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FIXED ENERGY INVERSE PROBLEM FOR EXPONENTIALLY DECREASING POTENTIALS *

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1. Introduction. In this paper we show that in two-body scattering the scattering matrix at a fixed energy determines real-valued exponentially decreasing potentials. This result has been proved by Novikov previously [3], see also [2], using a $\overline{\partial}$-equation. We present a different method, which combines a density argument and real analyticity in part of the complex momentum. The latter has been noted in [2]; here we give a short proof using contour deformations, similarly to [1, Section 1.5]. We thus prove:

THEOREM 1.1. *Suppose that $n \geq 3$, $V, V' \in e^{-\gamma_0|w|}L^\infty(\mathbb{R}^n_w;\mathbb{R})$ for some $\gamma_0 > 0$, and $\lambda > 0$. If $S_+(\lambda) = S'_+(\lambda)$, then $V = V'$. Here $S_+(\lambda)$, resp. $S'_+(\lambda)$ are the scattering matrices of $H = \Delta + V$ and $H' = \Delta + V'$ at energy λ.*

Theorem 1.1 for compactly supported potentials follows from an analogous result in [4] for the corresponding Dirichlet-to-Neumann map. See [5, Section 12], and the references given in these papers for a review of the relation between the Dirichlet-to-Neumann map and the fixed energy problem.

The general method follows [4], as discussed in [1]. We thus recall the construction of complex exponential solutions u_ρ, $\rho \in \mathbb{C}^n$ of $(H - \lambda)u_\rho = 0$, where $u_\rho(w) = e^{i\rho\cdot w}(1 + v_\rho(w))$, $\rho\cdot\rho = \lambda$, and $v_\rho \to 0$ in an appropriate sense as $\rho \to \infty$. These solutions exist for ρ outside an 'exceptional set' which is discrete in z. We also show that if we write $\rho = z\nu + \rho_\perp$, $\nu \in \mathbb{S}^{n-1}$, $\rho_\perp \in \mathbb{R}^n$ perpendicular to ν, and $z \in \mathbb{C}\setminus\mathbb{R}$, then for fixed ν, u_ρ is analytic in z and real analytic in $\rho_\perp$, hence extends to be analytic in a neighborhood of $\mathbb{R}^{n-1}\setminus\{0\}$ in $\mathbb{C}^{n-1}_{\rho_\perp}$. The exceptional set is then given by the zeros of an analytic function of z and $\rho_\perp$. We caution the reader that the extension of u_ρ to complex $\rho_\perp$ does not agree with $u_{z\nu+\rho_\perp}$ where $\rho_\perp$ is allowed to be complex; indeed v_ρ will merely lie in $e^{\gamma|w|}L^2(\mathbb{R}^n)$ for some $\gamma > 0$.

We use this in the inverse problem as follows. Let u_ρ, $u'_{\rho'}$ be exponential eigenfunctions of H, resp. H', as above. Now consider the pairing

$$\int_{\mathbb{R}^n} u_\rho(V - V')u'_{\rho'},$$

where $\rho = z\nu + \rho_\perp$, $\rho' = z'\nu + \rho'_\perp$, and ν is fixed. If u_ρ, u'_ρ are replaced by tempered distributional eigenfunctions of H and H', then a standard argument shows that $S_+(\lambda) = S'_+(\lambda)$ implies that the corresponding pairing vanishes. We employ a density argument to deduce that the pairing also vanishes for the complex exponentials provided that $|\operatorname{Im} z + \operatorname{Im} z'|$ is small and $\rho\cdot\rho = \lambda = \rho'\cdot\rho'$. We then let $\rho, \rho' \to \infty$. By analyticity, the pairing still vanishes. On the other hand, $v_\rho, v'_{\rho'} \to 0$, so for $\zeta = \rho - \rho' \in \mathbb{R}^n$ we deduce that $\int_{\mathbb{R}^n} e^{i\zeta\cdot w}(V - V') = 0$, i.e. the Fourier transform

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of $V - V'$, hence $V - V'$, vanish. In fact, this step will be slightly more complicated, because the density argument imposes restrictions on ζ, and we first deduce vanishing of the Fourier transform of $V - V'$ in a spherical shell of finite 'thickness', and then use the exponential decay of $V - V'$ to conclude that it is in fact identically zero.

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2. Exponential eigenfunctions. In this section we recall the construction of exponential solutions of $(H - \lambda)u = 0$ from [4]. First, for $\rho \in \mathbb{C}^n$, let

$$u_\rho^0(w) = e^{i\rho\cdot w}.$$

Thus, u_ρ^0 is an 'exponential eigenfunction' of Δ, namely

$$(\Delta - \lambda)u_\rho^0 = 0, \qquad \rho \cdot \rho = \lambda.$$

We assume everywhere that $n \geq 3$.

