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004

A CLASS OF THE QUASILINEAR PARABOLIC SYSTEMS ARISING IN POPULATION DYNAMICS *

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Abstract. This paper,using the duality technique and Hölder's inequality, proves the global existence of solutions for the quasilinear parabolic systems with Cross-diffusion effects and compectition interaction on any smooth bounded domain in R^2.

1. Introduction. The purpose of the present paper is to study the existence and uniqueness of $W_p^{2,1}$valued solutions for quasilinear parabolic systems arising in population dynamics as

$$\begin{cases} u_t = \triangle(a_1u + duv) + uF(u,v), & (x,t) \in \Omega \times (0,\infty). \\ v_t = \triangle(a_2v) + vG(u,v), & (x,t) \in \Omega \times (0,\infty). \\ \delta u + (1-\delta)u_\gamma = 0, & (x,t) \in \partial\Omega \times (0,\infty). \\ \delta v + (1-\delta)v_\gamma = 0, & (x,t) \in \partial\Omega \times (0,\infty). \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain in R^2 with smooth boundary $\partial\Omega$; $a_i(i = 1,2), d$ are positive constants; F, G are given C^1-functions;$\delta \in \{0,1\}$; γ denoting the outer unit normal on $\partial\Omega$; u_0 and $v_0$are initial functions which are assumed to satisfy

$$\begin{cases} u_0, v_0 \in H^{1+\varepsilon}(\Omega) \quad \text{for some} \quad 0 < \varepsilon < 1; \\ u_0(x), v_0(x) \geq 0, \qquad\qquad \forall x \in \Omega; \end{cases} \quad (H_0).$$

This systems has been introduced by Shigesada *et al.*[1] as a model of two competitive species which are interacting each other and migrating under self and cross-diffusion effects. The unknown functions u and v denote the population densities of the two species at time t and position $x \in \Omega$. The two boundary conditions show that the flow of an individual is tangential on the boundary $\partial\Omega$.

Masuda and Mimura (cf. [2]) proved, for the first time, global existence of a solution in the case that $\Phi \equiv 0$ and $\Omega \subset R^1$.

Yagi [3] studied the following systems.

$$\begin{cases} u_t = div\{\nabla[(\alpha_1 + \alpha_{11}u + \alpha_{12}v)u] \\ \qquad +\beta_1 u\nabla\Phi\} + c_1u - \gamma_{11}u^2 - \gamma_{12}uv, & \text{in } \Omega \times (0,\infty), \\ v_t = div\{\nabla[(\alpha_2 + \alpha_{21}u + \alpha_{22}v)v] \\ \qquad +\beta_2 v\nabla\Phi\} + c_2v - \gamma_{21}uv - \gamma_{22}v^2, & \text{in } \Omega \times (0,\infty), \\ [(\alpha_1 + \alpha_{11}u + \alpha_{12}v)u]_\gamma + \beta_1 u\Phi_\gamma = 0, & \text{on } \partial\Omega \times (0,\infty), \\ [(\alpha_2 + \alpha_{21}u + \alpha_{22}v)v]_\gamma + \beta_2 v\Phi_\gamma = 0, & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.2}$$

where $\alpha_i(i = 1,2)$ are positive constants,α_{ij} ,γ_{ij} and β_i ,$c_i(i,j = 1,2)$ are nonnegative constants . Φ is the C^1-function; In the case, global existence results are shown when one of the following conditions holds:

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(1) $0 < \alpha_{21} < 8\alpha_{11}$ and $0 < \alpha_{12} < 8\alpha_{22}$;
(2) $\alpha_{11} > 0, \alpha_{21} = \alpha_{22} = 0$.

We should note that the situation is quite different from our problem where $\alpha_{11} = \alpha_{21} = 0$.

