Methods and Applications of Analysis.
Vol. 9, No. 3, pp. 463–468, September 2002

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TURBULENCE WITHOUT PRESSURE: EXISTENCE OF THE INVARIANT MEASURE *

HENRY P. MCKEAN†

1. Introduction. A number of proofs have been offered of the fact that Burgers' equation, with Brownian external force, settles down, with time, into a statistically steady state: see, for instance, Sinai [1996], E-Khanin-Mazel-Sinai [2000], and Kuksin-Shirikyan [2001]. I propose a simple proof based on ideas of Döblin [1940] and Feller [1966]. The equation in question:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + e\frac{db}{dt}$$

represents an ∞-dimensional diffusion in the space of functions $v(x)$: $0 \le x < 1$ of period 1 say, with mean value $\int_0^1 v = 0$. The external force edb/dt is a sum of "modes" $e_n(x) \equiv$ a constant $e_n \times \sqrt{2}\sin/\cos(2\pi nx)$, indexed by $n \ge 1$, multiplied each by the differential of its private 1-dimensional standard Brownian motion $b_n(t) : 0 \le t < \infty$. It is assumed for the present proof that all modes are active, *i.e.* $e_n \ne 0$ for any $n \ge 1$, and that the force is smooth in respect to $0 \le x < 1$, *i.e.* that e_n vanishes rapidly; the second proviso permits you to realize the diffusion in the space $C^\infty[0, 1)$. The force competes with the restoring drift $(1/2)\partial^2 v/\partial x^2$, pulling back towards the origin as per $\int_0^1 vv'' = \int_0^1 (v')^2 \le 0$, and with the twist $v\partial v/\partial x$, so-called because $\int_0^1 v(vv') = 0$, the outcome being the statistical steady state cited at the start. The simplicity of the present method has its price: in particular, it *does not yield* the exponentially fast convergence of $F_t(v) \equiv E_v[F(v_t)]$ to the invariant mean $\int F(v)dM(v)$, which must be a consequence of the rapid return of the diffusion to the vicinity of $v \equiv 0$. Observe, in this connection,

$$d\int_0^1 v^2 = -\int_0^1 (v')^2\,dt + 2\int_0^1 ev\,db + \int_0^1 e^2\,dt$$

$$\le -4\pi^2\int_0^1 v^2\,dt + 2\int_0^1 ev\,db + \int_0^1 e^2\,dt$$

with the obvious result that, up to the passage time $T = \min(t : \int_0^1 v^2 = r^2)$,

$$e^{4\pi^2 t}r^2 \le e^{4\pi^2 t}\int_0^1 v^2 \le \int_0^1 v_0^2 + 2\int_0^t e^{4\pi^2 s}\int_0^1 ev\,db + \int_0^1 e^2\,\frac{e^{4\pi^2 t}-1}{4\pi^2},$$

which yields

$$E_v(e^{4\pi^2 T}) \le \frac{R^2}{r^2 - (1/4\pi^2)\int_0^1 e^2} \text{ for } R^2 = \int_0^1 v^2 > r^2 > \frac{1}{4\pi^2}\int_0^1 e^2.$$

*Received July 2, 2002; accepted for publication December 13, 2002.
†CIMS, 251 Mercer Street, New York, NY 10012, USA (mckean@cims.nyu.edu).

2. The Diffusion. The equation can be solved with the help of the Cole-Hopf substitution: if $w = exp[-\int_0^1 d\xi \int_\xi^x v(\eta)d\eta]$, then