For the Hamiltonian H, we then seek exponential solutions u of the form

$$u = u_\rho = e^{i\rho\cdot w}(1 + v_\rho),\ \rho\cdot\rho = \lambda,\ \rho \in \mathbb{C}^n, \tag{2.1}$$

where v_ρ is considered a perturbation. In fact, we will have $v_\rho \in L^2_r(\mathbb{R}^n)$ for all $r < 0$ (when v_ρ exists). Here $L^2_p = L^2_p(\mathbb{R}^n)$ denotes the $L^2(\mathbb{R}^n, \langle w\rangle^{2p}\,dw)$, $\langle w\rangle^s = (1 + |w|^2)^{s/2}$. Substituting u into $(H - \lambda)u = 0$, we obtain

$$(\Delta + 2\rho \cdot D_w + V)v_\rho = -V. \tag{2.2}$$

The construction given below works under power-decay assumptions on V, but we state it for exponentially decaying V, since v_ρ is real analytic in the appropriate components of ρ only in that case. So we assume that

$$V \in e^{-\gamma_0|w|}L^\infty(\mathbb{R}^n),\ \gamma_0 > 0. \tag{2.3}$$

Thus, we need to construct a right inverse $G(\rho)$ to

$$P(\rho) = \Delta + 2\rho \cdot D_w + V$$

that can be applied to rapidly decreasing functions. Once this is done,

$$u_\rho = e^{i\rho\cdot w}(1 - G(\rho)V)$$

is the solution to the original problem. Below we write

$$P_0(\rho) = \Delta + 2\rho \cdot D_w. \tag{2.4}$$

Since a right inverse $G_0(\rho)$ of $P_0(\rho)$ can be constructed explicitly, perturbation theory will give the existence of $G(\rho)$.

Namely, let

$$G_0(\rho) = \mathcal{F}^{-1}(|\xi|^2 + 2\rho\cdot\xi)^{-1}\mathcal{F},$$

so $P_0(\rho)G_0(\rho) = \mathrm{Id}$ e.g. on Schwartz functions. Thus, on the Fourier transform side $G_0(\rho)$ acts via multiplication by $(|\xi|^2 + 2\rho\cdot\xi)^{-1}$ which is in $L^1(\mathbb{R}^n)$. It is convenient to represent ρ as

$$\rho = z\nu + \rho_\perp,\ \rho_\perp \in \mathbb{R}^n,\ \nu \in \mathbb{S}^{n-1},\ z \in \mathbb{C},\ \rho_\perp \cdot \nu = 0.$$

We often identify $\operatorname{span}\{\nu\}^\perp$ with $\mathbb{R}^{n-1}$. For $z \notin \mathbb{R}$, this distribution is conormal to

$$S(\rho) = \{\xi \in \mathbb{R}^n : \ |\xi|^2 + 2\operatorname{Re}\rho\cdot\xi = 0,\ \operatorname{Im}\rho\cdot\xi = 0\} \\ = \{\xi \in \mathbb{R}^n : \ (\xi+\rho_\perp)^2 = \rho_\perp^2,\ \nu\cdot\xi = 0\}. \tag{2.5}$$

Note that $S(\rho)$ actually depends only on $\rho_\perp$ and ν, not on z.

Below we write $e^{\gamma\langle w\rangle}L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n; e^{-2\gamma\langle w\rangle}\,dw)$, and if m is an integer,

$$e^{\gamma\langle w\rangle}H^m(\mathbb{R}^n) = \{u \in e^{\gamma\langle w\rangle}L^2(\mathbb{R}^n) : \ D^\alpha u \in e^{\gamma\langle w\rangle}L^2(\mathbb{R}^n),\ |\alpha| \le m\}.$$

The latter is equivalent to $e^{-\gamma\langle w\rangle}u \in H^m(\mathbb{R}^n)$, hence the notation.

We first recall:

Proposition 2.1. *[4, Proposition 3.1], [6, Theorem 1.1]* $G_0(\rho) : L^2_p \to L^2_r$ *is bounded for* $p > 0$, $r < 0$, $r < p-1$. *Moreover, the norm of* $G_0(\rho)$ *as a bounded operator between these spaces goes to* 0 *as* $|\rho| \to \infty$.

Our central result is the following proposition.