In [4], Redinger given the global existence of the solution for the system (1.1) under the following assumption conditions

$$\begin{cases} (1) F(u,v) \equiv e_1 - h(u) - d_1 v, \; G(u,v) \equiv e_2 - d_2 u - b_2 v; \\ (2) h \in C^2(R,R), \text{and}, lim\ inf\ \ h(s)/s^\nu > 0. \\ \quad\quad \text{for some } \nu > 1. \\ \quad\quad \text{where the } e_i, d_i \text{ and } b_2 \text{ are positive constants .} \end{cases} \tag{1.3}$$

In this present paper, our main result can be stated as following

THEOREM. *Assume u_0 and v_0 satisfy (H_0),and, $F(u,v), G(u,v)$ to have the following properties:*

$$\begin{cases} (1) F(u,v) \equiv e_1 - h(u) - d_1 v, \; G(u,v) \equiv e_2 - d_2 u - b_2 v; \\ (2) h \in C^2(R,R), \text{and}, h(s) \geq A_0 s. \\ \quad\quad \text{where the } e_i, d_i \text{ ,} A_0 \text{and } b_2 \text{ are positive constants .} \end{cases} \tag{1.4}$$

Then, systems (1.1) has a unique global solution.

To prove this result , we will work in the framework of L_p and $W_p^{2,1}$and employ the L_p-requarity theory for the linear parabolic equations. Our basic tools, which help us to derive some *a priori* estimates, are the duality technique and the Gronwall's inequality.

Notation**: (see [5][8])**
$Q_T = \Omega \times [0,T]$.

$L_p(\Omega)$ is the Banach space consisting of all measurable functions on Ω that are p th-power($p \geq 1$) summable on Ω. The norm in it is defined by the equations

$$\|u\|_{p,\Omega} = (\int_\Omega |u(x)|^p\, dx)^{\frac{1}{p}} \text{ and } \|u\|_{\infty,\Omega} = vrimax\,|u|\,.$$

$W_p^{2,1}(Q_T)(p \geq 1)$ is the Banach space consisting of the elements of $L_p(Q_T)$ having generalized derivatives of the form $D_t^r D_x^s$ with r and s satisfying the inequality $2r+s \leq 2$. The norm in it is defined by the equality

$$\|u\|_{p,Q_T}^{(2)} = \sum_{2r+s\leq 2} \|D_t^r D_x^s u\|_{p,Q_T}.$$

2. Preliminaries.

2.1. Existence and uniqueness of local solutions. A local existence result for (1.1) is given by the following theorem (see Amann [6],Yagi [7]).

LEMMA 2.1. *Suppose that u_0 and v_0 satisfy (H_0), then, there exists a positive constant T^* such that (1.1) has a unique nonnegative solution (u, v), s.t.: $\forall\ 0 < T < T^*$,*

$$u, v \in C((0, T]; H^2(\Omega)) \cap C^1((0, T]; L^2(\Omega)).$$

2.2. The (backward) adjoint equation. In order to use a duality technique to obtain estimates on (u, v) in $W_p^{2,1}(Q_T)(p \geq 1)$, we need estimates on the solution χ of the (backward) adjoint equation

$$\begin{cases} \chi_t + a_0 \triangle \chi + \theta = 0\ , \text{ in } Q_T, \\ \chi_\gamma = 0, \text{ on } \partial\Omega \times (0, T), \\ \chi(\cdot, T) = 0, \text{ in } \Omega. \end{cases} \tag{2.1}$$

where $\theta \in L_p(Q_T)(q = \frac{p}{p-1}), \theta \geq 0; a_0 \in C^{0,\lambda}(Q_T)$ for some $\lambda > 0$,and, there exists a constant μ_0,s.t.:

$$a_0(x, t) \geq \mu_{0,} \text{ for all } (x, t) \in Q_T \tag{2.2}$$

We now state some well-known L_p-regularity results for (2.1).

LEMMA 2.2. *The adjoint equation (2.1) has a unique solution $\chi \in W_p^{2,1}(Q_T)$ with $\chi \geq 0$. If $\|\theta\|_{q,Q_T} = 1$, then there exists a constant $C = C(p, T)$ independent of θ, and, is continue to T, such that $\|\chi\|_{q,Q_T}^{(2)} \leq C$. Furthermore, C can be chosen so that*

$$\begin{cases} \text{(i) If } p > 2, \text{ then } \|\chi\|_{4q/(4-q),Q_T} \leq C, \text{ and, } \|\nabla\chi\|_{4q/(4-q),Q_T} \leq C, \\ \text{(ii) } \|\chi(\cdot, 0)\|_{q,\Omega} \leq C. \end{cases} \tag{2.3}$$

Proof. From [4](Theorem IV.9.1), there is a constant $C_0(p)$ independent T, such that

$$\|\chi\|_{q,Q_T}^{(2)} \leq C_0(p)\|\theta\|_{q,Q_T} = C_0(p).$$

Furthermore, there is a constant $C_1(p)$ independent T. such that

$$\|div\chi\|_{q^*,Q_T} \leq C_1(p)\|\chi\|_{q,Q_T}^{(2)},$$

where $q^* = 4q/(4 - q)$(see [4] lemma II 3.3).
Therefore, it follows that (2.3)(i) holds.
The same is true for (2.3)(ii) due again to [4] (lemma II 3.4).

