

UNIFORM BLOW-UP ESTIMATES FOR NONLINEAR HEAT EQUATIONS AND APPLICATIONS

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We consider the following nonlinear heat equation

$$(0.1) \quad \begin{cases} u_t &= \Delta u + |u|^{p-1}u \\ u(0) &= u_0, \end{cases}$$

where $u : (x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}$,

$$(0.2) \quad 1 < p, (N - 2)p < N + 2 \text{ and either } u_0 \geq 0 \text{ or } (3N - 4)p < 3N + 8.$$

More general vector-valued heat equations can be considered with similar results (see [MZ99] for more details) :

$$(0.3) \quad \begin{cases} u_t &= \Delta u + F(|u|)u \\ u(0) &= u_0, \end{cases}$$

where $u : \Omega \times [0, T) \rightarrow \mathbb{R}^M$, p satisfies (0.2), $\Omega = \mathbb{R}^N$ or Ω is a smooth bounded convex domain of \mathbb{R}^N , $F(|u|) \sim |u|^{p-1}$ as $|u| \rightarrow +\infty$, and $M \in \mathbb{N}$.

We are interested in the blow-up phenomenon for (0.1). Many authors have been interested in this topic. Let us mention for instance Friedman [Fri65], Fujita [Fuj66], Ball [Bal77], Bricmont and Kupiainen [BKL94], Chen and Matano [CM89], Galaktionov and Vázquez [GV95], Giga and Kohn [GK89], [GK87], [GK85], Herrero and Velázquez [HV93], [HV92].

In the following, we consider $u(t)$ a blow-up solution of (0.1) and denote its blow-up time by T . We aim at finding sharp uniform estimates at blow-up and specifying the blow-up behavior of $u(t)$. Such a study is done considering equation (0.1) in its self-similar form : for all $a \in \mathbb{R}^N$, we define

$$(0.4) \quad y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad w_{a,T}(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t).$$

Therefore, $w_{a,T} = w$ satisfies $\forall s \geq -\log T, \forall y \in \mathbb{R}^N$:

$$(0.5) \quad \frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1}w.$$

Let us introduce the following Lyapunov functional associated with (0.5) :

$$E(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla w|^2 + \frac{1}{2(p-1)} |w|^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho(y) dy$$

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where $\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}$.

The question is to estimate $w_{a,T}(s)$ as $s \rightarrow +\infty$, uniformly with respect to a , whether a is a blow-up point or not (a is called a blow-up point if there exists $(a_n, t_n) \rightarrow (a, T)$ such that $|u(a_n, t_n)| \rightarrow +\infty$).

Giga and Kohn showed that self-similar variables are convenient for describing the blow-up rate in the following sense : there exists $\epsilon_0(u_0) > 0$ such that $\forall s \geq s_0^*(u_0)$,

$$(0.6) \quad \epsilon_0 \leq |w(s)|_{L^\infty} \leq \frac{1}{\epsilon_0}.$$

We first aim at sharpening this result in order to obtain compactness properties in our problem.

1. A Liouville Theorem for equation (0.5). We are interested in classifying all global and bounded solutions of (0.5), for all subcritical p :

$$(1.1) \quad p > 1 \text{ and } (N - 2)p < N + 2.$$

We claim the following :

THEOREM 1. A Liouville Theorem for equation (0.5). *Assume (1.1) and consider w a solution of (0.5) defined for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$ such that $\forall (y, s) \in \mathbb{R}^N \times \mathbb{R}, |w(y, s)| \leq C$. Then, either $w \equiv 0$, or $w \equiv \kappa$ or $w(y, s) = \pm \varphi(s - s_0)$ where $\kappa = (p - 1)^{-\frac{1}{p-1}}$, $s_0 \in \mathbb{R}$ and $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$.*

REMARK. φ is in fact an L^∞ connection between two critical points of (0.5): κ and 0. Indeed,

$$\dot{\varphi} = -\frac{\varphi}{p-1} + \varphi^p, \quad \varphi(-\infty) = \kappa, \quad \varphi(+\infty) = 0.$$

REMARK. A similar classification result can be obtained with a solution w defined only on $(-\infty, s^*)$ (see [MZ98]).

