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UNIQUENESS RESULTS IN NONLINEAR ELLIPTIC PROBLEMS

MASSIMO GROSSI*

Abstract. In this paper we condider uniqueness results for the following nonlinear problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We will show as the geometry of the domain plays a crucial role in this context. We also discuss the nondegeneracy of the solution.

1. Introduction. In this survey we consider uniqueness results for semilinear elliptic problems of the type

$$(P) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain of $\mathcal{R}^N$, $N \geq 2$ and $f \in C^1(\mathcal{R})$. In this context the shape of the domain and the structure of nonlinearity f plays a crucial role. Indeed, even in the simple case of $f(s) = s^p$ we have multiplicity results for domains Ω with a "rich" topology or suitable geometry (see for example [3] and the references therein). So some restictions on the geometry of the domain is needed.

We point out that an important tool in the uniqueness results seems to be plaied by the linearized equation associated to (P), namely

$$(L) \qquad \begin{cases} -\Delta v = f'(u)v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where u is a solution of (P). We say that the solution u of (P) is nondegenerate if (L) admits only the trivial solution $v = 0$.

First uniqueness results for (P) were obtained when Ω is the ball, since in this case it is possible to reduce (P) to an ODE problem via the Gidas-Ni-Nirenberg theorem. Thanks to this reduction some uniqueness results for (P) were deduced for some special nonlinearities. We discuss this in Section 1. In Section 2 we consider the more difficult problem of a nonspherical domain. Of course, the previous approach does not work. At this stage the dimension of the space plays a role. Indeed, the results obtained in literature are weaker if the dimension of the space is greater that two. In this case, ($N \geq 3$), uniqueness results are obtained in perturbed cases, for example for special nonlinearities f_ε "close" to a suitable one. The reason of this restrictions relies on the difficulty to deduce qualitative properties of the solution of the linearized problem (L).

Finally in Section 4 we consider uniqueness results involving the critical Sobolev exponent.

*Dipartimento di Matematica, Università di Roma "La Sapienza", P.le A.Moro 2, 00185, Roma, Italy (grossi@mat.uniroma1.it). Supported by MURST, Project "Variazional Methods and Nonlinear Differential Equations".

2. The Radially Symmetric Case. We start this section recalling the following basic results, due to Gidas, Ni and Nirenberg (1979).

THEOREM 2.1. *Let us consider the problem (P) where Ω is convex and symmetric with respect to x_i, $1 \le i \le N$. Then u is symmetric with respect to x_i and $\frac{\partial u}{\partial x_i} < 0$ for $x_i > 0$.*

Proof. (see [17]).

COROLLARY 2.2. *Let us consider the problem (P) where Ω is the ball $|x| \le R$. Then $u = u(\rho)$ is radial and $u'(\rho) < 0$ for $\rho > 0$.*

Proof. (see [17]).

This Corollary is the starting point of the next uniqueness results in a ball. From this, Gidas, Ni and Nirenberg deduced the following result:

THEOREM 2.3. *Let us consider the problem (P) with $f(s) = s^p$, $1 < p < \frac{N+2}{N-2}$ and Ω is the ball $|x| \le R$. Then there exists only one solution to (P).*

Proof. By the previous Corollary we have that u solves the following ordinary differential equation,

$$\begin{cases} -u'' - \frac{N-1}{\rho}u' = u^p & \text{in } |x| \le R \\ u > 0 & \text{in } |x| \le R \\ u'(0) = u(R) = 0 \end{cases} \tag{2.1}$$

From the scaling invariance of (2.1) and the analiticity of the solution we get the uniqueness of the solution (see [18] for the details). □

Let us denote by $B_{P,r} = \{x \in \mathcal{R}^N \text{ such that } |x - P| \le r\}$ and $B_r = B_{0,r}$. Moreover let λ_1 be the first eigenvalue of $-\Delta$ in B_R with Dirichlet boundary conditions.

In the next years a lot of work was done to obtain uniqueness results for the more general problem

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } B_R \\ u > 0 & \text{in } B_R \\ u = 0 & \text{in } \partial B_R \end{cases} \tag{2.2}$$

where $\lambda < \lambda_1$ if $1 < p < \frac{N+2}{N-2}$ and $0 < \lambda < \lambda^*$ if $p = \frac{N+2}{N-2}$. Here $\lambda^* = \lambda_1$ for $N \ge 4$ and $\lambda^* = \frac{\lambda_1}{4}$ for $N = 3$.

