ON A HYPERCYCLE SYSTEM WITH NONLINEAR RATE

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Abstract. We study a $(N+1)$−hypercyclical reaction-diffusion system with nonlinear rate p. It is shown that there exists a critical threshold N_0 such that for $N \leq N_0$ the system is stable while for $N > N_0$ it becomes unstable. It is also shown that for large p, N_0 remains a constant: in fact for $p \geq p_0 \sim 4.35$, $N_0 = 5$ and for $p < p_0 \sim 4.35$, $N_0 = 4$.

1. Introduction. In this paper, we are concerned with the following reaction-diffusion system:

$$(1.1) \begin{cases} \frac{\partial X_i}{\partial t} = D_X \frac{\partial^2 X_i}{\partial x^2} - g_X X_i + M \sum_{j=1}^N k_{ij} X_i X_j^n, & i = 1, 2, \ldots, N, \quad x \in R, \\ \frac{\partial M}{\partial t} = D_M \frac{\partial^2 M}{\partial x^2} + k_M - g_M M - LM \sum_{i,j=1}^N k_{ij} X_i X_j^n, & x \in R, \end{cases}$$

where X_i denotes the concentration of the polymers, and M is the concentration of activated monomers. N is the number of different polymer species. The replication of each polymer X_i is catalysed by each X_j at a constant rate k_{ij}. Linear (non-catalytic) growth terms are neglected. The activated monomers are produced at a constant rate, k_M; g_X and g_M are decay rate constants. L is the number of monomers in each polymer, and D_X and D_M are constant diffusion coefficients. The exponent n is a positive number.

We assume that the coefficients k_{ij} are represented by a hypercyclical $N \times N$ matrix,

$$(k_{ij}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & k_0 \\ k_0 & 0 & \cdots & 0 & 0 \\ 0 & k_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & k_0 & 0 \end{pmatrix}_{N \times N} , \quad k_0 > 0.$$

When $n \neq 1$, We call (1.1) is a hypercycle system with nonlinear rate. The reason is the following: at each X_i, the kinetic reaction rate is given by

$$(1.2) \qquad \Gamma_i = -g_X + M \sum_{j=1}^N k_{ij} X_j^n.$$

When $n = 1$, we have a linear growth rate for Γ_i and the system is called classical hypercycle system. When $n < 1$, the growth rate is sublinear and $n > 1$ the growth rate is superlinear. Such nonlinear reaction rates were also introduced and studied in the one component case, i.e., $N = 1$, by many authors, see [18], [19], [20] and the references therein.

The classical hypercycle system arises as a spatial model concerning the origin of life similar to the one introduced by Eigen and Schuster [12]. A number of RNA-like polymers ("components") catalyse the replication of each other in a cyclic way.

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Examples in nature include Krebs and Bethe-Weizsäcker cycles. Eigen and Schuster argue that the hypercycle satisfies important criteria of natural selection: 1. Selective stability of each component due to favorable competition with error copies, 2. Cooperative behavior of the components integrated into the hypercycle, and 3. Favorable competition of the hypercycle unit with other less efficient systems.

Nonlinear rates are interesting to model different coupling strength of the various components, where a higher rate n corresponds to stronger coupling.

We consider stationary *cluster-like* solutions of (1.1). A cluster may loosely be defined as a region of high concentration $\sum_{i=1}^{N} X_i$ of the polymers and low concentration of the monomer, as monomers are consumed by the replication of polymers. (If the region shrinks to a point, this phenomenon is called point-condensation.)

Cluster solutions for (1.1) have been studied by numerous authors. For numerical aspects of clusters, see [3], [5], [6], [7] and the references therein. In [35], we first made a rigorous study on the existence and stability of cluster solutions of hypercycle system with linear rate ($n = 1$) in R^2: it was shown that for $N \leq 4$, cluster solutions are stable while for large N, cluster solutions become unstable. In [35], we analyzed the cluster solutions in R^1 for $n = 1$ and we found the exact threshold for N. More precise statements of the results of [35] will be stated later.

Let us first reduce the system (1.1) to standard form. Dividing by g_X and g_M, respectively, gives

$$\frac{1}{g_X}\partial_t X_i = \frac{D_X}{g_X}X_i'' - X_i + \frac{M}{g_X}\sum_{j=1}^{N} k_{ij}X_i X_j^n, x \in R,$$

$$\frac{1}{g_M}\partial_t M = \frac{D_M}{g_M}M'' + \frac{k_M}{g_M} - M - \frac{LM}{g_M}\sum_{ji,=1}^{N} k_{ij}X_i X_j^n, \ x \in R.$$

Rescaling $M = (k_M/g_M)\hat{M}$, $X_i = (g_M/L)^{1/(n+1)}\hat{X}_i$, we get

$$\frac{1}{g_X}\partial_t \hat{X}_i = \frac{D_X}{g_X}\hat{X}_i'' - \hat{X}_i + \frac{1}{g_X}\frac{k_M}{g_M}\hat{M}(\frac{g_M}{L})^{n/(n+1)}\sum_{j=1}^{N} k_{ij}\hat{X}_i \hat{X}_j^n,$$

$$\frac{1}{g_M}\partial_t \hat{M} = \frac{D_M}{g_M}\hat{M}'' + 1 - \hat{M} - \hat{M}\sum_{i,j=1}^{N} k_{ij}\hat{X}_i \hat{X}_j^n.$$

Rescaling space variables x and time variable t:

$$x = \sqrt{\frac{D_M}{g_M}}\hat{x}, \quad t = \frac{1}{g_M}\hat{t},$$

renaming constants:

$$A = \frac{k_M}{g_X g_M}(\frac{g_M}{L})^{\frac{n}{n+1}}, \quad \epsilon^2 = \frac{D_X}{D_M}\frac{g_M}{g_X}, \quad \tau = \frac{g_X}{g_M}$$

and dropping the hats, we finally arrive at the following standard form

(1.3) $\begin{cases} \partial_t X_i = \epsilon^2 X_i'' - X_i + AM\sum_{i=1}^{N} k_{ij}X_i X_j^n, & x \in R, \\ \tau\partial_t M = M'' + 1 - M - M\sum_{i,j=1}^{N} k_{ij}X_i X_j^n, & x \in R. \end{cases}$

We shall study (1.3) on the real line R for $\epsilon > 0$ small. Since existence and stability of solutions might depend on A we will treat it as a parameter. We look for solutions of (1.3) which are even:

$$X_i = X_i(|x|) \in H^1(R), \quad i = 1, \ldots, N,$$

$$1 - M(x) = 1 - M(|x|) \in H^1(R).$$

The stationary equation corresponding to (1.3) becomes

(1.4) $$\begin{cases} \epsilon^2 X_i'' - X_i + AM \sum_{j=1}^N k_{ij} X_i X_j^n = 0, & i = 1, \ldots, N, \quad \in R, \\ M'' + 1 - M - M \sum_{i=1,j}^N k_{ij} X_i X_j^n = 0, & x \in R. \end{cases}$$

We first construct cluster solutions to (1.4). To this end, we need to introduce some assumptions and notations.