Theorem 1 has the following corollary :

COROLLARY 1. A Liouville Theorem for equation (0.1). *Assume (1.1) and consider u a solution of (0.1) defined for all $(x, t) \in \mathbb{R}^N \times (-\infty, 0)$ such that $\forall (x, t) \in \mathbb{R}^N \times (-\infty, 0), |u(x, t)| \leq C(-t)^{-\frac{1}{p-1}}$. Then, either $u \equiv 0$, or $u(x, t) = \pm \kappa(T^* - t)^{-\frac{1}{p-1}}$ for some $T^* \geq 0$.*

The proofs can be found in [MZ99] and [MZ98]. The key tools in the proof are the following :

- i) A classification of all possible linear behaviors of $w(s)$ as $s \rightarrow -\infty$ in $L^2_\rho(\mathbb{R}^N)$ ($L^\infty_{loc}(\mathbb{R}^N)$),
- ii) The following geometric transformations which keeps (0.5) invariant :

$$w(y, s) \rightarrow w_{a,b}(y, s) = w(y + ae^{\frac{s}{2}}, s + b),$$

where $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$,

- iii) A blow-up criterion for (0.5) used for solutions close to the constant point κ (This criterion is also a blow-up criterion for (0.1) via the transformation (0.4)) :

If for some $s_0 \in \mathbb{R}$, $I(w(s_0)) > 0$ where

$$I(w) = -2E(w) + \frac{p-1}{p+1} \left(\int_{\mathbb{R}^N} |w(y)|^2 \rho(y) dy \right)^{\frac{p+1}{2}},$$

then $w(s)$ blows up in finite time.

REMARK. This criterion is sharp for solutions near constants. Indeed, if $w(s_0) \equiv C_0$, then

$$w \text{ blows-up in finite time} \Leftrightarrow |C_0| > \kappa \Leftrightarrow I(C_0) > 0.$$

REMARK. The proof of the Liouville Theorem strongly relies on the existence of a Lyapunov functional for equation (0.5) and can not be extended to other systems where the nonlinearity is not a gradient. In [Zaa], we go beyond this restriction and introduce new tools to prove a Liouville Theorem of the same type for the following system

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q.$$

2. Localization at blow-up. We assume again (0.2). The estimate (0.6) of Giga and Kohn gives compactness in the problem. Using a compactness procedure in the singular zone of \mathbb{R}^N (which is, say $\{y \mid |w(y, s)| \geq \frac{\kappa}{2}\}$), we find a solution satisfying the hypotheses of Theorem 1. Therefore, Δw is small with respect to $|w|^p$ in this singular zone (or equivalently, Δu is small with respect to $|u|^p$). A subcritical localization procedure introduced by Zaag [Zaa98] (under the level of the constant κ) allows us to propagate this estimate towards the intermediate zone between the singular and the regular one. We claim the following :

THEOREM 2. Comparison with the associated ordinary differential equation. Assume (0.2) and consider $T \leq T_0$ and $\|u_0\|_{C^2(\mathbb{R}^N)} \leq C_0$. Then, $\forall \epsilon > 0$, there is $C(\epsilon, C_0, T_0)$ such that $\forall (x, t) \in \mathbb{R}^N \times [0, T)$,

$$|u_t - |u|^{p-1}u| \leq \epsilon |u|^p + C.$$

REMARK. This way, we prove that the solution of the PDE (0.1) can be uniformly and globally in space-time compared to a solution of an ODE (localized by definition). Note that the condition $u(0) \in C^2(\mathbb{R}^N)$ is not restrictive because of the regularizing effect of the Laplacian.

REMARK. Many striking corollaries can be derived from this theorem. It implies in particular that no oscillation is possible near a blow-up point a , and that $|u(x, t)| \rightarrow +\infty$ as $(x, t) \rightarrow (a, T)$. Moreover, $\forall \epsilon_0 > 0$, there exists $t_0(\epsilon_0) < T$ such that for all $b \in \mathbb{R}^N$, if $|u(b, t)| \leq (1 - \epsilon_0)\kappa(T - t)^{-\frac{1}{p-1}}$ for some $t \in [t_0, T)$, then b is not a blow-up point (this specifies more precisely a former result by Giga and Kohn where $t_0 = t_0(\epsilon_0, a)$).