3. The estimates for u and v. We note the nonnegativity of u, v follows from theorem 2.1, and ,denote by T^* a maximal existence time of the solution (u, v) to (1.1)

LEMMA 3.1. *If $T^* < \infty$, then, under the assumptions (H_0) and (1.4), we have the following estimates*

$$\|v\|_{\infty,Q_{T^*}} \le \ Const. \qquad \|u\|_{4,Q_{T^*}} \le \ Const.$$

Proof. From the system (1.1),we see that v satisfies

$$\begin{cases} v_t \le \triangle (a_2 v) + e_2 v - b_2 v^2, & (x,t) \in \Omega \times (0, T^*). \\ \delta v + (1-\delta) v_\gamma = 0, & (x,t) \in \partial\Omega \times (0, T^*). \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

Hence, it follows from the comparison principle that v is bounded from above by a constant M_0.*i.e.*

$$\|v\|_{\infty,Q_{T^*}} \le M_0 \equiv \ \text{const.} \tag{3.1}$$

Integrating (1.1)-u equation over Q_T $(0 < T < T^*)$, results in

$$\begin{aligned} \int_\Omega u(T)dx \le & \int_\Omega u_0(x)dx + \int_0^T \int_{\partial\Omega} (a_1 u + duv)_\gamma \\ & + \int_{Q_T} [e_1 u - h(u)u] dxdt \\ \le & \int_\Omega u_0(x)dx + \int_0^T \int_{\partial\Omega} [(a_1 + dv)\, u_\gamma + duv_\gamma] \\ & + \int_{Q_T} (e_1 u - A_0 u^2) dxdt, \end{aligned} \tag{3.2}$$

From the boundary condition of the system (1.1), we have that

$$\int_0^T \int_{\partial\Omega} [(a_1 + dv)\, u_\gamma + duv_\gamma] \le 0, \tag{3.3}$$

Thus, (3.2) becomes that

$$\begin{aligned} \int_\Omega u(T)dx + \int_{Q_T} A_0 u^2 dxdt \le & \int_\Omega u_0(x)dx + \int_{Q_T} e_1 u dxdt, \\ \le & \int_\Omega u_0(x)dx + \int_{Q_T} e_1 u^2 dxdt, \end{aligned} \tag{3.4}$$

By use of Gronwall's inequality, (3.4) implies that

$$\|u\|_{2,Q_{T^*}} \le C(u_0, T^*). \tag{3.5}$$

Thus,

$$g_1(x,t) = v(e_2 - d_2 u - b_2 v) \in L_2(Q_{T^*}) \tag{3.6}$$

and, v solves the following equation

$$\begin{cases} v_t = \triangle (a_2 v) + g_1(x,t), & (x,t) \in \Omega \times (0,T^*); \\ \delta v + (1-\delta) v_\gamma = 0, & (x,t) \in \partial\Omega \times (0,T^*); \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{3.7}$$

From [4], we have that

$$v \in W_2^{2,1}(Q_{T^*}), \nabla v \in L_4(Q_{T^*}), \tag{3.8}$$

From system (1.1), (3.1),(3.3) and (3.7),we known that u solves the following equation

$$\begin{cases} u_t = h_1(x,t)\triangle u + h_2(x,t)\nabla u \\ \qquad +h_3(x,t)u + h_4(x,t), & (x,t) \in \Omega \times (0,T^*); \\ \delta v + (1-\delta) v_\gamma = 0, & (x,t) \in \partial\Omega \times (0,T^*); \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.9}$$

where,

$$\begin{cases} h_1(x,t) \equiv (a_1 + dv) \in L_\infty(Q_{T^*}) \\ h_2(x,t) \equiv 2d(\nabla v) \in L_4(Q_{T^*}) \\ h_3(x,t) \equiv d\triangle v + e_1 \in L_2(Q_{T^*}) \\ h_4(x,t) \equiv -h(u)u - d_1 vu \in L_1(Q_{T^*}) \end{cases} \tag{3.10}$$

