,p}} = \sup_{t>0} t^\sigma \|S(t)\varphi\|_{\dot{H}^{s,p}} < \infty\}.$$

Recently, Furioli [7] improved the result of Ribaud and Youssfi [20] by extending the range of the nonlinear growth parameter from $\alpha + 1$ to α with $\alpha < 1$.

For the critical Sobolev space $\dot{H}^{s_c}$ with $s_c = n/2 - 2/\alpha$, there is naturally the Besov space $\dot{B}^{s_c}_{2,\infty}$ such that the homogeneous function ψ given by (1.6) is in $\dot{B}^{s_c}_{2,\infty}$ but not in $\dot{H}^{s_c}$. In the case when $\alpha \in 2\mathbb{N} \setminus \{0\}$, global self-similar solutions have been established for the Cauchy problem (1.1)-(1.2) by Planchon in [18, 19] using the generalized Strichartz estimates in Lorentz spaces and nonlinear estimates in Besov spaces together with the paraproduct decomposition and the Littelewood-Paley characterization of Besov spaces. In this paper, we extend Planchon's result in [18, 19] to the case when $\alpha \notin 2\mathbb{N} \setminus \{0\}$. This we do by employing the mean difference characterization and multiplication in Besov spaces in deriving the a priori nonlinear estimates for the nonlinear growth parameter α.

2. Preliminaries.

2.1. Besov spaces. In this subsection we introduce some equivalent definitions and norms for Besov spaces needed in this paper. The reader is referred to the well-known books [1, 16, 25, 26] for details.

We first introduce the following equivalent norms for the Besov spaces $\dot{B}^s_{p,m}$ and $B^s_{p,m}$:

$$\|v\|_{\dot{B}^s_{p,m}} \simeq \sum_{|\alpha|=N} \left(\int_0^\infty t^{-m\sigma} \sup_{|y|\leq t} \| \triangle^2_y \, \partial^\alpha v\|_p^m \frac{dt}{t} \right)^{\frac{1}{m}}, \tag{2.1}$$

$$\|v\|_{B^s_{p,m}} \simeq \|v\|_p + \|v\|_{\dot{B}^s_{p,m}}, \tag{2.2}$$

where

$$\triangle^2_y \, v \triangleq \tau_y v + \tau_{-y} v - 2v, \qquad \tau_{\pm y} v(\cdot) = v(\cdot \pm y),$$

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, \cdots n,$$

$\alpha = (\alpha_1, \cdots, \alpha_n)$ and $s = N + \sigma$ with a nonnegative integer N and $0 < \sigma < 2$. In the special case when s is not an integer, (2.1) is also equivalent to the following norm:

$$\|v\|_{\dot{B}^s_{p,m}} \simeq \sum_{|\alpha|=[s]} \left(\int_0^\infty t^{-m(s-[s])} \sup_{|y|\leq t} \| \triangle_y \, \partial^\alpha v\|_p^m \frac{dt}{t} \right)^{\frac{1}{m}}, \tag{2.3}$$

where $\triangle_{\pm y} \, v(\cdot) = \tau_{\pm y} v - v$ and $[s]$ denotes the largest integer not larger than s. In the case when $m = \infty$, the norm $\|v\|_{\dot{B}^s_{p,\infty}}$ in the above definition should be modified as follows:

$$\|v\|_{\dot{B}^s_{p,\infty}} \simeq \sum_{|\alpha|=N} \sup_{t>0} \sup_{|y|\leq t} t^{-\sigma} \| \triangle^2_y \, \partial^\alpha v\|_p, \quad s \in \mathbb{R} \tag{2.4}$$

$$\|v\|_{\dot{B}^s_{p,\infty}} \simeq \sum_{|\alpha|=[s]} \sup_{t>0} \sup_{|y|\leq t} t^{-s-[s]} \| \triangle_y \, \partial^\alpha v\|_p, \quad s \notin \mathbb{Z}. \tag{2.5}$$

We now introduce the Paley–Littlewood definition of Besov spaces. Let $\hat{\varphi}_0 \in C_c^\infty(\mathbb{R})$ with

$$\hat{\varphi}_0(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2 \end{cases} \tag{2.6}$$

be the real-valued radial Bump function. It is easy to see that

$$\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi), \qquad j \in \mathbb{Z},$$
$$\hat{\psi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi) - \hat{\varphi}_0(2^{-j+1}\xi), \qquad j \in \mathbb{Z} \tag{2.7}$$

are also real-valued radial Bump functions satisfying that

$$\sup_{\xi \in \mathbb{R}^n} 2^{j|\alpha|} |\partial^\alpha \hat{\psi}_j(\xi)| < \infty, \quad j \in \mathbb{Z},$$
$$\sup_{\xi \in \mathbb{R}^n} 2^{j|\alpha|} |\partial^\alpha \hat{\varphi}_j(\xi)| < \infty, \quad j \in \mathbb{Z}.$$

We have the Littlewood–Paley decomposition:

$$\hat{\varphi}_0(\xi) + \sum_{j=0}^{\infty} \hat{\psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n, \tag{2.8}$$

$$\sum_{j \in \mathbb{Z}} \hat{\psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.9}$$

$$\lim_{j \to +\infty} \hat{\varphi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n. \tag{2.10}$$

For convenience, we introduce the following notations:

$$\triangle_j f = \mathcal{F}^{-1} \hat{\psi}_j \mathcal{F} f = \psi_j * f, \quad j \in \mathbb{Z}, \tag{2.11}$$
$$S_j f = \mathcal{F}^{-1} \hat{\varphi}_j \mathcal{F} f = \varphi_j * f, \quad j \in \mathbb{Z}. \tag{2.12}$$

Then we have the following Littlewood-Paley definition of Besov spaces:

$$B_{p,q}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \middle| \|f\|_{B_{p,q}^s} = \|S_0 f\|_p + \left(\sum_{j=1}^{\infty} 2^{jsq} \|\triangle_j f\|_p^q \right)^{\frac{1}{q}} \right.$$

$$= \|\varphi_0 * f\|_p + \left(\sum_{j=1}^{\infty} 2^{jsq} \|\psi_j * f\|_p^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$\dot{B}_{p,q}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \middle| \|f\|_{\dot{B}_{p,q}^s} = \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f\|_p^q \right)^{\frac{1}{q}} \right.$$

$$= \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\psi_j * f\|_p^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$\dot{B}_{p,\infty}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \middle| \|f\|_{\dot{B}_{p,\infty}^s} = \sup_{j \in \mathbb{Z}} 2^{js} \|\triangle_j f\|_p = \sup_{j \in \mathbb{Z}} 2^{js} \|\psi_j * f\|_p < \infty \right\},$$

$$\dot{B}_{p,\infty}^{-\sigma} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \middle| \|f\|_{\dot{B}_{p,\infty}^{-\sigma}} = \sup_{t>0} t^{\frac{\sigma}{2}} \|H(t) f\|_p < \infty \right\}, \quad \sigma > 0,$$

where $1 \leq p \leq \infty$, $1 \leq q < \infty$, $s \in \mathbb{R}$ and $H(t) = e^{t\Delta}$ is the heat semigroup.

