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WELL-POSEDNESS OF SCALAR CONSERVATION LAWS WITH SINGULAR SOURCES *

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Abstract. We consider scalar conservation laws with nonlinear singular sources with a concentration effect at the origin. We assume that the flux A is not degenerated and we study whether it is possible to define a well-posed limit problem. We prove that when A is strictly monotonic then the limit problem is well-defined and has a unique solution. The definition of this limit problem involves a layer which is shown to be very stable. But when A is not monotonic this problem can be unstable. Indeed we can construct two sequences of approximate solutions which converge to two different functions although their initial values coincide in the limit.

1. Introduction. This work is motivated by the lost of well-posedness which can appear in hyperbolic conservation laws with low regularity singular source terms. Especially, the Saint-Venant model of shallow water develops such a singular term when the bottom is discontinuous, a situation that occurs naturally in numerical analysis.

As a simplified model we consider the following scalar conservation law:

$$\frac{\partial u_\epsilon}{\partial t} + \frac{\partial A(u_\epsilon)}{\partial x} + Z_\epsilon(x)B(u_\epsilon) = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{1}$$

$$u_\epsilon(x, t = 0) = u_\epsilon^0(x), \quad u_\epsilon^0(x) \in L^1 \cap L^\infty(\mathbb{R}), \tag{2}$$

where the unknown functions $u_\epsilon \in L^\infty(\mathbb{R}\times]0, +\infty[)$ belongs to $\mathbb{R}$, A and B are smooth functions, and $Z_\epsilon \in L^1(\mathbb{R})$ converges to a Dirac mass:

$$\begin{cases} Z_\epsilon \text{ is supported in } [0, \epsilon] \\ \epsilon Z_\epsilon(\epsilon y) \longrightarrow K(y) \ \text{ in } L^1(\mathbb{R}). \end{cases} \tag{3}$$

For future reference we also introduce the notations:

$$\begin{aligned} &a(\xi) = A'(\xi), \\ &\lambda = \int_0^1 K(y)\,dy, \\ &a_0 = \sup|B'/A'| < +\infty, \\ &b_0 = \sup|B/A'| < +\infty, \\ &\Phi(u) = \int_0^u \frac{A'}{B}(v)\,dv \quad \text{(not always defined)}. \end{aligned}$$

As usual, we consider only solutions of (1) which fulfill the entropy conditions:

$$\frac{\partial S(u_\epsilon)}{\partial t} + \frac{\partial \eta(u_\epsilon)}{\partial x} + Z_\epsilon(x)S'(u_\epsilon)B(u_\epsilon) \leq 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{4}$$

for all convex entropy functions S and flux of entropy η related by:

$$\eta' = A'S'. \tag{5}$$

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We show in this paper the following result:

Theorem 1.1. *(Main Theorem) Assume $A', A'' > 0$ and that $u^0_\epsilon \to u^0 \in L^1 \cap L^\infty$ in $L^1(\mathbb{R})$. Then the full familly u_ϵ converges to a function $u \in C([0,\infty[, L^1(\mathbb{R}))$ which is characterized below. This limit does not depend on the details of Z_ϵ but only on $\lambda = \int K(y)\,dy$. The result is wrong if A' changes sign.*

The limit problem was first studied by Greenberg, Leroux, Baraille, and Noussair in [9] in the case $B =$ Cste. In [6], Gosse proposed a construction of the limit solutions for BV initial datas when Φ is one-to-one. This result uses the nonconservative products defined by Lefloch and Tzavaras in [11] and Raymond in [16]. Notice that even in this case, the uniqueness for the limit problem was not proved.

Since $Z_\epsilon(x) \to \lambda\delta(x)$, the solution u exhibits a jump at $x = 0$, and the limit of the product $Z_\epsilon B(u_\epsilon)$ can be defined in various ways depending of the microscopic structure of Z_ϵ and u_ϵ.

Our aim is to define in a proper way the limit equation when $\epsilon = 0$ in order to have a well-posed problem which is consistant with the problem (1), namely such that u_ϵ converges to the solution of this limit problem. Especially, a very remarkable point here is that the limit u is the same for all profiles Z_ϵ, even when they are not monotonic. The limit only depends on $\lambda = \int_0^1 K(y)\,dy$. We have made this choice in order to mimic the Saint-Venant system where the source term is of the form $z'(x)h$ where z is the bottom topography of the river and h is the unknown depth of the water. Indeed when the bottom has a discontinuity, the related source term has a Dirac behaviour. The result of Theorem 1.1 is quite pessimistic for this problem. Indeed it seems to indicate that for subsonic flow (which corresponds to the case A non monotonic) the problem will be ill-posed.

Seguin and Vovelle have some related results in [17] when a conservation law with a discontinuous flux modeling oil in porous media is studied, and a similar problem has been studied by Lewicka in [12] where well-posedness of a system of balance laws is considered.

Finally, notice that several numerical schemes have been recently proposed to solve conservation laws with sources since the work of Greenberg and Leroux [8] (see for instance [1, 4, 7, 10]).

The paper is organised as follows:

In section 2 we state two characterisations of the limit problem. The first one deals with a layer problem, and the second one use the Φ function. Notice that the latest one needs that Φ is one-to-one and hence is more restrictive. Then we state the precise theorem about convergence to the limit problem, uniqueness and stability of this limit problem, and equivalence of its two characterisations. We end this section providing a counter-example when A is not monotonic.

In Section 3 we study the layer characterisation. The main tools are the kinetic formulation of scalar conservation laws first studied by Brenier in [2] and Giga and Miyakawa in [5] and developped by Lions Perthame and Tadmor in [13], and the existence of strong traces for scalar conservation laws [18] (see [19] and the works of Chen and Rascle [3] for strong traces in time). The uniqueness proof is based on methods introduced by Perthame in [14].

Section 4 is devoted to the proof of the equivalence of the two characterisation when Φ is one-to-one.

Finally we give the proof of the ill-posedness when A' is not monotonic in the last section.

2. Assumptions and main results. Although the source term $B(u_\epsilon)Z_\epsilon(x)$ is a bounded measure, and therefore converges in $\mathcal{D}'$ to a Dirac mass, this information is too weak to describe the limiting process. In fact microscopic information is needed. We propose two ways to achieve it. The first characterisation of the limit problem reads:

PROBLEM 2.1. *(Layer Problem) We say that $u \in L^\infty(\mathbb{R}\times]0,+\infty[)$ is solution of Problem 2.1 with initial value $u^0 \in L^1 \cap L^\infty(\mathbb{R})$ and layer profile $K \in L^1(]0,1[)$ if it exists $\overline{u} \in L^\infty(]0,1[\times]0,+\infty[)$ such that*

$$\frac{\partial u}{\partial t} + \frac{\partial A(u)}{\partial x} = 0, \quad x \in]-\infty, 0[, \ t \in]0,+\infty[, \tag{6}$$

$$\frac{\partial u}{\partial t} + \frac{\partial A(u)}{\partial x} = 0, \quad x \in]0,+\infty[, \ t \in]0,+\infty[, \tag{7}$$

$$\frac{\partial A(\overline{u})}{\partial y} + K(y)B(\overline{u}) = 0, \quad y \in]0,1[, \ t \in]0,+\infty[, \tag{8}$$

$$\overline{u}(0+,t) = u(0-,t), \quad t \in]0,+\infty[, \tag{9}$$

$$\overline{u}(1-,t) = u(0+,t), \quad t \in]0,+\infty[, \tag{10}$$

$$u(x,0+) = u^0(x), \quad x \in \mathbb{R}, \tag{11}$$

where (6), (7), and (8) are endowed with the entropy conditions:

$$\frac{\partial S(u)}{\partial t} + \frac{\partial \eta(u)}{\partial x} \le 0, \quad x \in]-\infty, 0[, \ t \in]0,+\infty[, \tag{12}$$

$$\frac{\partial S(u)}{\partial t} + \frac{\partial \eta(u)}{\partial x} \le 0, \quad x \in]0,+\infty[, \ t \in]0,+\infty[, \tag{13}$$

$$\frac{\partial \eta(\overline{u})}{\partial y} + K(y)S'(\overline{u})B(\overline{u}) \le 0, \quad y \in]0,1[, \ t \in]0,+\infty[. \tag{14}$$

We introduce the function Φ:

$$\Phi(u) = \int_0^u A'(v)/B(v)\, dv$$

when it is well defined. When Φ is one-to-one, we can define the limit problem as two equations of conservation laws defined on the domains $\{x < 0\}$ and $\{x > 0\}$ and coupled by the trace values at $x = 0$, thus eliminating $\overline{u}$ in the layer.

PROBLEM 2.2. *(Gap problem) Fix $\lambda \in \mathbb{R}$. We say that the function $u \in L^\infty(\mathbb{R}\times]0,+\infty[)$ is solution of Problem 2.2 with initial value $u^0 \in L^\infty \cap L^1(\mathbb{R})$ and gap λ if it verifies:*

$$\frac{\partial u}{\partial t} + \frac{\partial A(u)}{\partial x} = 0, \quad x \in]-\infty, 0[, \ t \in]0,+\infty[, \tag{15}$$

$$\frac{\partial u}{\partial t} + \frac{\partial A(u)}{\partial x} = 0, \quad x \in]0,+\infty[, \ t \in]0,+\infty[, \tag{16}$$

$$\Phi(u(0+,t)) = \Phi(u(0-,t)) - \lambda, \quad t \in]0,+\infty[, \tag{17}$$

$$u(x,0) = u^0(x), \quad x \in \mathbb{R}, \tag{18}$$

where (15) and (16) are endowed with the entropy inequalities:

$$\frac{\partial S(u)}{\partial t} + \frac{\partial \eta(u)}{\partial x} \le 0, \quad x \in]-\infty, 0[, \ t \in]0,+\infty[, \tag{19}$$

$$\frac{\partial S(u)}{\partial t} + \frac{\partial \eta(u)}{\partial x} \le 0, \quad x \in]0,+\infty[, \ t \in]0,+\infty[, \tag{20}$$

for S convex function and η related to S by (5).

The paper is devoted to the proof of the following theorem.