Existence results for (2.2) are classical if $1 < p < \frac{N+2}{N-2}$. If $p = \frac{N+2}{N-2}$ in [5] it was proved that for $0 < \lambda < \lambda^*$ there exists at least one solution. Problem (2.2) is not scaling invariant and so it cannot be studied as (2.1). Indeed nontrivial ODE methods are used in order to prove the following results,

THEOREM 2.4. *Problem (2.2) admits a unique solution for any $\lambda < \lambda_1$ if $1 < p < \frac{N+2}{N-2}$ and $0 < \lambda < \lambda^*$ for $p = \frac{N+2}{N-2}$.*

A lot of authors give some contribution to this result. We only recall the papers [28],[1],[37],[34]. As we remarked, in this papers an important role is played by the nondegeneracity of the solution u.

3. The Nonradial Setting: the case $N = 2$.. In this section we will consider the more general case where Ω has not any radial symmetry. Of course the ODE methods of the previous section are not applicable anymore. On the other hand, Theorem 2.1 continues to have a central role.
First of all we recall the following result

THEOREM 3.1. *Let us consider the problem (P), with $N = 2$, $f(s) = s^p$ with $p > 1$. Here Ω is convex and symmetric with respect to x_i, $1 \leq i \leq N$. Then (P) admits only one solution.*

This result was proved by Dancer in [15] as a consequence of a general theorem contained also in [15] and of the known uniqueness result for the ball.

At this point we would like to quote here that the linearized operator L plays a crucial role in the uniqueness proof of [15]. Some properties of L were studied in [13].

Before proving the main result we need to recall a few facts about the maximum principle for second order elliptic operators of the form $Lu = \Delta u + c(x)u$ with $c(x) \in L^\infty(D)$, $u \in W^{2,N}_{loc} \cap C(\overline{D})$.

DEFINITION 3.2. *We say that the maximum principle holds for L in D if $Lu \leq 0$ in D and $u \geq 0$ on ∂D imply $u \geq 0$ in D.*

Two well known sufficient conditions for the maximum principle to hold are the following (see [20],[32])

$$c(x) \leq 0 \quad \text{in } D, \tag{3.1}$$

(3.2) there exists a function $g \in W^{2,N}_{loc} \cap C(\overline{D}), g > 0$ in $\overline{D}$ such that $Lg \leq 0$ in D

Now we denote by $\lambda_1(L, D)$ the principal eigenvalue of L in D. For the meaning and the properties of $\lambda_1(L, D)$ we refer to [6], where is also condiderd the case of nonsmooth boundary. In particular we have

PROPOSITION 3.3. *The principal eigenvalue $\lambda_1(L, D)$ is strictly decreasing in its dependence on D and on the coefficient $c(x)$. Moreover the "refined" maximum principle holds for L in D if and only if $\lambda_1(L, D)$ is positive.*

We refer to [6] for the definition of "refined" maximum principle which is a generalized formulation of the maximum principle in the case when one cannot prescribe boundary values of the functions involved.

It is important to notice that, by using this generalized definition of the first eigenvalue, it is possible to prove that also the following condition, which is slightly different from (3.2), is sufficient for the maximum principle to hold.

$$\begin{array}{l}\text{there exists } g \in W^{2,N}_{\text{loc}} \cap C(\overline{D}), g > 0 \text{ in } D \text{ such that } Lg \leq 0 \text{ in } D\\ \text{but } g \not\equiv 0 \text{ on some regular part of } \partial D\end{array} \tag{3.3}$$

We also recall the following sufficient condition for the maximum principle (see [5], [6]) PROPOSITION 3.4. *There exists $\delta > 0$, depending only on N, diam(D), $||c||_{L^\infty(D)}$ such that the maximum principle holds for L in any domain $D' \subset D$ with $|D'| < \delta$.*

Finally we remark that regardless of the sign of c if $Lu \leq 0$ in D and $u \geq 0$ in D then $u > 0$ in D unless $u \equiv 0$ (Strong Maximum Principle).
Now we consider a solution $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ of the problem (P). We are interested in studying the linearized problem We have the following theorem, which holds for $N \geq 2$.