Proposition 2.2. *Suppose that* $\gamma > 0$ *and fix* $\nu \in \mathbb{S}^{n-1}$. *Then there exists a neighborhood* U *of* $\mathbb{R}^{n-1}\setminus\{0\}$ *in* $\mathbb{C}^{n-1}$ *and an operator*

$$\mathcal{G}_0(z,\rho_\perp) : e^{-\gamma\langle w\rangle}L^2(\mathbb{R}^n) \to e^{\gamma\langle w\rangle}H^2(\mathbb{R}^n)$$

defined on $(\mathbb{C}\setminus\mathbb{R})\times U$ *such that* $\mathcal{G}_0$ *is analytic on* $(\mathbb{C}\setminus\mathbb{R})\times U$ *as a bounded operator between these spaces, and its restriction to* $(\mathbb{C}\setminus\mathbb{R})\times(\mathbb{R}^{n-1}\setminus\{0\})$ *is* $G_0(\rho)$, $\rho = z\nu+\rho_\perp$. *Thus, for* $z \in \mathbb{C}\setminus\mathbb{R}$, $\rho_\perp \in \mathbb{R}^{n-1}\setminus\{0\}$, *the operator* $G_0(\rho) : e^{-\gamma|w|}L^2 \to e^{\gamma|w|}L^2$ *is complex-analytic in* z, *real analytic in* $\rho_\perp$. *Moreover,* $G_0(\rho) \to 0$ *as a bounded operator on this space as* $|\rho| \to \infty$.

Proof. We fix some $(z^0, \rho^0_\perp)$, and show that $G_0(\rho)$ extends to be complex analytic in a neighborhood of this in $\mathbb{C}_z \times \mathbb{C}^{n-1}_{\rho_\perp}$. In fact, it is convenient to consider

$$R_0(\rho) = e^{-i\rho_\perp\cdot w}G_0(\rho)e^{i\rho_\perp\cdot w}.$$

Since the multipliers are holomorphic as maps

$$e^{\gamma\langle w\rangle}H^m(\mathbb{R}^n) \to e^{\gamma'\langle w\rangle}H^m(\mathbb{R}^n),\ \gamma < \gamma',$$

for $|\operatorname{Im}\rho_\perp|$ sufficiently small, and unitary for $\rho_\perp$ real, the original statement follows after we show that $R_0(\rho)$ extends analytically.

We do so by contour deformation on the Fourier transform side. Let $\xi = (\xi_\parallel, \xi_\perp)$ be the decomposition of ξ according to the decomposition $\operatorname{span}(\{\nu\}) \oplus \operatorname{span}(\{\nu\})^\perp$ of $\mathbb{R}^n$. Thus, $\mathcal{F}R_0(\rho)\mathcal{F}^{-1}$ is a multiplication operator by F^{-1} where

$$F(\xi, z, \rho_\perp) = |\xi-\rho_\perp|^2 + 2(\xi-\rho_\perp)\cdot\rho = \xi_\parallel^2 + 2z\xi_\parallel + \xi_\perp^2 - \rho_\perp^2. \tag{2.6}$$

Then

$$\operatorname{Im}F = 2\operatorname{Im}z\,\xi_\parallel,\ \operatorname{Re}F = \xi_\parallel^2 + 2\operatorname{Re}z\,\xi_\parallel + \xi_\perp^2 - \rho_\perp^2.$$

Thus the multiplication operator by F^{-1} is singular where $F = 0$, i.e. at

$$\tilde{S}(\rho) = \{\xi : \ \xi_\parallel = 0,\ \xi_\perp^2 = \rho_\perp^2\},$$

which is a sphere in the hyperplane $\xi_{\|} = 0$.

It is convenient to break up $G_0(\rho)$ into two pieces by introducing a cutoff $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ that is identically 1 near $\tilde{S}(\rho^0)$. For instance, we may take

$$\psi(\xi) = \phi(\xi_{\|}, \xi_\perp^2)$$

with $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, identically 1 near $(0, |\rho_\perp^0|^2)$. Then

$$R_0(\rho) = R_0'(\rho) + R_0''(\rho),\ R_0'(\rho) = \mathcal{F}^{-1}(|\xi|^2 + 2\xi\cdot\rho - \rho_\perp^2)^{-1}\psi(\xi)\mathcal{F}.$$

Then $R_0''(\rho)$ is a Fourier multiplier by the function $(1-\psi(\xi))F(\xi, z, \rho_\perp)^{-1}$, which is in fact a symbol of order -2, analytic in z and $\rho_\perp$ for $\operatorname{Im}\rho_\perp$ small, hence $R_0''(\rho)$ is analytic as a map $L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$.