$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + wf db + \frac{w}{2}\left[f^2 - \int_0^1 \left(\frac{w'}{w}\right)^2\right]$$

with $-f' = e$, $w > 0$ and $\int_0^1 \ell n w = 0$, and *this* equation yields to the Feynman-Kaç formula: $w(t,x) = Z^{-1}E_x[\mathfrak{z} w(0,x_t)]$, in which $x(t) : t \geq 0$ is an auxiliary 1-dimensional standard Brownian motion,

$$\mathfrak{z} = \exp[\int_0^t f(x_{t-s})db_s], \text{ and } Z = \exp\left[\int_0^1 \ell n E_\bullet(\mathfrak{z} w)\right]$$

is a normalizer to keep $\int_0^1 \ell n w \equiv 0$. The recipe may be re-expressed in terms of the auxiliary Brownian motion *tied* at $x(0) = 0$ and $x(t) = 0$. Then a simple application of Kolmogorov-Čentsov shows that the path w (and so also v) can be realized in the space of functions jointly of class $C[0,\infty)$ in respect to $t \geq 0$ and of class $C^\infty[0,1)$ in respect to $0 \leq x < 1$. In this way the diffusion is constructed: $v = -w'/w$. The aim is now to prove the existence of the limit $F_\infty(v) \lim_{t\uparrow\infty} E_v[F/v_t]$ and to identify it as the invariant mean $\int F(v)dM(v)$. Naturally, it is essential that the mass of the distribution of v not run out to ∞. I dispose of this at once by the estimate employed at the end of Section 1 which yields

$$E\left(\int_0^1 v^2\right) \leq e^{-4\pi^2 t}\int_0^1 v_0^2 + \int_0^1 e^2 \frac{1}{4\pi^2}(1 - e^{-4\pi^2 t})$$

whence

$$P\left(\int_0^1 v^2 > R^2\right) \leq R^{-2}\left[e^{-4\pi^2 t}r^2 + \frac{1}{4\pi^2}\int_0^1 e^2\right] \text{ with } r^2 = \int_0^1 v_0^2.$$

3. Equicontinuity. Let $\overset{\bullet}{v}(t,x)$ be the functional gradient $\partial v(t,x)/\partial v(0,y)$ for fixed $0 \leq y < 1$. You have $\partial\overset{\bullet}{v}/\partial t = (1/2)\partial^2\overset{\bullet}{v}/\partial x^2 - (\partial/\partial x)(v\overset{\bullet}{v})$ with $\overset{\bullet}{v}(0,x)dx =$ the unit mass at $x = y$, and this may be solved by a combination of Cameron-Martin and Feynman-Kaç: *to wit*,

$$\overset{\bullet}{v}(t,x) = E_x[e^{-\int_0^t v(t-s,\,x_s)dx_s - \frac{1}{2}\int_0^t v^2(t-s,x_s)ds - \int_0^t v'(t-s,x_s)ds}, x_t = y]^1$$

which reduces to

$$E_y[e^{\int_0^t v(s,x_s)dx_s - \frac{1}{2}\int_0^t v^2(s,x_s)ds}, x_t = x] \equiv E_y[\mathfrak{v}, x_t = x]$$

upon reversal of the auxiliary Brownian path as per $x(s) \to x(t-s)$ $(s \leq t)$. Now the chain rule in function space applied to $F_t(v) = E_v[F(v_t)] = {}^2BM[F(v_t)]$ with F of class $C^1[C[0,1) \to \mathbb{R}]$ and $v + \vartheta\Delta v$ in place of v, plain, yields

$$F_t(v + \Delta v) - F_t(v) = \int_0^1 d\vartheta \int_0^1 \Delta v(y)dy\ BM \int_0^1 \text{grad}F\ E_y[\mathfrak{v}, x_t = x]dx$$

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[1] $E[I, x_t = y]$ is short for the density $(\partial/\partial y)E[I, x_t \leq y]$.

[2] BM is the Brownian mean over the individual motions $b_n : n \geq 1$.

with grad F taken at v_t, so that

$$|F_t(v+\Delta v) - F_t(v)| \le |\text{ grad } F|_\infty \int_0^1 |\Delta v| dy E_y(\mathfrak{v}) \le |\text{ grad } F|_\infty |\Delta v|_\infty$$

in view of $E(\mathfrak{v}) \le 1$. This provides compactness, permitting you to choose $\alpha = \alpha_1 > \alpha_2 >$ *etc.* $\downarrow 0$ so as to make $G_\alpha(v) = \alpha \int_0^\infty e^{-\alpha t} F_t(v) dt$ converge to a function $G_0(v)$ of class $C[C[0,1) \to \mathbb{R}]$, uniformly on compact figures such as $K = (v : \int_0^1 (v')^2 \le R^2)$. I prefer this mode of convergence to the plain $\lim_{t\uparrow\infty} F_t(v)$ as it avoids a difficulty with the non-compactness of $C[0,1)$.