Let

(1.5) $$p = n + 1 > 1$$

and w be the unique solution of the following problem

(1.6) $$\begin{cases} w'' - w + w^p = 0, w > 0 \ \text{in} \ R, \\ w(0) = \max_{y \in R} w(y), w(y) \to 0 \ \text{as} \ |y| \to +\infty. \end{cases}$$

Put

(1.7) $$L_\epsilon := \frac{N}{2A^{1+\frac{1}{n}} k_0^{\frac{1}{n}}} \epsilon \int_R (w(y))^{n+1} dy.$$

If $\lim_{\epsilon \to 0} L_\epsilon < L_0 := (\frac{1}{n+1})^{\frac{1}{n}} \frac{n}{n+1}$, then the following equation has two solutions:

(1.8) $$\eta^{\frac{1}{n}}(1 - \eta) = L_\epsilon.$$

We denote the smaller one by η^s, where $0 < \eta^s < \frac{1}{n+1}$ and the larger one by η^l, where $1 > \eta^l > \frac{1}{n+1}$.

We now state the existence result. In fact, this is quite easy. We search for solutions of the following type

(1.9) $$X_i = X_0, \quad i = 1, \ldots, N.$$

Substituting (1.9) into (1.4), we see that (X_0, M) satisfies

(1.10) $$\begin{cases} \epsilon^2 X_0'' - X_0 + AMk_0 X_0^p = 0, x \in R, \\ M'' + 1 - M - Mk_0 N X_0^p = 0, x \in R. \end{cases}$$

Existence of solutions to (1.10) can be shown as in the case $p = 2$: the standard Gray-Scott model. Similar to the proof of Theorem 2.1 of [34], we can obtain the following existence theorem:

THEOREM 1.1.

Assume that

(1.11) $\epsilon << 1$

and

(1.12) $\epsilon << L_\epsilon, \lim_{\epsilon \to 0} L_\epsilon < L_0.$

Then problem (1.4) admits two solutions $(X^s_\epsilon, M^s_\epsilon) = (X^s_{\epsilon,1}, \ldots, X^s_{\epsilon,N}, M^s_\epsilon)$ *and* $(X^l_\epsilon, M^l_\epsilon) = (X^l_{\epsilon,1} \ldots, X^l_{\epsilon,N}, M^l_\epsilon)$ *with the following properties:*

(1) all components are even functions.

(2) $X^s_{\epsilon,i} = \frac{1}{AM^s_\epsilon(0)k_0)^{\frac{1}{n}}}(1 + o(1))w(\frac{|x|}{\epsilon})$, $i = 1, \ldots, N$, *where w is the unique solution of (1.6).*

(3) $M^s_\epsilon(x) \to 1$, $M^l_\epsilon(x) \to 1$ *for all $x \neq 0$ and $M^s_\epsilon(0)$, $M^l_\epsilon(0)$ satisfy*

(1.13)
$$M^s_\epsilon(0) \sim \eta^s, \quad M^l_\epsilon(0) \sim \eta^l,$$
$$0 < M^s_\epsilon(0) < M^l_\epsilon(0) < 1.$$

(4) There exist $a > 0, b > 0$ such that

(1.14)
$$1 - M^s_\epsilon(x) \leq Ce^{-a|x|}, \quad 1 - M^l_\epsilon(x) \leq Ce^{-a|x|},$$
$$X^s_{\epsilon,i}(x) \leq C\frac{1}{(AM^s_\epsilon(0)k_0)^{\frac{1}{n}}}e^{-b\frac{|x|}{\epsilon}}, \quad X^l_{\epsilon,i}(x) \leq C\frac{1}{(AM^l_\epsilon(0)k_0)^{\frac{1}{n}}}e^{-b\frac{|x|}{\epsilon}},$$

Finally, if $\lim_{\epsilon \to 0} L_\epsilon > L_0$, then there are no single-cluster solutions.

We note that existence of single-pulse solution has also been studied in [10].

The main goal of this paper is to study the stability and instability of the cluster solution constructed in Theorem 1.1. To this end, we first linearize the equations (1.4) around $(X^s_\epsilon, M^s_\epsilon)$ or $(X^l_\epsilon, M^l_\epsilon)$, respectively. From now on we omit the superscripts s or l where this is possible without confusing the reader. The linearized operator is as follows:

(1.15)
$$\mathcal{L}_\epsilon \begin{pmatrix} \phi_{\epsilon,i} \\ \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2 \phi''_{\epsilon,i} - \phi_{\epsilon,i} + AM_\epsilon \sum_{j=1}^N k_{ij}(\phi_{\epsilon,i}X^n_{\epsilon,j} + n\phi_{\epsilon,j}X_{\epsilon,i}X^{n-1}_{\epsilon,j}) \\ +A\psi_\epsilon \sum_{j=1}^N k_{ij}X_{\epsilon,i}X^n_{\epsilon,j} \\ \psi''_\epsilon - \psi_\epsilon - \psi_\epsilon \sum_{i,j=1}^N k_{ij}X_{\epsilon,i}X^n_{\epsilon,j} \\ -M_\epsilon \sum_{i,j=1}^N k_{ij}(\phi_{\epsilon,i}X^n_{\epsilon,j} + n\phi_{\epsilon,j}X_{\epsilon,i}X^{n-1}_{\epsilon,j}) \end{pmatrix},$$

where $i = 1, \ldots, N$. The eigenvalue problem becomes

(1.16) $\mathcal{L}_\epsilon \begin{pmatrix} \phi_{\epsilon,i} \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \lambda_\epsilon \phi_{\epsilon,i} \\ \tau \lambda_\epsilon \psi_\epsilon \end{pmatrix}, i = 1, \ldots, N.$

We consider $\mathcal{L}_\epsilon$ in the Sobolev space $(H^2(R)) \otimes H^2(R)$ and equip $(H^2(R))^N \oplus H^2(R)$ with the following norm

$$\|(X, u)\|^2_{(H^2(R))^N \oplus H^2(R)} = \|X(y)\|^2_{(H^2(R))^N} + \|u(x)\|^2_{H^2(R)}.$$

Certainly 0 is an eigenvalue of $\mathcal{L}_\epsilon$. The criterion for linearized stability of a cluster solution is that the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ (except for 0) lies in a left half plane

$\{\lambda \in \mathcal{C} : \text{Re } (\lambda) < -c_0\}$ where $c_0 > 0$, and that 0 is a simple eigenvalue, where $\mathcal{C}$ denotes the set of complex numbers.

In [36], the linear case $n = 1$ is studied and the following result is proved.

Theorem A: *Let $n = 1$. Assume that*

$$(1.17) \qquad \epsilon << 1, \quad \epsilon << L_\epsilon, \quad \lim_{\epsilon \to 0} L_\epsilon < \frac{1}{4}.$$

Let $(X_\epsilon^s, M_\epsilon^s)$ and $(X_\epsilon^l, X_\epsilon^l)$ be the solutions constructed in Theorem 1.1. Then for $\epsilon << 1$, we have the following.

(1) (stability) Assume that $N \leq 4$ and $\tau << 1$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly stable.

(2) (Instability) Assume that $N > 4$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly unstable.

(3) (Instability) $(X_\epsilon^l, M_\epsilon^l)$ is linearly unstable.

A natural question is the following: what is the effect of n on the stability of cluster solutions? What is the relation between n and the critical threshold? Will large n increase the critical threshold? We shall answer these questions affirmatively in this paper and prove the following theorem.