Beside the classical Besov spaces, we also need the so-called generalized Besov spaces.

DEFINITION 2.1. *For $s \in \mathbb{R}$, $1 \leq q \leq \infty$ and any Banach space E, define $\dot{B}_E^{s,q}$ as*

$$\dot{B}_E^{s,q} = \{f \mid \|f; \dot{B}_E^{s,q}\| = (\sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f\|_E^q)^{\frac{1}{q}} < \infty\} \qquad (2.13)$$

where $\triangle_j$ is the Littlewood-Paley operator on $\mathbb{R}^n$ defined in (2.11).

REMARK 2.1.
(i) One easily verifies that for $s < 0$, $\dot{B}_E^{s,q}$ can be characterized equivalently as

$$\dot{B}_E^{s,q} = \{f \mid \|f; \dot{B}_E^{s,q}\| = (\sum_{j \in \mathbb{Z}} 2^{jsq} \|S_j f\|_E^q)^{\frac{1}{q}} < \infty\},$$

where S_j is the Littlewood-Paley operator on $\mathbb{R}^n$ defined in (2.12).
(ii) Similarly to the classical Besov spaces, we have the following equivalent norms for $\dot{B}_E^{s,q}$:

$$\|f; \dot{B}_E^{s,q}\| \cong \sum_{|\alpha|=N} \left(\int_0^\infty t^{-q\sigma} \sup_{|y| \leq t} \|\triangle_y^2 \partial^\alpha f\|_E^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

$$\|f; \dot{B}_E^{s,\infty}\| \cong \sum_{|\alpha|=N} \sup_{|y|<t} t^{-\sigma} \|\triangle_y^2 \partial^\alpha f\|_E,$$

where $s = N + \sigma$ and $0 < \sigma < 2$. In the special case when $s \neq 0$, we also have

$$\|f; \dot{B}_E^{s,q}\| \cong \sum_{|\alpha|=[s]} \left(\int_0^\infty t^{-q(s-[s])} \sup_{|y| \leq t} \|\triangle_y \partial^\alpha f\|_E^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

$$\|f; \dot{B}_E^{s,\infty}\| \cong \sum_{|\alpha|=[s]} \sup_{|y|<t} t^{-(s-[s])} \|\triangle_y \partial^\alpha f\|_E,$$

where $\triangle_y$ and $\triangle_y^2$ denote the first and second order difference operators, respectively, as defined at the beginning of this subsection.
(iii) If E is the Lorentz space $L^{p,r}(\mathbb{R}^n)$, then

$$\dot{B}_{(p,r)}^{s,q} := \dot{B}_{L^{p,r}}^{s,q} = \{f \in L^{p,r} \mid \|f; \dot{B}_{(p,r)}^{s,q}\| = (\sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f\|_{(p,r)}^q)^{\frac{1}{q}} < \infty\}.$$

This norm is useful in the study of self-similar solutions. Another type of generalized Besov spaces can be obtained if E is taken as the Morrey space M_r^p.
(iv) Let $E = L^q(I; L^r)$ with $I = \mathbb{R}$ or $I \subset \mathbb{R}$ being an interval. Then we have

$$\mathcal{L}^q(I; \dot{B}_{r,\rho}^s) := \dot{B}_{L^q(I;L^r)}^{s,\rho}$$
$$= \{f \in L^q(I; L^r) \mid \|f; \dot{B}_{L^q(I;L^r)}^{s,\rho}\| = (\sum_{j \in \mathbb{Z}} 2^{js\rho} \|\triangle_j f\|_{L^q(I;L^r)}^\rho)^{\frac{1}{\rho}} < \infty\},$$

$$\mathcal{L}^q(I; \dot{B}_{r,\infty}^s) := \dot{B}_{L^q(I;L^r)}^{s,\infty}$$
$$= \{f \in L^q(I; L^r) \mid \|f; \dot{B}_{L^q(I;L^r)}^{s,\infty}\| = \sup_{j \in \mathbb{Z}} 2^{js} \|\triangle_j f\|_{L^q(I;L^r)} < \infty\},$$

where $1 \leq q \leq \infty$, $1 \leq r \leq \infty$ and $1 \leq \rho < \infty$.

REMARK 2.2. In the above definitions of Besov spaces, L^p-type spaces are chosen as the base spaces. More general Besov spaces can also be defined with the L^p space replaced by the Lorentz space $L^{p,r}$, the time-space Banach space $L^q(\mathbb{R}, L^r)$ and the Morrey space M_p^q.

2.2. Strichartz estimates. Consider the Cauchy Problem (1.1)-(1.2) with the data $\psi \in H^s$. It is well known that $S(t) = \mathcal{F}^{-1}\exp(-i|\xi|^2 t)\mathcal{F} = e^{it\triangle}$ only generates a C_0-semigroup in the Sobolev space H^s $(s \in \mathbb{R})$. Let $v(t,x) = \mathcal{F}^{-1}\exp(\pm i|\xi|^2 t)\mathcal{F}\psi(x)$ and define

$$\mathcal{E}_{q,s,p}(\lambda) = \|v(\lambda^2 t, \lambda x)\|_{L^q(\mathbb{R};\dot{H}^{s,p})},$$
$$\mathcal{E}_{s,2}(\lambda) = \|\psi(\lambda x)\|_{\dot{H}^s}.$$

Then $v \in L^q(\mathbb{R}^+; \dot{H}^{s,p})$ implies that the degree of the polynomials $\mathcal{E}_{q,s,p}(\lambda)$ and $\mathcal{E}_{s,2}(\lambda)$ of λ is the same so

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right). \tag{2.14}$$

Moreover, if

$$\begin{cases} 2 \leq p \leq 2^* = \dfrac{2n}{n-2}, & \text{for} \quad n > 2, \\ 2 \leq p < \infty, & \text{for} \quad n = 2, \\ 2 \leq p \leq \infty, & \text{for} \quad n = 1, \end{cases} \tag{2.15}$$

then $v \in L^q(\mathbb{R}; \dot{H}^{s,p})$ with q and p determined by (2.14). The pair (q, p) satisfying both (2.14) and (2.15) is called as an *admissible pair*. The set of all admissible pairs will be denoted by Λ.

