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# REGULARITY OF THE MINIMIZER FOR THE $D$-WAVE GINZBURG-LANDAU ENERGY *

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**Abstract.** We study the minimizer of the $d$-wave Ginzburg-Landau energy in a specific class of functions. We show that the minimizer having distinct degree-one vortices is Holder continuous. Away from vortex cores, the minimizer converges uniformly to a canonical harmonic map. For a single vortex in the vortex core, we obtain the $C^{\frac{1}{2}}$-norm estimate of the fourfold symmetric vortex solution. Furthermore, we prove the convergence of the fourfold symmetric vortex solution under different scales of $\delta$.

**1. Introduction.** In this paper, we investigate the minimizer of the $d$-wave Ginzburg-Landau energy

$$E_{\epsilon,\delta}(u) = \int_\Omega \frac{1}{2}\,|\nabla u|^2 + \frac{1}{4\,\epsilon^2}(1-|u|^2)^2 + \frac{1}{2}\,\delta\,|\partial_x\partial_y u|^2\,dx\,dy\,, \tag{1.1}$$

defined on a class of functions

$$V_g = \{u = u(x,y) :\ u \in H^1_g(\Omega;\mathbb{C})\,,\ \partial_x\partial_y u = h \in L^2(\Omega) \text{ in distribution sense}\} \tag{1.2}$$

with the norm $\|\cdot\|$ defined by $\|u\|^2 = \|u\|^2_{H^1} + \|\partial_x\partial_y u\|^2_{L^2}$ . Hereafter, $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$, $g : \partial\Omega \to S^1$ is a smooth map with degree $d \in \mathbb{N}$, $0 < \epsilon, \delta \ll 1$ are small parameters, and

$$H^1_g(\Omega;\mathbb{C}) = \{u \in H^1(\Omega;\mathbb{C}) :\quad u|_{\partial\Omega} = g\}\,.$$

The $d$-wave Ginzburg-Landau energy describes high-temperature superconductors. From [2], [7] and [8], we learned the $d$-wave Ginzburg-Landau energy without the magnetic field given by

$$G(u) = \int_{\mathbb{R}^2} \frac{1}{2}\,|\nabla u|^2 + \frac{1}{4}(1-|u|^2)^2 + \frac{1}{2}\,\beta\,|(\partial_x^2 - \partial_y^2)\,u|^2\,dx\,dy\,, \tag{1.3}$$

where $\beta$ is a positive constant. Rotating the coordinates by 45°, we may obtain

$$G(u) = \int_{\mathbb{R}^2} \frac{1}{2}\,|\nabla u|^2 + \frac{1}{4}(1-|u|^2)^2 + 2\,\beta\,|\partial_x\partial_y u|^2\,dx\,dy\,, \tag{1.4}$$

Hereafter, we assume that $|u| \to 1$ and all the derivatives of $u$ decay fast as $|(x,y)| \to \infty$. Such an assumption is consistent with the results in [9] and [19]. Then we may transform (1.4) into (1.1) up to some constants.

For the minimization of (1.1), we may use the standard direct method to obtain the energy minimizer $u^\delta_\epsilon$ of $E_{\epsilon,\delta}(\cdot)$ over the function class $V_g$. Here we have used the fact that both $H^1(\Omega)$ and $L^2(\Omega)$ are reflexive Banach spaces. The minimizer $u^\delta_\epsilon$ is a weak solution of

$$\Delta u + \frac{1}{\epsilon^2}\,(1-|u|^2)\,u - \delta\,\partial_x^2\partial_y^2\,u = 0 \quad \text{ on } \Omega\,, \tag{1.5}$$

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The highest derivative term is $\delta\,\partial_x^2\partial_y^2$ which is a degenerate elliptic operator. Such a term has a small divisor $\delta$ and may lose derivatives by the standard bootstrap argument for (1.5). This may cause the main difficulty to get the regularity of $u_\epsilon^\delta$. Until now, there is not any regularity theorem for (1.5). Now we state a general regularity theorem of $u_\epsilon^\delta$ as follows:

Theorem I. *Suppose $\delta = \delta_\epsilon$ is a positive constant which may depend on $\epsilon$. Then there exists a minimizer $u_\epsilon^\delta$ of (1.1) over $V_g$ such that*

$$\|u_\epsilon^\delta\|_{C^{\frac{1}{2}}(\bar{\Omega})} = O(\epsilon^{-1}\sqrt{1+\delta})\,. \tag{1.6}$$

For (1.5), we may rescale the spatial variables by $\epsilon$, and obtain

$$\Delta u + (1-|u|^2)\,u - \delta\,\epsilon^{-2}\,\partial_x^2\partial_y^2\,u = 0 \quad \text{on } \frac{1}{\epsilon}\,\Omega\,, \tag{1.7}$$

where $\frac{1}{\epsilon}\,\Omega \equiv \{(x,y) : (\epsilon\,x, \epsilon\,y) \in \Omega\}$. From physical literature (cf. [2] and [7]), the coefficient $\delta\,\epsilon^{-2}$ of $\partial_x^2\partial_y^2\,u$ is positive and bounded i.e. $0 < \delta = O(\epsilon^2)$ as $\epsilon \to 0+$. Hereafter, we only consider such a quantity for $\delta$. By the same argument of [16], we have

Theorem A. *Suppose $0 < \delta = O(\epsilon^2)$ as $\epsilon \to 0+$. Then there exists a minimizer $u_\epsilon$ of (1.1) over $V_g$ such that*
(i) *$u_\epsilon$ has $d$ degree-one vortices in $\Omega$,*
(ii) *$E_{\epsilon,\delta}(u_\epsilon) = \pi\,d\,\log\frac{1}{\epsilon} + O(1)$ as $\epsilon \to 0+$,*
(iii) *$u_\epsilon$ converges to $u_*$ (up to a subsequence) strongly in $L^2(\Omega)$ and weakly in $H^1_{loc}(\Omega\backslash\{a_1, \cdots, a_d\})$,*
(iv) *$(a_1, \cdots, a_d) \in \Omega^d$ is a global minimizer of the renormalized energy $W_g$ in [3].*

Here $u_*$ is a canonical harmonic map defined by

$$u_*(z) = \prod_{j=1}^{d} \frac{z-a_j}{|z-a_j|}\,e^{i\,h(z)}, \quad \forall z \in \Omega\,, \tag{1.8}$$

and $h$ is a real-valued harmonic function. For the product in (1.8), we have used the fact that $\mathbb{R}^2$ is equivalent to $\mathbb{C}$. Actually, the vortices of $u_\epsilon$ may arbitrarily tend to $a_j$'s (up to a subsequence) as $\epsilon$ goes to zero. Away from the vortex cores $B_\rho(a_j)$'s, we obtain a uniform convergence of $u_\epsilon$ as follows:

Theorem II. *Suppose $0 < \delta = O(\epsilon^2)$ as $\epsilon \to 0+$. Then for $\rho > 0$, the minimizer $u_\epsilon$ converges to $u_*$ (up to a subsequence) uniformly on $\Omega \setminus \cup_{j=1}^d B_\rho(a_j)$ as $\epsilon$ goes to zero, where $u_*$ and $(a_1, \cdots, a_d) \in \Omega^d$ are defined in Theorem A. Hereafter, $B_\rho(a_j)$ is the disk in $\mathbb{R}^2$ with radius $\rho$ and center at $a_j$.*