THEOREM 1.2. *Assume that*

$$(1.18) \qquad \epsilon << 1, \quad \epsilon << L_\epsilon, \quad \lim_{\epsilon \to 0} L_\epsilon < L_0.$$

Let $(X_\epsilon^s, M_\epsilon^s)$ and $(X_\epsilon^l, X_\epsilon^l)$ be the solutions constructed in Theorem 1.1. Let

$$(1.19) \qquad N_0 := \begin{cases} 4, n \leq n_0 \sim 3.35, \\ 5, n > n_0 \sim 3.35, \end{cases}$$

where n_0 satisfies

$$(\cos\frac{2\pi}{5})(4(n_0 + 2)\cos\frac{2\pi}{5} + n_0 + 4)^2((n_0 + 2)\cos\frac{2\pi}{5} + 2)$$

$$(1.20) \qquad -(n_0 + 2)n_0^2(\sin\frac{2\pi}{5})^2 = 0.$$

Then for $\epsilon << 1$, we have the following.

(1) (stability) Assume that $N \leq N_0$ and $\tau << 1$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly stable.

(2) (Instability) Assume that $N > N_0$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly unstable.

(3) (Instability) $(X_\epsilon^l, M_\epsilon^l)$ is linearly unstable.

Remarks: 1. As we see from the theorem, if p increases, the critical threshold can only grow by at most 1. This means that stability is only very marginally influenced by growing interaction strength which is a new and surprising fact as one would think that stronger interaction would improve stability.

2. As in [35], we may generalize the results in this paper to the case of a general matrix (k_{ij}). We will mention the results in the last section and present some examples.

It is interesting and important to know the exact threshold also to verify the validity of our model by experiment: It can now be studied if the thresholds given by theory and the one determined by experiments are the same. Furthermore, the agreement between theoretical values and numerically calculated ones for related models

play an important role in finding which model to choose preferably. (We refer to the works quoted at the end of the introduction for related numerical investigations, in particular to [5], where among others multi-cluster states in one space dimension have been computed numerically).

Let us conclude this introduction by mentioning some related results.

In [6] the parameter dependence of stability of clusters and spirals against parasites (i.e., rival polymers which receive catalytic support from the hypercycle but do not contribute to the catalysis of any other polymer) is studied numerically. A parasite may or may not destroy the hypercycle depending on the rate constants. In [7] clusters (for $N = 5$) are established numerically for the elementary N-hypercycle system,

In [5] for a closely related reaction-diffusion model the dependence of cluster states on diffusivities is shown numerically including the cluster size, their shape, and the distance between different clusters.

The effect of faulty replication on the hypercycle has been studied by an analysis of the geometry of bifurcations around steady states and numerical computations in the framework of an ODE reaction model [1].

For a cellular automata model it was shown numerically that a spiral wave structure may be stable against parasites [3]. The chaotic dynamics for this type of model has been investigated numerically in [17], [28].

There are a number of recent results on the special case $N = 1$, $n = 1$ of our model, which is then also called Gray-Scott system [13], [14]. We would like to recall them here. In [10], by using Mel'nikov method, Doelman, Kaper and Zegeling constructed single and multiple pulse solutions for (1.1) in the one-dimensional case with $D_M = 1, D_X = \delta^2 << 1$, where $X_i = X$. In their paper [10], it is assumed that $k_M = g_M \sim \delta^2, g_X \sim \delta^{2\alpha/3}$, $k_{11} = 1$, $L = 1$, where $\alpha \in [0, \frac{3}{2})$. In this case, they showed that $M = O(\delta^\alpha), X = O(\delta^{-\frac{\alpha}{3}})$. Later the stability of single and multiple pulse solutions in 1-D are obtained in [8], [9]. (The techniques are extended to other reaction-diffusion equations in [11].) Some related results on the existence and stability of solutions to the Gray-Scott model in 1-D can be found in [26] and [29].

In R^2 and R^3, Muratov and Osipov [21] have given some formal asymptotic analysis on the construction and stability of spiky solution. In [33], the system (1.1) for $N = 1$ is studied on the real axis in the shadow system case, namely, $D_M >> 1, D_X << 1$ and $k_M = g_M = O(1), g_X = O(1)$, $k_{11} = 1$, $L = 1$. The shadow system can be reduced to a single equation. For spike solutions of single equations and other systems, we refer to [15], [16], [27], [23], [24], [25], [31], [32], and the references therein.

In the general higher dimensional case rigorous existence and stability results on the Gray-Scott system have been established in [34]. The existence of one-spike solutions is proved. Their stability is established and rests upon the derivation and analysis of a related NLEP (nonlocal eigenvalue problem).

The structure of the paper is as follows:

In Section 2, we separate the eigenvalue problem into two cases: small eigenvalues and large eigenvalues. The small eigenvalue is shown to be 0 with dimension 1. The case of large eigenvalues is then reduced to a system of nonlocal eigenvalue problems (NLEP).

In Section 3, we analyze the system of NLEP and show that it can be reduced to two eigenvalue problem-one is local but with complex coefficients, another one is a NLEP.

In Section 4, we study the two eigenvalue problems and thus finish the proof of

Theorem 1.3.

In Section 5, we make we drop the condition that the system is a classical hypercycle and make some remarks about the general matrix case.

Finally, in Section 6, we discuss our results.

Throughout this paper, the letter C will always denote various generic constants which are independent of ϵ, for ϵ sufficiently small. The notation $A \sim B$ means that $\lim_{\epsilon \to 0} \frac{A}{B} = 1$ and $A = O(B)$ is defined as $|A| \leq C|B|$.

2. Reduction to a system of NLEP. Let (X_ϵ, M_ϵ) be one of the two solutions constructed in Section 1. We now study the eigenvalue problem associated with (X_ϵ, M_ϵ). We assume that

$$\epsilon << L_\epsilon, \lim_{\epsilon \to 0} L_\epsilon < L_0.$$

We need to analyze the following eigenvalue problem (letting $x = \epsilon y$)

$$(2.1) \quad \begin{cases} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + AM_\epsilon \sum_{j=1}^N k_{ij}(X_{\epsilon,j}^n \phi_{\epsilon,i} + n\phi_{\epsilon,j} X_{\epsilon,i} X_{\epsilon,j}^{n-1}) \\ + A\psi_\epsilon \sum_{j=1}^N k_{ij} X_{\epsilon,i} X_{\epsilon,j}^n = \lambda_\epsilon \phi_{\epsilon,i}, \ y \in R, \\ \Delta_x \psi_\epsilon - \psi_\epsilon - \psi_\epsilon \sum_{i,j=1}^N k_{ij} X_{\epsilon,i} X_{\epsilon,j}^n \\ - M_\epsilon \sum_{i,j=1}^N k_{ij}(X_{\epsilon,j}^n \phi_{\epsilon,i} + nX_{\epsilon,i} X_{\epsilon,j}^{n-1} \phi_{\epsilon,j}) = \tau\lambda_\epsilon \psi_\epsilon, \ x \in R, \\ \lambda_\epsilon \in \mathcal{C}. \end{cases}$$

We assume that $(\phi_{\epsilon,1}, ..., \phi_{\epsilon,N}, \psi_\epsilon) \in (H^2(R))^N \oplus H^2(R)$.