Set,

$$h_4^*(x,t) \equiv 0.$$

then,

$$h_4(x,t) \le h_4^*(x,t), \qquad \forall (x,t) \in Q_{T^*}$$

Let u^* solves the following equation

$$\begin{cases} u_t^* = h_1(x,t)\triangle u^* + h_2(x,t)\nabla u^* \\ \qquad +h_3(x,t)u^* + h_4^*(x,t), & \forall (x,t) \in \Omega \times (0,T^*); \\ \delta u^* + (1-\delta) u_\gamma^* = 0, & (x,t) \in \partial\Omega \times (0,T^*); \\ u^*(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.11}$$

By use of the comparison principle, we known that

$$0 \le u \le u^*, \qquad \forall (x,t) \in Q_{T^*}$$

From [4] and (3.11),we have that,$\forall \epsilon \in (0,1)$

$$u^* \in W_{2-\epsilon}^{2,1}(Q_{T^*}) \subset W_{2-\epsilon}^{1}(Q_{T^*}) \subset L_{3(2-\epsilon)/(1+\epsilon)}(Q_{T^*}), \tag{3.12}$$

Thus, taking $\epsilon = \frac{2}{7}$, we have that

$$u^* \in L_4(Q_{T^*}) \Rightarrow u \in L_4(Q_{T^*})$$

Furthermore ,we can also obtain the following estimate for v.

LEMMA 3.2. *If $T^* < \infty$, then, under the assumptions (H_0) and (1.4), there exists a positive constant $\lambda(T^*) > 0$, s.t.*

$$v \in C^{0,\lambda(T^*)}(\overline{Q}_{T^*})$$

Proof. From the Lemma 3.1, we have that

$$g_1(x,t) \equiv e_2 v - d_2 uv - b_2 v^2 \in L_4(Q_{T^*}). \tag{3.13}$$

Hence, v also solves the following parabolic equation

$$\begin{cases} v_t = a_2 \triangle v + g_1(x,t), & \text{in } Q_{T^*} \\ \delta v + (1-\delta) v_\gamma = 0, & \text{on } \partial\Omega \times (o, T^*) \\ v(x,0) = v_0(x), & \text{in } \Omega \end{cases} \tag{3.14}$$

From [4] , we have that

$$v \in W_4^{2,1}(Q_{T^*}) \subset W_4^1(Q_{T^*})$$

By use the embedding theorem (see [4]), there exists a positive constant $\lambda(T^*) > 0$ (Notice that $4 > N + 1 = 3$), s.t.

$$W_4^1(Q_{T^*}) \subset C^{0,\lambda(T^*)}(\overline{Q_{T^*}})$$

We have completed the proof of lemma 3.2

We are now ready to derive an important estimate for u. Our main methods are the duality technique and the Hölder inequality.

We first give two lemmas.

LEMMA 3.3. *Under the hypotheses (H_0) and (1.4), if $T^* < \infty$, then, for all $p > 1$, we have*

$$\|u\|_{p,Q_{T^*}} \le C(p, T^*).$$

Proof. Let φ satisfy the following backward equation

$$\begin{cases} \varphi_t + (a_1 + dv)\triangle\varphi + \theta = 0 \text{ in } Q_T, \\ \varphi_\gamma = 0 \text{ on } \partial\Omega \times (0,T), \\ \varphi(\cdot, T) = 0 \text{ in } \Omega, \end{cases} \tag{3.15}$$

Where, $\theta \in L_q(Q_T)(q = \frac{p}{p+1}, p > 2)$ with $\theta \geq 0$ and $\|\theta\|_{q,Q_T} = 1, (u, v)$ is the local solution of (1.1) in $\Omega \times [0, T^*), 0 < T < T^*$.