To estimate $u_\epsilon$ in the vortex cores $B_\rho(a_j)$'s, we may simplify the minimization problem by setting $\Omega = B_1(0)$, where $B_1(0)$ is the unit disk in $\mathbb{R}^2$ with center at the origin. Moreover, we consider a modified minimization problem given by

$$\text{Minimize } E_{\epsilon,\delta} \text{ over } W_0 = V_{g_0} \cap W\,, \tag{1.9}$$

where $g_0 \equiv e^{i\,\theta}$ on $\partial\Omega$ and

$$W = \{u = \sum_{k\in\mathbb{Z}} a_{1+4k}(r)\, e^{i\,(1+4k)\,\theta} \text{ on } \Omega\,, a_{1+4k}(r) \in \mathbb{R}\,, \forall r \in [0,1], k \in \mathbb{Z}\}\,.$$

Hereafter, $(r, \theta)$ denotes the polar coordinates in $\mathbb{R}^2$. The function space $W$ provides fourfold symmetry for the minimizer. Actually, fourfold symmetry is a characteristic of vortex states in $d$-wave superconductors (cf. [5], [7], [9]). Please note that the function space $W_0$ is a subspace of $V_{g_0}$. Hence we cannot assure that the energy minimizer $u_\epsilon$ of $E_{\epsilon,\delta}$ on $W_0$ is a weak solution of (1.5). In [11], we prove that $u_\epsilon$ is a weak solution of (1.5) and we obtain the $H^1$-norm estimate as follows:

Theorem B. *Assume that* $0 < \delta = O(\epsilon^N)$*, where* $N$ *is a positive constant independent of* $\epsilon$*. Then there exists a minimizer* $u_\epsilon$ *of (1.9) such that* $u_\epsilon$ *is a weak solution of (1.5) and*

$$u_\epsilon = u_0^\epsilon + v_\epsilon\,, \tag{1.10}$$

*where* $u_0^\epsilon \equiv f_0(\frac{r}{\epsilon})\, e^{i\,\theta}$ *is the unique energy minimizer of* $E_{\epsilon,0}$ *over* $H^1_{g_0}(\Omega)$ *(cf. [?]),* $v_\epsilon \in W \cap H^1_0(\Omega)$ *satisfies*

$$\|v_\epsilon\|_{L^2(\Omega)} = O(\sqrt{\delta}\epsilon^{-1})\,, \quad \|v_\epsilon\|_{H^1(\Omega)} = O(\sqrt{\delta}\epsilon^{-2})\,. \tag{1.11}$$

*Hereafter,* $u_\epsilon$ *is called the fourfold symmetric vortex solution of (1.5).*

From [6], [10], [12], one may know qualitative theorems of $u_0^\epsilon$. Then Theorem I implies that

$$\|v_\epsilon\|_{C^{\frac{1}{2}}(\bar{\Omega})} = O(1/\epsilon)\,. \tag{1.12}$$

The upper bound of (1.12) may tend to infinity as $\epsilon$ goes to zero. By the fourfold symmetry of $u_\epsilon$, we may improve the estimate (1.12) by

Theorem III. *Assume that* $0 < \delta = O(\epsilon^N)$ *as* $\epsilon$ *goes to zero, where* $N \geq 4$ *is a constant independent of* $\epsilon$*. Then*

$$\|v_\epsilon\|_{C^{\frac{1}{2}}(\bar{\Omega})} = O(\sqrt{\delta}\,\epsilon^{-2})\,, \tag{1.13}$$

Theorem III is essential to prove the stability of the fourfold symmetric vortex solution $u_\epsilon$. Actually, $H^1$-norm estimate (cf. Theorem B) cannot assure the stability of $u_\epsilon$. We may consider the associated quadratic form given by

$$Q_\epsilon(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 - \frac{1}{2\epsilon^2}(1 - |u_\epsilon|^2)|w|^2 + \frac{1}{\epsilon^2}(u_\epsilon \cdot w)^2 + \frac{1}{2}\delta|w_{xy}|^2\, dx\, dy\,, \tag{1.14}$$

for $w \in V_0$, where

$$V_0 = \{u = u(x,y) :\ u \in H^1_0(\Omega; \mathbb{C})\,,\ \partial_x\partial_y u = h \in L^2(\Omega) \text{ in distribution sense}\}\,,$$

and $Q_\epsilon(w) = \frac{1}{2}\frac{d^2}{dt^2}\, E_{\epsilon\,,\delta}(u_\epsilon + tw)|_{t=0}$ is the associated second variational form. In Corollary I, we will use (1.13) to prove $Q_\epsilon(w) > 0$ for $w \in V_0\,, \|w\|_{L^2} \neq 0\,,$ provided the parameter $\delta$ is sufficiently small. Therefore the fourfold symmetric vortex solution $u_\epsilon$ is stable if the parameter $\delta$ is sufficiently small.

From (1.13), it is remarkable that if $N > 4$, then $\|v_\epsilon\|_{C^{\frac{1}{2}}(\bar\Omega)} \to 0$ as $\epsilon$ goes to zero. Theorem III gives us the $C^{\frac{1}{2}}$-norm estimate of the perturbation term $v_\epsilon$ in $\Omega$. Can we have $C^\alpha$-norm, $\frac{1}{2} < \alpha < 1$, estimate of $v_\epsilon$? To answer this question, we state another result for the estimate of $v_\epsilon$ in $C^\alpha$-norm, $\frac{1}{2} < \alpha < 1$, as follows:

THEOREM IV. *Assume that* $0 < \delta = O(\epsilon^N)$ *as* $\epsilon$ *goes to zero, where* $N \geq 6$ *is a constant independent of* $\epsilon$. *Suppose*

$$\liminf_{\eta\to 0+} \eta^{-2} \int_{\{1-\eta\leq x^2+y^2\leq 1\}} |\partial_x^2 v_\epsilon|^2 + |\partial_y^2 v_\epsilon|^2 \, dx\, dy = O(\epsilon^{-6}) \,, \tag{1.15}$$

*for* $0 < \epsilon < \epsilon_0$, *where* $\epsilon_0$ *is a positive constant. Then for* $\frac{1}{2} < \alpha < 1$,

$$\|v_\epsilon\|_{C^\alpha(\bar\Omega)} = O(\sqrt{\delta}\, \epsilon^{-3}) \,. \tag{1.16}$$

Theorem IV implies that the $C^\alpha$-norm, $\frac{1}{2} < \alpha < 1$, estimate of $v_\epsilon$ may depend on the behavior of $\partial_x^2 v_\epsilon$ and $\partial_y^2 v_\epsilon$ near the boundary. It is remarkable that the boundary condition of $u_\epsilon$ is

$$\begin{cases} u_\epsilon & = g_0 \quad \text{on } \partial\Omega\,, \\ \partial_x\partial_y u_\epsilon & = 0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{1.17}$$

Hence the boundary condition of $v_\epsilon$ is

$$\begin{cases} v_\epsilon & = 0 \quad \text{on } \partial\Omega\,, \\ \partial_x\partial_y v_\epsilon & = -\partial_x\partial_y u_0^\epsilon \quad \text{on } \partial\Omega\,. \end{cases} \tag{1.18}$$

As $\epsilon$ goes to zero, $\partial_x\partial_y v_\epsilon|_{\partial\Omega} = -\partial_x\partial_y u_0^\epsilon|_{\partial\Omega} \sim -\partial_x\partial_y e^{i\,\theta} \neq 0$. Thus $\nabla^2 v_\epsilon$ may not tend to zero on $\partial\Omega$, and it is possible that $\partial_x^2 v_\epsilon$ and $\partial_y^2 v_\epsilon$ may have boundary layer on $\partial\Omega$. Therefore (1.15) is necessary to Theorem IV.