Since $X_{\epsilon,i} = X_0, n = p - 1, k_{ij} = k_0 \hat{k}_{ij} = k_0 \delta_{i,j+1}$ modulo N, problem (2.1) becomes

$$(2.2) \quad \begin{cases} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + Ak_0 M_\epsilon X_0^{p-1} k_0 \sum_{j=1}^N \hat{k}_{ij}(\phi_{\epsilon,i} + n\phi_{\epsilon,j}) \\ + Ak_0 \psi_\epsilon X_0^p = \lambda_\epsilon \phi_{\epsilon,i}, \\ \Delta\psi_\epsilon - \psi_\epsilon - Nk_0 \psi_\epsilon X_0^p \\ - M_\epsilon k_0 \sum_{i,j=1}^N \hat{k}_{ij}(\phi_{\epsilon,i} + n\phi_{\epsilon,j}) X_0^{p-1} = \tau\lambda_\epsilon \psi_\epsilon. \end{cases}$$

Let us first formally derive the limiting eigenvalue problems.

Since (X_0, M_ϵ) satisfies (1.10), we have

$$(2.3) \qquad X_0(y) \sim (AM_\epsilon(0)k_0)^{-\frac{1}{p-1}} w(y) \quad \text{in } H^1(R)$$

and

$$(2.4) \qquad M_\epsilon^{\frac{1}{p-1}}(0)(1 - M_\epsilon(0)) \sim L_\epsilon := \frac{Nk_0}{2(Ak_0)^{\frac{p}{p-1}}} \epsilon \int_R w(y)^p dy.$$

The eigenvalue problem is changed into

$$(2.5) \quad \begin{cases} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + \sum_{j=1}^N \hat{k}_{ij}(\phi_{\epsilon,i} + (p-1)\phi_{\epsilon,j})w^{p-1} \\ + Ak_0(AM_\epsilon(0)k_0)^{-\frac{p}{p-1}} \psi_\epsilon w^p \sim \lambda_\epsilon \phi_{\epsilon,i}, \\ \Delta\psi_\epsilon - \psi_\epsilon - Nk_0(AM_\epsilon(0)k_0)^{-\frac{p}{p-1}} \psi_\epsilon w^p \\ - M_\epsilon k_0(AM_\epsilon(0)k_0)^{-1} \sum_{i,j=1}^N \hat{k}_{ij}(w^{p-1}\phi_{\epsilon,i} + (p-1)w^{p-1}\phi_{\epsilon,j}) \sim \tau\lambda_\epsilon \psi_\epsilon. \end{cases}$$

From the equation for ψ_ϵ, we formally have (setting $\beta^2 = 1 + \tau\lambda$)

$$\psi_\epsilon(0) = \frac{1}{2\beta} \int_R e^{-\beta|x|} \left(-k_0 N(AM_\epsilon(0)k_0)^{-\frac{p}{p-1}} \psi_\epsilon w^p \right.$$

$$-M_\epsilon k_0 (A M_\epsilon(0) k_0)^{-1} \sum_{i,j=1}^{N} \hat{k}_{ij} (w^{p-1} \phi_{\epsilon,i} + (p-1) w^{p-1} \phi_{\epsilon,j}) \Big)$$

$$\sim \frac{1}{2\beta} \epsilon [-\psi_\epsilon(0) k_0 N (A M_\epsilon(0) k_0)^{-\frac{p}{p-1}} \int_R w^p(y) \, dy$$

$$-M_\epsilon(0) k_0 N (A M_\epsilon(0) k_0)^{-1} \int_R p(\sum_{i=1}^{N} \phi_{\epsilon,i}) w^{p-1} \, dy].$$

By (1.8), we have

$$\psi_\epsilon(0) \sim -\left(1 + \frac{1}{2\beta} k_0 N (A M_\epsilon(0) k_0)^{-\frac{p}{p-1}} \epsilon \int_R w^p \right)^{-1}$$

$$\left(\frac{1}{2\beta} M_\epsilon(0) k_0 N (A M_\epsilon(0) k_0)^{-1} \epsilon (\int_R w^{p-1} p(\sum_{i=1}^{N} \phi_{\epsilon,i})) \right)$$

$$\sim -(1 + \frac{1 - M_\epsilon(0)}{\beta M_\epsilon(0)})^{-1} \left(\frac{1}{2\beta} M_\epsilon(0) k_0 N (A M_\epsilon(0) k_0)^{-1} \epsilon (\int_R w^{p-1} p(\sum_{i=1}^{N} \phi_{\epsilon,i})) \right)$$

Substituting this relation into the equation for ϕ_i, we obtain the following nonlocal eigenvalue problem (NLEP):

$$(2.6) \qquad \Delta \phi_{\epsilon,i} - \phi_{\epsilon,i} + w^{p-1} \phi_{\epsilon,i} + (p-1) \sum_{j=1}^{N} \hat{k}_{ij} \phi_{\epsilon,j} w^{p-1}$$

$$- \frac{p(1 - M_\epsilon(0))}{\beta M_\epsilon(0) + 1 - M_\epsilon(0)} w^p \frac{\int_R w^{p-1} \sum_{i=1}^{N} \phi_{\epsilon,i}}{N \int w^p} \sim \lambda_\epsilon \phi_{\epsilon,i}.$$

Although we have formally obtained (2.7), however we can rigorously prove the following separation of eigenvalues.

THEOREM 2.1.
Let λ_ϵ be an eigenvalue of (2.2).
(1) Suppose that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. Then we have $\lambda_\epsilon = 0$ if ϵ is small enough and

$$(\phi_\epsilon, \psi_\epsilon) \in span \ \{(X'_\epsilon, M'_\epsilon)\}.$$

(2) Suppose that $\lambda_\epsilon \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of the following NLEP

$$(2.7) \qquad \Delta \phi_i - \phi_i + w^{p-1} \phi_i + (p-1) \sum_{j=1}^{N} \hat{k}_{ij} \phi_j w^{p-1}$$

$$- \frac{p(1 - \eta)}{\beta_0 \eta + 1 - \eta} w^p \frac{\int_R w^{p-1} \sum_{i=1}^{N} \phi_i}{N \int w^p} = \lambda_0 \phi_i, i = 1, ..., N$$

where $\eta = \lim_{\epsilon \to 0} M_\epsilon(0), \beta_0 = \sqrt{1 + \tau\lambda_0}$.

Proof.

(2) follows from asymptotic analysis.

To prove (1), we can proceed exactly as in Section 6 of [34], where stability of a single cluster state is studied. Let us denote the linear operator on the left hand side of (2.7) as $\mathcal{L}$, where $\mathcal{L} : (H^2(R))^N \to (L^2(R))^N$. The key point is to prove the following lemma:

LEMMA 2.2. *(1). Let ϕ be an eigenfunction of (2.7) with $\lambda_0 = 0$. Then we have*

$$\phi \in \mathcal{K}_0 := span \{w^{'}(y)\vec{e}_0\},$$

where $\vec{e}_0 = (1, \ldots, 1)^\tau$. (This implies that $Ker(\mathcal{L}) = \mathcal{K}_0$.)

(2). The operator $\mathcal{L}$ is an invertible operator if restricted as follows

$$\mathcal{L} : \mathcal{K}_0^{\perp,1} \to \mathcal{K}_0^{\perp,2},$$

where

$$\mathcal{K}_0^{\perp,1} = \{u \in (H^2(R))^N | \int_R uw^{'}(y)\vec{e}_0 = 0\},$$

$$\mathcal{K}_0^{\perp,2} = \{u \in (L^2(R))^N | \int_R uw^{'}(y)\vec{e}_0 = 0\}.$$

The proof of Lemma 2.2 is technical and is similar to Theorem 4.1 of [36].