To analyze $R_0'(\rho)$, it is convenient to introduce polar coordinates in $\xi_\perp$: $\xi_\perp = r\omega$, $|\omega| = 1$, $r \geq 0$. Then

$$F = \xi_{\|}^2 + 2z\xi_{\|} + r^2 - \rho_\perp^2.$$

Now, by Fubini's theorem $R_0(\rho)f = \mathcal{F}^{-1}F^{-1}\mathcal{F}f$ can be written as

$$(R_0'(\rho)f)(w) = (2\pi)^{-n}\int_{\mathbb{R}}\int_{\mathbb{S}^{n-2}}\int_0^\infty e^{ir\omega\cdot w_\perp}e^{i\xi_{\|}w_{\|}}(\xi_{\|}^2 + 2z\xi_{\|} + r^2 - \rho_\perp^2)^{-1}$$
$$\psi(\xi_{\|}, r\omega)(\mathcal{F}f)(\xi_{\|}, r\omega)\,dr\,d\omega\,d\xi_{\|}.$$

We divide the $\xi_{\|}$ integral into two pieces, corresponding to $\xi_{\|} \geq 0$ and $\xi_{\|} \leq 0$. In each piece, we then deform the contour of the r integral in a compact set disjoint from $\operatorname{supp}(1-\psi)$ near $r_0 = |\rho_\perp^0|$ in such a way that $\operatorname{Im} r^2 = 2\operatorname{Re} r\,\operatorname{Im} r$ and $\operatorname{Im} z\,\xi_{\|}$ have the same sign on the contour. Note that the integrand is analytic in r for $\operatorname{Im} r$ small and $\xi_{\|} \neq 0$.

Thus, suppose that $\operatorname{Im} z > 0$. For $\xi_{\|} > 0$, we deform the contour $[0, +\infty)_r$ near r_0 to a curve Γ_+ so that $\operatorname{Im} r \geq 0$ on Γ_+ and r_0 does *not* lie on the Γ_+. Now, F never vanishes along Γ_+, provided that $\rho_\perp$ is close to $\rho_\perp^0$. Thus, extending ψ to be 1 on $\Gamma_+ \setminus [0, +\infty)$, and using that $\mathcal{F}f$ extends to be analytic in a tube $\{\xi \in \mathbb{C}^n : |\operatorname{Im}\xi| < \gamma\}$,

$$(2\pi)^{-n}\int_0^\infty\int_{\mathbb{S}^{n-2}}\int_0^\infty e^{ir\omega\cdot w_\perp}e^{i\xi_{\|}w_{\|}}(\xi_{\|}^2 + 2z\xi_{\|} + r^2 - \rho_\perp^2)^{-1}$$
$$\psi(\xi_{\|}, r\omega)(\mathcal{F}f)(\xi_{\|}, r\omega)\,dr\,d\omega\,d\xi_{\|}$$
$$= (2\pi)^{-n}\int_0^\infty\int_{\mathbb{S}^{n-2}}\int_{\Gamma_+} e^{ir\omega\cdot w_\perp}e^{i\xi_{\|}w_{\|}}(\xi_{\|}^2 + 2z\xi_{\|} + r^2 - \rho_\perp^2)^{-1}$$
$$\psi(\xi_{\|}, r\omega)(\mathcal{F}f)(\xi_{\|}, r\omega)\,dr\,d\omega\,d\xi_{\|},$$

and on the right hand side we can allow $\rho_\perp$ to become complex, proving real analyticity of $R_0'(\rho)$ in $\rho_\perp$, and extending it as an analytic family of operators $\mathcal{R}_0'(z, \rho_\perp)$. This argument parallels the analytic continuation argument of [1, Chapter 1]. It is now easy to see that $\mathcal{R}_0'(z, \rho_\perp)$ maps into $e^{\gamma\langle w\rangle}H^2(\mathbb{R}^n)$; indeed, it maps into $e^{\gamma\langle w\rangle}C_\infty^\infty(\mathbb{R}^n)$, where $C_\infty^\infty(\mathbb{R}^n)$ is the space of smooth functions which are bounded with all derivatives.

For $\xi_{\|} < 0$ we proceed similarly, deforming the contour $[0, +\infty)_r$ near r_0 to a curve Γ_- so that $\operatorname{Im} r \geq 0$ on Γ_- and r_0 does *not* lie on the Γ_-. Again, we deduce real analyticity in $\rho_\perp$.