To understand more on the structure of a single vortex, we may rescale the spatial variables by $\epsilon$ i.e. we set $\tilde u_\epsilon(x,y) = u_\epsilon(\epsilon\, x, \epsilon\, y)$, for $(x,y) \in \frac{1}{\epsilon}\,\Omega$. Then $\tilde u_\epsilon$ is a weak solution of (1.7). For the convergence of $\tilde u_\epsilon$, we have

THEOREM V. *Let* $\tilde u_\epsilon(x,y) = u_\epsilon(\epsilon\, x, \epsilon\, y)$, *for* $(x,y) \in \frac{1}{\epsilon}\,\Omega$. *Then* $\tilde u_\epsilon$ *converges weakly (up to a subsequence) to* $\tilde u$ *in* $H^1_{loc}(\mathbb{R}^2)$. *Furthermore,* $\tilde u$ *is a weak solution of*

$$\Delta u + (1-|u|^2)u = 0 \quad \text{on } \mathbb{R}^2\,, \text{ if } \delta = o(\epsilon^2)\,, \tag{1.19}$$

$$\Delta u + (1-|u|^2)u - \lambda\,\partial_x^2\partial_y^2\, u = 0 \quad \text{on } \mathbb{R}^2\,, \text{ if } \delta = \lambda\epsilon^2\,, \tag{1.20}$$

*where* $\lambda$ *is a positive constant independent of* $\epsilon$. *In particular, if* $\delta = O(\epsilon^N)\,, N > 4$, *then* $\tilde u = f(r)\, e^{i\,\theta}$, *where* $f$ *is the unique solution of*

$$f'' + \frac{1}{r}\, f' - \frac{1}{r^2}\, f + (1-f^2)\, f = 0\,, \quad \forall r > 0\,, \tag{1.21}$$
$$f(0) = 0\,, f(+\infty) = 1\,.$$

The equation (1.19) has been investigated to find solutions with vortex structures (cf. [3], [4], [6], [12]). However, the uniqueness of (1.19) with $\lim_{|(x,y)|\to\infty} |u(x,y)-e^{i\theta}| = 0$ has not yet been proved. Hence it is still open that $\tilde{u} = f(r)\,e^{i\theta}$ if $0 < \delta = O(\epsilon^N)\,, 2 < N \leq 4$.

In the rest of this paper, we will prove Theorem I and introduce a general regularity theorem in Section 2. In Section 3, we will prove Theorem II. In Section 4 and 5, we will complete the proof of Theorem III, IV and V, respectively.

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**2. General Regularity Theorem.** In this section, we will provide a proof of Theorem I. To prove Theorem I, we need a crucial Lemma given by

LEMMA I. *Assume $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$. Let $u \in V_g$ satisfy*

$$\int_\Omega |\partial_x \partial_y\, u|^2 \leq A\,, \tag{2.1}$$

$$\|u\|^2_{H^1(\Omega)} \leq B\,, \tag{2.2}$$

*where $A, B$ are positive constant and $V_g$ is defined in (1.2). Then*

$$u \text{ is of } C^{\frac{1}{2}}(\bar{\Omega}) \quad \text{and } \|u\|_{C^{\frac{1}{2}}(\bar{\Omega})} \leq C\,\sqrt{A+B}\,, \tag{2.3}$$

*where $C$ is a positive constant depending only on $\Omega$.*

It is remarkable that Lemma I is a general regularity theorem for functions satisfying (2.1) and (2.2). Now we prove Lemma I as follows. From extension theorem (cf. [1]), we may extend the function $u$ on a cube $Q = [a,b] \times [\alpha, \beta]$ such that

$$\int_Q |\partial_x \partial_y\, u|^2 \leq C_0\, A\,, \tag{2.4}$$

$$\|u\|^2_{H^1(Q)} \leq C_0\, B\,, \tag{2.5}$$

where $C_0$ is a positive constant depending on $\Omega$, $a < b; \alpha < \beta$ are constants. By (2.5) and Fubini Theorem, there exists $x_0 \in [a,b]$ such that

$$\|u(x_0,\cdot)\|_{H^1([\alpha,\beta])} \leq B_0\,, \tag{2.6}$$

where $B_0 = 2C_0\, B/|b-a|$. Hence by Sobolev embedding, $u(x_0,\cdot) \in C^{\frac{1}{2}}([\alpha,\beta])$. Fix $x \in [a,b]$ arbitrarily. Without loss of generality, we may assume $u$ is smooth on $Q$.

Then

$$\begin{aligned}
&\int_\alpha^\beta u^2(x,y)+u_y^2(x,y)\,dy\\
&\le 2\int_\alpha^\beta u^2(x_0,y)\,dy+2\int_\alpha^\beta |u(x,y)-u(x_0,y)|^2\,dy\\
&\quad +2\int_\alpha^\beta u_y^2(x_0,y)\,dy+2\int_\alpha^\beta |u_y(x,y)-u_y(x_0,y)|^2\,dy \ \ \text{(by Triangle inequality)}\\
&=2\int_\alpha^\beta u^2(x_0,y)+u_y^2(x_0,y)\,dy+2\int_\alpha^\beta \Big|\int_{x_0}^x u_x(t,y)\,dt\Big|^2\,dy\\
&\quad +2\int_\alpha^\beta \Big|\int_{x_0}^x u_{xy}(t,y)\,dt\Big|^2\,dy\\
&\le 2B_0+2\int_\alpha^\beta |x-x_0|\int_{x_0}^x |u_x(t,y)|^2+|u_{xy}(t,y)|^2\,dt\,dy \quad \text{(by Holder inequality)}\\
&\le 2[B_0+|b-a|\,C_0\,(A+B)] \quad (\because (2.4),(2.5))\,,
\end{aligned}$$

where $u_y=\partial_y u$ and $u_{xy}=\partial_x\partial_y u$. Hence

$$\|u(x,\cdot)\|^2_{H^1([\alpha,\beta])}\le C_1\,(A+B)\,,$$

for $x\in[a,b]$, where $C_1$ is a positive constant depending only on $\Omega$ and $|b-a|$. Thus by Sobolev embedding,

$$\|u(x,\cdot)\|_{C^{\frac12}([\alpha,\beta])}\le C_2\,\sqrt{A+B}\,, \tag{2.7}$$

for $x\in[a,b]$, where $C_2$ is a positive constant depending only on $\Omega$, $|b-a|$ and $|\beta-\alpha|$. Similarly, we may obtain

$$\|u(\cdot,y)\|_{C^{\frac12}([a,b])}\le C_3\,\sqrt{A+B}\,, \tag{2.8}$$

for $y\in[\alpha,\beta]$, where $C_3$ is a positive constant depending only on $\Omega$, $|b-a|$ and $|\beta-\alpha|$. Therefore by (2.7) and (2.8), we may complete the proof of Lemma I.