THEOREM 2.1. *We assume that $A \in C^3(\mathbb{R})$, $A'' > 0$, $A' \geq 0$ and we fix $K \in L^1(\mathbb{R})$ supported in $[0,1]$.*

(i) Then there exists a unique $u \in L^\infty(\mathbb{R}\times]0,+\infty[)$ solution of Problem 2.1. We have:

$$\|u\|_{L^\infty} \leq \|u^0\|_{L^\infty} + b_0 \int_0^1 |K(y)|\,dy.$$

(ii) Denote by H_K the Heavyside type function:

$$H_K(x) = \begin{array}{l} \exp\left(a_0 \int |K(y)|\,dy\right) \quad \text{for } x < 0 \\ 1 \quad \text{for } x > 0. \end{array} \tag{21}$$

Then two solutions of Problem 2.1, for the same layer profile K, satisfy for every $t \geq 0$:

$$\int_{-\infty}^{+\infty} H_K(x)|u(t,x) - v(t,x)|\,dx \leq \int_{-\infty}^{+\infty} H_K(x)|u^0(x) - v^0(x)|\,dx.$$

(iii) Fix the layer profile K and consider a sequence u_n^0 uniformly bounded in $L^1 \cap L^\infty(\mathbb{R})$ which converges weakly to $u^0 \in L^1 \cap L^\infty(\mathbb{R})$, then the associated solution u_n of the Problem 2.1 converges strongly in $L^1_{\mathrm{loc}}(\mathbb{R}\times]0,+\infty[)$ to the solution u of Problem 2.1 with initial value u^0.

(iv) Assume that u_ϵ^0 is uniformly bounded in $L^1 \cap L^\infty(\mathbb{R})$, that u_ϵ^0 converges strongly to u^0 in $L^1(\mathbb{R})$ and that $Z_\epsilon \in L^1(\mathbb{R})$ satisfies (3), then the solution u_ϵ of (1)(2) (4) converges strongly in $L^1(\mathbb{R}\times]0,+\infty[)$ to the solution u of Problem 2.1.

(v) Assume that Φ is one-to-one. Then Problem 2.1 with layer profile K is equivalent to Problem 2.2 with gap $\lambda = \int K(y)\,dy$.

REMARKS.

1: The weak stability stated in *(iii)* means that the layer is very stable. That is why the limit problem can be defined by a non-linear expression on the trace of u at $x = 0-$ and $x = 0+$ in Problem 2.2 when Φ is defined and one-to-one. In particular this shows that the solution does not depend on the shape of the profile K but only on $\lambda = \int K(y)\,dy$.

2: We can precise somewhat the notion of traces used here. Equations (15)(19) (respectively (16)(20)) are in the sense of distribution in $]-\infty,0[\times]0,+\infty[$ (respectively in $]0,+\infty[\times]0,+\infty[$), namely using regular test functions compactly supported in those domains. In [18], it has been shown that for such solution there exists a unique trace reached by L^1_{loc} convergence at the boundary of the domain. Here we have denoted $u(0-,t)$ the trace on the boundary $x = 0$ of the domain $]-\infty,0[\times]0,+\infty[$, $u(0+,t)$ the trace on the boundary $x = 0$ of the domain $]0,+\infty[\times]0,+\infty[$, and $u(x,0)$ the trace on the the boundary $t = 0$. We recall here this result in our framework:

THEOREM 2.2. *Let I be an interval of $\mathbb{R}$ and assume that $A \in C^3(\mathbb{R})$, $A'' > 0$. Then for every solution $u \in L^\infty(I\times]0,+\infty[)$ to (15)(19) in $I\times]0,+\infty[$, and every* *end-point α of I there exists a function $u(\alpha,\cdot)$ in $L^\infty(]0,+\infty[)$ and a function $u(\cdot,0)$ in $L^\infty(I)$ such that, for every $R>0$, $T>0$:*

$$\operatorname*{ess\ lim}_{x\to\alpha,\ x\in I}\int_0^T |u(x,t)-u(\alpha,t)|\,dt = 0, \tag{22}$$

$$\operatorname*{esslim}_{t\to 0}\int_{I\cap]-R,R[} |u(x,t)-u(x,0)|\,dx = 0. \tag{23}$$

3: We consider a convex flux A in order to use this trace theorem. In fact, the proof of this theorem use the averaging lemmas where only the following relaxed condition is needed:

$$\begin{array}{l} for\ every\ (\tau,\zeta)\in\mathbb{R}\times\mathbb{R},\ (\tau,\zeta)\neq(0,0):\\ \mathcal{L}(\{\xi \mid \tau+\zeta.A'(\xi)=0\})=0, \end{array} \tag{24}$$

where $\mathcal{L}$ is the Lebesgue measure. All the results of this paper can be extended to this case.

4: The condition $A'\geq 0$ is not only technical. Indeed it can be *impossible* to define a well-posed limit problem if A is not one-to-one. We show in section 5 the following result:

THEOREM 2.3. *(Non monotonic $A(\cdot)$) Consider the equation*

$$\frac{\partial u_\epsilon}{\partial t}+\frac{\partial A(u_\epsilon)}{\partial x}-Z_{\epsilon,\lambda}(x)B(u_\epsilon)=0,\quad x\in\mathbb{R}, t\geq 0, \tag{25}$$

where $Z_{\epsilon,\lambda}\geq 0$ vanishes out of $]0,\epsilon[$, $\int_0^\epsilon Z_{\epsilon,\lambda}(x)\,dx=\lambda$. We assume that there is $u_0\in\mathbb{R}$, $\eta>0$ and $C>0$ such that

$$A'(v)<0 \quad \text{on }]u_0-\eta,u_0[, \tag{26}$$
$$A'(v)>0 \quad \text{on }]u_0,u_0+\eta[, \tag{27}$$

and $0\leq (A'/B)(u)\leq C$ for $u\in]u_0-\eta,u_0+\eta[$. Then for $\lambda>0$ small enough there exists $u_\epsilon^0(x)$ and $v_\epsilon^0(x)$ such that u_ϵ^0 and v_ϵ^0 converge strongly in L^1_{loc} to the same limit, but the corresponding solutions $u_\epsilon, v_\epsilon\in L^\infty(\mathbb{R}\times]0,+\infty[)$ to (25) converge strongly in L^1_{loc} to two different functions. In other words, the limit problem is unstable for strong topology.

We can take, for instance, $A(u)=u^2/2$, $B(u)=u$, $u_0=0$.

3. Study of the layer problem. As mentioned in the introduction, we use the following kinetic formulation of scalar conservation laws. We define $a(\xi)=A'(\xi)$ and for $v\in\mathbb{R}$:

$$\chi(v,\xi)=\begin{cases} 1 & \text{if } 0\leq\xi\leq v\\ 0 & \text{if } \xi \text{ is not between } 0 \text{ and } v\\ -1 & \text{if } v\leq\xi\leq 0.\end{cases}$$

In the following, we say that a function $f\in L^\infty(I\times\mathbb{R})$ is a χ-function if there exists a function $u\in L^\infty(I)$ such that $f=\chi(u,\xi)$. We first recall the following equivalence (see the result of Lions Perthame and Tadmor in [13]).

Theorem 3.1. *Let I_1, I_2 be two open intervals of $\mathbb{R}$, $\kappa = 0$ or 1, and $Z \in L^1(I_1)$. Then $u \in L^\infty(I_1 \times I_2)$ is solution of*

$$\kappa\partial_t u + \partial_x A(u) + Z(x)B(u) = 0, \tag{28}$$

$$\kappa\partial_t S(u) + \partial_x \eta(u) + Z(x)S'(u)B(u) \leq 0, \tag{29}$$

for every convex function S and entropy flux η ($\eta' = S'a$) if and only if there exists a nonnegative measure $m \in \mathcal{M}(I_1 \times I_2 \times \mathbb{R})$ such that the kinetic function $f(x,t,\xi) = \chi(u(x,t),\xi)$ is solution in $I_1 \times I_2 \times \mathbb{R}$ of:

$$\kappa\partial_t f + a(\xi)\partial_x f - Z(x)B(\xi)\left[\partial_\xi f - \delta_0(\xi)\right] = \partial_\xi m. \tag{30}$$

We can also give a weak kinetic formulation to Problem 2.1 in this framework:

Problem 3.1. *(Kinetic layer Problem) Let $K \in L^1(]0,1[)$ and $u^0 \in L^\infty \cap L^1(\mathbb{R})$. Consider a function $f \in L^\infty(\mathbb{R}\times]0,+\infty[\times\mathbb{R})$ supported in a strip of the form $\mathbb{R}\times]0,+\infty[\times[-L,L]$ with $L > 0$. Then we say that f is solution to Problem 3.1 with layer profile K if there exist $m_1 \in \mathcal{M}(]-\infty,0[\times]0,+\infty[\times\mathbb{R})$, $m_2 \in \mathcal{M}(]0,+\infty[\times]0,+\infty[\times\mathbb{R})$, $\overline{m} \in \mathcal{M}(]0,1[\times]0,+\infty[\times\mathbb{R})$, m_l and $m_r \in \mathcal{M}(]0,+\infty[\times\mathbb{R})$ nonnegative measures, $\overline{f} \in L^\infty(]0,1[\times]0,+\infty[\times\mathbb{R})$ supported in $[-L,L]$ with respect to ξ, and $\nu, \overline{\nu}$ nonnegative measures with $\int \nu(x,t,\xi)\,d\xi = 1$ and $\int \overline{\nu}(x,t,\xi)\,d\xi = 1$, such that:*

$$\begin{cases} \dfrac{\partial f}{\partial t} + a(\xi)\dfrac{\partial f}{\partial x} = \dfrac{\partial m_1}{\partial \xi}, \quad x \in]-\infty,0[,\ t \in]0,+\infty[,\ \xi \in \mathbb{R}, \\ \dfrac{\partial f}{\partial t} + a(\xi)\dfrac{\partial f}{\partial x} = \dfrac{\partial m_2}{\partial \xi}, \quad x \in]0,+\infty[,\ t \in]0,+\infty[,\ \xi \in \mathbb{R}, \\ 0 \leq \operatorname{sign}(\xi) f(x,t,\xi) \leq 1, \\ \dfrac{\partial f}{\partial \xi}(x,t,\xi) = \delta_0(\xi) - \nu(x,t,\xi), \end{cases} \tag{31}$$

$$\begin{cases} a(\xi)\dfrac{\partial \overline{f}}{\partial y} - K(y)B(\xi)\left[\dfrac{\partial \overline{f}}{\partial \xi} - \delta_0(\xi)\right] = \dfrac{\partial \overline{m}}{\partial \xi}, \quad y \in]0,1[,\ t \in]0,+\infty[,\ \xi \in \mathbb{R}, \\ 0 \leq \operatorname{sign}(\xi)\overline{f}(y,t,\xi) \leq 1, \\ \dfrac{\partial \overline{f}}{\partial \xi}(y,t,\xi) = \delta_0(\xi) - \overline{\nu}(y,t,\xi), \end{cases} \tag{32}$$

$$\begin{cases} a(\xi)\overline{f}(0+,t,\xi) = a(\xi)f(0-,t,\xi) + \partial_\xi m_l, \quad t \in]0,+\infty[,\ \xi \in \mathbb{R}, \\ a(\xi)f(0+,t,\xi) = a(\xi)\overline{f}(1-,t,\xi) + \partial_\xi m_r, \quad t \in]0,+\infty[,\ \xi \in \mathbb{R}, \\ f(x,0+,\xi) = \chi(u^0(x),\xi) \quad x \in \mathbb{R},\ \xi \in \mathbb{R}. \end{cases} \tag{33}$$

This statement uses the trace of f at $x = 0+$, $x = 0-$ and $t = 0+$. Hence, we first verify the existence of such traces. Moreover, since we will study non-linear terms of f, we are interesting in characterizing whether those traces are reached by L^1_{loc} convergence.