The last part follows from the preceeding proposition since $e^{-\gamma|w|}L^2 \subset L^2_p \subset L^2_r \subset e^{\gamma|w|}L^2$.

Instead of the explicit contour deformation, we could have used the partial Fourier transform in $w_\parallel$, to deduce that

$$G_0(\rho)f = e^{i\rho_\perp \cdot w}\mathcal{F}_\parallel^{-1}(\Delta_\perp + \xi_\parallel^2 + 2z\xi_\parallel - \rho_\perp^2)^{-1}\mathcal{F}_\parallel e^{-i\rho_\perp \cdot w} f$$

is real analytic in $\rho_\perp$ and analytic in z by inserting step functions $1 = H(\xi_\parallel) + H(-\xi_\parallel)$, and using the analyticity of

$$(\Delta_\perp - \sigma)^{-1} : e^{-\gamma|w_\perp|}L^2(\text{span}\{\nu\}^\perp) \to e^{\gamma|w_\perp|}L^2(\text{span}\{\nu\}^\perp)$$

in σ. ☐

Corollary 2.3. *Suppose that $\gamma, \gamma_0 > 0$. Then the operator*

$$e^{-\gamma_0\langle w\rangle}\mathcal{G}_0(z, \rho_\perp) \in \mathcal{B}(e^{-\gamma|w|}L^2, e^{-(\gamma_0-\gamma)\langle w\rangle}H^2)$$

is analytic in z and in $\rho_\perp$ as a bounded operator between these spaces.

Corollary 2.4. *Suppose that $V \in e^{-\gamma_0|w|}L^\infty$ and $\gamma_0 > 2\gamma$, and let U be as in Proposition 2.2. Then there exists a set*

$$\mathcal{E} \subset (\mathbb{C} \setminus \mathbb{R})_z \times U,$$

which is given by the zeros of an analytic function and whose intersection with

$$(\mathbb{C} \setminus \mathbb{R})_z \times (\mathbb{R}^{n-1} \setminus \{0\})_{\rho_\perp}$$

is bounded, such that $(\mathrm{Id} + V\mathcal{G}_0(z, \rho_\perp))^{-1}$ exists in the complement of $\mathcal{E}$, and in a neighborhood of every point where it exists, $(\mathrm{Id} + V\mathcal{G}_0(z, \rho_\perp))^{-1}$ is analytic with values in compact operators on $e^{-\gamma|w|}L^2$.

Proof. By the preceeding corollary, $e^{-\gamma_0\langle w\rangle}\mathcal{G}_0(z, \rho_\perp) : e^{-\gamma|w|}L^2 \to e^{-(\gamma_0-\gamma)\langle w\rangle}H^2$ is analytic in z and $\rho_\perp$. But the inclusion $e^{-(\gamma_0-\gamma)\langle w\rangle}H^2 \hookrightarrow e^{-\gamma|w|}L^2$ is compact, and $e^{\gamma_0\langle w\rangle}V \in L^\infty$, so $V\mathcal{G}_0(z, \rho_\perp)$ is an analytic family of compact operators on $e^{-\gamma|w|}L^2$. Moreover, as $|z| \to \infty$ or $|\rho_\perp| \to \infty$, $\rho_\perp$ real, $V\mathcal{G}_0(z, \rho_\perp) = VG_0(\rho) \to 0$ in norm. Thus, the conclusion follows by analytic Fredholm theory. ☐

We write

$$\mathcal{G}(z, \rho_\perp) = \mathcal{G}_0(z, \rho_\perp)(\mathrm{Id} + V\mathcal{G}_0(z, \rho_\perp))^{-1}, \; G(\rho) = G_0(\rho)(\mathrm{Id} + VG_0(\rho))^{-1}.$$

We immediately deduce the following result.

Proposition 2.5. *Suppose that $V \in e^{-\gamma_0|w|}L^\infty$,*

$$v_{z,\rho_\perp} = -\mathcal{G}(z, \rho_\perp)V.$$

Then

$$((\mathbb{C} \setminus \mathbb{R}) \times U) \setminus \mathcal{E} \ni (z, \rho_\perp) \mapsto v_\rho$$

is an analytic function, with values in $e^{\gamma|w|}L^2$, for any $\gamma > 0$.

Corollary 2.6. *Let $\nu \in \mathbb{S}^{n-1}$. Suppose that $V, V' \in e^{-\gamma_0|w|}L^\infty$, and let $\mathcal{E}$, $\mathcal{E}'$ be the exceptional sets of these two potentials. Then for $(z, \rho_\perp) \notin \mathcal{E}$, $(z', \rho'_\perp) \notin \mathcal{E}'$ the pairing*

$$\int_{\mathbb{R}^n} u_\rho (V - V') u'_{\rho'} \tag{2.7}$$

converges if $|\operatorname{Im} z + \operatorname{Im} z'| < \gamma_0$, and is analytic in $z, z', \rho_\perp, \rho'_\perp$.