Now we want to prove Theorem I. From the standard direct method, it is easy to obtain a minimizer $u_\epsilon^\delta\in V_g$ of (1.1). Let $u_\epsilon^0$ be a minimizer of the energy functional

$$E_{\epsilon,0}(u)=\int_\Omega \frac12|\nabla u|^2+\frac{1}{4\epsilon^2}\,(1-|u|^2)^2$$

on $H_g^1(\Omega)$. From [3], we learned the quantitative properties of $u_\epsilon^0$. Then it is easy to check that

$$E_{\epsilon,0}(u_\epsilon^0)=\pi\,d\,\log\frac1\epsilon+O(1)\,, \tag{2.9}$$

$$\int_\Omega |\partial_x\partial_y\,u_\epsilon^0|^2=O(\epsilon^{-2})\,. \tag{2.10}$$

Hence

$$\begin{aligned}
E_{\epsilon,0}(u_\epsilon^0)+\frac12\delta\int_\Omega|\partial_x\partial_y\,u_\epsilon^\delta|^2 &\le E_{\epsilon,0}(u_\epsilon^\delta)+\frac12\delta\int_\Omega|\partial_x\partial_y\,u_\epsilon^\delta|^2\\
&=E_{\epsilon,\delta}(u_\epsilon^\delta)\\
&\le E_{\epsilon,\delta}(u_\epsilon^0)\\
&=E_{\epsilon,0}(u_\epsilon^0)+\frac12\delta\int_\Omega|\partial_x\partial_y\,u_\epsilon^0|^2\\
&=E_{\epsilon,0}(u_\epsilon^0)+\delta\,O(\epsilon^{-2}) \quad (\because (2.10))\,.
\end{aligned}$$

Hence

$$\int_\Omega |\partial_x\partial_y\, u_\epsilon^\delta|^2 = O(\epsilon^{-2})\,. \tag{2.11}$$

Moreover,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u_\epsilon^\delta|^2 &\le E_{\epsilon,\delta}(u_\epsilon^\delta)\\ &\le E_{\epsilon,\delta}(u_\epsilon^0)\\ &= \pi\, d\, \log\frac{1}{\epsilon} + O(\delta\epsilon^{-2}) \quad (\because (2.9), (2.10))\,.\end{aligned}$$

Thus

$$\|u_\epsilon^\delta\|^2_{H^1(\Omega)} = O(\log\frac{1}{\epsilon} + \delta\epsilon^{-2})\,. \tag{2.12}$$

Therefore by (2.11), (2.12) and Lemma I, we may complete the proof of Theorem I.

**3. Minimizer with Multiple Vortices.** In this section, we assume $0 < \delta = O(\epsilon^2)$ as $\epsilon$ goes to zero. From Theorem A in Section 1, we obtain a minimizer $u_\epsilon$ having $d$ degree-one vortices near $a_j, j = 1, \cdots, d$. By Theorem I, the minimizer $u_\epsilon$ is of $C^{\frac{1}{2}}(\bar\Omega)$ and $\|u_\epsilon\|_{C^{\frac{1}{2}}(\bar\Omega)} = O(\epsilon^{-1})$. Such an upper bound is unbounded as $\epsilon$ tends to zero, and cannot assure any strong convergence of $u_\epsilon$. The main purpose of this section is to prove Theorem II and get a bounded estimate for the $C^{\frac{1}{2}}$-norm of $u_\epsilon$ away from the vortex cores.

Now we want to prove Theorem II. Let $\rho > 0$ be a small constant and $\Omega_\rho \equiv \Omega \setminus \cup_{j=1}^d B_\rho(a_j)$, where $a_j$'s are defined in Theorem A. We may define a comparison map $v_\epsilon$ given by

$$v_\epsilon = \begin{cases} u_\epsilon & \text{in } B_\rho(a_j)\,, j = 1, \cdots, d\,,\\ w_\epsilon & \text{in } \Omega_\rho\,,\end{cases} \tag{3.1}$$

where $w_\epsilon$ is the minimizer of the energy functional

$$E_{\epsilon,0}(w;\Omega_\rho) = \int_{\Omega_\rho} \frac{1}{2}|\nabla w|^2 + \frac{1}{4\epsilon^2}(1-|w|^2)^2$$

over the function class $H^1_{\tilde g}(\Omega_\rho)$. Here the boundary condition $\tilde g$ is defined by

$$\tilde g = \begin{cases} g & \text{on } \partial\Omega\,,\\ u_\epsilon & \text{on } \partial B_\rho(a_j)\,, j = 1, \cdots, d\,.\end{cases}$$

From Theorem A and [3], we may obtain quantitative properties of $w_\epsilon$. Now we define the following energy functionals:

$$\begin{aligned}&E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j))\\ &= \sum_{j=1}^d \int_{B_\rho(a_j)} \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{4\,\epsilon^2}(1-|u_\epsilon|^2)^2 + \frac{1}{2}\,\delta\,|\partial_x\partial_y u_\epsilon|^2\,,\end{aligned} \tag{3.2}$$

$$E_{\epsilon,\delta}(w_\epsilon;\Omega_\rho) = \int_{\Omega_\rho} \frac{1}{2}|\nabla w_\epsilon|^2 + \frac{1}{4\,\epsilon^2}(1-|w_\epsilon|^2)^2 + \frac{1}{2}\,\delta\,|\partial_x\partial_y w_\epsilon|^2\,. \tag{3.3}$$

Then

$$
\begin{aligned}
&E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(w_\epsilon;\Omega_\rho) + \frac{1}{2}\delta\int_{\Omega_\rho} |\partial_x\partial_y u_\epsilon|^2 \\
&\le E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(u_\epsilon;\Omega_\rho) + \frac{1}{2}\delta\int_{\Omega_\rho} |\partial_x\partial_y u_\epsilon|^2 \\
&= E_{\epsilon,\delta}(u_\epsilon) \\
&\le E_{\epsilon,\delta}(v_\epsilon) \\
&= E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,\delta}(w_\epsilon;\Omega_\rho) \\
&= E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(w_\epsilon;\Omega_\rho) + \frac{1}{2}\delta\int_{\Omega_\rho} |\partial_x\partial_y w_\epsilon|^2 \,,
\end{aligned}
$$

i.e.

$$\int_{\Omega_\rho} |\partial_x\partial_y u_\epsilon|^2 \le \int_{\Omega_\rho} |\partial_x\partial_y w_\epsilon|^2 \,. \tag{3.4}$$

Similarly,

$$
\begin{aligned}
&E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + \frac{1}{2}\int_{\Omega_\rho} |\nabla u_\epsilon|^2 \\
&\le E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(u_\epsilon;\Omega_\rho) + \frac{1}{2}\delta\int_{\Omega_\rho} |\partial_x\partial_y u_\epsilon|^2 \\
&= E_{\epsilon,\delta}(u_\epsilon) \\
&\le E_{\epsilon,\delta}(v_\epsilon) \\
&= E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,\delta}(w_\epsilon;\Omega_\rho) \,,
\end{aligned}
$$

i.e.