Lemma 3.1. *Let I_1, I_2 be two intervals of $\mathbb{R}$ and $f \in L^\infty(I_1 \times I_2 \times \mathbb{R})$ with $0 \leq \operatorname{sign}(\xi) f \leq 1$ solution to:*

$$b(\xi)\frac{\partial f}{\partial y_1} = \frac{\partial m}{\partial \xi} + \frac{\partial g_1}{\partial y_2} + g_2, \tag{34}$$

where m, g_1, g_2 are locally bounded measures on $I_1 \times I_2 \times \mathbb{R}$, and b is regular such that $b(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}$. Then there is a distributional limit $f(\alpha, \cdot, \cdot)$ for a end point α of I_1, namely:

$$\operatorname{ess}\lim_{y_1 \to \alpha} \int_{I_2} \int_{\mathbb{R}} f(y_1, y_2, \xi) \phi(y_2, \xi)\, d\xi\, dy_2 = \int_{I_2} \int_{\mathbb{R}} f(\alpha, y_2, \xi) \phi(y_2, \xi)\, d\xi\, dy_2,$$

for every functions $\phi \in L^1(I_2 \times \mathbb{R})$.

When $f(\alpha, y_2, \xi) = \chi(u(\alpha, y_2), \xi)$ a.e. for some function $u(\alpha, \cdot) \in L^\infty(I_2)$, then $f(\alpha, \cdot, \cdot)$ is reached by L^1_{loc} convergence, namely

$$\operatorname{ess}\lim_{y_1 \to \alpha} \int_{K} \int_{\mathbb{R}} |f(y_1, y_2, \xi) - f(\alpha, y_2, \xi)|\, d\xi\, dy_2 = 0$$

for every bounded interval $K \subset I_2$.

We choose $I_1 =]-\infty, 0[$, $I_2 =]0, +\infty[$, $(y_1, y_2) = (x, t)$, $g_2 = 0$, $g_1 = -f$, $b(\xi) = a(\xi)$ then this lemma ensures the existence of a space trace $f(0-, \cdot, \cdot)$. Taking $I_1 =]0, +\infty[$, we can also define $f(0+, \cdot, \cdot)$. Now if we choose $I_1 =]0, 1[$, $I_2 =]0, +\infty[$, $(y_1, y_2) = (y, t)$, $f = \overline{f}$, $b(\xi) = a(\xi)$, $g_1 = 0$, $g_2 = -K(y)B'(\xi)\overline{f} - K(y)B(0)\delta_0(\xi)$ and $m = \overline{m} + K(y)B(\xi)\overline{f}$ we can define $\overline{f}(0+, \cdot, \cdot)$ and $\overline{f}(1-, \cdot, \cdot)$. Finally if we choose $I_1 =]0, +\infty[$, $I_2 =]0, +\infty[$ (or $]-\infty, 0[$), $(y_1, y_2) = (t, x)$, $b(\xi) = 1$, $g_2 = 0$, $g_1 = -a(\xi)f$, $m = m_1$ (or m_2) we can define $f(\cdot, 0+, \cdot)$, and since by hypothesis this value is a χ- function, we see that the initial value is reached by L^1_{loc} convergence.

We postpone the proof of this lemma in the appendix. We show now the following theorem:

THEOREM 3.2. *We assume that $A \in C^3(\mathbb{R})$ $A'' > 0$ and $A' \geq 0$.*

(i) For $u^0 \in L^\infty \cap L^1(\mathbb{R})$ and $K \in L^1(]0, 1[)$ we denote

$$L_0 = \|u^0\|_{L^\infty} + b_0 \int_0^1 |K(y)|\, dy.$$

Then there exists a unique $f \in L^\infty(\mathbb{R}\times]0, +\infty[\times\mathbb{R})$ solution to Problem 3.1. The function f is supported in $[-L_0, L_0]$ with respect to ξ, $f = \chi(u, \xi)$ for some $u \in L^\infty(\mathbb{R}\times]0, +\infty[)$, and u is the unique solution to Problem 2.1.

(ii) Consider u_1, u_2 solutions to Problem 2.1 with initial values u_1^0, u_2^0 and with the same layer profile K. We have for every $t \in]0, +\infty[$ (see Theorem 2.1)

$$\int_{-\infty}^{+\infty} H_K(x) |u_1(x, t) - u_2(x, t)|\, dx \leq \int_{-\infty}^{+\infty} H_K(x) |u_1^0(x) - u_2^0(x)|\, dx$$

where H_K is defined by (21).

(iii) Consider u_n solution of Problem 2.1 with initial value u_n^0 and same layer profile K such that u_n^0 converges weakly to u^0 in $L^\infty *$, then u_n converges strongly in L^1_{loc} to u solution of Problem 2.1 with initial value u^0 and profile K.

(iv) Finally let u_ϵ be solution of (1)(2)(4), where Z_ϵ satisfies (3), and assume that u_ϵ^0 converges strongly to u^0 in L^1_{loc}, then u_ϵ converges strongly to u, solution of Problem 2.1 with initial value u^0 and layer profile K.

REMARK. Point (i) implies that the unique solution to Problem 2.1 verifies:

$$\|u\|_{L^\infty} \leq \|u^0\|_{L^\infty} + b_0 \int_0^1 |K(y)|\, dy.$$

Indeed, from the definition to χ, we have

$$|u(x,t)| = \left| \int_0^{u(x,t)} f(x,t,\xi)\, d\xi \right| \le L_0,$$

since f is supported in $[-L_0, L_0]$ with respect to ξ and $\|f\|_{L^\infty} = 1$.

The proof of this theorem uses the kinetic formulation and is based on Perthame's methods in [14]. Except for the initial condition, the kinetic formulation of Problem 3.1 is linear and so stable by weak compactness. This allows to pass to the weak limit. The only difficulty is to see that the conditions on the traces are stable by weak convergence which is not obvious in the formulation to Problem 2.1. This result is proved in Lemma 3.2 whose proof is posponed to the apendix.

LEMMA 3.2. *(Kinetic weak stability of traces) Let I be an open interval of $\mathbb{R}$ and let $f_n \in L^\infty(I\times]0,+\infty[\times\mathbb{R})$, $\|f_n\|_{L^\infty} \le 1$, solutions to:*

$$a(\xi)\frac{\partial f_n}{\partial x} = \frac{\partial m_n}{\partial \xi} + \frac{\partial g_n^1}{\partial t} + g_n^2, \tag{35}$$

where m_n are locally uniformly bounded measures on $I\times]0,+\infty[\times\mathbb{R}$ and g_n^1, g_n^2 are uniformly bounded in $L^\infty(I\times]0,+\infty[\times\mathbb{R})$. Consider a end point α of I. We assume that f_n, g_n^1, g_n^2 converge weakly in $L^\infty*$ to f, g^1, g^2 and that m_n converges to $m \in \mathcal{M}((I\cup\{\alpha\})\times]0,+\infty[\times\mathbb{R})$ in the sense of measure in $(I\cup\{\alpha\})\times]0,+\infty[\times\mathbb{R}$, namely, for every $\phi \in C^0$ compactly supported in $(I\cup\{\alpha\})\times]0,+\infty[\times\mathbb{R}$ (notice that we may have $\phi(\alpha,\cdot,\cdot) \neq 0$):

$$\lim_{n\to+\infty} \int_{(I\cup\{\alpha\})} \int_0^{+\infty} \int_{\mathbb{R}} \phi(x,t,\xi) m_n(dx,dt,d\xi)$$
$$= \int_{(I\cup\{\alpha\})} \int_0^{+\infty} \int_{\mathbb{R}} \phi(x,t,\xi) m(dx,dt,d\xi).$$

Then there exists $m_\alpha \in \mathcal{M}(]0,+\infty[\times\mathbb{R})$ such that $(m - \delta_\alpha(x) m_\alpha) \perp \delta_\alpha(x)$, $f_n(\alpha,\cdot,\cdot)$ converges to a function $L^\infty(]0,+\infty[\times\mathbb{R})$ in L^∞ $w*$ and the trace of f defined in Lemma 3.1 satisfies for α left end point (respectively right end point)

$$a(\xi) f(\alpha,\cdot,\cdot) = \lim_{n\to+\infty} a(\xi) f_n(\alpha,\cdot,\cdot) + \partial_\xi m_\alpha \quad (\text{respectively} - \partial_\xi m_\alpha).$$

We are now ready to show Theorem 3.2.

Proof of Theorem 3.2. We divide the proof in several steps.
(i) Uniqueness of the solution.
We define

$$J(f_1, f_2) = |f_1 - f_2|^2 + (|f_1| - f_1^2) + (|f_2| - f_2^2). \tag{36}$$

It verifies the following property:

LEMMA 3.3. *Let f_1, $f_2 \in L^1(\mathbb{R})$ be such that:*

$$\begin{aligned} \partial_\xi f_1 &= \delta_0(\xi) - \nu_1, \\ \partial_\xi f_2 &= \delta_0(\xi) - \nu_2, \end{aligned} \tag{37}$$

with ν_1, ν_2 nonnegative measure, $\int \nu_1 \, d\xi = \int \nu_2 \, d\xi = 1$. Then $J(f_1, f_2) = 0$ if and only if there exists $u \in \mathbb{R}$ such that $f_1 = f_2 = \chi(u, \cdot)$.