THEOREM 3.5. *Let u be a solution of (P) with $f(0) \geq 0$ and assume that Ω is convex in the x_1- direction and symmetric with respect to the hyperplane $x_1 = 0$. Then any solution v of (L) is symmetric in x_1, i.e. $v(x_1, x_2, \ldots, x_N) = v(-x_1, x_2, \ldots, x_N)$.*

Proof. The proof is the same as the one shown in a lecture of L. Nirenberg in a slightly different case (see also the remark after the proof).

Let us denote a point x in $\mathcal{R}^N$ by (x_1, y), $y \in \mathcal{R}^{N-1}$. Applying the symmetry result of Gidas, Ni, Nirenberg to problem (P) we know that u is symmetric with respect to x_1 and $\frac{\partial u}{\partial x_1} > 0$ in $\Omega_1^- = \{x = (x_1, y) \in \Omega \quad \text{such that } x_1 < 0\}$.
We consider the operator

$$L = \Delta + f'(u) \tag{3.4}$$

and want to prove that the maximum principle holds for L in Ω_1^-. To do this we show that the sufficient condition (3.3) is satisfied.
If we set

$$g = \frac{\partial u}{\partial x_1} \quad \text{in } \Omega_1^- \tag{3.5}$$

we have that g satisfies (3.3) since by the Hopf Lemma $\frac{\partial u}{\partial x_1} \not\equiv 0$ on $\partial\Omega \cap \partial\Omega_1^-$ (note that we have assumed $f(0) \geq 0$). So the maximum principle holds for L in Ω_1^-.
Now we consider the function

$$\psi(x) = v(x_1, y) - v(-x_1, y), \qquad x = (x_1, y) \in \Omega_1^- \tag{3.6}$$

where v is a solution of L. By easy calculation, using that u is symmetric in x_1, we get

$$\begin{cases} L\psi = 0 & \text{in } \Omega_1^- \\ \psi = 0 & \text{in } \partial\Omega_1^- \end{cases}$$

and hence $\psi \equiv 0$ in Ω_1^- because of the maximum principle. So v is symmetric in x_1. □

REMARK 3.6. *Let us consider the following eigenvalue problem*

$$\begin{cases} -\Delta v + v = f'(u)v + \mu v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where u is a solution of P.
If $\mu < 0$ in [2] it is shown that v is symmetric in x_1.

Of course if Ω is a ball, the previous theorem gives the radial symmetry of v. This was already shown by Lin and Ni in [27], using a different argument, for any $\mu \leq 0$.

We make now some important remarks about the nodal set of v that will also be used in the sequel. Let us set

$$\mathcal{N} = \overline{\{x \in \Omega \text{ such that } v(x) = 0\}}$$

$$\widetilde{\Omega} = \{x \in \Omega : v(x) \neq 0\}$$

$$\Omega_i^- = \{x = (x_1, \dots, x_N) \in \Omega \text{ such that } x_i < 0\} \quad i = 1, \dots, N$$

We have

THEOREM 3.7. *The following properties hold*
i) there cannot exist any component of $\widetilde{\Omega}$ all contained in one Ω_i^-, $i = 1, \dots, N$.
ii) if $N = 2$ then the origin $(0, \dots, 0)$ does not belong to $\mathcal{N}$.
iii) if $N = 2$ then $\mathcal{N} \cap \partial\Omega = \emptyset$.