4. $G_0(v)$ **is Constant in Respect to** v**.** The point is that the diffusion comes close to the origin $v \equiv 0$ so that the path emanating from that place is typical; this is the idea of Döblin [1940]. Let a small number r and a big number R be fixed, let K be the compact figure $(v : (\int_0^1 ev)^2 \le r^2$ & $\int_0^1 (v')^2 \le R^2)$, and let T be the smaller of the passage time to K and an adjustable integer $N = 1, 2, 3$ *etc.* Then

$$G_\alpha(v) = \alpha E_v [\int_0^T e^{-\alpha t} F_t(v) dt] + E_v[e^{-\alpha T} G_\alpha(v_T)]$$

implies 1) $G_0(v) = E_v[G_0(v_T)]$ since $T \le N$; 2) the same with T now equal to the passage time to K, by making $N \uparrow \infty$; and 3) $G_0(v) = G_0(0)$ by making $r \downarrow 0$ so that K shrinks to the origin. It is here that the proviso $e_n \ne 0 (n \ge 1)$ is used. Of course 2) is correct only if the passage time to K is finite with probability one. This is so provided R is big enough.

Proof. If, for some small r and big R, the passage time T is infinite, then for every $t \ge 0$, either $(\int_0^1 ev)^2 > r^2$ or $\int_0^1 (v')^2 > R^2$. Let E be the set of times $s \le t$ when $(\int_0^1 ev)^2 > r^2$ and E' its complement, on which you must have $\int_0^1 (v')^2 > R^2$. Two cases arise.

Case 1: $\int_0^\infty e^{4\pi^2 t} \left(\int_0^1 ev\right)^2 dt < \infty$. Then

$$d\int_0^1 v^2 = -\int_0^1 v'^2\, dt + 2\int_0^1 ev\, db + \int_0^1 e^2\, dt,$$

$$\le -\frac{1}{2}\int_0^1 (v')^2\, dt - 2\pi^2 \int_0^1 v^2\, dt + 2\int_0^1 ev\, db + \int_0^1 e^2\, dt,$$

and the resulting estimate

$$e^{2\pi^2 t}\int_0^1 v^2 \le \int_0^1 v_0^2 - \frac{1}{2}\int_0^t e^{2\pi^2 s}\int_0^1 (v')^2\, ds + 2\int_0^t e^{2\pi^2 s}\int_0^1 ev\, db + \int_0^1 e^2 \times \frac{e^{2\pi^2 t}}{2\pi^2}$$

implies[3]

$$\int_0^t ds\, e^{2\pi^2 s}\int_0^1 (v')^2 \le \int_0^1 e^2 \times e^{2\pi^2 t} \quad \text{for } t\uparrow\infty.$$

But now

$$2e^{2\pi^2 t}\int_0^1 e^2 \ge \int_0^t e^{2\pi^2 s}(\int_0^1 ev)^2 + \int_0^t e^{2\pi^2 s}\int_0^1 (v')^2$$

$$\ge r^2\int_E e^{4\pi^2 s} + R^2\int_{E'} e^{2\pi^2 s}$$

cannot be balanced as $t\uparrow\infty$ if R is too big in comparison to $\int_0^1 e^2$, no matter how small the fixed number $r>0$ may be.

