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A NOTE TO THE REGULARITY OF SOLUTIONS FOR THE EVOLUTION P-LAPLACIAN EQUATIONS

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In this note we consider the following Cauchy problem:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = div(|\nabla u|^{p-2}\nabla u) & (x,t) \in Q_T = R^N \times (0,T) \\ u(x,0) = u_0(x) & x \in R^N. \end{cases}$$

where $p > 2,\ u_0 \in L^1_{loc}(R^N)$ satisfies that there exist constants $r > 0$ and $\rho_0 > 0$ such that

$$(2)\qquad |||u_0|||_{r,\rho_0} \equiv \sup_{x_0 \in R^N} \int_{B_{\rho_0}(x_0)} |u_0(x)|dx < \infty, \quad B_\rho(x_0) = \{|x - x_0| < \rho\}.$$

It is well known that there exists a solution $u \in C^\alpha_{loc}(Q_T) \cap L^\infty_{loc}(Q_T),\ \nabla u \in C^{\beta,\frac{\beta}{2}}_{loc}(Q_T)$ to (1) (see [C],[DF],[DH]). The proofs of $\nabla u \in C^{\beta,\frac{\beta}{2}}_{loc}(Q_T)$ are very complex and difficult. In this note we use another approach to prove the Hölder continuity of ∇u. We prove $u_t \in L^\infty_{loc}(Q_T),\ \nabla u \in C^{\beta,\frac{\beta}{1+\beta}}_{loc}(Q_T)$, where the Hölder index to t is great than $\frac{\beta}{2}$.

DEFINITION. A function $u(x,t)$ defined in Q_T is called a weak solution of (1), if $u \in C^\alpha_{loc}(Q_T) \cap L^p(0,T : W^{1,p}_{loc}(R^N) \cap L^\infty(Q_T)$ $\alpha \in (0,1)$ and for any $\phi(x,t) \in C^1(\overline{Q_T})$ $\phi = 0$ if $|x|$ large enough,

$$(3)\qquad \begin{aligned} &\int_{R^N} u(x,t)\phi(x,t)dx + \int_0^t \int_{R^N} [-u\phi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\phi]dxdt \\ =\ &\int_{R^N} u_0(x)\phi(x,0)dx. \end{aligned}$$

We obtain the following result.

THEOREM 1. Let $u_0 \geq 0$, $|||u_0(x)|||_{r,\rho_0} < \infty$ for some $\rho_0 > 0$. Then there exist constants $C > 0$ and $\beta \in (0,1)$ such that the solution of (1) satisfies

$$(4)\qquad \sup_{R^N} |u_t(x,t)| \leq Ct^{-\frac{N(p-1)+p}{\kappa}} |||u_0|||^{\frac{p}{\kappa}}_{r,\rho}, \quad \nabla u \in C^{\beta,\frac{\beta}{1+\beta}}(Q_T),$$

where $\kappa = N(p-2) + p$.

Proof. let u be the solution of (1). Acoording to [WZYL], for $\forall\delta \in (0,T)$, $u(x,t+\delta)$ is the limit of the solutions of the following boundary value problems

$$(5)\qquad \begin{cases} \dfrac{\partial v}{\partial t} = div((|\nabla v|^{p-2})\nabla v) & (x,t) \in B_n \times (0,T-\delta) \\ v(x,t) = u(x,\delta) & (x,t) \in \partial B_n \times (0,T-\delta) \\ v(x,0) = u(x,\delta) & x \in B_n \end{cases}$$

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where $B_n = \{|x| < n\}$. Let v_n be the solution of (5). Set

$$w_n(x,t) = \lambda^\gamma v_n(x, \lambda t), \quad \lambda > 1, \quad \gamma = \frac{1}{p-2}.$$

Then w_n satisfies

$$(6) \qquad \begin{cases} \dfrac{\partial w}{\partial t} = div(|\nabla w|^{p-2}\nabla w) & (x,t) \in B_n \times (0, \frac{T-\delta}{\lambda}) \\ w(x,t) = \lambda^\gamma u(x,\delta) & (x,t) \in \partial B_n \times (0, \frac{T-\delta}{\lambda}) \\ w(x,0) = \lambda^\gamma u(x,\delta) & x \in B_n \end{cases}$$