This means that $J(f_1, f_2)$ specifies the distance between f_1 and f_2 and whether those functions are χ-functions.

Proof. If $J(f_1, f_2) = 0$, since f_1, f_2 are valued in $[-1, 1]$, $f_1 = f_2$ and $|f_1| = f_1^2$. Hence $f_1 = f_2$ is valued in $\{-1, 0, 1\}$, and there exists $u \in \mathbb{R}$ such that $\nu_1 = \nu_2 = \delta_u(\xi)$. Since $f_1 \in L^1(\mathbb{R})$, we have $f_1(\xi) = 0$ for $|\xi| > |u|$. Therefore, integrating (37) leads to $f_1 = f_2 = \chi(u, \cdot)$. Conversely, if $f_1 = f_2 = \chi(u, \cdot)$, then $f_1 = f_2$ is valued in $\{-1, 0, 1\}$ so $J(f_1, f_2) = 0$. □

We introduce two solutions f_1 and f_2 of Problem 3.1 with initial values u_1^0, u_2^0 and same layer profile.

Since f_1 and f_2 are valued in $[-1, 1]$, $J(f_1, f_2)$ is the sum of three nonnegative terms. Notice that, since $\mathrm{sign}(f_1) = \mathrm{sign}(f_2) = \mathrm{sign}(\xi)$, we have the equality:

$$J(f_1, f_2) = -2 f_1 f_2 + (f_1 + f_2)\mathrm{sign}(\xi). \tag{38}$$

Let us first show the following proposition which state the equations satisfied by $J(f_1, f_2)$ and $J(\overline{f}_1, \overline{f}_2)$.

PROPOSITION 3.1. *we have in the sense of distribution:*

$$\partial_t \int H_K(x) J(f_1, f_2) \, d\xi + \partial_x \int a(\xi) H_K(x) J(f_1, f_2) \, d\xi \leq 0 \tag{39}$$

on $]-\infty, 0[\times]0, +\infty[$ *and on* $]0, +\infty[\times]0, +\infty[$,

$$\partial_y \int a(\xi) J(\overline{f}_1, \overline{f}_2) \, d\xi \leq a_0 |K(y)| \int a(\xi) J(\overline{f}_1, \overline{f}_2) \, d\xi \tag{40}$$

on $]0, 1[$*, and:*

$$\begin{aligned} \int a(\xi) J(f_1(0+), f_2(0+)) \, d\xi &\leq \int a(\xi) J(\overline{f}_1(1-), \overline{f}_2(1-)) \, d\xi \\ \int a(\xi) J(f_1(0-), f_2(0-)) \, d\xi &\geq \int a(\xi) J(\overline{f}_1(0+), \overline{f}_2(0+)) \, d\xi. \end{aligned} \tag{41}$$

for $t \in]0, +\infty[$.

Proof. We can write from (38) (see [14] for a justification):

$$\begin{aligned} \partial_t J(f_1, f_2) &= (\mathrm{sign}(\xi) - 2f_1)\partial_t f_2 + (\mathrm{sign}(\xi) - 2f_2)\partial_t f_1, \\ \partial_x J(f_1, f_2) &= (\mathrm{sign}(\xi) - 2f_1)\partial_x f_2 + (\mathrm{sign}(\xi) - 2f_2)\partial_x f_1. \end{aligned}$$

Since $H_K(x)$ is constant on $]0, +\infty[$ and on $]-\infty, 0[$, from (31) we find:

$$\begin{aligned} &\frac{\partial}{\partial t}\{H_K(x) J(f_1, f_2)\} + a(\xi)\frac{\partial}{\partial x}\{H_K(x) J(f_1, f_2)\} \\ &= H_K(x)\left[(\mathrm{sign}(\xi) - 2f_1)\partial_\xi m_2 + (\mathrm{sign}(\xi) - 2f_2)\partial_\xi m_1\right], \end{aligned}$$

on $]0, +\infty[\times]0, +\infty[\times\mathbb{R}$ and on $]-\infty, 0[\times]0, +\infty[\times\mathbb{R}$. Then, integrating with respect to ξ and noticing that $\partial_\xi(\mathrm{sign}(\xi) - 2f) = 2\nu \geq 0$, it follows:

$$\begin{aligned} &\partial_t \textstyle\int H_K(x) J(f_1, f_2) \, d\xi + \partial_x \int a(\xi) H_K(x) J(f_1, f_2) \, d\xi \\ = \; &-2H_K(x) \textstyle\int (\nu_1 m_2 + \nu_2 m_1) \, d\xi \leq 0 \end{aligned} \tag{42}$$

on $]-\infty,0[\times]0,+\infty[\times\mathbb{R}$ and on $]0,+\infty[\times]0,+\infty[\times\mathbb{R}$ in the sense of distribution. Notice that this inequality can be rigourously obtained using regularization as in [14]. In the same way, we write:

$$\begin{aligned}\partial_y J(\overline{f}_1,\overline{f}_2) &= (\mathrm{sign}(\xi)-2\overline{f}_1)\partial_y\overline{f}_2+(\mathrm{sign}(\xi)-2\overline{f}_2)\partial_y\overline{f}_1\\ \partial_\xi J(\overline{f}_1,\overline{f}_2) &= (\mathrm{sign}(\xi)-2\overline{f}_1)\left[\partial_\xi\overline{f}_2-\delta_0(\xi)\right]\\ &\qquad +(\mathrm{sign}(\xi)-2\overline{f}_2)\left[\partial_\xi\overline{f}_1-\delta_0(\xi)\right]\end{aligned}$$

since $\partial_\xi J(\overline{f}_1,\overline{f}_2)=-(1/2)\partial_\xi\left[(2\overline{f}_1-\mathrm{sign}(\xi))(2\overline{f}_2-\mathrm{sign}(\xi))\right]$. And so, we find on $]0,1[\times]0,+\infty[$:

$$\begin{aligned}&\partial_y\int a(\xi)J(\overline{f}_1,\overline{f}_2)\,d\xi+K(y)\int B'(\xi)J(\overline{f}_1,\overline{f}_2)\,d\xi\\ =\quad &-2\int(\overline{\nu}_1\overline{m}_2+\overline{\nu}_2\overline{m}_1)\,d\xi\le 0.\end{aligned}$$

Since $a(\xi)\ge 0$ and $a_0=\mathrm{Sup}|B'/a|$ this leads to:

$$\partial_y\int a(\xi)J(\overline{f}_1,\overline{f}_2)\,d\xi\le a_0|K(y)|\int a(\xi)J(\overline{f}_1,\overline{f}_2)\,d\xi.$$

We set $f(0-)$ for $f(0-,\cdot,\cdot)$ and $\overline{f}(0+)$ for $\overline{f}(0+,\cdot,\cdot)$. The expression (38) of $J(f_1,f_2)$ leads to:

$$\begin{aligned}&J(f_1(0-),f_2(0-))-J(\overline{f}_1(0+),\overline{f}_2(0+))\\ &=[\mathrm{sign}(\xi)-2f_1(0-)][f_2(0-)-\overline{f}_2(0+)]\\ &\quad+[\mathrm{sign}(\xi)-2\overline{f}_2(0+)][f_1(0-)-\overline{f}_1(0+)].\end{aligned}$$

So, from (33) we get:

$$\begin{aligned}&\int a(\xi)J(f_1(0-),f_2(0-))\,d\xi-\int a(\xi)J(\overline{f}_1(0+),\overline{f}_2(0+))\,d\xi\\ &=2\int(\nu_1(0+)m_{2l}+\overline{\nu}_2(0-)m_{1l})\,d\xi\\ &\ge 0.\end{aligned}$$

Doing the same thing at the interface $y=1-/x=0+$, we find:

$$\int a(\xi)J(f_1(0+),f_2(0+))\,d\xi\le\int a(\xi)J(\overline{f}_1(1-),\overline{f}_2(1-))\,d\xi.$$

□

Since J is nonlinear, we need to show that the traces at $x=0+,0-$, $y=0+,1-$ are strongly reached by L^1_{loc} convergence. Assume at this stage the following proposition (proved in *(ii)*).

PROPOSITION 3.2. *Let f be solution of Problem 3.1. Then $f(0+,\cdot,\cdot)$, $f(0-,\cdot,\cdot)$, $\overline{f}(0+,\cdot,\cdot)$ and $\overline{f}(1-,\cdot,\cdot)$ are strongly reached by L^1_{loc} convergence.*

Then integrating (39) on $]0,+\infty[$ and $]-\infty,0[$ and (40) on $]0,1[$ gives:

$$\frac{d}{dt}\int_{-\infty}^{0}\int_{\mathbb{R}} H_K(x)J(f_1,f_2)\,d\xi\,dx \le -H_K(0-)\int_{\mathbb{R}} a(\xi)J(f_1(0-),f_2(0-))\,d\xi,$$
$$\frac{d}{dt}\int_{0}^{+\infty}\int_{\mathbb{R}} H_K(x)J(f_1,f_2)\,d\xi\,dx \le H_K(0+)\int_{\mathbb{R}} a(\xi)J(f_1(0+),f_2(0+))\,d\xi,$$
$$\int a(\xi)J(\overline{f}_1(1-),\overline{f}_2(1-))\,d\xi \le \exp(a_0\int_0^1 |K(z)|\,dz)\int a(\xi)J(\overline{f}_1(0+),\overline{f}_2(0+)).$$

Notice that $H_K(0-)=\exp(a_0\int_0^1|K(z)|\,dz)$, $H_K(0+)=1$. So, summing the two first inqualities and using the third one with (41) leads to:

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\int_{\mathbb{R}} H_K(x)J(f_1,f_2)\,d\xi\,dx \le 0. \tag{43}$$

Since $f_1(x,0+,\xi)=f_2(x,0+,\xi)$ are χ-function, thanks to Lemma 3.1, the initial values are strongly reached by L^1 convergence. So, for almost every $t\ge 0$:

$$\left[\int H_K(x)J(f_1,f_2)\,d\xi\,dx\right](t) \le \int H_K(x)J(\chi(u_1^0,\cdot),\chi(u_2^0,\cdot))\,d\xi\,dx.$$

If $u_1^0=u_2^0$ then Lemma 3.3 ensures that $J(f_1,f_2)=0$ almost everywhere, and so $f_1(t,x,\xi)=f_2(t,x,\xi)=\chi(u(t,x),\xi)$. Theorem 3.1 implies that u is the unique solution of Problem 2.1 with initial value u^0 and layer profile K. Now if $u_1^0\ne u_2^0$, since f_1 and f_2 are χ-functions we have:

$$J(f_1,f_2)=|f_1-f_2|=\mathbf{1}_{\{v \text{ between} u_1 \text{ and } u_2\}}$$

and so:

$$\int J(f_1,f_2)\,d\xi=|u_1(t,x)-u_2(t,x)|.$$

Then equation (43) is equivalent to:

$$\frac{d}{dt}\int_{\mathbb{R}} H_K(x)|u_1(x,t)-u_2(x,t)|\,dx \le 0, \tag{44}$$

for every u_1,u_2 solution of Problem 2.1.
(ii) Proof of Proposition 3.2.
Study on $x\in]-\infty,0[$.