Proof. i) Suppose that there exists a component D of $\widetilde{\Omega}$ all contained in Ω_i^- and $v > 0$ in D. Then $\lambda_1(L, D) = 0$ (where L is the operator defined in (3.4)) since v is an eigenfunction of L in D corresponding to the zero eigenvalue and does not change sign in D (being $v = 0$ on $\partial\Omega$ we have $v = 0$ on $\partial\Omega$). On the other hand, in the proof of the previous theorem we have shown that L satisfies the maximum principle in Ω_i^- and this implies, by Proposition 3.3, that $\lambda_1(L, \Omega_i^-) > 0$. Then, by monotonicity, also $\lambda_1(L, D)$ should be positive which gives a contradiction.
ii) We will show that if $v(0) = 0$ then $v \equiv 0$. Suppose $v(0) = 0$ and $v \not\equiv 0$ and set $U_0 = \Omega$. Since $v \not\equiv 0$ and $v(0) = 0$ by the Strong Maximum Principle it cannot be $v \leq 0$ in Ω, so that $U_0^+ = \{x \in U_0 \ : \ v(x) > 0\}$ is open and nonempty. Choose a component A_1 of U_0^+. If S_i, $i = 1, 2$ is the operator that sends a point to the symmetric one with respect to the x_1-axis, we have that $S_i(A_1)$ is also a component of U_0^+ because of the symmetry of v. It cannot happen that $A_1 \cap S_1(A_1) = \emptyset$ or $A_1 \cap S_2(A_1) = \emptyset$ for otherwise A_1 or $S_1(A_1)$ would be contained in Ω_1^-, which is impossible by (i). So $A_1 = S_1(A_1) = S_2(A_1)$ is symmetric with respect to the coordinate axes and is open and connected, therefore arcwise connected. If we choose four symmetric points P_j, $j \in \{1, \dots, 4\}$ and join them with simple poligonal curves symmetric in pairs, we can costruct a simple closed poligonal curve $C_1 \subset A_1$ which is symmetric with respect to the axes. By the Jordan Curve Theorem $U_0 \setminus C_1$ has two components and, because C_1 is symmetric, the origin belongs to the component which has not ∂U_0 as part of the boundary. Let us denote by U_1 the component that contains 0 and call it the interior of C_1, while by the exterior of C_1 we mean the other component. On $\partial U_1 = C_1$ we have $v > 0$, so that $v \not\equiv 0$ in U_1 and, by the Strong Maximum Principle, it is not possible that $v \geq 0$ in U_1, since $v(0) = 0$, so that $U_1^- = \{x \in \Omega_1 \ : \ v(x) < 0\}$ is open and nonempty. Taking a component A_2 of U_1^- we observe that $v = 0$ on ∂A_2 because $v \geq 0$ on ∂U_1 so that A_2 is also a component of $\widetilde{\Omega}$. As before we can costruct a closed symmetric simple curve $C_2 \subset A_2$ and in the interior U_2 of C_2 (the component of $U_1 \setminus C_2$ to which the origin belongs) we can choose a component A_3 of $U_2^+ = \{x \in U_2 \ : \ v(x) > 0\}$ which is also a component of $\widetilde{\Omega}$. Moreover A_3 is disjoint from A_1 because A_1 contains $C_1 = \partial\Omega_1$ which belongs to the exterior of C_2. Proceeding in this way we obtain infinitely many disjoint components $\{A_n\}_{n \geq 1}$ of $\widetilde{\Omega}$.

This is not possible because by Proposition 3.4 there exists $\delta > 0$ such that $|A_n| \geq \delta$ for each n, otherwise by the Maximum Principle v would be 0 in A_n, since $v = 0$ on ∂A_n and $Lv = 0$ in A_n with $L = \Delta - \lambda + f'(v)$. Hence there are only finitely many components A_n which gives a contradiction.
iii) We will show that in a neighboorhood of $\partial\Omega$ we have $v > 0$ or $v < 0$. Suppose the contrary and choose a component A_1 of $U_0^+ = \{x \in U_0 \ : \ v(x) > 0\}$. Since $v = 0$ on $\partial\Omega$ we have $v = 0$ on ∂A_1 and as in (ii) we costruct a closed simple curve $C_1 \subset A_1$ symmetric with respect to the axes. In the exterior U_1 of C_1, i.e. in the component containing $\partial\Omega$ there are points where $v < 0$ by what we assumed. So we can costruct a closed simple curve $C_2 \subset A_2$ where A_2 is a nonempty component of $U_1^- = \{x \in U_1 \ : \ v(x) < 0\}$. Proceeding as in the proof of (ii) we obtain infinitely many components of $\tilde{\Omega}$ which is not possible by Proposition 3.4, as we remarked before. □

REMARK 3.8. *If Ω is a ball in $\mathcal{R}^N$, the properties i) - iii) are easy consequences of the radial symmetry of v.*

Now we consider two solutions u_1 and u_2 of the problem P and set

$$\mathcal{M} = \overline{\{x \in \Omega \text{ such that } u_1(x) = u_2(x)\}} \quad , \quad \widehat{\Omega} = \{x \in \Omega \text{ such that } u_1 \neq u_2\}$$

The next theorem contains some information on $\mathcal{M}$ and a uniqueness result.