Case 2: $\int_0^\infty e^{4\pi^2 t}(\int_0^1 ev)^2 dt = \infty$. You have

$$e^{2\pi^2 t}\int_0^1 v^2 \le \int_0^1 v_0^2 - \frac{1}{2}\int_0^t e^{2\pi^2 s}\int_0^1 (v')^2\, ds + 2\int_0^t e^{2\pi^2 s}\int_0^1 ev\, db + \int_0^1 e^2 \times \frac{e^{2\pi^2 t}}{2\pi^2}$$

as before, and an application of the law of the iterated logarithm to the Brownian integral produces the over-estimate of the right side by

$$\int_0^1 v_0^2 - \frac{1}{2}\int_0^t e^{2\pi^2 s}\int_0^1 (v')^2 - 2\sqrt{\int_0^t e^{4\pi^2 s}\left(\int_0^1 ev\right)^2 \times \ell n\, \ell n\ (\text{ditto})} + \int_0^1 e^2 \times e^{2\pi^2 t},$$

valid *i.o.* as $t\uparrow\infty$[4], so that, *i.o.*,

$$N\times\sqrt{\int_0^t e^{4\pi^2 s}\left(\int_0^1 ev\right)^2} + \frac{1}{2}\int_0^t e^{2\pi^2 s}\int_0^1 (v')^2 + \le \int_0^1 v_0^2 + \int_0^1 e^2 \times e^{2\pi^2 t}$$

for any $N = 1, 2, 3$ *etc.* you like, and

$$Nr\sqrt{\int_E e^{4\pi^2 s}} + \frac{R^2}{2}\int_{E'} e^{2\pi^2 s} \le 2\int_0^1 e^2 \times e^{2\pi^2 t}\ \ i.o.$$

But then $\int_{E'} e^{2\pi^2 s}$ is small compared to $e^{2\pi^2 t}$, R being large, so that

$$\int_E e^{4\pi^2 s} = \frac{e^{4\pi^2 t}-1}{4\pi^2} - \int_{E'} e^{4\pi^2 s} \ge \frac{e^{4\pi^2 t}-1}{4\pi^2} - e^{2\pi^2 t}\int_{E'} e^{2\pi^2 s}$$

is comparable to $(1/4\pi^2)e^{4\pi^2 t}$, and the preceding display may be unbalanced by choice of N.

[3] $\int_0^\infty I db$ is finite if $\int_0^\infty I^2 dt < \infty$ for any non-anticipating I.

[4] The point is that $\int_0^t I db$ looks like a standard 1-dimensional Brownian motion run with the clock $\int_0^t I^2$.

5. Identification of $G_0(0)$. To complete the proof, it is necessary to know that $G_0(0)$ does not depend upon the mode of approach of α to 0^+. Then $G_0(0) = \int F(v)dM(v)$ with invariant M: in fact, G_α formed with $F_t(v) = E_v[F(v_t)]$ in place of F is nothing but $E_v[G_\alpha(v_t)]$ with the old G_α so that

$$\int F_t(v)dM(v) = E_v[G_0(v_t)] = G_0(0) = \int F(v)dM(v),$$

as advertised. The uniqueness of the invariant measure is now self evident, too. The omitted identification of $G_0(0)$ is simple. Take $F \geq 0$ and let it vanish off the compact figure $K = (v : \int_0^1 (v')^2 \leq R^2)$. This is harmless to the generality of F, R being adjustable. Let m_α be the maximum of G_α; obviously, $m_\alpha \downarrow m_0 \geq 0$ as $\alpha \downarrow 0$ and $G_0 \leq m_0$. It is to be proved that $G_0 \equiv m_0$.

Proof. Let T be the passage time to K. Then, with the cut-off in F, $F(v_t) = 0$ for $t \leq T$, and $G_\alpha(v) = E_v[e^{-\alpha T} G_\alpha(v_T)]$; in particular, G_α peaks at some place $v_\alpha \epsilon K$. Now, with $\alpha =$ the old α_n of §3 and $n \uparrow \infty$, you have $m_\alpha = G_\alpha(v_\alpha)$, and the convergence of $G_\alpha(v)$ to the constant $G_0(0)$, which is uniform on the compact K, implies $m_0 = G_0(v_0)$ for some $v_0 \epsilon K$. Then $m_0 = G_0(0)$ ——— in short, the full $\lim_{\alpha \downarrow 0} G_\alpha(v) = m_0$ exists. This nice trick is adapted from Feller [1966].

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