We choose nonincreasing functions $\phi_\eta(x)$, $\phi_\eta(x)=1$ for $x<-\eta$, $\phi_\eta(x)=0$ for $x>-\eta/2$. We find from (39) (with $f_1=f_2=f$):

$$\begin{aligned}&\frac{d}{dt}\int H_K(x)\phi_\eta(x)J(f,f)\,d\xi\,dx\\ \le\ &\int a(\xi)H_K(x)\phi_\eta'(x)J(f,f)\,dx\,d\xi \le 0,\end{aligned}$$

since $a(\xi)\ge 0$. So passing to the limit when η goes to 0 we find:

$$\frac{d}{dt}\int_{-\infty}^{0}\int_{\mathbb{R}} H_K(x)J(f,f)\,d\xi\,dx \le 0. \tag{45}$$

Thanks to Lemma 3.1 and (33), the initial data is reached by $L^1_{\rm loc}$ convergence. So using (45), we find that $J(f,f) = 2(|f| - f^2) = 0$ for almost every $(x,t,\xi) \in]-\infty,0[\times]0,+\infty[\times\mathbb{R}$ (since $J(f(x,0+,t),f(x,0+,t)) = 0$). Lemma 3.3 ensures that there exists a function $u \in L^\infty(]-\infty,0[\times]0,+\infty[)$ such that $f(x,t,\xi) = \chi(u(x,t),\xi)$. So, from Theorem 3.1 and (31), u is solution of (6),(12) and thanks to Theorem 2.2, $u(0-,t)$ is strongly reached by $L^1_{\rm loc}$ convergence and so it is the same for $f(0-,t,\xi)$. In particular $J(f(0-),f(0-)) = 0$.

Study at the interface $x = 0-/y = 0+$.

From (41) we find:

$$\int a(\xi) J(\overline{f}(0+),\overline{f}(0+))\, d\xi \leq \int a(\xi) J(f(0-),f(0-))\, d\xi = 0.$$

But $a(\xi) > 0$, so $J(\overline{f}(0+,t,\xi),\overline{f}(0+,t,\xi)) = 0$ for almost every $(t,\xi) \in]0,+\infty[\times\mathbb{R}$. As before we conclude that $\overline{f}(0+,\cdot,\cdot)$ is a χ-function thanks to Lemma 3.3 and from Lemma 3.1, that $\overline{f}(0+,\cdot,\cdot)$ is reached by $L^1_{\rm loc}$ convergence.

Study on $]0,1[$.

Since $\overline{f}(0+,\cdot,\cdot)$ is reached by $L^1_{\rm loc}$ convergence, from (40), we get for almost every y:

$$\int a(\xi) J(\overline{f},\overline{f})\, d\xi\Big|_y \leq \exp(a_0 \int_0^y |K(z)|\, dz) \int a(\xi) J(\overline{f},\overline{f})\, d\xi\Big|_{y=0} \leq 0. \tag{46}$$

So thanks to Lemma 3.3 there exists $\overline{u} \in L^\infty(]0,1[\times]0,+\infty[)$ such that $\overline{f}(y,t,\xi) = \chi(\overline{u}(y,t),\xi)$. So, from Theorem 3.1 and (32), $\overline{u}$ is solution of (8), (14). From (8) we see that the trace of $A(\overline{u})$ at $y = 1-$ is strongly reached in $L^1_{\rm loc}$. This implies that $\overline{u}(1-,t)$ is strongly reached by $L^1_{\rm loc}$ convergence since A is one-to-one ($A' > 0$). Hence the same holds true for $\overline{f}(1-,t,\xi)$. In particular $J(\overline{f}(1-),\overline{f}(1-)) = 0$.

Study at the interface $x = 0+/y = 1-$.

From (41) we find:

$$\int a(\xi) J(f(0+),f(0+))\, d\xi \leq 0. \tag{47}$$

As above we conclude that $f(0+,t,\xi)$ is reached by L^1 convergence.

(iii) Existence of the solution

Let us consider u_ϵ solution of (1) (2)(4) with Z_ϵ satisfying (3). We assume in addition that u^0_ϵ converges strongly to u^0 in $L^\infty(\mathbb{R})$. We set:

$$\chi_\epsilon(x,t,\xi) = \chi(u_\epsilon(x,t),\xi), \quad x \in \mathbb{R},\ t \geq 0,\ \xi \in \mathbb{R}.$$

From Theorem 3.1 and equations (1) (4), there exists a nonnegative measure m_ϵ such that:

$$\partial_t \chi_\epsilon + a(\xi)\partial_x \chi_\epsilon - Z_\epsilon(x) B(\xi)\,[\partial_\xi \chi_\epsilon - \delta_0(\xi)] = \partial_\xi m_\epsilon. \tag{48}$$

We denote

$$L_\epsilon = \|u^0_\epsilon\|_{L^\infty} + b_0 \int_{\mathbb{R}} |Z_\epsilon(y)|\, dy.$$

Notice that L_ϵ converges to

$$L_0 = \|u^0\|_{L^\infty} + b_0 \int_0^1 |K(y)|\, dy.$$

Let us show that χ_ϵ is supported in $[-L_\epsilon, L_\epsilon]$ with respect to ξ. Consider

$$\begin{aligned}\psi_\epsilon(x,\xi) &= \mathbf{1}_{\{\xi \geq \|u_\epsilon^0\|_{L^\infty}\}} \quad \text{for } x \leq 0,\\ &= \mathbf{1}_{\left\{\xi \geq \|u_\epsilon^0\|_{L^\infty} + b_0 \int_0^x |Z_\epsilon(y)|\, dy\right\}} \quad \text{for } x \geq 0.\end{aligned}$$

We have

$$\begin{aligned}&\partial_\xi \psi_\epsilon \geq 0\\ &\partial_x \psi_\epsilon + b_0 |Z_\epsilon(x)| \partial_\xi \psi_\epsilon = 0.\end{aligned}$$

By definition to b_0:

$$A'(\xi) b_0 |Z_\epsilon(x)| \geq -B'(\xi) Z_\epsilon(x),$$

so

$$a(\xi)\partial_x \psi_\epsilon - B'(\xi) Z_\epsilon(x) \partial_\xi \psi_\epsilon \leq 0.$$

Notice that $\psi_\epsilon \chi_\epsilon \geq 0$ since $\chi_\epsilon \geq 0$ for $\xi \geq 0$. So

$$\begin{aligned}&\partial_t(\psi_\epsilon \chi_\epsilon) + a(\xi)\partial_x(\psi_\epsilon \chi_\epsilon) - B(\xi) Z_\epsilon(x) \partial_\xi(\psi_\epsilon \chi_\epsilon) - \partial_\xi(\psi_\epsilon m_\epsilon) + (\partial_\xi \psi_\epsilon) m_\epsilon\\ &= [a(\xi)\partial_x \psi_\epsilon - B(\xi) Z_\epsilon(x) \partial_\xi \psi_\epsilon]\, \chi_\epsilon\\ &\leq 0.\end{aligned}$$

Notice that $-\int \partial_\xi \psi_\epsilon \, dm_\epsilon \leq 0$, so Gronwall Lemma yields:

$$\int \psi_\epsilon(x,\xi) \chi_\epsilon(x,t,\xi)\, dx\, d\xi \leq \left(\int \psi_\epsilon(x,\xi) \chi(u_\epsilon^0(x),\xi)\, dx\, d\xi\right) \exp(\|B' Z_\epsilon\|_{L^\infty} t) = 0,$$

which leads to $\chi_\epsilon(x,t,\xi) = 0$ for $\xi \geq \sup\left(\|u_\epsilon^0\|_{L^\infty}, \|u_\epsilon^0\|_{L^\infty} + \int_0^x |Z_\epsilon(y)|\, dy\right)$. In the same way we find that $-\psi_\epsilon(x,-\xi)\chi_\epsilon \geq 0$ and

$$\int -\psi_\epsilon(x,-\xi) \chi_\epsilon(t,x,\xi)\, dx\, d\xi \leq 0.$$

Hence $\chi_\epsilon(x,t,\xi) = 0$ for $\xi \leq \inf\left(-\|u_\epsilon^0\|_{L^\infty}, -\|u_\epsilon^0\|_{L^\infty} - \int_0^x |Z_\epsilon(y)|\, dy\right)$. Finally χ_ϵ is supported in $[-L_\epsilon, L_\epsilon]$ with respect to ξ.

For every nonnegative regular function ϕ compactly supported in $\mathbb{R} \times \mathbb{R}^+$ we have:

$$\begin{aligned}\int \phi(x,t)\, dm_\epsilon(x,t,\xi) &= \int \partial_t \phi(x,t) \chi_\epsilon \, dt\, dx\, d\xi\\ &\quad + \int \partial_x \phi(x,t) \xi a(\xi) \chi_\epsilon \, dt\, dx\, d\xi\\ &\quad - \int \phi(x,t) \partial_\xi(\xi B(\xi)) Z_\epsilon(x) \chi_\epsilon \, dt\, dx\, d\xi\\ &\leq 2|L_\epsilon| \|\phi\|_{W^{1,1}(\mathbb{R}^+ \times \mathbb{R})} \sup_{[-L_\epsilon, L_\epsilon]} (|\xi| + |\xi a(\xi)|)\\ &\quad + 2|L_\epsilon| \|\phi\|_{L^\infty} \int_0^1 |Z_\epsilon(x)|\, dx \sup_{[-L_\epsilon, L_\epsilon]} |\partial_\xi(\xi B(\xi))|\\ &\leq C(\phi).\end{aligned}$$

Therefore m_ϵ is locally bounded, uniformly with respect to ϵ.