Set $g_n = w_n - v_n$. By Comparison principle $g_n \geq 0$ and

$$(7) \qquad \begin{aligned} &\textstyle\int_{B_n} g_n(x,t)\phi(x,t)dx - \int_0^t \int_{B_n} g_n \phi_t dx d\tau + \int_0^t \int_{B_n} (|\nabla w_n|^{p-2}\nabla w_n - \\ &-|\nabla v_n|^{p-2}\nabla v_n) \cdot \nabla\phi dx d\tau = \textstyle\int_{B_n} (\lambda^\gamma - 1) u(x,\delta)\phi(x,0)dx \end{aligned}$$

where $\phi \in C^1(\overline{B_n \times (0,T)})$ $\phi = 0$ near ∂B_n. Notice that (see [DH])

$$\|u(x,\delta)\|_{L^\infty(R^N)} \leq C\delta^{-\frac{N}{\kappa}} |||u_0|||_{r,\rho_0}^{\frac{p}{\kappa}}.$$

In (7), we take

$$\phi = (g_n - k)_+ \quad k = (\lambda^\gamma - 1)\|u(x,\delta)\|_{L^\infty(R^N)}.$$

Using Steklov averaging process, we get

$$\begin{aligned} &\textstyle\int_{B_n} (g_n - k)_+^2 dx \\ &+2\textstyle\int_0^t \int_{B_n \cap \{w > k\}} (|\nabla w_n|^{p-2}\nabla w_n - |\nabla v_n|^{p-2}\nabla v_n)(\nabla w_n - \nabla v_n) dx d\tau = 0. \end{aligned}$$

This implies $g_n \leq k$ *a.e. on* $B_n \times (0, \frac{T-\delta}{\lambda})$. Thus

$$(8) \qquad 0 \leq \lambda^\gamma v_n(x,\lambda t) - v_n(x,t) \leq (\lambda^\gamma - 1)\|u(x,\delta)\|_{L^\infty(R^N)}.$$

Divided (8) by $\lambda - 1$ and let $\lambda \to 1^+$, we get

$$|\gamma v_n(x,t) + t v_{nt}(x,t)| \leq \gamma \|u(x,\delta)\|_{L^\infty(R^N)}.$$

This inequality implies

$$|u_t(x, t+\delta)| \leq \frac{C\||u(x,\delta)\||_{L^\infty(R^N)}}{t}.$$

Let $\delta \to t$, we get the first estimate of (4).

We now prove the second estimate of (4). Notice that for fixed $t \in (0,T)$ $u(x,t)$ is a solution of the following elliptic equations

$$div(|\nabla u|^{p-2}\nabla u) = u_t(x,t) \quad x \in R^N.$$

By [T], there exist constants $\beta \in (0,1)$, $C > 0$ dependent only on $|u_t|_{L^\infty}$, $|u|_{L^\infty}$ such that

$$(9) \qquad |\nabla u(x_1,t) - \nabla u(x_2,t)| \leq C|x_1 - x_2|^\beta.$$

We now prove that ∇u is Hölder cotinuous to t. For convernience, we assume that u is a smooth solution, otherwise by uniqueness of solution we can consider the regularized problem. Take the x_j-derivative in (1) to obtain

$$\frac{\partial u_{x_j}}{\partial t} = (div(|\nabla u|^{p-2}\nabla u))_{x_j} \tag{10}$$