The rest of the proof is exactly the same as in Section 6 of [33]. For the sake of limited space, we omit the details here.

3. Analysis of system of NLEP. In this section we analyze the nonlinear eigenvalue problem (NLEP) which we have obtained in Section 2. To this end, we introduce two eigenvalue problems: the first is the following eigenvalue problem with complex coefficients

$$(3.1) \qquad \begin{cases} \Delta\phi - \phi + w^{p-1}\phi + (p-1)\sigma w^{p-1}\phi = \lambda\phi \\ \sigma = \sigma_R + \sqrt{-1}\sigma_I = e^{i\theta}, \theta \in (-\pi, \pi], \quad \phi \in H^1(R), \end{cases}$$

where w is defined by (1.6).

The second is a nonlocal eigenvalue problem (NLEP):

$$(3.2) \quad \Delta\phi - \phi + pw^{p-1}\phi - \frac{p(1-\eta)}{\eta\sqrt{1+\tau\lambda} + 1 - \eta} \frac{\int_R w^{p-1}\phi}{\int_R w^p} w^p = \lambda\phi, \phi \in H^2(R)$$

where

$$0 < \eta < 1, \tau \geq 0, \lambda \in \mathcal{C}, \lambda = \lambda_R + i\lambda_I, \lambda_R \geq 0$$

and we take principal branch for $\sqrt{1+\tau\lambda}$.

We show that the study of NLEP (2.7) can be reduced to the study of (3.1) and (3.2). We say an eigenvalue problem is stable if there exists a constant $c_0 > 0$ such that all eigenvalues λ we have $Re(\lambda) < -c_0$. We say it is unstable if there exists an eigenvalue λ with $Re(\lambda) > 0$.

We then have the following

LEMMA 3.1. *Suppose that (3.2) with $\eta = \lim_{\epsilon \to 0} M_\epsilon(0)$ has no Hopf bifurcation (as given in Lemma 4.6 below). Then (2.7) is stable if and only if both (3.1) (with $\sigma = e^{-\frac{2j\pi\sqrt{-1}}{N}}, j = 1, ..., N-1$) and (3.2) (with $\eta = \lim_{\epsilon \to 0} M_\epsilon(0)$) are stable.*

Proof.

(1) Suppose (3.1) and (3.2) are stable. We now show that (2.7) is stable, too: Asuming that there exists $\lambda_0 \geq 0$ such that (2.7) holds we show that

$$\phi_i = 0, \quad i = 1, ..., N.$$

We first take care of the nonlocal terms. Adding all equations for $i = 1, ..., N$, we get

$$\Delta(\sum_{i=1}^{N} \phi_i) - (\sum_{i=1}^{N} \phi_i) + pw^{p-1}(\sum_{i=1}^{N} \phi_i)$$

$$-p\frac{1-\eta}{\beta\eta + 1 - \eta}\frac{\int_R(\sum_{i=1}^{N} \phi_i)w^{p-1}}{\int_R w^p}w^p = \lambda_0\phi_i.$$

Since (3.2) is stable and we have no Hopf bifurcation, we have

(3.3) $$\sum_{i=1}^{N} \phi_i = 0.$$

Suppose (3.3) holds so the nonlocal terms in (NLEP) all vanish. We end up with the following:

(3.4) $$\Delta\phi_i - \phi_i + w^{p-1}\phi_i + (p-1)\sum_{j=1}^{N} \hat{k}_{ij}\phi_j w^{p-1} = \lambda_0\phi_i.$$

After diagonalizing $\hat{k}_{ij}$ (keeping the notation for ϕ_i) we get

(3.5) $$\Delta\phi_i - \phi_i + (1 + (p-1)e^{2\pi i\sqrt{-1}/N})w^{p-1}\phi_i = \lambda_0\phi_i,$$

Since (3.1) is stable, we have $\phi_i = 0$. Therefore (2.7) is stable.

(2) Suppose (3.2) is unstable. Then there exists an eigenfunction $\phi_0 \neq 0$ with an eigenvalue λ_0 to (3.2) such that $\text{Re}(\lambda_0) > 0$. Now we take $\phi_1 = ... = \phi_N = \phi_0$ in (2.7) and we see that (2.7) also admits the eigenvalue λ_0. So (2.7) is unstable.

On the other hand, suppose (3.2) is not unstable and (3.1) is not stable. Since (3.2) has no Hopf bifurcations, (3.2) is stable. Then similar to (1), we must have

$$\sum_{i=1}^{N} \phi_i = 0$$

and so all the nonlocal terms vanish. We are left with the following local eigenvalue problem

(3.6) $$\Delta\phi_i - \phi_i + w^{p-1}\phi_i + (p-1)\sum_{j=1}^{N} \hat{k}_{ij}w^{p-1}\phi_j = \lambda\phi_i, \sum_{i=1}^{N} \phi_i = 0.$$

It is easy to see that (3.6) is not stable because (3.1) is not stable. Lemma 3.1 is thus proved. $\square$

4. Study of the two eigenvalue problems. In this section, we study two eigenvalue problems (3.1) and (3.2) derived from Section 3. The analysis presented in this section is the key estimate for this paper.

To study (3.1) and (3.2), we first collect some important properties associated with the function w. We first study some local eigenvalue problems.

LEMMA 4.1. *(1) The linear operator*

$$\begin{cases} L_0\phi := \phi'' - \phi + pw^{p-1}\phi, \\ \qquad \phi \in H^1(R) \end{cases}$$

has the kernel

$$Ker\,(L_0) = span\,\left\{w'(y)\right\}.$$

(2) The eigenvalue problem (EVP)

$$(EVP) \qquad \begin{cases} \phi'' - \phi + \mu w^{p-1}\phi = 0, \\ \qquad \phi \in H^1(R) \end{cases}$$

admits the following set of eigenvalues

$$\mu_1 = 1,\ v_1 = span\,\{w\},$$

$$\mu_2 = p,\ v_2 = Ker\,(L_0),$$

$$\mu_3 > p.$$

(3) If $\mu_R > 0$, then the following eigenvalue problem

$$\begin{cases} \phi'' - \phi + w^{p-1}\phi + \mu_R w^{p-1}\phi = \lambda\phi, \\ \qquad \mu_R > 0,\ \phi \in H^1(R) \end{cases}$$

admits a positive (principal) eigenvalue λ_1 such that

$$-\lambda_1 = \inf_{\phi \in H^1(R)\setminus\{0\}} \frac{\int_R (\phi')^2 + \phi^2 - (1+\mu_R)w^{p-1}\phi^2}{\int_R \phi^2} < 0.$$

(4) Let ϕ (complex-valued) satisfy the following eigenvalue problem

$$\begin{cases} \phi'' - \phi + w^{p-1}\phi + (p-1)\sigma w^{p-1}\phi = \lambda\phi \\ \qquad Re(\sigma) \le 0, \quad \phi \in H^1(R), \quad \lambda \ne 0. \end{cases}$$