THEOREM 3.9. *Suppose that f is convex. Then we have*

$$\text{(3.7)} \qquad \begin{array}{c} \textit{there cannot exist any component } D \textit{ of} \\ \widehat{\Omega} \textit{ all contained in one } \Omega_i^-,\ i = 1, \dots, N. \end{array}$$

$$\text{(3.8)} \qquad \textit{if } N = 2 \quad \textit{then} \quad \mathcal{M} \cap \partial\Omega = \emptyset$$

$$\text{(3.9)} \qquad \textit{if } N = 2 \quad \textit{and} \quad \max_{x \in \overline{\Omega}} u_1(x) = \max_{x \in \overline{\Omega}} u_2(x) \quad \textit{then} \quad u_1 \equiv u_2$$

Proof. Set $w(x) = u_1(x) - u_2(x),\ x \in \Omega$. Since f is convex w satisfies

$$\text{(3.10)} \qquad \begin{cases} \Delta w + f'(u_2) w \leq 0 & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega. \end{cases}$$

and

$$\text{(3.11)} \qquad \begin{cases} \Delta w + f'(u_1) w \geq 0 & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega. \end{cases}$$

First we notice that if $w \geq 0$ by (3.10) and the strong maximum principle $w > 0$ in Ω so that $\Omega = \widehat{\Omega}$. Thus we assume that w changes sign in Ω. To prove (3.7) let us argue by contradiction supposing that there exists a component D of $\widehat{\Omega}$ all contained in Ω_i^- for some $i \in \{1, \dots, N\}$ and $w > 0$ in D.
Since in Theorem (3.5) we proved that in Ω_i^- the maximum principle holds for the operators $L_i = \Delta - \lambda + f'(u_i) \quad i = 1, 2$, by Proposition 2.1 we have that $\lambda_1(L_1, \Omega_i^-) > 0$, for $i = 1, 2$. Hence also $\lambda_1(L_1, D) > 0$ and, again by Proposition (3.3), the "refined" maximum principle holds for L_1 in D. This last fact together with (3.11) would imply

that $w \le 0$ in D against what we assumed. If instead we suppose $w \le 0$ in D then we argue in the same way using the operator L_2 and (3.10).
To prove (3.8) it is enough to observe that, by the Gidas, Ni and Nirenberg symmetry result, u_1 and u_2 are symmetric in any x_i and hence so is w. Thus arguing as in iii) of the previous theorem the assumption $\mathcal{M} \cap \partial\Omega \neq \emptyset$ would bring a contradiction.
Finally, to prove (3.9), we notice that, again by the Gidas, Ni and Nirenberg result, $\max_{x\in\overline{\Omega}} u_i(x) = u_i(0)$, $i = 1, 2$; therefore if the two maxima coincide the origin belongs to $\mathcal{M}$. As in ii) of Theorem 3.7 this gives a contradiction. □
Now we prove a generalization of (3.9) of Theorem 3.9.
Let Ω be as before and $N = 2$. Let us call a function $u \in C^1(\overline{\Omega})$ symmetric and monotone if u is symmetric in x_1, x_2 and $\frac{\partial u}{\partial x_i} > 0$ in Ω_i^-, $i = 1, 2$ and let $f : \mathcal{R} \longrightarrow \mathcal{R}$ be a C^1-function.

THEOREM 3.10. *Suppose that $N = 2$, f is convex and $u_1, u_2 \in C^3(\Omega) \cap C^1(\overline{\Omega})$ are symmetric and monotone functions that satisfy the equation*

$$-\Delta u + \lambda u = f(u) \qquad in \quad \Omega \tag{3.12}$$

If $u_1(0) = u_2(0)$ and $u_1 \le u_2$ on $\partial\Omega$ then u_1 and u_2 coincide.