It is easy to check that

$$u(t,x) = S(t)\psi(x) + \int_0^t S(t-\tau)g(\tau,x)d\tau \stackrel{\triangle}{=} S(t)\psi(x) + \mathcal{G}g(t,x)$$

solves the following Cauchy problem for the linear Schödinger equation

$$iu_t + \triangle u = g(t,x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad u(0,x) = \psi(x), x \in \mathbb{R}^n.$$

Let $I = \mathbb{R}$ or $I \subset \mathbb{R}$ be any interval with $0 \in \bar{I}$ and let (q, p), (q_j, p_j) $(j = 1, 2)$ be admissible pairs. As a result of the TT^*-method and interpolation (see [8, 13]), we have the following Strichartz estimates:
(i) For any $\psi \in L^2$, $S(\cdot)\psi \in L^q(I; L^p) \cap C(I; L^2)$ and

$$\|S(\cdot)\psi\|_{L^q(I;L^p)} \leq C\|\psi\|_{L^2}. \tag{2.16}$$

(ii) For $g \in L^{q_2'}(I; L^{p_2'})$, $\mathcal{G}g \in C(I; L^2) \cap L^{q_1}(I; L^{p_1})$ and

$$\|\mathcal{G}g; L^{q_1}(I; L^{p_1})\| \leq C\|g; L^{q_2'}(I; L^{p_2'})\|. \tag{2.17}$$

Since $(-\triangle)^{-\frac{\rho}{2}}$ is a holomorphic mapping from $\dot{B}_{p,\ell}^s$ into $\dot{B}_{p,\ell}^{s+\rho}$ and $B_{p,\ell}^s = \dot{B}_{p,\ell}^s \cap L^p$ when $s > 0$, we have the following Strichartz estimates in Besov spaces:

(i) For $\psi \in \dot{H}^s$, $S(\cdot)\psi \in C(I; \dot{H}^s) \cap L^q(I; \dot{B}^s_{p,2})$ and

$$\|S(\cdot)\psi; L^q(I; \dot{B}^s_{p,2})\| \leq C\|\psi\|_{\dot{H}^s}, \quad (q,p) \in \Lambda. \tag{2.18}$$

(ii) For (q_1,p_1), $(q_2,p_2) \in \Lambda$ and $g \in L^{q_2'}(I; \dot{B}^s_{p_2',2})$, $\mathcal{G}g \in C(I; \dot{H}^s) \cap L^{q_1}(I; \dot{B}^s_{p_1,2})$ and

$$\|\mathcal{G}g; L^{q_1}(I; \dot{B}^s_{p_1,2})\| \leq C\|g; L^{q_2'}(I; \dot{B}^s_{p_2',2})\|. \tag{2.19}$$

(iii) For (q,p), (q_1,p_1), $(q_2,p_2) \in \Lambda$, $\psi \in H^s$ and $g \in L^{q_2'}(I; B^s_{p_2',2})$, we have $S(\cdot)\psi \in C(I; H^s) \cap L^q(I; B^s_{p,2})$, $\mathcal{G}g \in C(I; H^s) \cap L^{q_1}(I; B^s_{p_1,2})$ and

$$\|S(\cdot)\psi; L^q(I; B^s_{p,2})\| \leq C\|\psi\|_{H^s}, \tag{2.20}$$

$$\|\mathcal{G}g; L^{q_1}(I; B^s_{p_1,2})\| \leq C\|g; L^{q_2'}(I; B^s_{p_2',2})\|. \tag{2.21}$$

The following generalized Strichartz estimates follow directly from the Strichartz estimates and interpolation theorem (see [13, 18] for details).

PROPOSITION 2.1. *Let $I = \mathbb{R}$ or $I \subset \mathbb{R}$ be an interval with $0 \in \bar{I}$ and let $S(t) = e^{it\triangle}$.*
(i) [13] *For $(q,r) \in \Lambda$ we have*

$$\|S(\cdot)\psi; L^{q,2}(I; L^{r,2})\| \lesssim \|\psi\|_2, \tag{2.22}$$

$$\|\int_{s<t} S(t-s)f(x,s)ds; L^{\infty}(I; L^2)\| \lesssim \|f; L^{q',2}(I; L^{r',2})\|, \tag{2.23}$$

$$\|\int_{s<t} S(t-s)f(x,s)ds; L^{q,2}(I; L^{r,2})\| \lesssim \|f; L^{q',2}(I; L^{r',2})\|, \tag{2.24}$$

where $\lesssim$ denotes the presence of a constant depending on q and r.
(ii) [18] *Let $\frac{4}{n} < \alpha < \frac{4}{n-2}$ and $\beta = \dfrac{2\alpha(\alpha+2)}{4-(n-2)\alpha}$. Then*

$$\|S(t)\psi; L^{\beta,\infty}(I; L^{\alpha+2})\| \lesssim \|\psi\|_{\dot{B}^{s_c}_{2,\infty}}, \quad s_c = \frac{n}{2} - \frac{2}{\alpha}, \tag{2.25}$$

where $\lesssim$ denotes the presence of a constant depending on $\alpha + 2$ and β.

3. Self-similar solutions in $\dot{B}^{s_c}_{2,\infty}$ with $s_c = n/2 - 2/\alpha$. In this section we establish existence of self-similar solutions in $\dot{B}^{s_c}_{2,\infty}$ ($s_c = n/2 - 2/\alpha$) for the Cauchy problem (1.1)-(1.2) or equivalently its integral form:

$$u(x,t) = S(t)\psi(x) - i\mu \int_0^t S(t-s)|u(s,x)|^{\alpha}u(s,x)ds. \tag{3.1}$$

In the case when $\alpha \in 2\mathbb{N}$, global self-similar solutions have been established by Planchon in [18, 19] for the Cauchy problem (1.1)-(1.2). More precisely, where, for $(q,r) \in \Lambda$,

$$G_{q,r}(\mathbb{R} \times \mathbb{R}^n) = \{v(t,x) \mid \sup_j 2^{js_c}\|\triangle_j v\|_{L^{q,2}(\mathbb{R};L^{r,2})} < \infty\}$$

with $L^{q,\infty}$ and $L^{q,2}$ standing for the Lorentz spaces and $\triangle_j$ ($j \in \mathbb{Z}$) being the Littlewood-Paley operator defined in the last subsection, Planchon established the following result (see [18, 19]).

THEOREM 3.1. [18, 19] *Let $n \geq 3$, $\alpha \geq 4/(n-2)$ and $\alpha \in 2\mathbb{N}$. For $(q,r) \in \Lambda$ (including the endpoint case, i.e. $(q,r) = (2, 2n/(n-2)) \in \Lambda$) there exists an $\varepsilon > 0$ such that if $\psi \in \dot{B}_{2,\infty}^{s_c}$ with $\|\psi; \dot{B}_{2,\infty}^{s_c}\| < \varepsilon$ then the problem $(1.1)-(1.2)$ has a unique global solution $u \in C_*(\mathbb{R}; \dot{B}_{2,\infty}^{s_c})$ with*

$$u \in L^\infty(\mathbb{R}; \dot{B}_{2,\infty}^{s_c}) \cap \bigcap_{(q,r) \in \Lambda} G_{q,r}(\mathbb{R} \times \mathbb{R}^n). \tag{3.2}$$

Our main result, which generalizes Theorem 3.1 by removing the restriction $\alpha \in 2\mathbb{N}$, can be stated as follows.