Then

$$Re(\lambda) \le -c_0 < 0.$$

Proof. The proof is similar to Lemma 3.1 of [36]. We omit the details. □

We are ready to study the first eigenvalue problem (3.1). We consider θ as a parameter. By Lemma 4.1 (3) and a perturbation argument, for $|\theta|$ small, there is

an unstable eigenvalue λ for problem (3.1), i.e. $\lambda = \lambda_R + i\lambda_I$ where $\lambda_R > 0$. On the other hand, by Lemma 4.1 (4), for $|\theta| \geq \frac{\pi}{2}$, problem (3.1) has only stable eigenvalues, i.e. $\lambda = \lambda_R + i\lambda_I$ where $\lambda_R < 0$. Now if we vary θ, then there must be a point θ^h such that for $\theta = \theta^h$, problem (3.1) has a Hopf bifurcation, i.e. there is an eigenvalue $\lambda = i\lambda_I$. Let us now compute θ^h. That is

LEMMA 4.2. *Let ϕ (complex-valued) satisfy the eigenvalue problem (3.1). Then there exists some θ^h with $\theta^h = \arccos \sigma_R$, where σ_R is the unique zero with $0 < \sigma_R < 1$ of the following polynomial*

$$(4.1) \quad g(\sigma) := \sigma_R(4(p+1)\sigma_R + p + 3)^2(p+1)\sigma_R + 2) - (p+1)(p-1)^2(1 - \sigma_R^2)$$

such that (1) If

$$|\theta| > \theta^h,$$

then

$$Re(\lambda) \leq -c_0 < 0.$$

(2) If

$$|\theta| < \theta^h,$$

then there exists an eigenvalue λ with $Re(\lambda) > 0$.
    (3) If $|\theta| = \theta^h$, then there exists an eigenvalue λ with $\lambda = i\lambda_I$.

Proof. We are looking for a Hopf bifurction for problem (3.1). Therefore we have to solve

$$(4.2) \qquad \Delta\phi - \phi + (1 + (p-1)\sigma)w^{p-1}\phi = \lambda\phi$$

with

$$\lambda = \sqrt{-1}\lambda_I$$

(i.e. the real part λ_R of λ vanishes) and

$$\sigma = \sigma_R + \sqrt{-1}\sigma_I, \quad |\sigma|^2 = \sigma_R^2 + \sigma_I^2 = 1.$$

Let

$$\gamma = \sqrt{1 + \lambda}, \mu = 1 + (p-1)\sigma, \phi = w^\gamma F.$$

Then F satisfies

$$(4.3) \qquad F'' + 2\gamma\frac{w'}{w}F' + (\mu - (\gamma + \frac{2}{p+1}\gamma(\gamma-1)))w^{p-1}F = 0.$$

Next we introduce the following new variable

$$(4.4) \qquad\qquad z = \frac{1}{2}(1 - \frac{w'}{w}).$$

Then

$$\frac{w'}{w} = 1 - 2z, w^{p-1} = 2(p+1)z(1-z), \frac{dz}{dx} = (p-1)z(1-z).$$

This yields the following equation for F as function of z

$$(4.5) \qquad z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,$$

where

$$(4.6) \quad a+b+1 = 2 + \frac{4\gamma}{p-1}, ab = 2\frac{2\gamma(\gamma-1) - (\mu-\gamma)(p+1)}{(p-1)^2}, c = 1 + \frac{2\gamma}{p-1}.$$

The solutions to (4.5) are standard hypergeometric functions. Now there are two solutions to (4.5):

$$F(a, b; c; z), z^{1-c}F(a-c+1, b-c+1; 2-c; z).$$

Since by our construction, F is regular at $z = 0$. At $z = 1$, $F(a, b; c; z)$ has a singularity

$$\lim_{z \to 1} (1-z)^{-(c-a-b)} F(a, b; c; z) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

where $c - a - b = -\frac{2\gamma}{p-1}$. Note that since $\lambda = \sqrt{1 + i\lambda_I}$, the real part of γ is positive. So a solution that is regular at both $z = 0$ and $z = 1$ can only exist if $\Gamma(x)$ has a pole at a or b, respectively. In other words, $a = 0, -1, -2, ...$ or $b = 0, -1, -2, ...$.

From (4.6), we compute that

$$a = \frac{2\gamma}{p-1} - \alpha$$

or

$$b = \frac{2\gamma}{p-1} - \alpha,$$

where α satisfies

$$(4.7) \qquad \alpha^2 + \alpha - \frac{2(p+1)}{(p-1)^2}\mu = 0.$$

By symmetry we may assume that $a = \frac{2\gamma}{p-1} - \alpha = -l, l \geq 0$ and $\alpha = \alpha_R + \sqrt{-1}\alpha_I$. So we to solve the system

$$(4.8) \qquad \begin{cases} \alpha_R^2 + \alpha_R - \alpha_I^2 - \frac{2(p+1)}{(p-1)^2}(1 + (p-1)\sigma_R) = 0 \\ \frac{2\gamma}{p-1} = \alpha - l \end{cases}$$

Since we take the principal branch for $\gamma = \sqrt{1 + i\lambda_I}$, it follows that

$$\alpha > l.$$

Moreover we have

$$\frac{4}{(p-1)^2} = (\alpha_R - l)^2 - \alpha_I^2$$

which implies that

$$(4.9) \qquad \alpha_R \geq l + \frac{2}{p-1}$$

On the other hand, we have

$$\frac{4}{(p-1)^2} = (\alpha_R - l)^2 - \alpha_I^2 = \alpha_R^2 - \alpha_I^2 - 2l\alpha_R + l^2$$

$$= -(2l+1)\alpha_R + l^2 + \frac{2(p+1)}{(p-1)^2}(1 + (p-1)\sigma_R).$$

So we obtain

$$\alpha_R = \frac{1}{2l+1}(l^2 + \frac{2}{p-1} + \frac{2(p+1)}{p-1}\sigma_R).$$

By (4.9), we have

$$\frac{1}{2l+1}(l^2 + \frac{2}{p-1} + \frac{2(p+1)}{p-1}\sigma_R) \geq l + \frac{2}{p-1}$$

which is impossible unless $l = 0$ or $l = 1$. For $l = 1$ we just recover the case $l = 0$ with the eigenfunction w' given by Lemma 4.1 (1). This clearly does not correspond to Hopf bifurcation.

For Hopf bifurcation to occur we must have $a = 0$ or $b = 0$. In this case, we have

$$\alpha_R = \frac{2}{p-1} + \frac{2(p+1)}{p-1}\sigma_R, \alpha_I = \frac{2(p+1)}{(p-1)(2\alpha_R+1)}\sigma_I.$$

Substituting this relation into (4.8) we obtain that σ_R must be a zero of the polynomial g defined by (4.1).

In summary, Hopf bifurcation can occur only at the point σ_R^h such that $g(\sigma_R^h) = 0$. Since such a point is unique, we conclude that for $|\theta| < \theta^h = \arccos\sigma_R^h$, there is unstable eigenvalues and for $|\theta| > \theta^h$, all eigenvalues are stable. $\square$

Let us now analyze the polynomial $g(\sigma_R)$ for $\sigma = e^{\frac{2\pi\sqrt{-1}}{N}}$. We note that as $p \to +\infty$, the zeroes of g approach the zeroes of

$$(4.10) \qquad g_0(\sigma_R) = \sigma_R^2(4\sigma_R + 1)^2 - (1 - \sigma_R^2).$$

The zero of g_0 is approximately 0.3726. Thus as p becomes large, $N_{critical}$-the critical threshold- can not exceed 6. In fact, one can compute explicitly that in the case $p > n_0 + 1$, where n_0 is given by (1.20), we have $g(\cos(\frac{2\pi}{N})) < 0$ for $N \leq 5$ and $g(\cos(\frac{2\pi}{N})) > 0$ for $N \geq 6$. In the case $p < n_0 + 1$, we have $g(\cos(\frac{2\pi}{N})) < 0$ for $N \leq 4$ and $g(\cos(\frac{2\pi}{N})) > 0$ for $N \geq 5$. That is we have the following corollary

COROLLARY 4.3. *For $N \leq N_0$, the problem (3.1) is stable, while for $N > N_0$, problem (3.1) is unstable, where*

$$N_0 = \begin{cases} 4 & \text{if } p \leq p_0, \\ 5 & \text{if } p \geq p_0 \end{cases}$$

and $p_0 \sim 4.35$.