Proof. As in the proof of Theorem 3.5 we deduce that the operators $L = \Delta - \lambda + f'(u_i)$, $i = 1, 2$ satisfy the maximum principle in Ω_j^-, $j = 1, 2$.
Since the difference $w = u_1 - u_2$ satisfies a linear equation $\Delta w - \lambda w + c(x)w = 0$ with $c \in L^\infty(\Omega)$ and $f \in C^1$ we have that Proposition 3.4 and the strong maximum principle apply to w. Arguing as in Theorem 3.7 we first deduce that cannot exist any component D of $\widehat{\Omega} = \{x \in \Omega \ : \ u_1 \neq u_2\}$ such that $u_1 = u_2$ on ∂D and cointained in Ω_j^-, $j = 1, 2$.
Then we can follow exactly the proof of Theorem 3.7 with the only remark that in the first step we choose a component A_1 of $\Omega_0^+ = \{x \in \Omega \ : \ w(x) > 0\}$ and we have $w = 0$ on ∂A_1, because of the hypothesis $w(x) \le 0$ on $\partial\Omega$. So A_1 is also a component of $\widehat{\Omega}$ with $u_1 = u_2$ on ∂A_1. The same property holds, by construction, also for the other components A_2, A_3; therefore we conclude as in Theorem 3.7.

REMARK 3.11. *If Ω is a ball then any solution u of (P) is radial and hence the claim of Theorem 3.10 follows immediately from the theory of ordinary differential equation. Therefore this result can be seen as a generalization of the uniqueness theorem for an o.d.e.*
Nevertheless it is instructive to see how we can get very easily this result in a ball without using the underlaying ordinary equation but exploiting only maximum principles. Therefore suppose $\Omega = B_R(0) \subset \mathcal{R}^N$ and $u_i \in C^2(\overline{\Omega})$, $i = 1, 2$, satisfying $-\Delta u = f(u)$ in Ω. Let us prove that if $u_1(0) = u_2(0)$ then $u_1 \equiv u_2$. In fact the difference $w = u_1 - u_2$ satisfies a linear equation $\Delta w + c(x)w = 0$. By Proposition 2.2 there exists $\delta > 0$ such that if $0 \le r_1 < r_2 < R$ and $r_2 - r_1 < \delta$ then the Maximum Principle holds for $\Delta + c$ in $B_{r_2} \setminus B_{r_1}$. We claim that u_1 and u_2 coincide on ∂B_r for any $r < \delta$. In fact it cannot be $u_1 > u_2$ on ∂B_r because by Proposition 3.4 and the strong maximum principle it would be $u_1 > u_2$ on B_r, against the assumption $u_1(0) = u_2(0)$. In the same way it is not possible that $u_1 < u_2$ on ∂B_r. So $u_1 \equiv u_2$ in $\overline{B}_\delta$. Making the same reasoning in $B_{\frac{3}{2}\delta} \setminus B_{\frac{1}{2}\delta}$ (that has ∂B_δ in the interior) we get $u_1 \equiv u_2$ in $B_{\frac{3}{2}\delta}$ and after a finite number of steps we get $u_1 \equiv u_2$ in B_R. □

4. The Nonradial Setting: The Case $N \geq 3$. In this section we consider uniqueness problems of (P) for suitable nonspherical domains $\Omega \subset \mathcal{R}^N$, $N \geq 3$. In this context the uniqueness results are weaker than the previous section. Indeed, since the topology of the nodal zones of the solution of (L) with $N \geq 3$ is more complicated than the corresponding two-dimensional case, it seems very difficult to obtain nondegeneracity results for solution of (P). We only know uniqueness results to solution of (P) for perturbed problem, i.e. when the nonlinearity $f = f_\epsilon$ and the solution u_ϵ converges to a solution of a limit problem. A first example is the following result, due to E.N. Dancer.

THEOREM 4.1. *Let us consider a solution of the following problem*

$$\begin{cases} -\varepsilon\Delta u + u = u^p & in\ \Omega \\ u > 0 & in\ \Omega \\ u = 0 & in\ \partial\Omega. \end{cases} \tag{4.1}$$

Here $1 < p < \frac{N+2}{N-2}$ and Ω satisfies some geometrical assumptions (see [14]): Then, for ε small enough, there exists only one solution to (4.1).