THEOREM 3.2.
(i) *Let*

$$\begin{cases} 4/n < \alpha < \infty, \text{ for } n = 2, \\ 4/n < \alpha < 4/(n-2), \text{ for } n \geq 3. \end{cases}$$

For $(q,r) \in \Lambda$ and $(\beta, \alpha+2)$ satisfying that

$$\frac{2}{\beta} = n\left(\frac{1}{2} - \frac{s_c}{n} - \frac{1}{\alpha+2}\right)$$

(this is, $(\beta, \alpha+2)$ is an s_c-admissible pair), there exists an $\varepsilon > 0$ such that if $\psi \in \dot{B}_{2,\infty}^{s_c}$ with $\|\psi; \dot{B}_{2,\infty}^{s_c}\| < \varepsilon$ then the problem $(1.1)-(1.2)$ or (3.1) has a unique global solution $u \in C_*(\mathbb{R}; \dot{B}_{2,\infty}^{s_c})$ with

$$u \in L^\infty(\mathbb{R}; \dot{B}_{2,\infty}^{s_c}) \cap L^{\beta,\infty}(\mathbb{R}; L^{\alpha+2,\infty}) \cap \bigcap_{(q,r) \in \Lambda} G_{q,r}(\mathbb{R} \times \mathbb{R}^n). \tag{3.3}$$

In particular, $u \in G_{2,\frac{2n}{n-2}}(\mathbb{R} \times \mathbb{R}^n)$ in the case when $n \geq 3$.

(ii) *Let $3 \leq n \leq 5$, $\alpha \geq 4/(n-2)$ and $\alpha \notin 2\mathbb{N}$. For $(q,r) \in \Lambda$ (including the endpoint case, i.e. $(q,r) = (2, 2n/(n-2)) \in \Lambda$) there exists an $\varepsilon > 0$ such that if $\psi \in \dot{B}_{2,\infty}^{s_c}$ with $\|\psi; \dot{B}_{2,\infty}^{s_c}\| < \varepsilon$ then the problem $(1.1)-(1.2)$ has a unique global solution $u \in C_*(\mathbb{R}; \dot{B}_{2,\infty}^{s_c})$ satisfying (3.2).*

(iii) *Let $n \geq 6$ and let $4/(n-2) \leq \alpha < \alpha_-$ or $\alpha_+ < \alpha < \infty$ with*

$$\alpha_- = \frac{n - \sqrt{n^2 - 32}}{4}, \qquad \alpha_+ = \frac{n + \sqrt{n^2 - 32}}{4}.$$

For $(q,r) \in \Lambda$ (including the endpoint case, i.e. $(q,r) = (2, 2n/(n-2)) \in \Lambda$) there exists an $\varepsilon > 0$ such that if $\psi \in \dot{B}_{2,\infty}^{s_c}$ with $\|\psi; \dot{B}_{2,\infty}^{s_c}\| < \varepsilon$ then the problem $(1.1)-(1.2)$ has a unique global solution $u \in C_*(\mathbb{R}; \dot{B}_{2,\infty}^{s_c})$ satisfying (3.2).

REMARK 3.1. (i) Let $\psi(x) = \lambda^{\frac{2}{\alpha}} \psi(\lambda x)$ with $\lambda > 0$ in Theorem 3.2. Then the problem $(1.1)-(1.2)$ or (3.1) has a unique global self-similar solution $u \in C_*(\mathbb{R}; \dot{B}_{2,\infty}^{s_c})$ satisfying Theorem 3.2.

(ii) Theorem 3.2 only deals with the case $\alpha > 4/n$. If, instead of using the Besov space $\dot{B}_{2,\infty}^{s_c}(\mathbb{R}^n)$, we use the function space $E_{s_c,p}(\mathbb{R}^n)$ then the restriction $\alpha > 4/n$ can be removed (see [5, 7, 20]).

(iii) The condition on α in Theorem 3.2 (iii) is related to the regularity issue $s_c < \alpha$.

To prove Theorem 3.2 we need the following lemma which was proved in [21].

LEMMA 3.1. *Let $\sigma_p = n \cdot \max(0, \frac{1}{p} - 1)$ and $m \in \mathbb{N}$ with $m \geq 2$. Suppose*

$$\min_{k=1,\cdots,m} \sum_{k \neq j} \frac{1}{r_k} < 1, \qquad \frac{1}{p} = \frac{1}{p_j} + \sum_{k \neq j} \frac{1}{r_k}, \qquad j = 1, 2, \cdots m.$$

If $s > \sigma_p$, then there exists a constant $C > 0$ such that

$$\|f_1 \cdots f_m; \dot{B}_{p,q}^s\| \leq C \sum_{j=1}^{m} \left(\|f_j; \dot{B}_{p_j,q}^s\| \prod_{k \neq j} \|f_k; L^{r_k}\| \right)$$

for all $(f_1, \cdots, f_m) \in \prod_{j=1}^{m} \dot{B}_{p,q}^s \cap L^{r_j}$.

Proof of Theorem 3.2. We first prove (i). For convenience, we only consider the case $t > 0$. The case $t < 0$ can be shown similarly.

Let

$$X = \begin{cases} L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c}) \cap L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty}) \cap \mathcal{L}^{\gamma(\alpha+2),2}(\mathbb{R}^+; \dot{B}_{L^{\alpha+2,2}}^{s_c,\infty}), & \text{for } n = 2, \\ L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c}) \cap L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty}) \cap \mathcal{L}^2(\mathbb{R}^+; \dot{B}_{L^{\frac{2n}{n-2}}}^{s_c,\infty}), & \text{for } n \geq 3, \end{cases}$$

where $\beta = 2\alpha(\alpha+2)/[4-(n-2)\alpha]$, $s_c = n/2 - 2/\alpha$ and $(\gamma(\alpha+2), \alpha+2) \in \Lambda$, and define the complete metric space $(\mathcal{X}, d)$ by

$$\mathcal{X} = \{u \in X \mid \|u\|_X \leq M\},$$
$$d(u, v) = \|u - v; L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})\|,$$

where M is a constant depending on $\|\psi; \dot{B}_{2,\infty}^{s_c}\|$ and is to be determined later. Set $f(u) = -i\mu|u|^\alpha u$ and define $\mathcal{T}u$ as the right-hand side of the integral equation (3.1), that is,

$$\mathcal{T}u(t,x) := S(t)\psi(x) + \mathcal{G}f(u(t,x))$$
$$= S(t)\psi(x) - i\mu \int_0^t S(t-s)|u(s,x)|^\alpha u(s,x)ds. \qquad (3.4)$$

We now prove that for some small constant M the mapping $\mathcal{T}$ is contractive from $\mathcal{X}$ into itself.