3. Optimal L^∞ estimates at blow-up. We still assume (0.2). Using estimate (0.6) and the Liouville Theorem, we make a compactness argument to get the following sharp estimates :

THEOREM 3. L^∞ refined estimates for $w(s)$. Assume that (0.2) holds. Then, there exist positive constants C_i for $i = 1, 2, 3$ such that if u is a solution of (0.1)

which blows-up at time T and satisfies $u(0) \in C^3(\mathbb{R}^N)$, then $\forall \epsilon > 0$, there exists $s_1(\epsilon) \geq -\log T$ such that $\forall s \geq s_1, \forall a \in \mathbb{R}^N$,

$$\begin{aligned} \|w_{a,T}(s)\|_{L^\infty} &\leq \kappa + \left(\frac{N\kappa}{2p} + \epsilon\right)\frac{1}{s}, & \|\nabla w_{a,T}(s)\|_{L^\infty} &\leq \frac{C_1}{\sqrt{s}}, \\ \|\nabla^2 w_{a,T}(s)\|_{L^\infty} &\leq \frac{C_2}{s}, & \|\nabla^3 w_{a,T}(s)\|_{L^\infty} &\leq \frac{C_3}{s^{3/2}}. \end{aligned}$$

REMARK. In the case $N = 1$, Herrero and Velázquez (Filippas and Kohn also) proved some related estimates, using a Sturm property introduced in particular by Chen et Matano (the number of space oscillations is a decreasing function of time).

REMARK. The constant $\frac{N\kappa}{2p}$ is optimal (see Herrero and Velázquez, Bricmont and Kupiainen, Merle and Zaag).

4. Different notions of blow-up profiles and the stability question. We assume (0.2). We consider $a \in \mathbb{R}^N$, a blow-up point of $u(t)$, solution of (0.1). From translation invariance, we can assume $a = 0$. We would like to know whether $u(t)$ (or $w_{0,T}(s)$ defined in (0.4)) has a universal behavior or not, as $t \rightarrow T$ (or $s \rightarrow +\infty$).

Filippas, Kohn, Liu, Herrero et Velázquez prove that w behaves in two distinct ways :

- either $\forall R > 0, \sup_{|y| \leq R} \left| w(Qy, s) - \left[\kappa + \frac{\kappa}{2ps} \left(l - \frac{1}{2} \sum_{k=1}^l y_k^2 \right) \right] \right| = O\left(\frac{1}{s^{1+\delta}}\right)$ as $s \rightarrow +\infty$, for some $\delta > 0$ where $l \in \{1, \dots, N\}$, Q is a $N \times N$ orthogonal matrix and I_l is the $l \times l$ identity matrix.

- or $\forall R > 0, \sup_{|y| \leq R} |w(y, s) - \kappa| \leq C(R)e^{-\lambda_0 s}$ for some $\lambda_0 > 0$.

From a physical point of view, these results do not tell us much about the transition between the singular zone ($w \geq \alpha$ where $\alpha > 0$) and the regular one ($w \simeq 0$). In [MZ99], we specify this transition by proving the existence of a profile in the variable $z = \frac{y}{\sqrt{s}}$.

THEOREM 4. Existence of a blow-up profile for equation (0.1). Assume (0.2) holds. There exists $l \in \{0, 1, \dots, N\}$ and a $N \times N$ orthogonal matrix Q such that $w(Q(z)\sqrt{s}, s) \rightarrow f_l(z)$ uniformly on compact sets $|z| \leq C$, where $f_l(z) = (p - 1 + \frac{(p-1)^2}{4p} \sum_{i=1}^l |z_i|^2)^{-\frac{1}{p-1}}$ if $l \geq 1$ and $f_0(z) = \kappa = (p - 1)^{-\frac{1}{p-1}}$.

This result has been proved by Velázquez in [Vel92]. However, the convergence speed depends on the considered blow-up point in [Vel92], whereas it is uniform in [MZ99]. This uniformity allows us to derive the stability of the profile f_N in [FKMZ].

Using renormalization theory, Bricmont and Kupiainen prove in [BK94] the existence of a solution of (0.5) such that

$$\forall s \geq s_0, \forall y \in \mathbb{R}^N, |w(y, s) - f_N\left(\frac{y}{\sqrt{s}}\right)| \leq \frac{C}{\sqrt{s}}.$$

Merle and Zaag prove the same result in [MZ97], thanks to a technique of finite-dimension reduction. They also prove the stability of such a behavior with respect to initial data, in a neighborhood of the constructed solution.