Proof. We consider the strip $|\operatorname{Im} z + \operatorname{Im} z'| < \gamma_1 < \gamma_0$, $\gamma_1 > 0$. Let $\gamma \in (0, (\gamma_0 - \gamma_1)/2)$. Then $1 + v_\rho$, $1 + v'_{\rho'}$ are analytic in $(z, z', \rho_\perp, \rho'_\perp)$ with values in $e^{\gamma|w|}L^2$. Hence,

$$u_\rho (V - V') u'_{\rho'} = e^{i(\rho+\rho')\cdot w}(V - V')(1 + v_\rho)(1 + v'_{\rho'})$$

is analytic in $(z, z', \rho_\perp, \rho'_\perp)$ with values in $L^1(\mathbb{R}^n)$. Integration preserves analyticity and proves the result. ∎

3. Density of generalized eigenfunctions. In this section we relate tempered distributional eigenfunctions of $H = \Delta + V$ to its exponential eigenfunctions, constructed in the previous section.

We first introduce some notation. For $\lambda > 0$, the free incoming Poisson operator is given by

$$\tilde{P}_+(\lambda)g = c \int_{\mathbb{S}^{n-1}} e^{-i\sqrt{\lambda}\, w\cdot\omega} g \, d\omega_a, \quad g \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ c = \lambda^{\frac{n-1}{4}} e^{-\frac{n-1}{4}\pi i}(2\pi)^{-\frac{n-1}{2}}.$$

The Poisson operator of H is then

$$P_+(\lambda)g = \tilde{P}_+(\lambda)g - R(\lambda + i0)((H - \lambda)\tilde{P}_+(\lambda)g) = \tilde{P}_+(\lambda)g - R(\lambda + i0)V\tilde{P}_+(\lambda)g.$$

Note that for $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $V\tilde{P}_+(\lambda)g$ is Schwartz, in fact decays exponentially, hence $R(\lambda + i0)$ can be applied to it. For $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$,

$$P_+(\lambda)g = e^{-i\sqrt{\lambda}|w|}g_- + e^{i\sqrt{\lambda}|w|}g_+ + L^2(\mathbb{R}^n),\ g_+, g_- \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ g_- = g. \tag{3.1}$$

For such g, $P_+(\lambda)g$ is characterized by the property that it is the unique solution u of $(H - \lambda)u = 0$ which is of the form (3.1). The scattering matrix is then the operator

$$S_+(\lambda) : \mathcal{C}^\infty(\mathbb{S}^{n-1}) \to \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ S_+(\lambda)g_- = g_+.$$

There is also an incoming Poisson operator $P_-(\lambda)$ which is characterized by the fact that for $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $P_-(\lambda)g$ is the unique solution u of $(H - \lambda)u = 0$ of the form

$$P_-(\lambda)g = e^{-i\sqrt{\lambda}|w|}g_- + e^{i\sqrt{\lambda}|w|}g_+ + L^2(\mathbb{R}^n),\ g_+, g_- \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ g_+ = g. \tag{3.2}$$

In particular, for $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$,

$$\overline{P_-(\lambda)g} = P_+(\lambda)\overline{g}. \tag{3.3}$$

The S-matrix is related to the Poisson operator via the following boundary pairing.

PROPOSITION 3.1. *[1, Lemma 2.2] Suppose that $\lambda > 0$, and $V \in e^{-\gamma_0|w|}L^\infty$, $\gamma_0 > 0$. Suppose that $(H-\lambda)u_+ \in L^2_s$, $(H-\lambda)u_- \in L^2_s$, $s > 1/2$, and*

$$u_+ = e^{-i\sqrt{\lambda}|w|}g_{+-} + e^{i\sqrt{\lambda}|w|}g_{++} + L^2,$$
$$u_- = e^{-i\sqrt{\lambda}|w|}g_{--} + e^{i\sqrt{\lambda}|w|}g_{-+} + L^2,$$

$g_{\pm\pm} \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. Then

$$\langle u_+, (H-\lambda)u_-\rangle - \langle (H-\lambda)u_+, u_-\rangle = 2i\sqrt{\lambda}(\langle g_{++}, g_{-+}\rangle - \langle g_{+-}, g_{--}\rangle). \tag{3.4}$$