First it follows from Proposition 2.1 and the Littlewood-Paley characterization of Besov spaces that

$$\|S(\cdot)\psi\|_X \leq C\|\psi; \dot{B}_{2,\infty}^{s_c}\|. \qquad (3.5)$$

Next from the Strichartz estimates (i.e. L^p-$L^{p'}$ estimates) and the generalized

Young inequality it follows that

$$\|\mathcal{G}(f(u) - f(v)); L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})\|$$

$$\leq \left\| \int_0^t |t - s|^{-n(\frac{1}{2} - \frac{1}{\alpha+2})} \|f(u) - f(v)\|_{L^{\frac{\alpha+2}{\alpha+1},\infty}} ds \right\|_{L^{\beta,\infty}}$$

$$\leq \left\| \int_0^t |t - s|^{-n(\frac{1}{2} - \frac{1}{\alpha+2})} \|u - v\|_{L^{\alpha+2,\infty}} (\|u\|_{L^{\alpha+2,\infty}}^{\alpha} + \|v\|_{L^{\alpha+2,\infty}}^{\alpha}) ds \right\|_{L^{\beta,\infty}}$$

$$\lesssim [\|u; L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})\|^{\alpha} + \|v; L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})\|^{\alpha}]$$

$$\times \|u - v\|_{L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})}, \tag{3.6}$$

where use has been made of the fact that

$$\frac{1}{\beta} + 1 = \frac{\alpha + 1}{\beta} + \frac{1}{q}, \quad \text{or} \quad q = \frac{2(\alpha + 2)}{n\alpha}.$$

In particular, we have

$$\|\mathcal{G}f(u); L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})\| \lesssim \|u; L^{\beta,\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})\|^{\alpha+1}. \tag{3.7}$$

Since $(\gamma(\alpha + 2), \alpha + 2) = (\dfrac{4(\alpha + 2)}{n\alpha}, \alpha + 2) \in \Lambda$ and $0 < s_c < 1$, it is easy to obtain by (3.5)-(3.7) and Proposition 2.1 that

$$\|\mathcal{G}f(u); L^{\infty}(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c})\| \vee \|\mathcal{G}f(u); \mathcal{L}^{\gamma(\alpha+2),2}(\mathbb{R}^+; \dot{B}_{L^{\alpha+2,2}}^{s_c,\infty})\|$$

$$\lesssim \|f(u); \dot{B}_{L^{\gamma(\alpha+2)',2}(\mathbb{R}^+; L^{(\alpha+2)',2})}^{s_c,\infty}\|$$

$$\cong \sup_{|y|\leq\tau} \tau^{-s_c} \| \triangle_y f(u(t,\cdot)); L^{\gamma(\alpha+2)',2}(\mathbb{R}^+; L^{(\alpha+2)',2})\|$$

$$\leq \sup_{|y|\leq\tau} \tau^{-s_c} \left\| \| \triangle_y u\|_{L^{\alpha+2,2}} \|u\|_{L^{\alpha+2,\infty}}^{\alpha} \right\|_{L^{\gamma(\alpha+2)',2}}$$

$$\leq \sup_{|y|\leq\tau} \tau^{-s_c} \| \triangle_y u\|_{L^{\gamma(\alpha+2),2}(\mathbb{R}^+; L^{\alpha+2,2})} \|u\|_{L^{\gamma(\alpha+2),\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})}^{\alpha}$$

$$\leq C\|u\|_{\mathcal{L}^{\gamma(\alpha+2),2}(\mathbb{R}^+; \dot{B}_{L^{\alpha+2,2}}^{s_c,\infty})} \|u\|_{L^{\gamma(\alpha+2),\infty}(\mathbb{R}^+; L^{\alpha+2,\infty})}^{\alpha}, \tag{3.8}$$

where $a \vee b = \max(a, b)$. In particular, when $n \geq 3$, by the Hölder inequality and the Sobolev embedding theorem $L^{\infty}(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c}) \hookrightarrow L^{\infty}(\mathbb{R}^+; L^{\frac{n\alpha}{2},\infty})$ we have

$$\left\| f(u); \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n+2},2})}^{s_c,\infty} \right\| = \sup_{|y|<\tau} \tau^{-s_c} \| \triangle_y f(u)\|_{L^2(\mathbb{R}^+; L^{\frac{2n}{n+2},2})}$$

$$\lesssim \sum_{\pm} \sup_{|y|<\tau} \tau^{-s_c} \| \triangle_{\pm y} u\|_{L^2(\mathbb{R}^+; L^{\frac{2n}{n-2},2})} \|(|\tau_{\pm y} u|^{\alpha} + |u|^{\alpha}); L^{\infty}(\mathbb{R}^+; L^{\frac{n}{2},\infty})\|$$

$$\lesssim \left\| u; \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n-2},2})}^{s_c,\infty} \right\| \|u; L^{\infty}(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c})\|^{\alpha}.$$

Thus we obtain that

$$\|\mathcal{T}u\|_X \leq C(\|\psi; \dot{B}_{2,\infty}^{s_c}\| + \|u\|_X^{\alpha+1}),$$

$$d(\mathcal{T}u, \mathcal{T}v) \leq C[\|u\|_X^{\alpha} + \|v\|_X^{\alpha}] d(u, v), \quad u, v \in \mathcal{X}.$$

Let $M = 2C\|\psi; \dot{B}_{2,\infty}^{s_c}\|$ be small enough (e.g. $M < [1/(2C)]^{1/\alpha}$) so that $\mathcal{T}$ is a contraction mapping from $\mathcal{X}$ into itself. By the Banach contraction mapping principle

we conclude that there is a unique solution $u \in \mathcal{X}$ for the problem (1.1)-(1.2) or equivalently (3.1). As a direct consequence of the Strichartz estimates, we have $u \in G_{q,r}(\mathbb{R}^+ \times \mathbb{R}^n)$ for any $(q,r) \in \Lambda$. We now prove that $u \in C_*(\mathbb{R}^+; B_{2,\infty}^{s_c})$, that is, $u(t,x)$ is weakly continuous at $t = 0$, by employing Planchon's idea [18]. From the localized Strichartz estimates (which can be obtained from Proposition 2.1) and the following Bernstein inequality

$$2^{js_c - 2(\frac{\alpha+1}{\alpha+2} - \frac{1}{2})j} \|\triangle_j(f(u)\|_2 \lesssim 2^{js_c} \|\triangle_j f(u)\|_{L^{\frac{\alpha+2}{\alpha+1},2}},$$

it follows on noting (3.8) that

$$\begin{aligned}
2^{-j\frac{4}{\alpha(\alpha+2)}} \|\triangle_j \mathcal{G}f(u)\|_2 &\lesssim \int_0^t 2^{-j\frac{4}{\alpha(\alpha+2)}} \|\triangle_j f(u)\|_2 ds \\
&\lesssim t^{\frac{\alpha}{2(\alpha+2)}} \|f(u); \mathcal{L}^{\gamma(\alpha+2)',2}(\mathbb{R}^+; \dot{B}_{L^{(\alpha+2)',2}}^{s_c,\infty})\| \\
&\lesssim t^{\frac{\alpha}{2(\alpha+2)}} \|u\|_X^{\alpha+1}.
\end{aligned}$$

This implies the weak continuity of $u(t,x)$ at $t = 0$, which completes the proof of (i).