We set:

$$
\begin{aligned}
&f_\epsilon(x,t,\xi) = \chi(u_\epsilon(x,t),\xi) \quad \text{for } x \in]-\infty,0[, \ t \geq 0, \ \xi \in \mathbb{R}\\
&f_\epsilon(x,t,\xi) = \chi(u_\epsilon(x+\epsilon,t),\xi) \quad \text{for } x \in]0,+\infty[, \ t \geq 0, \ \xi \in \mathbb{R}\\
&\overline{f}_\epsilon(y,t,\xi) = \chi(u_\epsilon(\epsilon y,t),\xi) \quad \text{for} y \in]0,1[, \ t \geq 0, \ \xi \in \mathbb{R}.
\end{aligned}
$$

Since $Z_\epsilon(x) = 0$ for $x \notin]0,\epsilon[$ we have:

$$
\frac{\partial f_\epsilon}{\partial t} + a(\xi)\frac{\partial f_\epsilon}{\partial x} = \frac{\partial m_{1\epsilon}}{\partial \xi} \quad x \in]-\infty,0[, \ t \in]0,+\infty[, \ \xi \in \mathbb{R} \tag{49}
$$

$$
\frac{\partial f_\epsilon}{\partial t} + a(\xi)\frac{\partial f_\epsilon}{\partial x} = \frac{\partial m_{2\epsilon}}{\partial \xi} \quad x \in]0,+\infty[, \ t \in]0,+\infty[, \ \xi \in \mathbb{R}, \tag{50}
$$

and:

$$
\epsilon\partial_t \overline{f}_\epsilon + a(\xi)\partial_y \overline{f}_\epsilon - \epsilon Z_\epsilon(\epsilon y)B(\xi)\left[\partial_\xi \overline{f}_\epsilon - \delta_0(\xi)\right] = \partial_\xi \overline{m}_\epsilon, \quad y \in]0,1[, \quad t \geq 0, \ \xi \in \mathbb{R}, \tag{51}
$$

where $m_{1\epsilon}$ (respectively $m_{2\epsilon}$ $\overline{m}_\epsilon$) is the restriction of m_ϵ to $x \in]-\infty,0[$ (respectively $]\epsilon,+\infty[$, $]0,\epsilon[$). Since $f_\epsilon, \overline{f}_\epsilon$ are χ-functions, there exist nonnegative measures, $\nu_\epsilon, \overline{\nu}_\epsilon$ such that:

$$
\int \nu_\epsilon(x,t,\xi)\,d\xi = 1, \tag{52}
$$

$$
\int \overline{\nu}_\epsilon(y,t,\xi)\,d\xi = 1, \tag{53}
$$

$$
\frac{\partial \overline{f}_\epsilon}{\partial \xi}(y,t,\xi) = \delta_0(\xi) - \overline{\nu}_\epsilon(y,t,\xi), \tag{54}
$$

$$
\frac{\partial f_\epsilon}{\partial \xi}(x,t,\xi) = \delta_0(\xi) - \nu_\epsilon(x,t,\xi). \tag{55}
$$

Thanks to Theorem 2.2, $u_\epsilon(x,0+)$ is reached by L^1_{loc} convergence so:

$$
f_\epsilon(x,0+,\xi) = \chi(u_\epsilon(x,0+),\xi) \quad x \in \mathbb{R}, \ \xi \in \mathbb{R}. \tag{56}
$$

Finally we have:

$$
a(\xi)\overline{f}_\epsilon(0+,t,\xi) = a(\xi)f_\epsilon(0-,t,\xi) + \partial_\xi m_{l\epsilon} \quad t \in]0,+\infty[\ \xi \in \mathbb{R}, \tag{57}
$$

$$
a(\xi)f_\epsilon(0+,t,\xi) = a(\xi)\overline{f}_\epsilon(1-,t,\xi) + \partial_\xi m_{r\epsilon} \quad t \in]0,+\infty[\ \xi \in \mathbb{R}, \tag{58}
$$

where for every $T, R > 0$:

$$
\begin{aligned}
m_{l\epsilon}([0,T]\times[-R,R]) &= m_\epsilon(\{0\}\times[0,T]\times[-R,R]),\\
m_{r\epsilon}([0,T]\times[-R,R]) &= m_\epsilon(\{\epsilon\}\times[0,T]\times[-R,R]).
\end{aligned}
$$

Equation (57) can be obtained multiplying equation (48) by $\theta_\eta(x)$ and letting η going to 0 where $\theta'(x)$ is an odd function and for $x \geq 0$ $\theta'_\eta(x) = \rho_\eta(x)$ with ρ_η is a classical mollifier. We obtain equation (58) in the same way.

There exist a sequence $\epsilon_n \to 0$ functions $f \in L^\infty(\mathbb{R}\times]0,+\infty[\times\mathbb{R})$, $\overline{f} \in (]0,1[\times]0,+\infty[\times\mathbb{R})$, supported in $[-L_0, L_0]$ with respect to ξ. nonnegative measures

$m_1 \in \mathcal{M}(]-\infty,0[\times]0,+\infty[\times\mathbb{R})$, $m_2 \in \mathcal{M}(]0,+\infty[\times]0,+\infty[\times\mathbb{R})$, and $\overline{m} \in \mathcal{M}]0,1[\times]0,+\infty[\times\mathbb{R})$ such that:

$$\begin{aligned}
&\lim_{n\to+\infty} f_{\epsilon_n} = f \quad \text{in } L^\infty W*,\\
&\lim_{n\to+\infty} \overline{f}_{\epsilon_n} = \overline{f} \quad \text{in } L^\infty W*,\\
&\lim_{n\to+\infty} m_{1\epsilon_n} = m_1 \quad \text{in } \mathcal{M}(]-\infty,0[\times]0,+\infty[\times\mathbb{R}),\\
&\lim_{n\to+\infty} m_{2\epsilon_n} = m_2 \quad \text{in } \mathcal{M}(]0,+\infty[\times]0,+\infty[\times\mathbb{R}),\\
&\lim_{n\to+\infty} \overline{m}_{\epsilon_n} = \overline{m} \quad \text{in } \mathcal{M}(]0,1[\times]0,+\infty[\times\mathbb{R}).
\end{aligned}$$

So there exists two nonnegative measures ν, $\overline{\nu}$ limit of ν_{ϵ_n}, $\overline{\nu}_{\epsilon_n}$ such that:

$$\int \nu(x,t,\xi)\,d\xi = 1, \tag{59}$$

$$\int \overline{\nu}(y,t,\xi)\,d\xi = 1, \tag{60}$$

$$\frac{\partial \overline{f}}{\partial \xi}(y,t,\xi) = \delta_0(\xi) - \overline{\nu}(y,t,\xi), \tag{61}$$

$$\frac{\partial f}{\partial \xi}(x,t,\xi) = \delta_0(\xi) - \nu(x,t,\xi). \tag{62}$$

Since $\epsilon Z_\epsilon(\epsilon\cdot)$ converges strongly to K, passing to the limit in (49)(50)(51) gives:

$$\frac{\partial f}{\partial t} + a(\xi)\frac{\partial f}{\partial x} = \frac{\partial m_1}{\partial \xi} \quad x \in]-\infty,0[,\ t \in]0,+\infty[,\ \xi \in \mathbb{R}, \tag{63}$$

$$\frac{\partial f}{\partial t} + a(\xi)\frac{\partial f}{\partial x} = \frac{\partial m_2}{\partial \xi} \quad x \in]0,+\infty[,\ t \in]0,+\infty[,\ \xi \in \mathbb{R}, \tag{64}$$

$$a(\xi)\partial_y \overline{f} - K(y)B(\xi)\left[\partial_\xi \overline{f} - \delta_0(\xi)\right] = \partial_\xi \overline{m}, \quad y \in]0,1[,\ t \geq 0,\ \xi \in \mathbb{R}. \tag{65}$$

Thanks to Lemma 3.2 passing to the limit in (57)(58) gives that there exists m_l, m_r nonnegative measures such that:

$$a(\xi)\overline{f}(0+,t,\xi) = a(\xi)f(0-,t,\xi) + \partial_\xi m_l, \quad t \in]0,+\infty[,\ \xi \in \mathbb{R}, \tag{66}$$

$$a(\xi)f(0+,t,\xi) = a(\xi)\overline{f}(1-,t,\xi) + \partial_\xi m_r, \quad t \in]0,+\infty[,\ \xi \in \mathbb{R}. \tag{67}$$

Since u_ϵ^0 converges strongly to u^0, $\chi(u_\epsilon^0(x),\xi)$ converges strongly to $\chi(u^0(x),\xi)$. Using Lemma 3.2, we find that $f(x,0+,\xi) - \chi(u^0(x),\xi) = \partial_\xi m_0$ where m_0 is a nonnegative measure. So:

$$\begin{aligned}
&\int \xi\left[f(x,0+,\xi) - \chi(u^0(x),\xi)\right]\,dx\,d\xi = -\int m_0(dx,d\xi) \leq 0,\\
&-\int u^0(x)\left[f(x,0+,\xi) - \chi(u^0(x),\xi)\right]\,dx\,d\xi = 0.
\end{aligned} \tag{68}$$

Since

$$\partial_\xi(f(x,0+,\xi) - \chi(u^0(x),\xi)) = \delta_{u^0(x)}(\xi) - \nu(x,0+,\xi),$$

summing the two terms of (68) and integrating by parts lead to:

$$\frac{1}{2}\int (\xi - u^0(x))^2\,d\nu(dx,0+,d\xi) \leq 0,$$

so $\nu(x,0+,\xi)=\delta_{u^0(x)}(\xi)$. Hence $f(x,0+,\cdot)=\chi(u^0(x),\cdot)$. So f is the unique solution of Problem 3.1 with initial value u^0 and layer profile K. So $f(x,t,\xi)=\chi(u(x,t),\xi)$ where $u(x,t)$ is the unique solution of Problem 2.1 with initial value u^0 and layer profile K. Since the values of $\mathrm{sign}(\xi)f$ are 0 or 1, in fact the convergence is strong in L^1_{loc}. So, since the limit is unique, the entire family f_ϵ converges to f when ϵ converges to 0. Finally, since $u_\epsilon=\int f_\epsilon\,d\xi$ and $u=\int f\,d\xi$, u_ϵ converges strongly to u in L^1_{loc}.
(iv) Weak stability.