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla u_\epsilon|^2 \le E_{\epsilon,\delta}(w_\epsilon;\Omega_\rho) \,. \tag{3.5}$$

Hence by (3.4), (3.5) and [3], we obtain

$$\int_{\Omega_\rho} |\partial_x\partial_y u_\epsilon|^2 \le K_0 \,, \tag{3.6}$$

$$\int_{\Omega_\rho} |\nabla u_\epsilon|^2 \le K_0 \,, \tag{3.7}$$

where $K_0$ is a positive constant depending on $\Omega_\rho$. Thus (3.6), (3.7) and Lemma I imply that

$$\|u_\epsilon\|_{C^{\frac{1}{2}}(\bar\Omega_\rho)} \le K_1 \,, \tag{3.8}$$

where $K_1$ is a positive constant depending on $\Omega_\rho$. Therefore by (3.8) and Arzela-Ascoli Theorem, we may complete the proof of Theorem II. Note that (3.8) provides a bounded $C^{\frac{1}{2}}$-norm estimate of $u_\epsilon$ on $\Omega_\rho$.

**4. Estimate of a Single Vortex.** In this section, we assume $\Omega = B_1(0)$ is a unit disk with center at the origin, and $0 < \delta = O(\epsilon^N)$ as $\epsilon$ goes to zero, where $N \ge 4$

is a constant independent of $\epsilon$. From Theorem B and using energy comparison, we have

$$\begin{aligned}&E_{\epsilon,0}(u_0^\epsilon)+\tfrac{1}{2}\delta\int_\Omega|\partial_x\partial_y u_\epsilon|^2\\ &\le E_{\epsilon,0}(u_\epsilon)+\tfrac{1}{2}\delta\int_\Omega|\partial_x\partial_y u_\epsilon|^2\\ &=E_{\epsilon,\delta}(u_\epsilon)\\ &\le E_{\epsilon,\delta}(u_0^\epsilon)\\ &=E_{\epsilon,0}(u_0^\epsilon)+\tfrac{1}{2}\delta\int_\Omega|\partial_x\partial_y u_0^\epsilon|^2\,,\end{aligned}$$

i.e.

$$\int_\Omega|\partial_x\partial_y u_\epsilon|^2\le\int_\Omega|\partial_x\partial_y u_0^\epsilon|^2\,. \tag{4.1}$$

Then by (4.1), (1.10) and Holder inequality, we obtain

$$\int_\Omega|\partial_x\partial_y v_\epsilon|^2=O(\epsilon^{-2})\,. \tag{4.2}$$

Here we have used some properties of $u_0^\epsilon$ (cf. [6], [12]).

From (1.11), (4.2) and extension theorem (cf. [1]), we may extend $v_\epsilon$ to a cube $Q_1=[-1,1]\times[-1,1]$ such that

$$\|v_\epsilon\|^2_{H^1(Q_1)}=O(\sqrt{\delta}\,\epsilon^{-2})\,, \tag{4.3}$$

$$\int_{Q_1}|\partial_x\partial_y v_\epsilon|^2=O(\epsilon^{-2})\,. \tag{4.4}$$

Here we have used the fact that $\delta=O(\epsilon^N), N\ge 4$ as $\epsilon$ goes to zero.

Now we want to prove Theorem III. Without loss of generality, we may assume $v_\epsilon$ is smooth on $Q_1$ and satisfies

$$\int_{Q_1}|\partial_x v_\epsilon|^2\le\sqrt{\delta}\,\epsilon^{-2}\,, \tag{4.5}$$

$$\int_{Q_1}|\partial_y v_\epsilon|^2\le\sqrt{\delta}\,\epsilon^{-2}\,, \tag{4.6}$$

$$\int_{Q_1}|\partial_x\partial_y v_\epsilon|^2\le\epsilon^{-2}\,. \tag{4.7}$$

By (4.6) and Fubini Theorem, there exists $x_0\in[-1,1]$ such that

$$\int_{-1}^1|\partial_y v_\epsilon(x_0,y)|^2\,dy\le 2\sqrt{\delta}\,\epsilon^{-2}\,. \tag{4.8}$$

From Theorem B, $u_\epsilon$ is of fourfold symmetry on $\Omega$ i.e. $v_\epsilon$ is of fourfold symmetry on

$Q_1$. Hence we may set $x_0 \in [0,1]$. Moreover,

$$
\begin{aligned}
&\int_{-1}^{1} |\partial_y v_\epsilon(x,y)|^2\, dy \\
&\le 2\int_{-1}^{1} |\partial_y v_\epsilon(x,y) - \partial_y v_\epsilon(x_0,y)|^2\, dy \\
&\quad + 2\int_{-1}^{1} |\partial_y v_\epsilon(x_0,y)|^2\, dy \quad \text{(by triangle inequality)} \\
&= 2\int_{-1}^{1} |\int_{x_0}^{x} \partial_x\partial_y\, v_\epsilon(s,y)\, ds|^2\, dy + 2\int_{-1}^{1} |\partial_y v_\epsilon(x_0,y)|^2\, dy \\
&\le 2|x-x_0| \int_{Q_1} |\partial_x\partial_y\, v_\epsilon|^2 + 2\int_{-1}^{1} |\partial_y v_\epsilon(x_0,y)|^2\, dy \quad \text{(by Holder inequality)} \\
&\le 6\sqrt{\delta}\,\epsilon^{-2} \quad (\because (4.7),(4.8))\,,
\end{aligned}
$$

for $|x-x_0| \le \sqrt{\delta}$, i.e.

$$\int_{-1}^{1} |\partial_y v_\epsilon(x,y)|^2\, dy \le 6\sqrt{\delta}\,\epsilon^{-2} \quad \text{for } x \in I_1 \equiv [x_0-\sqrt{\delta}, x_0+\sqrt{\delta}] \cap [0,1]\,. \tag{4.9}$$

Let $Q_2 = ([-1,1]\backslash I_1) \times [-1,1]$. Then (4.6) implies that

$$\int_{Q_2} |\partial_y v_\epsilon|^2\, dx\, dy \le \sqrt{\delta}\,\epsilon^{-2}\,. \tag{4.10}$$

Hence by Fubini Theorem and the fourfold symmetry of $v_\epsilon$, there exists $x_1 \in [0,1]\backslash I_1$ such that

$$\int_{-1}^{1} |\partial_y v_\epsilon(x_1,y)|^2\, dy \le 2\sqrt{\delta}\,\epsilon^{-2}\,. \tag{4.11}$$

As for (4.9), we obtain

$$\int_{-1}^{1} |\partial_y v_\epsilon(x,y)|^2\, dy \le 6\sqrt{\delta}\,\epsilon^{-2} \quad \text{for } x \in I_2 \equiv [x_1-\sqrt{\delta}, x_1+\sqrt{\delta}] \cap [0,1]\backslash I_1\,. \tag{4.12}$$

By induction, there exist $x_k \in [0,1]\backslash \cup_{j=1}^{k} I_j$ such that

$$\int_{-1}^{1} |\partial_y v_\epsilon(x_k,y)|^2\, dy \le 2\sqrt{\delta}\,\epsilon^{-2}\,, \tag{4.13}$$

$$\int_{-1}^{1} |\partial_y v_\epsilon(x,y)|^2\, dy \le 6\sqrt{\delta}\,\epsilon^{-2}\,, \quad \text{for } x \in I_{k+1}\,, \tag{4.14}$$

for $k = 0,1,2,\cdots N$, where

$$I_{k+1} \equiv [x_k-\sqrt{\delta}, x_k+\sqrt{\delta}] \cap [0,1]\backslash \cup_{j=1}^{k} I_j\,.$$