REMARK 3.2. *This is stated for $V \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ in [1]. However, if u_+, u_- are as above, then $Vu_+u_- \in e^{-\gamma|w|}L^1$ for $\gamma < \gamma_0$, hence the conclusion is equivalent to the corresponding statement with $H-\lambda$ replaced by $\Delta - \lambda$.*

Let $R(\lambda') = (H-\lambda')^{-1}$ for $\lambda' \in \mathbb{C}\setminus\mathbb{R}$. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, and apply this proposition with

$$u_- = R(\lambda - i0)f = e^{-i\sqrt{\lambda}|w|}g_{--} + L^2,\ u_+ = P_+(\lambda)g.$$

We deduce that

$$\langle u_+, f\rangle = -2i\sqrt{\lambda}\langle g, g_{--}\rangle. \tag{3.5}$$

Our density result is the following.

PROPOSITION 3.3. *Suppose that $V \in e^{-\gamma_0|w|}L^\infty$, and let $0 < \gamma < \gamma' < \gamma_0$. Then the set*

$$\mathcal{F} = \{P_+(\lambda)g_+ :\ g_+ \in \mathcal{C}^\infty(\mathbb{S}^{n-1})\}$$

is dense in the nullspace of $H-\lambda$ on $e^{\gamma|w|}L^2$ in the topology of $e^{\gamma'|w|}L^2$.

Proof. Suppose that $f \in e^{-\gamma'|w|}L^2$ is orthogonal to $\mathcal{F}$. Let $u_- = R(\lambda - i0)f$. By (3.5), for all $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $\langle g, g_{--}\rangle = 0$ since $\langle f, P_+(\lambda)g\rangle$ vanishes by assumption. But $u_- = R_0(\lambda - i0)f'$, $f' = f - VR(\lambda - i0)f \in e^{-\gamma|w|}L^2$. Thus, $\mathcal{F}u_-$ is the product of an analytic function, namely $\mathcal{F}f'$, and $(|\xi|^2 - (\lambda - i0))^{-1}$. Thus, $u_- \in L^2$ implies that $\mathcal{F}f'$ vanishes on the sphere $|\xi| = \sqrt{\lambda}$. Hence $\mathcal{F}f' = (\xi^2 - \lambda)\phi$, with ϕ analytic in the strip $|\operatorname{Im}\xi| < \gamma'$. Thus, $u_- \in e^{-\gamma|w|}L^2$ for $\gamma < \gamma'$. Thus for $v \in e^{\gamma|w|}L^2$ with $(H-\lambda)v = 0$,

$$\langle f, v\rangle = \langle (H-\lambda)u_-, v\rangle = \langle u_-, (H-\lambda)v\rangle = 0,$$

i.e. f is orthogonal to the nullspace of $H-\lambda$ on $e^{\gamma|w|}L^2$. Thus, $\mathcal{F}$ is dense in this nullspace. □

Our approach to the inverse problem relies on relating the S-matrices to the pairing (2.7). Thus, we consider two operators H and H' induced by potentials V and V' respectively, and show that the equality of the S-matrices at a fixed energy λ implies the vanishing of an analogous pairing. For this we use the following consequence of Proposition 3.1 applied with Δ in place of H.

PROPOSITION 3.4. *Suppose that $\lambda > 0$. Let $u_+ = P_+(\lambda)g_+$, $u_- = P'_-(\lambda)g_-$. Then*

$$\langle u_+, (\Delta-\lambda)u_-\rangle - \langle (\Delta-\lambda)u_+, u_-\rangle = 2i\sqrt{\lambda}(\langle S_+(\lambda)g_+, g_-\rangle - \langle g_+, S'_-(\lambda)g_-\rangle). \tag{3.6}$$

Corollary 3.5. *Suppose* $\lambda > 0$, $S_+(\lambda) = S'_+(\lambda)$. *Let* $u_+ = P_+(\lambda)g_+$, $u_- = P_-(\lambda)g_-$. *Then*

$$\int_{\mathbb{R}^n} (V - V')u_+\overline{u_-} = 0. \tag{3.7}$$

Similarly, if $u_+ = P_+(\lambda)g_+$, $u_- = P_+(\lambda)g_-$, *then.*

$$\int_{\mathbb{R}^n} (V - V')u_+u_- = 0. \tag{3.8}$$

Proof. (3.7) follows from the preceeding proposition since $S'_-(\lambda)^* = S'_+(\lambda)$. Then (3.8) follows from (3.7) by applying the latter with g_- replaced by $\overline{g_-}$ and using (3.3). □