Consider a sequence $u^0_n\in L^\infty(\mathbb{R})$ and $u^0\in L^\infty(\mathbb{R})$ such that u^0_n converges weakly to u^0 in $L^\infty*$. We consider f_n the solution of Problem 3.1 with a fixed profile K and initial value u^0_n. By weak compactness there exists $f\in L^\infty(\mathbb{R}\times]0,+\infty[\times\mathbb{R})$, $\overline{f}\in L^\infty(]0,1[\times]0,+\infty[\times\mathbb{R})$ and a subsequence n_p such that f_{n_p} and $\overline{f}_{n_p}$ converge weakly to f and $\overline{f}$ in $L^\infty*$ and $f,\overline{f}$ are solutions of (31) (32) (33). To ensure that f is the solution to Problem 3.1 with layer profile K and initial value u^0 we have to show that:

$$f(x,0+,\xi)=\chi(u^0(x),\xi).$$

We denote $u(x,t)=\int f(x,t,\xi)\,d\xi$ and $u_n(x,t)=\int f_n(x,t,\xi)\,d\xi$. Thanks to Lemma 3.2, $\int f(x,0+,\xi)\,d\xi=u^0(x)$. So we have $u(x,0+)=u^0(x)$. Using the averaging lemmas (see [15]) on $(]-\infty,0[\times]0,+\infty[\times\mathbb{R})$ and on $(]0,+\infty[\times]0,+\infty[\times\mathbb{R})$, we find that u_n converges strongly to u in L^1_{loc}. But u_n verifies (6)(12) on $(]-\infty,0[\times]0,+\infty[\times\mathbb{R})$ and on $(]0,+\infty[\times]0,+\infty[\times\mathbb{R})$. So passing to the limit u verifies the same equation and thanks to Theorem 2.2 $u(x,0+)$ is reached by L^1_{loc} convergence. Moreover $f(x,t,\xi)=\chi(u(x,t),\xi)$ and so $f(x,0+,\xi)=\chi(u^0(x),\xi)$.

4. Equivalence between gap and layer problems. We show in this section the following proposition which implies the equivalence between gap and layer problems.

PROPOSITION 4.1. *We assume that $A\in C^3(\mathbb{R})$ $A''>0$, $A'\geq 0$ and Φ is one-to-one. For $u^0\in L^\infty\cap L^1(\mathbb{R})$ and $\lambda\in\mathbb{R}$ there exists a unique solution to Problem 2.2. Moreover it coincides with the solution to the Problem 2.1 with layer profile K whenever $\int_0^1 K(y)\,dy=\lambda$. In particular, two solutions u_1,u_2 verify for every $t\in]0,+\infty[$ and every such layer profile K*

$$\int_{-\infty}^{+\infty} H_K(x)|u_1(x,t)-u_2(x,t)|\,dx\leq\int_{-\infty}^{+\infty} H_K(x)|u^0_1(x)-u^0_2(x)|\,dx.$$

This proposition is a consequence of the following Lemma:

LEMMA 4.1. *Assume that $A'>0$ and Φ is one-to-one. For every $\overline{u}^0\in\mathbb{R}$ and $K\in L^1(]0,1[)$ we define $u\in L^\infty(]0,1[)$ by:*

$$\Phi(u(y))+\int_0^y K(z)\,dz=\Phi(\overline{u}^0).$$

Then $\chi(u(y),\xi)$ is the unique function $\overline{f}\in L^\infty(]0,1[\times\mathbb{R})$ such that there exists a nonnegative measure $\overline{\nu}$ with $\int\overline{\nu}\,d\xi=1$ and a nonnegative measure $\overline{m}$ such that

$$\overline{f}(0+,\xi)=\chi(\overline{u}^0,\xi) \tag{69}$$
$$0\leq\mathrm{sign}(\xi)\overline{f}(y,\xi)\leq 1 \tag{70}$$
$$\partial_\xi\overline{f}=\delta_0(\xi)-\overline{\nu}(y,\xi) \tag{71}$$
$$a(\xi)\partial_y\overline{f}-K(y)B(\xi)\left[\partial_\xi\overline{f}-\delta_0(\xi)\right]=\partial_\xi\overline{m}\quad\text{on }]0,1[\times\mathbb{R}. \tag{72}$$

We first prove this Lemma:

Proof. The function $\chi(u(y),\xi)$ is constant on the characteristics $\Phi(\xi) + \int_0^y K(z)\,dz = \mathrm{Cst}$, so it is solution of the Problem with $\overline{m} = 0$. Next, assume that there are two solutions of this problem f_1, f_2. As in the proof of Theorem 3.2 we show that

$$\partial_y \int_{\mathbb{R}} a(\xi) J(f_1, f_2)\,d\xi + K(y) \int_{\mathbb{R}} B'(\xi) J(f_1, f_2)\,d\xi \leq 0.$$

Since the initial value is a χ-function, thanks to Lemma 3.1 the initial value is reached by L^1_{loc} convergence so for every $y \in]0,1[$:

$$\begin{aligned} &\int_{\mathbb{R}} a(\xi) J(f_1, f_2)\,d\xi \Big|_y \\ &\leq \exp(a_0 \int_0^y |K|(z)\,dz) \int_{\mathbb{R}} a(\xi) J(f_1, f_2)\,d\xi \Big|_0 = 0. \end{aligned}$$

So $\chi(u(y,\xi)$ is the unique solution. □

Now we can show the Proposition 4.1:

Proof. Consider the solution of Problem 2.1 with initial value u^0 and layer Profile K such that $\int_0^1 K(y)\,dy = \lambda$. Thanks to Theorem 3.2, $f(x,t,\xi) = \chi(u(x,t),\xi)$ is solution of the Problem 3.1 and we have seen in its demonstration that $\overline{f}(0+,t,\xi) = \chi(u(0-,t),\xi)$ for almost every $t \in]0,+\infty[$. So using the above lemma we find that for almost every t,

$$\overline{f}(y,t,\xi) = \chi(\overline{u}(y,t),\xi),$$

where

$$\Phi(\overline{u}(y,t)) + \int_0^y K(z)\,dz = \Phi(u(0-,t)).$$

Finally $\overline{u}(1-,t) = u(0+,t)$, so u is solution of Problem 2.2.

Conversely, if u is solution of Problem 2.2 then we introduce $f(x,t,\xi) = \chi(u(x,t),\xi)$, $\overline{f}(y,t,\xi) = \chi(\overline{u}(y,t),\xi)$, with

$$\Phi(\overline{u}(y,t)) + \int_0^y K(z)\,dz = \Phi(u(0-,t)).$$

It is easy to see that f is solution of Problem 3.1 and so u is solution of Problem 2.1. □

5. Ill-posedness when A is not monotonic.

In this section we give the proof of Theorem 2.3. This result shows that if the sign of $a(\xi)$ is not constant then we cannot define a well-posed limit problem considering only the limit initial value since the result depends on the value of the initial value in the layer $x \in]0,\epsilon[$.

Proof. Since $0 < A'/B \leq C$ we can define the function Φ on $]u_0 - \eta, u_0 + \eta[$ by:

$$\Phi(u) = \int_{u_0}^u \frac{A'}{B}(v)\,dv, \tag{73}$$

and this function is increasing (and so one-to-one). Hence, for every $0 < \lambda < \inf(|\Phi(u_0+\eta)|;|\Phi(u_0-\eta)|)$ we can define $u_\epsilon^0, v_\epsilon^0 \in L^\infty(\mathbb{R})$:

$$u_\epsilon^0(x) = \begin{cases} u_0 \quad \text{on } \mathbb{R}\backslash]0,\epsilon[\\ \Phi^{-1}(\Phi(u_0) + \displaystyle\int_0^x Z_{\epsilon,\lambda}(y)\,dy) \quad \text{on }]0,\epsilon[, \end{cases}$$

$$v_\epsilon^0(x) = \begin{cases} u_0 \quad \text{on } \mathbb{R}\backslash]0,\epsilon[\\ \Phi^{-1}(\Phi(u_0) + \displaystyle\int_0^x Z_{\epsilon,\lambda}(y)\,dy - \lambda) \quad \text{on }]0,\epsilon[. \end{cases}$$

Notice that $u_\epsilon^0 > u_0$ and $v_\epsilon^0 < u_0$ on $]0,\epsilon[$. So $u_1 = u_\epsilon^0(\epsilon-) = \Phi^{-1}(\Phi(u_0)+\lambda) > u_0$ and $v_1 = v_\epsilon^0(0+) = \Phi^{-1}(\Phi(u_0)-\lambda) < u_0$. The function u_ϵ^0 is regular on $]-\infty,\epsilon[\cup]\epsilon,+\infty[$ and verifies on those intervals:

$$A'(u_\epsilon^0)\partial_x u_\epsilon^0 - Z_{\epsilon,\lambda}(x)B(u_\epsilon^0) = 0.$$

It has a discontinuity at $x = \epsilon$ with:

$$\begin{aligned} &u_\epsilon^0(\epsilon-) = u_1 > u_0 \\ &\sigma_1 = \frac{A(u_1) - A(u_0)}{u_1 - u_0} > 0. \end{aligned}$$

So the solution $u_\epsilon(x,t)$ of (25) with initial value u_ϵ^0 has a shock with speed σ_1 and is defined by:

$$\begin{aligned} u_\epsilon(x,t) &= u_\epsilon^0(x) \quad \text{on }]-\infty,\epsilon[\\ &= u_1 \quad \text{on }]\epsilon,\sigma_1 t[\\ &= u_0 \quad \text{on }]\sigma_1 t,+\infty[. \end{aligned}$$

In the same way we see that v_ϵ^0 is regular on $]-\infty,0[\cup]0,+\infty[$ and is a stationary solution on those two intervals. It has a discontinuity at $x = 0$ with:

$$\begin{aligned} &v_\epsilon^0(0+) = v_1 < u_0 \\ &\sigma_2 = \frac{A(v_1) - A(u_0)}{v_1 - u_0} < 0. \end{aligned}$$

So the solution $v_\epsilon(x,t)$ of (25) with initial value v_ϵ^0 has a shock with speed σ_2 and is defined by :

$$\begin{aligned} v_\epsilon(x,t) &= v_\epsilon^0(x) \quad \text{on }]0,+\infty[\\ &= v_1 \quad \text{on }]\sigma_2 t,0[\\ &= u_0 \quad \text{on }]-\infty,\sigma_2 t[. \end{aligned}$$

Passing to the limit when ϵ goes to 0, we find that u_ϵ^0 and v_ϵ^0 converge strongly in L^1 to the constant function equal to u_0 and that v_ϵ and u_ϵ converge strongly in L^1 respectively to the function u and v defined by:

$$\begin{aligned} u(x,t) &= u_0 \quad \text{on }]-\infty,0[\cup]\sigma_1 t,+\infty[\\ &= u_1 \quad \text{on }]0,\sigma_1 t[\\ v(x,t) &= u_0 \quad \text{on }]-\infty,\sigma_2 t[\cup]0,+\infty[\\ &= v_1 \quad \text{on }]\sigma_2 t,0[. \end{aligned}$$

It is clear that $u \neq v$. ☐

Appendix A. Proof of Lemma 3.1.