Here $N = O(1/\sqrt{\delta})$ is a positive integer such that

$$\cup_{k=1}^{N} I_k = [0,1]\,. \tag{4.15}$$

Note that the interiors of $I_k$'s are disjoint to each other. Similarly, by (4.5) and the same argument as (4.13)-(4.15), we may obtain $y_l \in [0,1]\setminus \cup_{k=1}^{l} J_k$ such that

$$\int_{-1}^{1} |\partial_x v_\epsilon(x,y_l)|^2\,dx \le 2\sqrt{\delta}\,\epsilon^{-2}\,, \tag{4.16}$$

$$\int_{-1}^{1} |\partial_x v_\epsilon(x,y)|^2\,dx \le 6\sqrt{\delta}\,\epsilon^{-2}\,, \qquad \text{for } y \in J_{l+1}\,, \tag{4.17}$$

for $l = 0,1,2,\cdots M$, where

$$J_{l+1} \equiv [y_l - \sqrt{\delta}, y_l + \sqrt{\delta}] \cap [0,1]\setminus \cup_{k=1}^{l} J_k\,.$$

Here $M = O(1/\sqrt{\delta})$ is a positive integer such that

$$\cup_{l=1}^{M} J_l = [0,1]\,. \tag{4.18}$$

Note that the interiors of $J_l$'s are disjoint to each other. From (4.14), (4.15) and Sobolev embedding,

$$\|v_\epsilon(x,\cdot)\|_{C^{\frac{1}{2}}([-1,1])} \le C_0\,\sqrt{\delta}\,\epsilon^{-2}\,, \tag{4.19}$$

for $x \in [0,1]$, where $C_0$ is a positive constant independent of $x, y, \epsilon$ and $\delta$. Similarly, by (4.17), (4.18) and Sobolev embedding,

$$\|v_\epsilon(\cdot,y)\|_{C^{\frac{1}{2}}([-1,1])} \le C_0\,\sqrt{\delta}\,\epsilon^{-2}\,, \tag{4.20}$$

for $y \in [0,1]$. Thus (4.19) and (4.20) imply that

$$\|v_\epsilon\|_{C^{\frac{1}{2}}([0,1]\times[0,1])} \le C_1\,\sqrt{\delta}\,\epsilon^{-2}\,, \tag{4.21}$$

where $C_1$ is a positive constant independent of $\epsilon$ and $\delta$. Therefore by (4.21) and the fourfold symmetry of $v_\epsilon$, we may complete the proof of Theorem III.

Now we want to prove Theorem IV. From (1.5) and (1.10), we have

$$\Delta v_\epsilon - \delta\partial_x^2\partial_y^2\, v_\epsilon = \delta\partial_x^2\partial_y^2\, u_0^\epsilon + N_\epsilon(v_\epsilon)\,, \tag{4.22}$$

where

$$N_\epsilon(v_\epsilon) = -\epsilon^{-2}\,[-(1-|u_0^\epsilon|^2)v_\epsilon + 2(u_0^\epsilon\cdot v_\epsilon)u_0^\epsilon + 2(u_0^\epsilon\cdot v_\epsilon)v_\epsilon + |v_\epsilon|^2 u_0^\epsilon + |v_\epsilon|^2 v_\epsilon]\,. \tag{4.23}$$

By (1.11) and (4.23), it is easy to check that

$$\|N_\epsilon(v_\epsilon)\|_{L^2(\Omega)} = O(\sqrt{\delta}\epsilon^{-3})\,. \tag{4.24}$$

Here we have used the assumption that $0 < \delta = O(\epsilon^N)$, $N \ge 6$ as $\epsilon$ goes to zero. Hence by (4.22), (4.24) and [3], we obtain

$$\|\Delta v_\epsilon - \delta\partial_x^2\partial_y^2\, v_\epsilon\|_{L^2(\Omega)} = O(\sqrt{\delta}\epsilon^{-3})\,. \tag{4.25}$$

Let $P \in C_0^\infty(\Omega)$ be a real-valued test function defined later. Then using integration by parts, we have

$$\begin{aligned}\int_\Omega P\,\Delta v_\epsilon\cdot(\Delta v_\epsilon-\delta\partial_x^2\partial_y^2\,v_\epsilon) &= \int_\Omega P|\Delta v_\epsilon|^2-\delta\int_\Omega P\,\Delta v_\epsilon\cdot\partial_x^2\partial_y^2\,v_\epsilon\\ &= \int_\Omega P|\Delta v_\epsilon|^2-\delta\int_\Omega P\,\partial_x^2 v_\epsilon\cdot\partial_x^2\partial_y^2\,v_\epsilon-\delta\int_\Omega P\,\partial_y^2 v_\epsilon\cdot\partial_x^2\partial_y^2\,v_\epsilon\\ &= \int_\Omega P|\Delta v_\epsilon|^2+\delta\int_\Omega P|\partial_x^2\partial_y v_\epsilon|^2-\frac{1}{2}\partial_y^2 P\,|\partial_x^2 v_\epsilon|^2\\ &\quad+\delta\int_\Omega P|\partial_y^2\partial_x v_\epsilon|^2-\frac{1}{2}\partial_x^2 P\,|\partial_y^2 v_\epsilon|^2\,,\end{aligned}$$

i.e.

$$\begin{aligned}\int_\Omega P\,\Delta v_\epsilon\cdot(\Delta v_\epsilon-\delta\partial_x^2\partial_y^2\,v_\epsilon) &= \int_\Omega P|\Delta v_\epsilon|^2+\delta\int_\Omega P\,(|\partial_x^2\partial_y v_\epsilon|^2+|\partial_y^2\partial_x v_\epsilon|^2)\\ &\quad-\tfrac{1}{2}\delta\int_\Omega \partial_y^2 P\,|\partial_x^2 v_\epsilon|^2+\partial_x^2 P\,|\partial_y^2 v_\epsilon|^2\,.\end{aligned} \tag{4.26}$$

Now we define the test function $P$ by

$$P(x,y)=e^{-\frac{2\eta}{1-x^2-y^2}}\,,\quad \text{for } (x,y)\in\Omega\,, \tag{4.27}$$

where $\eta>0$ is a small parameter. Then it is easy to check that

$$\partial_x^2 P\,,\partial_y^2 P\ \le\ 0 \quad \text{for } x^2+y^2\le 1-\eta\,, \tag{4.28}$$

$$\partial_x^2 P\,,\partial_y^2 P\ \le \alpha_0\eta^{-2} \text{ for } 1-\eta\le x^2+y^2\le 1\,, \tag{4.29}$$

where $\alpha_0>0$ is a universal constant independent of $\eta$. Hence (4.26), (4.28) and (4.29) imply that

$$\int_\Omega P\,\Delta v_\epsilon\cdot(\Delta v_\epsilon-\delta\partial_x^2\partial_y^2\,v_\epsilon)\ge\int_\Omega P|\Delta v_\epsilon|^2-\frac{1}{2}\alpha_0\delta\eta^{-2}\int_{1-\eta\le x^2+y^2\le 1}|\partial_x^2 v_\epsilon|^2+|\partial_y^2 v_\epsilon|^2\,. \tag{4.30}$$