We next study the NLEP (3.2). We first recall the following lemma

LEMMA 4.4. *[37] Consider the following eigenvalue problem*

$$(4.11) \qquad \Delta\phi - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_R w^{p-1}\phi}{\int_R w^p}w^2 = \lambda\phi, \phi \in H^2(R).$$

Then if $\gamma \leq p$, we have $Re(\lambda) < -c_1 < 0$ for some $c_1 > 0$.

From Lemma (4.4), it follows immediately we have the following

LEMMA 4.5. *Consider the eigenvalue problem (3.2).*
(1) If $\tau << 1$ and $0 < \eta < \frac{1}{p}$. Let $\lambda_0 \neq 0$ be an eigenvalue of (3.2). Then we have $Re(\lambda_0) \leq -c_1$ for some $c_1 > 0$.
(2) Suppose that $\frac{1}{p} < \eta < 1$, then problem (3.2) admits a real eigenvalue λ_0 with $\lambda_0 \geq c_2 > 0$ for some $c_2 > 0$.

Proof.
(1). When $\tau = 0$, we have

$$\frac{p(1-\eta)}{\eta\sqrt{1+\tau\lambda}+1-\eta} = p(1-\eta) > p-1$$

if $0 < \eta < \frac{1}{p}$. By Lemma 4.4, we must have that $\lambda_R < -c_1 < 0$. The case $\tau << 1$ follows from a perturbation argument.

(2). Assume that $\frac{1}{p} < \eta < 1$. By Lemma 4.1 (3), L_0 has a positive eigenvalue $\mu_1 > 0$. Consider the following function

$$h(\alpha) = \int_R ((L_0 - \alpha)^{-1}w^{p-1})w^{p-1}.$$

It is easy to see that

$$h'(\alpha) = \int_R ((L_0 - \alpha)^{-2}w^{p-1})w^{p-1} = \int_R [(L_0 - \alpha)^{-1}w^{p-1}]^2 > 0$$

and

$$\lim_{\alpha \to \mu_1} h(\alpha) = +\infty.$$

Next we consider the function

$$(4.12) \qquad \rho(\lambda) = \frac{\eta\sqrt{1+\tau\lambda}+1-\eta}{p(1-\eta)} - 1 - (\int_R w^p)^{-1}\lambda h(\lambda).$$

Note that

$$\rho(0) = \frac{1}{p(1-\eta)} - 1 > 0$$

since $\frac{1}{p} < \eta < 1$. On the other hand,

$$\lim_{\lambda \to \mu_1-} \rho(\lambda) = -\infty.$$

Hence there must exist an $\lambda_0 \in (0, \mu_1)$ such that $\rho(\lambda_0) = 0$. It is easy to see that this $\lambda_0 > 0$ is an eigenvalue of (3.2). $\square$

In the general case $\tau > 0$, $0 < \eta \leq \frac{1}{p}$, there are no analytic results for problem (3.2). Fortunately, we can use hypergeometric functions and generalized hypergeometric functions to reduce problem (3.2) to a computable problem. Such an idea has already been used in [8]. However, here we propose a different transformation so that the eigenvalue problem becomes computable more easily. We recall that by Lemma 4.3 (2) for $\tau = 0$, all eigenvalues are stable. So if we vary τ, either we obtain stability or Hopf bifurcation. All we need is to compute when Hopf bifurcation occurs.

Let us first introduce the so-called generalized Gauss function. Let $a_1, a_2, ..., a_A$ and $b_1, b_2, ..., b_B$ be two sequences of numbers. Consider the following series

$$(4.13) \qquad 1 + \frac{a_1 a_2 ... a_A}{b_1 b_2 ... b_B} \frac{z}{1!} + \frac{(a_1 + 1)(a_2 + 1)...(a_A + 1)}{(b_1 + 1)(b_2 + 1)...(b_B + 1)} \frac{z^2}{2!} + ...$$

$$\equiv {}_A F_B \left\{ \begin{array}{cccc} a_1, & a_2, & ..., & a_A \quad ; \\ & & & \quad\quad z \\ b_1, & b_2, & ..., & b_B \quad ; \end{array} \right\}$$

${}_A F_B$ is called generalized Gauss function or generalized hypergeometric function.

Now we have the following lemma. The proof is similar to Lemma 3.4 of [36]. So we omit the details.

LEMMA 4.6. *Let* $\lambda = \sqrt{-1} \lambda_I$ *be an eigenvalue of problem (3.2). Then* λ *is a solution of the following algebraic equation*

$$\frac{(3p-1)(p^2 - 1 - \lambda)}{p(p^2 - 1)} \frac{\eta \sqrt{1 + \tau\lambda} + 1 - \eta}{p(1 - \eta)}$$

$$(4.14) \qquad = {}_4 F_3 \left\{ \begin{array}{cccc} 1, \; \frac{1}{2}, & \frac{2p}{p-1}, & \frac{r}{p-1} + 1 \quad ; \\ & & & \quad\quad 1 \\ \frac{p - \sqrt{1+\lambda}}{p-1} + 1, & \frac{p + \sqrt{1+\lambda}}{p-1} + 1, & \frac{r}{p-1} + \frac{3}{2} \quad ; \end{array} \right\}$$

By Lemma (4.6), problem (3.2) can be solved by using Mathematica. We will not produce any numerical results here. The readers can refer to [8] for some numerical results.

5. General Matrix Case. Theorems 1.1 and 1.3 can be extended to more general matrices (k_{ij}).

Let us consider system (1.3):

$$(5.1) \qquad \left\{ \begin{array}{l} \partial_t X_i = \epsilon^2 X_i'' - X_i + AM \sum_{i=1}^{N} k_{ij} X_i X_j^n, \quad x \in R, \\ \tau \partial_t M = M'' + 1 - M - M \sum_{i,j=}^{N} k_{ij} X_i X_j^n, \quad x \in R, \end{array} \right.$$

where (k_{ij}) is a general matrix. To ensure existence, we put the following symmetric condition

$$(5.2) \qquad \sum_{i=1}^{N} k_{ij} = \sum_{j=1}^{N} k_{ij} = k_0.$$

Then Theorem 1.1 holds true without any change.

The main problem is the stability. To this end, we need to put an extra assumption

$$(5.3) \qquad \text{(H1)} \qquad [1 + \text{spec}(\mathcal{B})] \cap \text{spec}(\text{EVP}) = \{p\},$$

where $\mathcal{B} = (\hat{k}_{ij})$ and the EVP is defined in Lemma 4.1.

The following is our main result on stability.