Proof. Since b is regular, $b(\xi)f \in L^\infty_{\mathrm{loc}}$. For every $\phi \in L^1(I_2 \times \mathbb{R})$ with compact support we set:

$$F_\phi(y_1) = \int_{I_2}\int_{\mathbb{R}} b(\xi)f(y_1,y_2,\xi)\phi(y_2,\xi)\,d\xi\,dy_2.$$

Let $\{\phi_n\}$ be a dense countable collection of regular compactly supported functions in $L^1_{\mathrm{loc}}(I_2 \times \mathbb{R})$. From (34), F_{ϕ_n} lies in $BV_{\mathrm{loc}}(I_1)$ for every n and so has a left-hand-side limit and a right-hand-side limit at every point. At the endpoint α, we denote this (right or left) limit by:

$$F_{\phi_n}(\alpha\pm).$$

Since $\{\phi_n\}$ is countable, there exists a measurable set $\Omega \subset I_1$ such that $\mathcal{L}(I_1 \setminus \Omega) = 0$ and such that:

$$\lim_{y_1 \to \alpha, y_1 \in \Omega} F_{\phi_n}(y_1) = F_{\phi_n}(\alpha\pm), \quad \text{for every } n.$$

We now consider a sequence $y_k \to \alpha, y_k \in \Omega$. Since we have $|f(y_k,\cdot,\cdot)| \leq 1$, there exists a subsequence still denoted y_k and a function $h \in L^\infty(I_2 \times \mathbb{R})$ such that $f(y_k,\cdot,\cdot)$ converges weakly to h in $L^\infty*$, namely, for every $\phi \in L^1(I_2 \times \mathbb{R})$:

$$\lim_{y_k \to y_1} \int f(y_k,y_2,\xi)\phi(y_2,\xi)\,dy_2\,d\xi = \int h(y_2,\xi)\phi(y_2,\xi)\,dy_2\,d\xi.$$

Especially for every n we deduce that:

$$\int b(\xi)h(y_2,\xi)\phi_n(y_2,\xi)\,dy_2\,d\xi = F_{\phi_n}(\alpha\pm).$$

Since $\{\phi_n\}$ is dense in $L^1_{\mathrm{loc}}(I_2 \times \mathbb{R})$ and $b(\xi) \neq 0$ for almost every ξ, this defines a unique function h, $|h| \leq 1$, which we denote $f(\alpha,\cdot,\cdot)$ and which does not depend on the sequence y_k. This proves the first statement of the Lemma. Next, assume there exists a function $u(\alpha,\cdot) \in L^\infty(I_2)$ such that $f(\alpha,y_2,\xi) = \chi(u(\alpha,y_2),\xi)$. First we have:

$$f^2(y_1,\cdot,\cdot) - f^2(\alpha,\cdot,\cdot) = (f(y_1) - f(\alpha))^2 + 2f(\alpha)(f(y_1) - f(\alpha)),$$

and since $f(y_1,\cdot,\cdot)$ converges weakly to $f(\alpha,\cdot,\cdot)$ we have for every $R > 0$:

$$\begin{aligned}&\lim_{y_1 \to 0} \int_{-R}^{R}\int_{-R}^{R} f^2(y_1,y_2,\xi)\,dy_2\,d\xi \\ &\geq \int_{-R}^{R}\int_{-R}^{R} f^2(\alpha,y_2,\xi)\,dy_2\,d\xi.\end{aligned}$$

but the first term is less that $\int \mathrm{sign}(\xi)f(y_1,y_2,\xi)\,dy_1\,d\xi$ since $0 \leq \mathrm{sign}(\xi)f \leq 1$. This converges to $\int \mathrm{sign}(\xi)f(\alpha)\,dy_2\,d\xi$ which is equal to the second term since $f(\alpha,\cdot,\cdot)$ is a χ-function. So finally,

$$\lim_{y_1 \to 0} \|f(y_1,\cdot,\cdot)\|_{L^2_{\mathrm{loc}}} = \|f(\alpha,\cdot,\cdot)\|_{L^2_{\mathrm{loc}}}.$$

and since $f(y_1,\cdot,\cdot)$ converges to $f(\alpha,\cdot,\cdot)$ weakly in $L^2_{\rm loc}$, the convergence is also strong. $\square$

Appendix B. Proof of Lemma 3.2.

Proof. Consider the case when α is a left end point of I (the proof is similar for right end point). Choose $\beta>\alpha$ such that $]\alpha,\beta[\subset I$. First, since $\|f_n(\alpha,\cdot,\cdot)\|_{L^\infty}\le 1$, there exists a subsequence (still denoted f_n) and a function $h\in L^\infty(]0,+\infty[\times\mathbb{R})$ such that $f_n(\alpha,\cdot,\cdot)$ converges to h in L^∞ $w*$. For every regular function $\psi(x)$ with support in $[\alpha,\beta[$ and every regular function $\phi(t,\xi)$ compactly supported we set:

$$\lambda_n^{\phi,\psi}=\psi(\alpha)\int_0^{+\infty}\int_{-\infty}^{+\infty}a(\xi)\phi(t,\xi)f_n(\alpha,t,\xi)\,d\xi\,dt$$
$$\lambda^{\phi,\psi}=\psi(\alpha)\int_0^{+\infty}\int_{-\infty}^{+\infty}a(\xi)\phi(t,\xi)h(t,\xi)\,d\xi\,dt.$$

The sequence $\lambda_n^{\phi,\psi}$ converges to $\lambda^{\phi,\psi}$ when n tends to $+\infty$. Moreover we have from (35):

$$\begin{aligned}\lambda_n^{\phi,\psi}=&-\int_\alpha^\beta\int_0^{+\infty}\int_{\mathbb{R}}\psi'(x)\phi(t,\xi)a(\xi)f_n(x,t,\xi)\,d\xi\,dt\,dx\\&+\int\psi(x)\partial_\xi\phi(t,\xi)m_n(dt,dx,d\xi)\\&+\int\psi(x)\partial_t\phi(t,\xi)g_n^1(x,t,\xi)\,dx\,dt\,d\xi\\&-\int\psi(x)\phi(t,\xi)g_n^2(x,t,\xi)\,dx\,dt\,d\xi.\end{aligned}$$

Passing to the limit we find:

$$\begin{aligned}\lambda^{\phi,\psi}=&-\int_\alpha^\beta\int_0^{+\infty}\int_{\mathbb{R}}\psi'(x)\phi(t,\xi)a(\xi)f(x,t,\xi)\,d\xi\,dt\,dx\\&+\int\psi(x)\partial_\xi\phi(t,\xi)m(dt,dx,d\xi)\\&+\int\psi(x)\partial_t\phi(t,\xi)g^1(x,t,\xi)\,dx\,dt\,d\xi\\&-\int\psi(x)\phi(t,\xi)g^2(x,t,\xi)\,dx\,dt\,d\xi.\end{aligned}$$

This expression characterize h in a unique way independently of the subsequence. So the limit is unique and the entire sequence f_n converges. By Radon-Nikodým Theorem we can split $m=\tilde m+\delta_\alpha(x)m_\alpha$ with $\tilde m\perp\delta_\alpha(x)$ where $m_\alpha\in\mathcal{M}(]0,+\infty[\times\mathbb{R})$ and $\tilde m$ can be seen has a measure on $I\times]0,+\infty[\times\mathbb{R}$. Passing to the limit in (35) we find:

$$a(\xi)\partial_x f=\partial_\xi\tilde m+\partial_t g_1+g_2,\quad x\in I,\ t>0,\ \xi\in\mathbb{R}.$$

Then multiplying by $\phi(t,\xi)\psi(x)$ and integrating we have:

$$\begin{aligned}
&\int a(\xi)f(\alpha,t,\xi)\phi(t,\xi)\psi(\alpha)\,dt\,d\xi\\
&= -\int_\alpha^\beta\int_0^{+\infty}\int_{\mathbb{R}}\psi'(x)\phi(t,\xi)a(\xi)f(x,t,\xi)\,dx\,dt\,d\xi\\
&\quad+\int\psi(x)\partial_\xi\phi(t,\xi)\tilde{m}(dt,dx,d\xi)\\
&\quad+\int\psi(x)\partial_t\phi(t,\xi)g^1(x,t,\xi)\,dx\,dt\,d\xi\\
&\quad-\int\psi(x)\phi(t,\xi)g^2(x,t,\xi)\,dx\,dt\,d\xi.
\end{aligned}$$

This leads to:

$$\begin{aligned}
&\int a(\xi)f(\alpha,t,\xi)\phi(t,\xi)\,dt\,d\xi\\
&=\int a(\xi)h(t,\xi)\phi(t,\xi)\,dt\,d\xi-\int\partial_\xi\phi(t,\xi)m_\alpha(dt,d\xi).
\end{aligned}$$

□

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