We now prove (ii) and (iii). Note first that $s_c = n/2 - 2/\alpha \geq 1$. Let

$$Y = L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c}) \cap \mathcal{L}^2(\mathbb{R}^+; \dot{B}_{L^{\frac{2n}{n-2},2}}^{s_c,\infty})$$

and let us introduce the complete metric space

$$\mathcal{Y} := \left\{ u \in Y \mid \|u\|_Y \leq M \right\}$$

with the metric

$$d(u,v) = \|u - v\|_Y,$$

where M is a constant depending on $\|\psi; \dot{B}_{2,\infty}^{s_c}\|$ and to be determined later. Consider, in the metric space $\mathcal{Y}$, the operator $\mathcal{T}$ defined by (3.4). It is easy to see by using the localized Strichartz estimates in frequency and the Littlewood-Paley characterization of Besov spaces that

$$\|S(t)\psi\|_Y \leq C\|\psi; \dot{B}_{2,\infty}^{s_c}\|. \tag{3.9}$$

Now, for the nonlinear term $f(u)$ we have the following estimates:

$$\left\| f(u); \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n+2},2})}^{s_c,\infty} \right\| \lesssim \|u; \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n-2},2})}^{s_c,\infty}\| \|u; L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c})\|^\alpha, \tag{3.10}$$

$$\left\| f'(u); \dot{B}_{L^2(\mathbb{R}^+; L^{\ell,2})}^{s_c,\infty} \right\| \lesssim \|u; \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n-2},2})}^{s_c,\infty}\| \|u; L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c})\|^{\alpha-1}, \tag{3.11}$$

where $\ell = \frac{2n\alpha}{\alpha(n+2)-4}$ and $s_c = n/2 - 2/\alpha < \alpha$ under the assumptions in (ii) and (iii). We first prove (ii) and (iii) under the assumption that the estimates (3.10) and (3.11) hold and then show that the estimates (3.10) and (3.11) are actually true.

From Proposition 2.1 and (3.10) it follows that

$$\|\mathcal{G}f(u); Y\| \lesssim \|u; \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n-2},2})}^{s_c,\infty}\| \|u; L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c})\|^\alpha. \tag{3.12}$$

On the other hand, since

$$f(u) - f(v) = \int_0^1 f'(u + \theta(u - v))d\theta(u - v),$$

it follows from Lemma 3.1, (3.11) and the embedding relation $\dot{B}_{2,\infty}^{s_c} \hookrightarrow L^{n\alpha/2}$ that

$$\|\mathcal{G}(f(u) - f(v)); Y\| \lesssim \left\| u - v; \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n-2}, 2})}^{s_c, \infty} \right\| \sum_{w=u,v} \|w; L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c})\|^\alpha$$

$$+ \left\| u - v; L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c}) \right\| \sum_{w=u,v} \|w; \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n-2}, 2})}^{s_c, \infty}\|$$

$$\times \sum_{w=u,v} \|w; L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c})\|^{\alpha-1}. \tag{3.13}$$

Thus, by (3.9), (3.12) and (3.13), we have

$$\|\mathcal{T}u\|_Y \leq C \left(\|\psi; \dot{B}_{2,\infty}^{s_c}\| + \|u; Y\|^{\alpha+1} \right),$$
$$d(\mathcal{T}u, \mathcal{T}v) \leq C[\|u\|_Y^\alpha + \|v\|_Y^\alpha] \cdot d(u, v), \quad u, v \in \mathcal{Y}.$$

Let $M = 2C\|\psi; \dot{B}_{2,\infty}^{s_c}\|$ be small enough (e.g. $M < [1/(2C)]^{1/\alpha}$) so that $\mathcal{T}$ is a contraction mapping from $\mathcal{Y}$ into itself. By the Banach contraction mapping principle we conclude that there is a unique solution $u \in \mathcal{Y}$ for the problem (1.1)-(1.2) or equivalently (3.1). As a direct consequence of the Strichartz estimates, we have $u \in G_{q,r}(\mathbb{R}^+ \times \mathbb{R}^n)$ for any $(q, r) \in \Lambda$. The weak continuity of $u(t, x)$ at $t = 0$ can be proved by using Planchon's idea [18] (cf. the proof of (i)). This proves (ii) and (iii).

To complete the proof of Theorem 3.2 we now need to prove the estimates (3.10) and (3.11). First it is easy to verify that $f(z) = |z|^\alpha z \in C^{[s_c]}(\mathbb{C}, \mathbb{C})$ with

$$f^{(k)}(0) = 0, \quad |f^{(k)}(z)| \leq C|z|^{\alpha+1-k}, \quad 0 \leq k \leq [s_c],$$

$$|f^{([s_c])}(z_1) - f^{([s_c])}(z_2)| \leq \begin{cases} C(|z_1|^{\alpha-[s_c]} + |z_2|^{\alpha-[s_c]})|z_1 - z_2|, & \text{for } \alpha \geq [s_c] \\ |z_1 - z_2|^{\alpha+1-[s_c]}, & \text{for } \alpha < [s_c]. \end{cases}$$

Without loss of generality, we may assume that f is a real-valued function. We first prove (3.10). We need the following equivalent norms for Besov spaces:

$$\left\| v; \dot{B}_{L^2(\mathbb{R}^+; L^{\frac{2n}{n+2}, 2})}^{s, \infty} \right\| = \sum_{|\alpha|=[s]} \sup_{|y|<\tau} \tau^{-(s-[s])} \|\triangle_y \partial^\alpha v\|_{L^2(\mathbb{R}^+; L^{\frac{2n}{n+2}, 2})},$$

$$\cong \sum_{|\alpha|=N} \sup_{|y|<\tau} \tau^{-\sigma} \|\triangle_y^2 \partial^\alpha v\|_{L^2(\mathbb{R}^+; L^{\frac{2n}{n+2}, 2})},$$

where $s = N + \sigma$ with a nonnegative integer N and $0 < \sigma < 2$ and $\triangle_y^2$ is as defined in (2.1). We distinguish between the following two cases.

Case 1. $1 \leq s_c < 2$. Since

$$\triangle_y^2 f(u) = f'(u) \triangle_y^2 u + \sum_\pm \triangle_{\pm y} u \int_0^1 [f'(\lambda \tau_{\pm y} u + (1 - \lambda)u) - f'(u)]d\lambda,$$

and noting that $\alpha > 1$, it follows that

$$
|\triangle_y^2 f(u)| \lesssim |f'(u)|| \triangle_y^2 u| + \sum_\pm |\triangle_{\pm y} u|^2 (|\tau_{\pm y} u|^{\alpha-1} + |u|^{\alpha-1}).
$$

Note that $[s_c] = 1$, $\dfrac{n+2}{2n} = \dfrac{n-2}{2n} + \alpha \dfrac{2}{n\alpha}$ and

$$
\frac{n+2}{2n} - \frac{s_c}{n} = \left(\frac{1}{\chi_1} - \frac{s_c - [s_c] + \varepsilon}{n}\right) + \left(\frac{1}{\chi_2} - \frac{[s_c] - \varepsilon}{n}\right) + \frac{2(\alpha - 1)}{n\alpha},
$$

where $\varepsilon > 0$ is a sufficient small constant and

$$
\chi_1 = \frac{2n}{n - 3 + \varepsilon}, \quad \chi_2 = \frac{2n}{n - s_c + 1 - \varepsilon}.
$$