Multiplying $u-$ equation of (1.1) by φ and integrating the product in Q_T, we have

$$\int_{Q_T} \varphi u_t dxdt = \int_{Q_T} \varphi \triangle[(a_1 + dv)u]dxdt + \int_{Q_T} \varphi u[e_1 - h(u) - d_1 v]dxdt, \quad (3.16)$$

i.e.

$$\begin{aligned} &-\int_{Q_T} u\varphi_t dxdt - \int_{\Omega} \varphi(\cdot, 0)u_0 dx \quad (3.17)\\ \leq &\int_{\partial\Omega\times(0,.T)} \{\varphi[(a_1 + dv)u]_\gamma - (a_1 + dv)u\varphi_\gamma\} \\ &+ \int_{Q_T} (a_1 + dv)u\triangle\varphi dxdt + \int_{Q_T} e_1\varphi udxdt, \end{aligned}$$

From the lemma 3.2, and the boundary condition of the system (1.1),and (3.15), we obtain that

$$\int_{Q_T} \theta udxdt \leq \int_{\Omega} \varphi(\cdot, 0)u_0 dx + \int_{Q_T} e_1\varphi udxdt \quad (3.18)$$

By lemma 3.1, (3.18) becomes that

$$\int_{Q_T} \theta udxdt \leq C(u_0, p, T) + C(M_0, p, T) \|u\|_{4p/(4+p),Q_T}. \quad (3.19)$$

We write

$$H(\lambda) = \frac{1}{1-\lambda}[\frac{4p}{4+p} - \lambda p], \lambda \in (0, \frac{4}{4+p}), \quad (3.20)$$

then

$$\frac{dH(\lambda)}{d\lambda} = \frac{1}{(1-\lambda)^2} \cdot \frac{-p^2}{4+p} < 0,$$

and

$$H(\frac{4}{4+p}) = 0, \quad (3.21)$$

Therefore, there exists λ_0, s.t.

$$H(\lambda_0) \leq 1, \lambda_0 \in (0, \frac{4}{4+p}). \quad (3.22)$$

$$\|u\|_{4p/(4+p),Q_T} = \left\| u^{\frac{4p}{4+p}-\lambda_0 p+\lambda_0 p} \right\|_{1,Q_T}^{(4+p)/4p} \tag{3.23}$$
$$= \left\| u^{(1-\lambda_0)H(\lambda_0)} u^{\lambda_0 p} \right\|_{1,Q_T}^{(4+p)/4p}$$
$$= \left\| u^{H(\lambda_0)} \right\|_{1,Q_T}^{(1-\lambda_0)(4+p)/4p} \|u^p\|_{1,Q_T}^{\lambda_0(4+p)/4p},$$

while by lemma 3.1 and (3.23) it is seen that

$$\left\| u^{H(\lambda_0)} \right\|_{1,Q_T}^{(1-\lambda_0)(4+p)/4p} \leq C_p(T^*), 0 < T < T^*, \tag{3.24}$$

where $C_p(T^*)$ is a positive constant depending on T^* and p.

Hence, (3.23) becomes

$$\|u\|_{4p/(4+p),Q_T} \leq C_p(T^*) \|u^p\|_{1,Q_T}^{\lambda_0(4+p)/4p}, \tag{3.25}$$

this then shows, in view of (3.25), that we can conclude from (3.19) that

$$\int_{Q_T} \theta u dx dt \leq C(u_0,p,T) + C(M_0,p,T) \|u\|_{4p/(4+p),Q_T} \tag{3.26}$$
$$\leq C(u_0,p,T) + C(M_0,p,T^*) \|u\|_{p,Q_T}^{\lambda_0(4+p)/4}.$$

From the process of the proof, it is easy to see that we can choose $C(u_0,p,T)$ and $C(M_0,p,T^*)$ are continuous functions depending on T. Therefore, (3.26) can be written that

$$\int_{Q_T} \theta u dx dt \leq C(u_0,p,T^*) + C(M_0,p,T^*) \|u\|_{p,Q_T}^{\lambda_0(4+p)/4}. \tag{3.27}$$

Noting that $\lambda_0 \in (0, \frac{4}{4+p}), \theta \in L_q(Q_T)$ is arbitrary and $\theta \geq 0$, then, (3.24) implies that

$$\|u\|_{p,Q_T} \leq C(u_0,p,T^*), \forall 0 < T < T^*. \tag{3.28}$$

Thus, we can obtain the following estimates for u

$$\|u\|_{p,Q_{T^*}} \leq C(u_0,p,T^*). \tag{3.29}$$

We complete the proof.

We now establish the *a priori* estimates in $W_p^{2,1}(Q_{T^*})$.