THEOREM 5.1. *Assume that*

$$(5.4) \qquad \epsilon << 1, \quad \epsilon << L_\epsilon, \quad \lim_{\epsilon \to 0} L_\epsilon < L_0,$$

and that assumption (H1) holds. Let $(X_\epsilon^s, M_\epsilon^s)$ and $(X_\epsilon^l, X_\epsilon^l)$ be the solutions given in Theorem 1.1.

Let $\sigma = \sigma_R + \sqrt{-1}\sigma_I$ be an eigenvalue of $(\hat{k}_{ij})$ and let the polynomial g be defined as in (4.1).

Then for $\epsilon << 1$, we have the following.

(1) (stability) Suppose that $\tau << 1$. Assume that $\sigma = 1$ is a simple eigenvalue and that for all σ with $\sigma_R > 0$, we have $g(\sigma) < 0$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly stable.

(2) (Instability) Assume that either $\sigma = 1$ is not simple or there exists $\sigma \neq 1$ with $\sigma_R > 0$ such that $g(\sigma) > 0$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly unstable.

(3) (Instability) $(X_\epsilon^l, M_\epsilon^l)$ is linearly unstable.

The proof of Theorem 5.1 is the same as that of Theorem 1.2. We omit the details. Note that the analysis in Sections 2 - 4 deals with general matrices $\hat{k}_{ij}$ and is not restricted to merely the hypercycle case $\hat{k}_{ij} = \delta_{i,j+1}$ modulo N. Let us now apply Theorem 5.1 to some interesting examples.

Our first example is the following cyclical bi-diagonal matrix

$$(5.5) \qquad (k_{ij}) = k_0 \begin{pmatrix} 1-\alpha & \alpha & 0 & ... & 0 \\ 0 & 1-\alpha & \alpha & ... & 0 \\ 0 & 0 & 1-\alpha & ... & 0 \\ ... & ... & ... & ... & \alpha \\ \alpha & 0 & ... & 0 & 1-\alpha \end{pmatrix}_{N \times N} , \quad k_0 > 0.$$

It is easy to calculate that the eigenvalues are $\sigma = 1 - \alpha(1 - e^{2\pi j \sqrt{-1}/N})$, $j = 1, \ldots, N$ and are all simple.

We substitute σ into the polynomial and compute the critical threshold $N_{critical}$. It turns out that $N_{critical}$ depends on both α and p: $N_{critical}$ will increase of the order α as α increases but $N_{critical}$ increases only slowly in p. In fact, let us fix α and consider the case $p \to +\infty$. Then as $p \to +\infty$, the zeroes of f approach the zeroes of the polynomial

$$g_0(\sigma) := \sigma_R^2 (4\sigma_R + 1)^2 - \sigma_I^2$$

which was defined in (4.10) above. Substituting $\sigma_R = 1 - \alpha + \alpha\cos(\theta), \sigma_I = \alpha\sin(\theta)$ into (4.10), we obtain the following

$$(5.6) \quad \rho(\theta, \alpha) = (1 - \alpha + \alpha\cos(\theta))^2 (5 - 4\alpha + 4\cos(\theta))^2 - \alpha^2(1 - \cos^2(\theta)) = 0.$$

Note that for α large and $\alpha\theta = O(1)$

$$\theta \sim \frac{5}{\alpha}$$

which by $\theta = 2\pi/N_0$ implies that

$$N_0 \sim \frac{2\pi}{5}\alpha.$$

On the other hand for $p = 2$ we get under the same assumptions

$$N_0 \sim \frac{2\pi\sqrt{3}}{17\sqrt{5}}.$$

In both cases the critical threshold N_0 grows linearly in *alpha* but it is bigger by the factor $\frac{17}{\sqrt{15}} \sim 4.39$ in the first case. Thus in this case large p changes quantitative but not the qualitative behavior of N_0. This shows a more striking change of behavior than in the hypercycle case.

Our second example is a system with $(N-1)$ interactions.

$$(5.7) \qquad (k_{ij}) = k_0 \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 \end{pmatrix}_{N \times N} , \quad k_0 > 0.$$

The eigenvalues of $(\hat{k}_{ij})$ are $\sigma = 1$ (which is simple) and $\sigma = 0$. The small cluster state is stable for *all* N independent of the rate p.

From all the previous examples, we see as a general trend that if the system is not too much dominated by diagonal terms we have stability. Otherwise, a parasite emerges. This means that cooperative behavior in contrast with self-enhancement is needed to stabilize the cluster.

For large p stability in increased somewhat. We point to the second example where the stability threshold N_0 for large α grows linearly in α and large p can improve N_0 by a constant of about 4.39. In the case $\alpha > 1$ (which means that the diagonal becomes negative and the off-diagonal elements are positive and bigger than the diagonal), this describes self-inhibition coupled with cooperative enhancement and leads to particularly good stability.

Furthermore, the second and the third example indicate that coupling between more and more different components X_i also improves stability. Note that in the last example the system can be arbitrarily large.

6. Discussion. We have studied a general system of $N + 1$ equations with nonlinear rate n describing the interaction of N polymer species which catalyse each other in a hypercyclic way and are all composed of the same type of monomer. In the special case $N = 1$, $n = 1$ the system reduces to the well-known Gray-Scott system.

We study the case of single-cluster solutions in the whole 1-D space. These are in some sense the simplest concentrated solutions in 1-D. This case appears to be relevant if the early biochemical reactions take place in very thin lines for example on the edges of rocks.

Besides the existence proof we provide the first rigorous results on stability for cluster states of a hypercyclic system with nonlinear rate. Namely, we establish the exact threshold between stability and instability in terms of the system size and the nonlinear rate n. It is shown that as $n \leq 3.35$, the stability regimes extends exactly in the range $N = 1, 2, 3, 4$. If $n > 3.35$, the stability regime extends to $N = 1, 2, 3, 4, 5$. This shows that the maximum critical threshold for hypercycle system with nonlinear rate is 5. This result might be important for making predictions about the outcome of experiments in biological applications and also for the testing of the validity of the models used. Furthermore, now a comparison of this theoretical result with numerical computations becomes possible.

We have also studied reaction-diffusion systems with nonlinear rate and general connection matrix (k_{ij}). There we have observed that the nonlinear rate can help increase the critical threshold $N_{critical}$ when there is large self-inhibition (Example 1, Section 5). We show that large self-inhibition and to some lesser extent large n can help stabilize large reaction system.

So how can very large systems be stabilized?

One possibility is to increase self-inhibition and the nonlinear rate, as in Example 1, Section 5.

Another possibility which is frequently observed in nature is by the formation of a block-diagonal structure. Then, since the spectra of different blocks are independent, they can for example be chosen as small hypercycles which are stable up to size 5. On the other hand, by assembling a large number of blocks the system can become arbitrarily large and still be stable.

In fact, it is much simpler to create systems which display block-diagonal structure with only a few interactions than systems with many components catalysing each other. Therefore block-diagonal structures are frequently observed and are very important for natural phenomena.

Finally, let us recall attention to the point made in the introduction numerically it is known that parasites may destroy stable cluster states. Our results complement the picture by the rigorously proved fact that even pure cluster states may turn unstable if they become two large. This implies that the hypercycle although it has some very preferable properties (see the beginning of the introduction) on the other hand it has an inherent instability behavior which may be an obstruction to the evolution of large biological systems.

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