From the Hölder inequality and the embedding

$$
\dot{B}_{L^2(\mathbb{R}^+;L^{\frac{2n}{n+2},2})}^{s_c,\infty} \hookrightarrow \dot{B}_{L^2(\mathbb{R}^+;L^{\chi_1,2})}^{s_c-[s_c]+\varepsilon,\infty}, \quad \dot{B}_{2,\infty}^{s_c} \hookrightarrow \dot{B}_{\chi_2,\infty}^{[s_c]-\varepsilon} \hookrightarrow \dot{B}_{L^{\chi_2,\infty}}^{[s_c]-\varepsilon,\infty}, \quad \dot{B}_{2,\infty}^{s_c} \hookrightarrow L^{\frac{n\alpha}{2},\infty},
$$

it is easy to see that

$$
\begin{aligned}
\left\| f(u); \dot{B}_{L^2(\mathbb{R}^+;L^{\frac{2n}{n+2},2})}^{s_c,\infty} \right\| &= \sup_{|y|<\tau} \tau^{-\sigma} \| \triangle_y^2 f(u) \|_{L^2(\mathbb{R}^+;L^{\frac{2n}{n+2},2})} \\
&\lesssim \sup_{|y|<\tau} \tau^{-\sigma} \| \triangle_y^2 u \|_{L^2(\mathbb{R}^+;L^{\frac{2n}{n-2},2})} \|u\|_{L^{\frac{n\alpha}{2},\infty}}^{\alpha} \\
&\quad + \sup_{|y|<\tau} \tau^{-(s_c-[s_c]+\varepsilon)} \| \triangle_y u \|_{L^2(\mathbb{R}^+;L^{\chi_1,2})} \\
&\quad \times \sup_{t\in\mathbb{R}^+} \{ \sup_{|y|<\tau} \tau^{-([s_c]-\varepsilon)} \| \triangle_y u \|_{L^{\chi_2,\infty}} \|u\|_{L^{\frac{n\alpha}{2},\infty}}^{\alpha-1} \} \\
&\lesssim \left\| u; \dot{B}_{L^2(\mathbb{R}^+;L^{\frac{2n}{n-2},2})}^{s_c,\infty} \right\| \|u\|_{L^{\frac{n\alpha}{2},\infty}}^{\alpha} + \| u; \dot{B}_{L^2(\mathbb{R}^+;L^{\chi_1,2})}^{s_c-[s_c]+\varepsilon,\infty} \| \\
&\quad \times \sup_{t\in\mathbb{R}^+} \left(\| u; \dot{B}_{L^{\chi_2,\infty}}^{[s_c]-\varepsilon,\infty} \| \|u\|_{L^{\frac{n\alpha}{2},\infty}}^{\alpha-1} \right) \\
&\lesssim \left\| u; \dot{B}_{L^2(\mathbb{R}^+;L^{\frac{2n}{n-2},2})}^{s_c,\infty} \right\| \| u; L^\infty(\mathbb{R}^+; \dot{B}_{2,\infty}^{s_c}) \|^\alpha.
\end{aligned}
$$

Thus (3.10) is proved in this case.

    Case 2. $s_c \geq 2$. Let $s_c = [s_c] - 1 + \sigma$. Then $1 \leq \sigma < 2$ and $[s_c] - 1 \geq 1$. A direct calculation using the Leibniz rule of derivatives gives that for $|\gamma| = [s_c] - 1$,

$$
\partial^\gamma f(u) = \sum_{k=1}^{[s_\alpha]-1} \sum_{\substack{\beta_1 + \cdots + \beta_k = \gamma, \\ |\beta_j| \geq 1}} \frac{\gamma!}{k! \prod_{j=1}^k \beta_j!} f^{(k)}(u) \prod_{j=1}^k \partial^{\beta_j} u. \tag{3.14}
$$

For simplicity set

$$
\begin{aligned}
F(u) &= f^{(k)}(u), \quad v = \tau_y u, \quad w = \tau_{-y} u, \\
u_j &= \partial^{\beta_j} u, \quad v_j = \partial^{\beta_j} v, \quad w_j = \partial^{\beta_j} w.
\end{aligned}
$$

Then the second-order difference of the term $f^{(k)}(u) \prod_{j=1}^{k} \partial^{\beta_j} u$ can be computed as

$$\triangle_y^2 \left(f^{(k)}(u) \prod_{j=1}^{k} \partial^{\beta_j} u \right) = F(v) \prod_{j=1}^{k} v_j + F(w) \prod_{j=1}^{k} w_j - 2F(u) \prod_{j=1}^{k} u_j$$

$$= F(u) \left(\prod_{j=1}^{k} v_j + \prod_{j=1}^{k} w_j - 2 \prod_{j=1}^{k} u_j \right) + (F(v) - F(u)) \left(\prod_{j=1}^{k} v_j - \prod_{j=1}^{k} u_j \right)$$

$$+ (F(u) - F(w)) \left(\prod_{j=1}^{k} u_j - \prod_{j=1}^{k} w_j \right) + (F(v) + F(w) - 2F(u)) \prod_{j=1}^{k} u_j$$

$$= F(u) \sum_j \triangle_y^2 u_j \cdot \prod_{a<j} v_a \prod_{b>j} u_b - F(u) \sum_j \sum_{i<j} (\triangle_{-y} u_j \cdot \triangle_y u_i) \prod_{a<j} v_a \prod_{\substack{b<j \\ b \neq i}} u_b$$

$$+ F(u) \sum_j \sum_{i>j} (\triangle_{-y} u_j \cdot \triangle_{-y} u_i) \prod_{\substack{a<i \\ a \neq j}} u_a \prod_{b>i} w_b$$

$$+ (F(v) - F(u)) \sum_j \triangle_y u_j \prod_{a<j} v_a \prod_{b>j} u_b$$

$$- (F(u) - F(w)) \sum_j \triangle_{-y} u_j \prod_{a<j} u_a \prod_{b>j} w_b + F'(u) \triangle_y^2 u \prod_j u_j$$

$$+ \int_0^1 (F'(\theta v + (1-\theta)u) - F'(u)) d\theta (v-u) \prod_j u_j$$

$$+ \int_0^1 (F'(\theta w + (1-\theta)u) - F'(u)) d\theta (w-u) \prod_j u_j = \sum_{k=1}^{8} I_k.$$

We now evaluate each I_k.