Lemma 3.4. *Under the hypotheses of lemma 3.3, we have*

$$u, v \in W_p^{2,1}(Q_{T^*}), \forall p > 1.$$

Proof. Because of lemma 3.3, we have that $g_1(x,t) \in L_p(Q_{T^*})$ $(p > 1)$, and $, v$ solves the equation (3.14). From [5], we have that

$$v \in W_p^{2,1}(Q_{T^*}), \forall p > 1. \tag{3.30}$$

From $u-$equation of systems (1.1), we known that u solves equation (3.10). By use of the lemma 3.3, we have also that

$$u \in W_p^{2,1}(Q_{T^*}), \qquad \forall p > 1 \tag{3.31}$$

4. The existence of global solutions. Now, we can begin to prove the existence theorem of global solutions.

THEOREM 4.1. *Assume that* (H_0) *and (1.4) are satisfied, then, (1.1) has a unique solution in* $\Omega \times (0, \infty)$.

Proof. The proof consists of four steps.

Step 1:

Suppose for contradiction that T^* ¡ ∞, from lemma 3.4, it follows that

$$u, v \in W_p^{2,1}(Q_{T^*}), \forall p > 1. \tag{4.1}$$

Now, let us differentiate the $u-$equation of (1.1) in t, and take the product with $(a_1 + dv)u_t$. Integrating the product on $\Omega \times [\delta_0,t]$ with a fixed $\delta_0 (0 < \delta_0 < t < T^*)$, we obtain that

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\times[\delta_0,t]} (a_1 + dv)\,(u_t^2)_t \\ &= \int_{\Omega\times[\delta_0,t]} (a_1 + dv)\,u_t\{\Delta[(a_1 + dv)\,u]\}_t \\ &\quad + \int_{\Omega\times[\delta_0,t]} (a_1 + dv)\,u_t(e_1u_t - uh'(u)u_t - d_1vu_t - duv_t). \end{aligned} \tag{4.2}$$

From (4.1), the last term of (4.2) has the following estimate

$$\int_{\Omega\times[\delta_0,t]} (a_1 + dv)\,u_t(e_1u_t - uh'(u)u_t - d_1vu_t - duv_t) \le C(p, T^*, \delta_0). \tag{4.3}$$

We have that the (4.2) becomes that

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\times[\delta_0,t]} (a_1 + dv)\,(u_t^2)_t \\ &\le C(p, T^*, \delta_0) + \int_{\partial\Omega\times[\delta_0,t]} (a_1 + dv)\,u_t[(a_1 + dv)\,u_\gamma + duv_\gamma]_t \\ &- \int_{\Omega\times[\delta_0,t]} \nabla[(a_1 + dv)\,u_t]\nabla[(a_1 + dv)\,u]_t \\ &= C(p, T^*, \delta_0) + \int_{\partial\Omega\times[\delta_0,t]} (a_1 + dv)\,u_t\{(a_1 + dv)\,(u_\gamma)_t + dv_tu_\gamma + du_tv_\gamma + du(v_\gamma)_t\} \\ &- \int_{\Omega\times[\delta_0,t]} [(a_1 + dv)\,\nabla u_t + du_t\nabla v][(a_1 + dv)\,\nabla u_t + dv_t\nabla u + du_t\nabla v + du\nabla v_t]. \end{aligned} \tag{4.4}$$

Next, let us differentiate the second equation in (1.1) in t, and, multiplying by $(1 + u^2)v_t$, and integrating the product in $\Omega \times [\delta_0, t]$, we can obtain that

$$\begin{aligned} &\frac{1}{2}\int_{\Omega} \left(1 + u^2\right) v_t^2(t) \\ &\le \int_{\Omega\times[\delta_0,t]} a_2(1 + u^2)v_t\Delta v_t + \int_{\Omega\times[\delta_0,t]} (1 + u^2)v_t(e_2v_t - d_2uv_t - d_2vu_t - 2b_2vv_t) \\ &\le C(p, T^*, \delta_0) + \int_{\partial\Omega\times[\delta_0,t]} a_2(1 + u^2)v_t(v_\gamma)_t - \int_{\Omega\times[\delta_0,t]} a_2\nabla[(1 + u^2)v_t]\nabla v_t. \end{aligned} \tag{4.5}$$