From (4.24), (4.27) and Holder inequality, we have

$$\begin{aligned}\int_\Omega P\,\Delta v_\epsilon\cdot(\Delta v_\epsilon-\delta\partial_x^2\partial_y^2\,v_\epsilon) &\le \|P\,\Delta v_\epsilon\|_{L^2(\Omega)}\,\|\Delta v_\epsilon-\delta\partial_x^2\partial_y^2\,v_\epsilon\|_{L^2(\Omega)}\\ &\le \frac{1}{2}\|P\,\Delta v_\epsilon\|^2_{L^2(\Omega)}+\frac{1}{2}\|\Delta v_\epsilon-\delta\partial_x^2\partial_y^2\,v_\epsilon\|^2_{L^2(\Omega)}\\ &\le \frac{1}{2}\int_\Omega P\,|\Delta v_\epsilon|^2\,dx\,dy+O(\delta\epsilon^{-6})\quad(\because (4.24),(4.27))\,,\end{aligned}$$

i.e.

$$\int_\Omega P\,\Delta v_\epsilon\cdot(\Delta v_\epsilon-\delta\partial_x^2\partial_y^2\,v_\epsilon)\le\frac{1}{2}\int_\Omega P\,|\Delta v_\epsilon|^2\,dx\,dy+O(\delta\epsilon^{-6})\,. \tag{4.31}$$

Hence (4.30) and (4.31) imply

$$\int_\Omega P\,|\Delta v_\epsilon|^2\le\alpha_0\delta\eta^{-2}\int_{1-\eta\le x^2+y^2\le 1}|\partial_x^2 v_\epsilon|^2+|\partial_y^2 v_\epsilon|^2\,dx\,dy+O(\delta\epsilon^{-6})\,. \tag{4.32}$$

Hence by (1.15), (4.32) and Fatou's Lemma,

$$\int_\Omega |\Delta v_\epsilon|^2 = O(\delta\,\epsilon^{-6})\,. \tag{4.33}$$

Therefore by (4.33) and standard theorems for $W^{2,p}$ estimate and Sobolev embedding, we may complete the proof of Theorem IV.

REMARK. From [11], the minimizer $u_\epsilon$ of (1.9) is a weak solution of (1.5). However, $u_\epsilon$ may not be a global minimizer of $E_{\epsilon\,,\delta}$ on $V_{g_0}$. We will prove that $u_\epsilon$ is a local minimizer of $E_{\epsilon\,,\delta}$ on $V_{g_0}$ if $\delta$ is sufficiently small. From Theorem B and Theorem III, (1.14) becomes

$$\begin{aligned} Q_\epsilon(w) &= \int_\Omega \frac{1}{2}|\nabla w|^2 - \frac{1}{2\epsilon^2}(1-|u_\epsilon^0|^2)|w|^2 + \frac{1}{\epsilon^2}(u_\epsilon^0\cdot w)^2 + \frac{1}{2}\delta|w_{xy}|^2\,dx\,dy \\ &\quad + \epsilon^{-2}\,O(\|v_\epsilon\|_{L^\infty} + \|v_\epsilon\|_{L^\infty}^2)\,\|w\|_{L^2} \quad (\because (1.10)) \\ &\geq \int_\Omega \frac{1}{2}|\nabla w|^2 - \frac{1}{2\epsilon^2}(1-|u_\epsilon^0|^2)|w|^2 + \frac{1}{\epsilon^2}(u_\epsilon^0\cdot w)^2\,dx\,dy \\ &\quad + O(\sqrt{\delta}\epsilon^{-4})\,\|w\|_{L^2}^2\,, \quad (\because (1.13)) \end{aligned}$$

i.e.

$$Q_\epsilon(w) \geq Q_\epsilon^0(w) + O(\sqrt{\delta}\epsilon^{-4})\,\|w\|_{L^2}^2\,, \tag{4.34}$$

for $w \in V_0$, where

$$Q_\epsilon^0(w) \equiv \int_\Omega \frac{1}{2}|\nabla w|^2 - \frac{1}{2\epsilon^2}(1-|u_\epsilon^0|^2)|w|^2 + \frac{1}{\epsilon^2}(u_\epsilon^0\cdot w)^2\,dx\,dy\,.$$

By (4.34) and [17], we have

$$Q_\epsilon(w) > 0 \quad \text{for } w \in V_0\,, \|w\|_{L^2} \neq 0\,, \tag{4.35}$$

provided

$$0 < \delta = o(\lambda_{1,\epsilon}^2\epsilon^8)\,, \quad 0 < \lambda_{1,\epsilon} = o(1)\,, \tag{4.36}$$

where $o(1)$ is a small quantity which tends to zero as $\epsilon$ goes to zero. Actually, $\lambda_{1,\epsilon}$ is the minimization of $Q_\epsilon^0(w)$ for $w \in H_0^1(\Omega)\,, \|w\|_{L^2} = 1$(cf. [11], [17], [18]). Therefore we have

COROLLARY I. *Under the same assumptions as Theorem III and (4.36), the fourfold symmetric vortex solution $u_\epsilon$ is stable.*

**5. Proof of Theorem V.** From Theorem A (ii), we obtain

$$\int_\Omega e_\epsilon(u_\epsilon) = \pi\log\frac{1}{\epsilon} + O(1)\,, \tag{5.1}$$

where $e_\epsilon$ is the energy density of $E_{\epsilon,0}$ and is defined by

$$e_\epsilon(u) \equiv \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1-|u|^2)^2\,.$$

We let

$$\hat{u}_\epsilon = \begin{cases} u_\epsilon & \text{if } |u_\epsilon| \le 1\,, \\ u_\epsilon/|u_\epsilon| & \text{if } |u_\epsilon| > 1\,. \end{cases}$$

Then $\hat{u}_\epsilon$ satisfies

$$|\hat{u}_\epsilon| \le 1 \quad \text{in } \Omega\,, \tag{5.2}$$

and

$$\int_\Omega e_\epsilon(\hat{u}_\epsilon) \le \int_\Omega e_\epsilon(u_\epsilon) \le \pi \log \frac{1}{\epsilon} + M_0\,, \tag{5.3}$$

where $M_0$ is a positive constant independent of $\epsilon$. By (5.2), (5.3) and the proof of Proposition 1.1 in [15], $\hat{u}_\epsilon$ has only one essential zero $a^\epsilon$ in $\Omega$ and $\deg(\frac{\hat{u}_\epsilon}{|\hat{u}_\epsilon|}, \partial B) = 1$, where $B = B_{\epsilon^\alpha}(a^\epsilon)$ and $\alpha \in (0,1)$. Without loss of generality, we may assume that $a^\epsilon = 0$. Now we want to prove

LEMMA II. *For each $R_0 > 1$, there exists a positive constant $C$ depending only on $R_0$ such that*

$$\int_{\Omega \setminus B_{\epsilon R_0}(0)} e_\epsilon(\hat{u}_\epsilon) \ge \pi \log \frac{1}{\epsilon} - C(R_0)\,, \qquad \textit{as } \epsilon \to 0+\,. \tag{5.4}$$