4. Inverse results: Proof of Theorem 1.1. Let $\lambda > 0$, and suppose that

$$V, V' \in e^{-\gamma_0|w|}L^\infty,\ \gamma_0 > 0.$$

Fix $\zeta \in \mathbb{R}^n$ such that $|\zeta| > 2\sqrt{\lambda}$, and let $\nu \in \mathbb{S}^{n-1}$ be orthogonal to ζ, and let $\mu \in \mathbb{S}^{n-1}$ orthogonal to both ζ and ν. For t real, $t > \sqrt{\frac{1}{4}|\zeta|^2 - \lambda}$, let

$$\begin{aligned} \rho = \rho(t) &= \frac{\zeta}{2} + (t^2 - \frac{1}{4}|\zeta|^2 + \lambda)^{1/2}\mu + it\nu, \\ \rho' = \rho'(t) &= \frac{\zeta}{2} - (t^2 - \frac{1}{4}|\zeta|^2 + \lambda)^{1/2}\mu - it\nu, \end{aligned} \tag{4.1}$$

so $\rho \cdot \rho = \lambda = \rho' \cdot \rho'$. By Corollary 2.6, the integral

$$\int_{\mathbb{R}^n} u_\rho(V - V')u'_{\rho'} \tag{4.2}$$

converges for all t, and is meromorphic in t in a neighborhood of $(\sqrt{\frac{1}{4}|\zeta|^2 - \lambda}, +\infty)$. We use a density argument, Proposition 3.3, and Corollary 3.5 to show that this integral actually vanishes if $S_+(\lambda) = S'_+(\lambda)$ and

$$2\sqrt{\lambda} < |\zeta| < \sqrt{4\lambda + \gamma_0^2}. \tag{4.3}$$

Indeed, for $\sqrt{\frac{1}{4}|\zeta|^2 - \lambda} < t < \gamma < \gamma_0/2$, $u'_{\rho'}$ can be approximated by $P_-(\lambda)g_-$ in $e^{\gamma|w|}L^2$ due to Proposition 3.3. Similarly, u_ρ can be approximated by $P_+(\lambda)g_+$ in $e^{\gamma|w|}L^2$. On the other hand, $V - V'$ lies in $e^{-\gamma_0|w|}L^2$. Hence the product can be approximated in L^1 by a product which takes the form of the integrand of (3.8). The equality of the S-matrices implies that (3.8) vanishes, hence so does (4.2), i.e. we deduce the following result.

Proposition 4.1. *Suppose that* $\lambda > 0$, $V, V' \in e^{-\gamma_0|w|}L^\infty$, $S_+(\lambda) = S'_+(\lambda)$. *Then for* ζ *satisfying* (4.3), ρ, ρ' *given by* (4.1) *with* $(z, \rho_\perp) \notin \mathcal{E}$, $(z, \rho_\perp) \notin \mathcal{E}'$,

$$\int_{\mathbb{R}^n} u_\rho(V - V')u'_{\rho'} = 0 \tag{4.4}$$

for $\sqrt{\frac{1}{4}|\zeta|^2 - \lambda} < t < \gamma_0/2$.

The pairing in (4.4) is meromorphic in t with $\operatorname{Re} t > \sqrt{\frac{1}{4}|\zeta|^2 - \lambda}$ and $|\operatorname{Im} t|$ sufficiently small. It vanishes on an interval inside this domain by the proposition. Thus, (4.4) holds for all $t > \sqrt{\frac{1}{4}|\zeta|^2 - \lambda}$. Then as $t \to \infty$, the integral on the left hand side of (4.4) converges to

$$\int_{\mathbb{R}^n} (V - V')u^0_\rho u^0_{\rho'} = \int_{\mathbb{R}^n} (V - V')\, e^{i\zeta \cdot w}\, dw \tag{4.5}$$

since $v_\rho \to 0$, $v'_\rho \to 0$ in $e^{\gamma|w|}L^2(\mathbb{R}^n)$ for any $\gamma > 0$ and $V - V' \in e^{-\gamma_0|w|}L^\infty(\mathbb{R}^n)$ with $\gamma_0 > 0$. But this is the Fourier transform of $V - V'$, evaluated at ζ. Hence the vanishing of (4.5) shows that the Fourier transform of $V - V'$ vanishes on the shell (4.3). Since this Fourier transform is real analytic, as $V - V' \in e^{-\gamma_0|w|}L^\infty$, we deduce that it vanishes everywhere, hence $V = V'$. This completes the proof of Theorem 1.1.

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