From the boundary conditions of systems (1.1), we have that

$$\begin{cases} (1)\ \delta=1: u=v=0 \Rightarrow u_t=v_t=0, \forall(x,t)\in\partial\Omega\times[\delta_0,t] \\ (2)\ \delta=0: u_\gamma=v_\gamma=0 \Rightarrow (u_\gamma)_t=(v_\gamma)_t=0, \forall(x,t)\in\partial\Omega\times[\delta_0,t] \end{cases} \tag{4.6}$$

Then, (4.4) (4.5) become that

$$\frac{1}{2}\int_{\Omega\times[\delta_0,t]}(a_1+dv)\,(u_t^2)_t \le C(p,T^*,\delta_0)- \tag{4.7}$$
$$-\int_{\Omega\times[\delta_0,t]}[(a_1+dv)\,\nabla u_t+du_t\nabla v][(a_1+dv)\,\nabla u_t+dv_t\nabla u+du_t\nabla v+du\nabla v_t],$$
$$\frac{1}{2}\int_{\Omega}\left(1+u^2\right)v_t^2(t)\le C(p,T^*,\delta_0)-\int_{\Omega\times[\delta_0,t]}a_2\nabla[(1+u^2)v_t]\nabla v_t, \tag{4.8}$$

we known that

$$-\int_{\Omega\times[\delta_0,t]}[(a_1+dv)\,\nabla u_t+du_t\nabla v][(a_1+dv)\,\nabla u_t+dv_t\nabla u+du_t\nabla v+du\nabla v_t]$$
$$=-\int_{\Omega\times[\delta_0,t]}\{(a_1+dv)^2\,|\nabla u_t|^2+(a_1+dv)\,[dv_t\nabla u+2du_t\nabla v \tag{4.9}$$
$$+d\,(a_1+dv)\,u\nabla v_t\nabla u_t+d^2u_t\nabla vu\nabla v_t\},$$
$$\le-\int_{\Omega\times[\delta_0,t]}(1-\varepsilon)\,(a_1+dv)^2\,|\nabla u_t|^2-$$
$$-d\,(a_1+dv)\,u\nabla u_t\nabla v_t-\varepsilon\int_{\Omega\times[\delta_0,t]}|\nabla(v_t)|^2+C(p,T^*,\delta_0,\varepsilon). \tag{4.10}$$

here, $\varepsilon\in(0,1)$ is arbitrary.

For (4.7), we have the similar estimates

$$-\int_{\Omega\times[\delta_0,t]}a_2\nabla[(1+u^2)v_t]\nabla v_t \tag{4.11}$$
$$\le -a_2(1-\varepsilon)\int_{\Omega\times[\delta_0,t]}[(1+u^2)\,|\nabla v_t|^2+C(p,T^*,\delta_0,\varepsilon).$$

From, (4.7)-(4.11), we obtain that, for any $\delta_0<t<T^*, \varepsilon\in(0,1), k>0$

$$\frac{1}{2}\int_{\Omega}(a_1+dv)\,u_t^2(t)+\frac{k}{2}\int_{\Omega}\left(1+u^2\right)v_t^2(t) \tag{4.12}$$
$$\le-\int_{\Omega\times[\delta_0,t]}\{(1-\varepsilon)\,(a_1+dv)^2\,|\nabla u_t|^2+d\,(a_1+dv)\,u\nabla u_t\nabla v_t+\varepsilon\int_{\Omega\times[\delta_0,t]}|\nabla(v_t)|^2$$
$$-ka_2(1-\varepsilon)\int_{\Omega\times[\delta_0,t]}[(1+u^2)\,|\nabla v_t|^2+C(p,T^*,\delta_0).$$
$$=-\int_{\Omega\times[\delta_0,t]}\{(1-\varepsilon)\,(a_1+dv)^2\,|\nabla u_t|^2+d\,(a_1+dv)\,u\nabla u_t\nabla v_t$$
$$+[ka_2(1-\varepsilon)-\varepsilon](1+u^2)\,|\nabla v_t|^2\}+C(p,T^*,\delta_0).$$

Taking $\varepsilon = min\{\frac{1}{2}, \frac{d^2}{2}\}, k = \frac{4d^2}{a_2}$,we have that

$$\frac{1}{2}\int_\Omega (a_1+dv)\, u_t^2(t) + \frac{d^2}{a_2}\int_\Omega \left(1+u^2\right) v_t^2(t) \le -\int_{\Omega\times[\delta_0,t]} \frac{1}{2}(a_1+dv)^2 |\nabla u_t|^2$$
$$-\int_{\Omega\times[\delta_0,t]} d\,(a_1+dv)\, u\nabla u_t \nabla v_t + d^2(1+u^2)\,|\nabla v_t|^2 + C(p,T^*,\delta_0). \tag{4.13}$$

i.e.