*Proof of Lemma II.* Fix $R_0 > 1$ arbitrarily. Let $\tilde{\epsilon} = \epsilon R_0$. Then (5.3) implies

$$\int_\Omega e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \le \int_\Omega e_\epsilon(\hat{u}_\epsilon) \le \pi \log \frac{1}{\tilde{\epsilon}} + M_0'\,, \tag{5.5}$$

where $M_0' = M_0 + \pi \log R_0$. Again, by (5.2), (5.5) and the proof of Proposition 1.1 in [15], $\hat{u}_\epsilon$ has only one essential zero $a^\epsilon$ in $\Omega$ and $\deg(\frac{\hat{u}_\epsilon}{|\hat{u}_\epsilon|}, \partial B') = 1$, where $B' = B_{\tilde{\epsilon}^{\tilde{\alpha}}}(a^\epsilon)$ and $\tilde{\alpha} \in (0,1)$. Moreover, by the same argument of Lemma 2.2 in [14] and [15], we obtain

$$\int_{\Omega \setminus B'} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \ge \pi \tilde{\alpha} \log \frac{1}{\tilde{\epsilon}} - M_1\,, \tag{5.6}$$

and

$$\int_{B'} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \le \pi (1 - \tilde{\alpha}) \log \frac{1}{\tilde{\epsilon}} + M_1\,, \tag{5.7}$$

where $M_1$ is a positive constant independent of $\epsilon$.

Now we claim that

$$\int_{B' \setminus B_{\tilde{\epsilon}}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \ge \pi (1 - \tilde{\alpha}) \log \frac{1}{\tilde{\epsilon}} - K\,, \tag{5.8}$$

where $K$ is a positive constant independent of $\epsilon$. Without loss of generality, we may assume that $B' = B_{\theta_0}(0), \theta_0 = \tilde{\epsilon}^{\tilde{\alpha}}$. Then (5.7) implies that

$$\int_{B_{\theta_0}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \le \pi \log \frac{\theta_0}{\tilde{\epsilon}} + M_1\,. \tag{5.9}$$

Moreover, we may rescale the spatial variable and rewrite (5.7) as

$$\int_{B_1(0)} e_{\epsilon_1}(\hat{u}_\epsilon) \le \pi\,(1-\tilde{\alpha})\,\log\frac{1}{\tilde{\epsilon}} + M_1\,, \tag{5.10}$$

where $\epsilon_1 = \tilde{\epsilon}^{1-\tilde{\alpha}}$. By (5.10) and Fubini theorem (cf. [14]), there exists $\theta_1 \in (\tilde{\epsilon}^{2\tilde{\alpha}}, \tilde{\epsilon}^{\tilde{\alpha}})$ such that

$$\theta_0\theta_1 \int_{\partial B_{\theta_0\theta_1}(0)} e_{\tilde{\epsilon}}\,(\hat{u}_\epsilon) \le C(\tilde{\alpha}, M_1)\,,$$

and that

$$\deg(\frac{\hat{u}_\epsilon}{|\hat{u}_\epsilon|}, \partial B_{\theta_0\theta_1}(0)) = 1\,.$$

Hence by the same argument of Lemma 2.2 in [14] and [15], we have

$$\int_{B_{\theta_0}(0)\backslash B_{\theta_0\,\theta_1}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \ge \pi\,\log\frac{1}{\theta_1} - M_2\,, \tag{5.11}$$

and

$$\int_{B_{\theta_0\,\theta_1}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \le \pi\,\log\frac{\theta_0\,\theta_1}{\tilde{\epsilon}} + M_1 + M_2\,, \tag{5.12}$$

where $M_2$ is a positive constant satisfying $M_2 \le C_0\,\theta_0$. Here $C_0$ is a positive constant independent of $\epsilon$. Thus by induction, we may obtain $\theta_1, \cdots, \theta_m \in (\tilde{\epsilon}^{2\tilde{\alpha}}, \tilde{\epsilon}^{\tilde{\alpha}})$ such that $\tilde{\epsilon} = \theta_0\theta_1\cdots\theta_m$ and

$$\int_{B_{\theta_0\cdots\theta_{k-1}}(0)\backslash B_{\theta_0\cdots\theta_k}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \ge \pi\,\log\frac{1}{\theta_k} - M_{k+1}\,, \tag{5.13}$$

and

$$\int_{B_{\theta_0\cdots\theta_k}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \le \pi\,\log\frac{\theta_0\cdots\theta_k}{\epsilon} + \sum_{j=1}^{k+1} M_j\,, \tag{5.14}$$

for $k = 1, \cdots, m$, where $M_j$'s are positive constants satisfying $M_{k+1} \le C_0\,\theta_0^k$ for $k \ge 0$. Note that $\sum_{j=1}^{k+1} M_j \le M_1 + C_0 \sum_{j=1}^{\infty} \theta_0^j \le C_1$, where $C_1$ is a positive constant independent of $\epsilon$ and $k$. Therefore by (5.13), we may obtain (5.8). By (5.6) and (5.8), we obtain (5.4) and complete the proof of Lemma II. Here we have used the fact that $\int_{\Omega\backslash B_{\tilde{\epsilon}}(0)} e_\epsilon(\hat{u}_\epsilon) \ge \int_{\Omega\backslash B_{\tilde{\epsilon}}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon)$ because of $R_0 > 1$.

From Lemma II, it is obvious that

$$\int_{\Omega\backslash B_{\epsilon\,R_0}(0)} e_\epsilon(u_\epsilon) \ge \pi\log\frac{1}{\epsilon} - C(R_0)\,, \tag{5.15}$$

Hence (5.1) and (5.15) imply that

$$\int_{B_{\epsilon\,R_0}(0)} e_\epsilon(u_\epsilon) \le K(R_0)\,, \quad \forall R_0 > 1\,, \tag{5.16}$$

where $K$ is a positive constant depending only on $R_0$. Since $\tilde{u}_\epsilon(x,y) = u_\epsilon(\epsilon\, x, \epsilon\, y)$, for $(x,y) \in \frac{1}{\epsilon}\,\Omega$, then (5.16) implies that

$$\int_{B_{R_0}(0)} e_1(\tilde{u}_\epsilon) \leq K(R_0)\,, \quad \forall R_0 > 1\,. \tag{5.17}$$

Thus $\|\tilde{u}_\epsilon\|_{H^1(B_{R_0}(0))} \leq K'(R_0)$ for all $R_0 > 1$, where $K'$ is a positive constant independent of $\epsilon$. Therefore we may obtain that $\tilde{u}_\epsilon$ converges to $\tilde{u}$ weakly in $H^1_{loc}(\mathbb{R}^2)$ as $\epsilon$ goes to zero(up to a subsequence). Moreover, by (1.7), it is obvious that $\tilde{u}$ is a weak solution of (1.19) if $\delta = o(\epsilon^2)$, and $\tilde{u}$ is a weak solution of (1.20) if $\delta = \lambda\epsilon^2$, where $\lambda$ is a positive constant independent of $\epsilon$. By Theorem III, [6] and [12], we may obtain $\tilde{u} = f(r)\, e^{i\,\theta}$ if $\delta = O(\epsilon^N)\,, N > 4$, where $f$ satisfies (1.21), and we complete the proof of Theorem V.

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