Estimation of I_1 and I_6. Let

$$d_0 = (\alpha + 1 - k) \left(\frac{1}{2} - \frac{s_c}{n} \right),$$

$$d_a = \frac{1}{2} - \frac{s_c - |\beta_a|}{n}, \quad 1 \le a \le k \quad \text{and} \quad a \neq j,$$

$$d_j = \frac{n-2}{2n} - \frac{s_c - (|\beta_j| + \sigma)}{n} \triangleq \frac{1}{\chi_1}.$$

Since $1/2 - s_c/n > 0$, it follows that $d_a > 0$ for $a = 0, 1, \cdots, k$. It is easy to verify that

$$d_0 + \sum_{a=1}^{k} d_a = \frac{n+2}{2n}.$$

Noting that the Besov spaces are translation invariant, we obtain by the Hölder inequality and Sobolev's embedding theorem that

$$\|I_1\|_{L^{\frac{2n}{n+2},2}} \le \|u\|_{\dot{B}_{2,\infty}^{s_c}}^{\alpha} \| \triangle_y^2 u_j \|_{L^{\chi_1,2}}. \tag{3.15}$$

Similarly, and letting

$$d_0 = (\alpha - k)\left(\frac{1}{2} - \frac{s_c}{n}\right),$$

$$d_a = \frac{1}{2} - \frac{s_c - |\beta_a|}{n}, \quad a = 1, \cdots, k,$$

$$d_{k+1} = \frac{n-2}{2n} - \frac{s_c - \sigma}{n} \triangleq \frac{1}{\chi_2},$$

we have

$$\|I_6\|_{L^{\frac{2n}{n+2},2}} \le \|u\|_{\dot{B}^{s_c}_{2,\infty}}^\alpha \| \triangle_y^2 u_j\|_{L^{\chi_2,2}}, \tag{3.16}$$

where use has been made of the fact that

$$d_0 + \sum_{a=1}^{k+1} d_a = \frac{n+2}{2n}.$$

We may estimate I_2 and I_3 similarly. In fact, let

$$d_0 = (\alpha + 1 - k)\left(\frac{1}{2} - \frac{s_c}{n}\right),$$

$$d_a = \frac{1}{2} - \frac{s_c - |\beta_a|}{n}, \quad a \ne i, j,$$

$$d_i = \frac{1}{2} - \frac{s_c - (|\beta_i| + \frac{\sigma}{2})}{n} \triangleq \frac{1}{\chi_3},$$

$$d_j = \frac{n-2}{2n} - \frac{s_c - (|\beta_j| + \frac{\sigma}{2})}{n} \triangleq \frac{1}{\chi_4}.$$

Then it is easy to verify that $d_a > 0$ for $a = 0, 1, \cdots, k$ and

$$d_0 + \sum_{a=1}^{k} d_a = \frac{n+2}{2n},$$

so it follows from the Hölder inequality and the Sobolev embedding theorem that

$$\|I_2\|_{L^{\frac{2n}{n+2},2}}, \|I_3\|_{L^{\frac{2n}{n+2},2}} \le \|u\|_{\dot{B}^{s_c}_{2,\infty}}^{\alpha-1} \| \triangle_{\pm y} u_i\|_{L^{\chi_3,2}} \| \triangle_{\pm y} u_j\|_{L^{\chi_4,\infty}} \tag{3.17}$$

To estimate I_4 and I_5, let

$$d_0 = (\alpha - k)\left(\frac{1}{2} - \frac{s_c}{n}\right),$$

$$d_a = \frac{1}{2} - \frac{s_c - |\beta_a|}{n}, \quad 1 \le a \le k \quad \text{and} \quad a \ne j,$$

$$d_j = \frac{n-2}{2n} - \frac{s_c - (|\beta_j| + \frac{\sigma}{2})}{n} \triangleq \frac{1}{\chi_5},$$

$$d_{k+1} = \frac{1}{2} - \frac{s_c - \frac{\sigma}{2}}{n} \triangleq \frac{1}{\chi_6}.$$

Then arguing in exactly the same way as in deriving (3.15)-(3.17) it can be obtained that

$$\|I_4\|_{L^{\frac{2n}{n+2},2}}, \|I_5\|_{L^{\frac{2n}{n+2},2}} \le \|u\|_{\dot{B}^{s_c}_{2,\infty}}^{\alpha-1} \| \triangle_{\pm y} u_j\|_{L^{\chi_5,2}} \| \triangle_{\pm y} u\|_{L^{\chi_6,\infty}}. \tag{3.18}$$

Finally, I_7 and I_8 can be estimated similarly as above by letting

$$d_a = \frac{1}{2} - \frac{s_c - |\beta_a|}{n}, \quad a = 1, \cdots, k,$$

$$d_{k+1} = \frac{n-2}{2n} - \frac{s_c - \frac{\sigma}{\alpha+1-k}}{n} \triangleq \frac{1}{\chi_7}$$

$$d_{k+2} = \frac{1}{2} - \frac{s_c - \frac{\sigma}{\alpha+1-k}}{n} \triangleq \frac{1}{\chi_8}$$

and noting that $d_a > 0$ for $a = 1, \cdots, k+2$ and

$$\sum_{a=1}^{k+1} d_a + \frac{\alpha - k}{d_{k+2}} = \frac{n+2}{2n},$$

and we have

$$\|I_7\|_{L^{\frac{2n}{n+2},2}}, \ \|I_8\|_{L^{\frac{2n}{n+2},2}} \le \|u\|_{\dot{B}_{2,\infty}^{s_c}}^k \|\bigtriangleup_{\pm y} u\|_{L^{\chi_7,2}} \|\bigtriangleup_{\pm y} u\|_{L^{\chi_8},\infty}^{\alpha-k}. \tag{3.19}$$

Noting that

$$\dot{B}_{L^{\frac{2n}{n-2},2}}^{s_c,\infty} \hookrightarrow \dot{B}_{L^{\chi_1,2}}^{\sigma+|\beta_j|,\infty}, \qquad \dot{B}_{L^{\frac{2n}{n-2},2}}^{s_c,\infty} \hookrightarrow \dot{B}_{L^{\chi_2,2}}^{\sigma,\infty},$$

$$\dot{B}_{L^{\frac{2n}{n-2},2}}^{s_c,\infty} \hookrightarrow \dot{B}_{L^{\chi_3,2}}^{|\beta_j|+\frac{\sigma}{2},\infty}, \qquad \dot{B}_{2,\infty}^{s_c} \hookrightarrow \dot{B}_{\chi_4,\infty}^{|\beta_j|+\frac{\sigma}{2}},$$

$$\dot{B}_{L^{\frac{n-2}{2n},2}}^{s_c,\infty} \hookrightarrow \dot{B}_{L^{\chi_5,2}}^{|\beta_j|+\frac{\sigma}{2},\infty}, \qquad \dot{B}_{2,\infty}^{s_c} \hookrightarrow \dot{B}_{\chi_6,\infty}^{\frac{\sigma}{2}},$$

$$\dot{B}_{L^{\frac{2n}{n-2},2}}^{s_c,\infty} \hookrightarrow \dot{B}_{L^{\chi_7,2}}^{\frac{\sigma}{\alpha+1-k},\infty}, \qquad \dot{B}_{2,\infty}^{s_c} \hookrightarrow \dot{B}_{\chi_8,\infty}^{\frac{\sigma}{\alpha+1-k}},$$

the estimate (3.10) follows from (3.14)-(3.19) and the equivalent norm of Besov spaces.

The estimate (3.11) can be shown similarly as above by noting that

$$\frac{1}{\ell} - \frac{s_c}{n} = (\alpha - 1)\left(\frac{1}{2} - \frac{s_c}{n}\right) + \frac{n-2}{2n} - \frac{s_c}{n}.$$

This completes the proof of Theorem 3.2. □

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