In [Zaa98] and [Vel92], it is proved that in this case, $u(x, t) \rightarrow u^*(x)$ as $t \rightarrow T$ uniformly on $\mathbb{R}^N \setminus \{0\}$ and that $u^*(x) \sim \left[\frac{8p|\log|x||}{(p-1)^2|x|^2} \right]^{\frac{1}{p-1}}$ as $x \rightarrow 0$.

One interesting problem is to relate all known blow-up profiles' notions : profiles for $|y|$ bounded, $\frac{|y|}{\sqrt{s}}$ bounded or $x \simeq 0$. We prove in the following that all these descriptions are equivalent, in the case of single point blow-up with a non degenerate profile (generic case). This answers many questions which arose in former works.

THEOREM 5. Equivalence of blow-up behaviors at a blow-up point. *Assume (0.2) and consider a an isolated blow-up point of $u(t)$ solution of (0.1). The following behaviors of $u(t)$ and $w_{a,T}(s)$ (defined in (0.4)) are equivalent :*

- i) $\forall R > 0, \sup_{|y| \leq R} \left| w(y, s) - \left[\kappa + \frac{\kappa}{2ps} (N - \frac{1}{2}|y|^2) \right] \right| = o\left(\frac{1}{s}\right)$ as $s \rightarrow +\infty$,
- ii) $\forall R > 0, \sup_{|z| \leq R} |w(z\sqrt{s}, s) - f_0(z)| \rightarrow 0$ as $s \rightarrow +\infty$ with $f_0(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$,
- iii) $\exists \epsilon_0 > 0$ such that for all $|x - a| \leq \epsilon_0, u(x, t) \rightarrow u^*(x)$ as $t \rightarrow T$ and $u^*(x) \sim \left[\frac{8p|\log|x-a||}{(p-1)^2|x-a|^2} \right]^{\frac{1}{p-1}}$ as $x \rightarrow a$.

A further application of the Liouville Theorem is the stability of the behavior described in Theorem 5, with respect to perturbations in initial data. Using a dynamical system approach, we prove in [FKMZ], with Fermanian the following :

THEOREM 6. Stability of the blow-up profile. *Assume (1.1) and consider $\tilde{u}(t)$ a blow-up solution of (0.1) with initial data \tilde{u}_0 which blows-up at $t = \tilde{T}$ at only one point $\tilde{a} = 0$ and satisfies (0.6). Assume that*

$$\text{for all } |x| \geq R \text{ and } t \in [0, \tilde{T}), \quad |\tilde{u}(x, t)| \leq M.$$

and that the function $\tilde{w}_{0,\tilde{T}}(y, s)$ defined in (0.4) satisfies uniformly on compact sets of \mathbb{R}^N

$$(4.1) \quad \tilde{w}_{0,\tilde{T}}(y, s) - \kappa \underset{s \rightarrow +\infty}{\sim} \frac{\kappa}{2ps} (N - \frac{|y|^2}{2}).$$

Then, there is a neighborhood \mathcal{V} in L^∞ of \tilde{u}_0 such that for all $u_0 \in \mathcal{V}$ the solution of (0.1) with initial data u_0 blows-up at time $T = T(u_0)$ at a unique point $a = a(u_0)$ and the function $w_{a,T}(y, s)$ defined in (0.4) satisfies uniformly on compact sets of \mathbb{R}^N

$$w_{a,T}(y, s) - \kappa \underset{s \rightarrow +\infty}{\sim} \frac{\kappa}{2ps} (N - \frac{|y|^2}{2}).$$

Moreover, $(a(u_0), T(u_0))$ goes to $(0, \tilde{T})$ as u_0 goes to \tilde{u}_0 .

REMARK. This results generalizes the stability result of [MZ97]. Note that unlike most applications of the Liouville Theorem, this result is valid for all subcritical p . In [FKZ], the same result is proved (only under the condition (0.2)), by a completely different approach based on the Liouville Theorem and on [MZ97].

REMARK. In [FKMZ], we prove the stability with respect to initial data of the blow-up behavior with the minimal speed

$$(4.2) \quad \|u(t)\|_{L^\infty} \leq C(T - t)^{-\frac{1}{p-1}}$$

for all subcritical p (that is under the condition (1.1)). Note that this result is obvious under the weaker assumption (0.2), for Giga and Kohn proved in [GK87] that all blow-up solutions satisfy (4.2). No blow-up rate estimate is known if

$$u_0 \text{ has no sign and } (3N - 4)p \geq 3N - 8.$$

Therefore, our result is meaningful in this last case.

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