$$\frac{1}{2}\int_\Omega (a_1+dv)\, u_t^2(t) + \frac{d^2}{a_2}\int_\Omega \left(1+u^2\right) v_t^2(t) \tag{4.14}$$
$$+\int_{\Omega\times[\delta_0,t]} \{\frac{1}{16}(a_1+dv)^2 |\nabla u_t|^2 + \int_{\Omega\times[\delta_0,t]} \frac{d^2}{8}(1+u^2)\,|\nabla v_t|^2 \le C(p,T^*,\delta_0).$$

Step 2:
Multiplying the first equation of (1.1) by u^p and integrating the product in $Q_t(0 < t < T^*)$, we have

$$\frac{1}{p+1}\int_\Omega u^{p+1}(t) = \frac{1}{p+1}\int_\Omega u_0^{p+1} + \int_{Q_t} u^p \,\triangle\, [(a_1+dv)\,u] \tag{4.15}$$
$$+\int_{Q_t} u^{p+1}\{e_1 - h(u) - d_1 v\}$$
$$\text{(from lemma 3.4)} \le C(u_0, v_0, p, T^*)$$

From the second equation of (1.1), a similar inequality for v is obtained

$$\frac{1}{p+1}\int_\Omega v^{p+1}(t) \le C(u_0, v_0, p, T^*). \tag{4.16}$$

Hence, from the systems (1.1) and the above estimates, we have, for all $0 < t < T^*$,

$$\| \{\Delta[(a_2+dv)u]\}(t) \|_{2,\Omega} + \| (\Delta v)(t) \|_{2,\Omega} \le C(u_0, T^*). \tag{4.17}$$

Furthermore, differentiating the $v-$equation in (1.1) in t, and take the scalar product with $v_t^{2p+1} (p \ge 1)$. Then integration of the product on [0,t]($0 < t < T^*$) yields that

$$\| (v_t)(t) \|_{2p+2,\Omega} \le C(u_0, p, T^*), \text{ for all } 0 < t < T^*, \tag{4.18}$$

thus

$$\| (\triangle v)(t) \|_{2p+2,\Omega} \le C(u_0, p, T^*), \text{ for all } 0 < t < T^*. \tag{4.19}$$

In view of the embedding theorem

$$H^1(\Omega) = W_2^1(\Omega) \subset L_4(\Omega).$$

We have that

$$\begin{cases} \| \{\nabla[(a_2+dv)u]\}(t) \|_{4,\Omega} \leq C(u_0,v_0,T^*), & \text{for all } 0<t<T^*, \\ \| (\nabla v)(t) \|_{p,\Omega} \leq C(u_0,v_0,T^*), & \text{for all } 0<t<T^* \end{cases} \tag{4.20}$$

Thus

$$\| (\nabla u)(t) \|_{p,\Omega} \leq C(u_0,v_0,T^*), \text{ for all } 0<t<T^*, \tag{4.21}$$

Noting that

$$|\Delta u| \leq |(a_2+dv)\Delta u| \leq |\Delta[(a_2+dv)u]| + 2d\,|\nabla u \nabla v| + d\,|u \triangle v|$$

then

$$\begin{aligned} &\| \Delta u \|_{2,\Omega}^2 \leq \| \Delta[(a_2+dv)u \|_{2,\Omega}^2 + 4d^2(\| \nabla u \|_{4,\Omega}^4 + \| \nabla v \|_{4,\Omega}^4) \\ &+d^2(\| u \|_{4,\Omega}^4 + \| \Delta v \|_{4,\Omega}^4) \leq C(u_0,T^*), \text{ for all } 0<t<T^*, \end{aligned} \tag{4.22}$$

Hence, the norms $\| u(t) \|_{H^2(\Omega)}$and $\| v(t) \|_{H^2(\Omega)}$ remain bounds as $t \to T^*$.

Step 3:

From [8], then the solution u, v, can be extended to a nonnegative solution beyond the T^*. Hence, (1.1) has a global solution, we complete the proof.

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