Let $x_0 \in R^N$, $0 < t_1 \leq t_2$, $\Delta t = t_2 - t_1, B(\Delta t) = B_{(\Delta t)^\delta}(x_0)$. Integrating (10) over $B(\Delta t) \times (t_1, t_2)$ and by integrating by parts, we get

$$\begin{aligned} & \int_{B(\Delta t)}(u_{x_j}(x,t_2) - u_{x_j}(x,t_1))dx \\ = & \int_{t_1}^{t_2}\int_{\partial B(\Delta t)}(div(|\nabla u|^{p-2}\nabla u))_{x_j}dxdt \\ = & \int_{t_1}^{t_2}\int_{\partial B(\Delta t)} div(|\nabla u|^{p-2}\nabla u)\nu_j d\sigma dt \\ = & \int_{t_1}^{t_2}\int_{\partial B(\Delta t)} u_t\nu_j d\sigma dt. \end{aligned} \tag{11}$$

where $\nu = (\nu_1, \nu_2, ..., \nu_N)$ is the unit outward normal vector of $\partial B(\Delta t)$. By the mean value theorem, there exists $x^* \in B(\Delta t)$ such that

$$|u_{x_j}(x^*, t_2) - u_{x_j}(x^*, t_1)| \leq C(\Delta t)^{1-\delta}. \tag{12}$$

Combining (9) and (12) and taking $\delta = \frac{1}{1+\beta}$, we get

$$\begin{aligned} & |u_{x_j}(x_0,t_2) - u_{x_j}(x_0,t_1)| \leq |u_{x_j}(x_0,t_2) - u_{x_j}(x^*,t_2)| \\ & +|u_{x_j}(x^*,t_2) - u_{x_j}(x^*,t_1)| + |u_{x_j}(x^*,t_1) - u_{x_j}(x_0,t_1)| \leq C(\Delta t)^{\frac{\beta}{1+\beta}}. \end{aligned}$$

Therefore $u_{x_j} \in C_{loc}^{\beta,\frac{\beta}{1+\beta}}(R^N)$ and Theorem 1 is proved.

REMARK 1. If the initial value u_0 is bounded, Theorem 1 holds for u_0 of variable sign. In fact if u is a solution of (1), by the uniqueness of solution $v = u + \|u_0\|_{L^\infty(R^N)}$ is a nonnegative solution of (1) with initial value $u_0 + \|u_0\|_{L^\infty(R^N)}$. Thus Theorem 1 holds for v, so does u.

REMARK 2. For the first boundary value problem, similar theorem holds. Consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = div(|\nabla u|^2\nabla u) & (x,t) \in \Omega \times (0,T) \\ u(x,t) = \psi(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & x \in R^N \end{cases} \tag{13}$$

where $\Omega \in R^N$ is a smooth bounded region.

THEOREM 2. Let $u_0 \in L^\infty(R^N)$, $\psi, \psi_t \in L^\infty(\partial\Omega \times (0,T))$. Then the solution u of (13) satisfies

$$|u_t(x,t)| \leq \frac{C}{t}, \quad \nabla u \in C_{loc}^{\beta,\frac{\beta}{1+\beta}}(\Omega \times (0,T)).$$

Proof. Without loss of generality, we assume $u \geq 0$, and u large enough, otherwise replace u by $u + C$, $C > \|u\|_{L^\infty}$. Set

$$v(x,t) = \lambda^\gamma u(x, \lambda t), \quad \lambda > 1, \quad \gamma = \frac{1}{p-2}.$$

Then v is the solution of (13) with

$$v(x,t) = \lambda^\gamma \psi(x, \lambda t) \ \ (x,t) \in \partial\Omega \times (0,T), \quad v(x,0) = \lambda^\gamma u_0(x) \ \ x \in R^N.$$

Notice that if $\lambda - 1$ is small enough, ψ large enough, we have

$$\lambda^\gamma \psi(x, \lambda t) - \psi(x,t) = (\lambda^\gamma - 1)\psi(x, \lambda t) + \psi_t(x, \xi)(\lambda - 1)t$$
$$= (\lambda - 1)(\tfrac{\lambda^\gamma - 1}{\lambda - 1}\psi(x, \lambda t) + t\psi_t(x, \xi)) > 0.$$

By comparison principle $\lambda^\gamma u(x, \lambda t) \geq u(x,t)$. Hence similar to the proof in Theorem 1, we can prove Theorem 2.

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