Proof. See [14]. □

Now we come back to the nonlinearity $f(s) = s^p$. We recall that, by Pohozaev's identity, in this case there is no solutions to (P) in star-shaped domains for $p \geq \frac{N+2}{N-2}$. Concerning the uniqueness we have the following result (see [22])

THEOREM 4.2. *Let us consider the problem*

$$\begin{cases} -\Delta u = N(N-2)u^p & in\ \Omega \\ u > 0 & in\ \Omega \\ u = 0 & in\ \partial\Omega. \end{cases} \tag{4.2}$$

Let $\Omega \subset \mathcal{R}^N$, $N \geq 3$, be a bounded smooth domain satisfying

$$\Omega \text{ is convex in the } x_i \text{ direction, } i = 1, .., N. \tag{4.3}$$

and

$$\Omega \text{is symmetric with respect to the hyperplanes } x_i = 0,\ i = 1, .., N. \tag{4.4}$$

Then there exists $\varepsilon > 0$ such that for any $p \in]\frac{N+2}{N-2} - \varepsilon, \frac{N+2}{N-2}[$ there is only one solution to (4.2). Moreover this solution is nondegenerate, i.e. the linear equation

$$\begin{cases} -\Delta v = pN(N-2)u^{p-1}v & in\ \Omega \\ v = 0 & in\ \partial\Omega. \end{cases} \tag{4.5}$$

admits only the trivial solution $v \equiv 0$.

In order to prove the previous theorem we need to know with great care the asymptotic behaviour of the solutions of (4.2) as $p \to \frac{N+2}{N-2}$. We start with the following result

PROPOSITION 4.3. *If Ω satisfies (4.3) and (4.4) then*

$$\frac{\int_\Omega |\nabla u_n|^2}{\left(\int_\Omega |u_n|^{p_n+1}\right)^{\frac{2}{p_n+1}}} \to S_N \ as\ n \to \infty, \tag{4.6}$$

where S_N is the best Sobolev constant in $\mathcal{R}^N$.

Proof. In the proof of this theorem we use a blow up technique as in the paper of Gidas and Spruck (see [19]) and some important results of [26].
Since Ω verifies (4.3) and (4.4), using the Pohozaev identity (see [31]) it is not difficult to prove that

$$u_n(0) = \|u_n\|_\infty \to \text{ as } n \to \infty\infty \tag{4.7}$$

Let us define

$$\widetilde{u}_n(x) = \frac{1}{\|u_n\|_\infty} u_n\Big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\Big), \quad \widetilde{u}_n : \Omega_n = \|u_n\|_\infty^{\frac{p_n-1}{2}} \cdot \Omega \to \mathcal{R} \tag{4.8}$$

By easy calculation $\widetilde{u}_n$ satisfies

$$\begin{cases} -\Delta \widetilde{u}_n = pN(N-2)\widetilde{u}_n^{p_n} & \text{in } \Omega_n \\ \widetilde{u}_n > 0 & \text{in } \Omega_n \\ \widetilde{u}_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{4.9}$$

Notice that $\widetilde{u}_n(0) = 1$, $0 < \widetilde{u}_n(x) \le 1$ for $x \in \Omega_n$ and Ω_n converges to $\mathcal{R}^N$ (by the notation $\Omega_n \to \mathcal{R}^N$ we mean that for any $K \subset \mathcal{R}^N$ we have $\Omega_n \supset K$ for n large). Again by elliptic theory $\widetilde{u}_n \to U$ in $C^1(K)$ for every compact set K of $\mathcal{R}^N$, and U solves

$$\begin{cases} -\Delta U = N(N-2)U^{\frac{N+2}{N-2}} & \text{in } \mathcal{R}^N \\ 0 \le U \le 1 & \text{in } \mathcal{R}^N \\ U(0) = 1 \end{cases} \tag{4.10}$$

The solution of (4.10) is unique (see [11]) and

$$U(x) = \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}} \tag{4.11}$$

Now by using some estimates contained in [26] we will prove that

$$\int_\Omega |u_n|^{p_n+1} \to \int_{\mathcal{R}^N} U(x)^{\frac{2N}{N-2}} \tag{4.12}$$

First of all we recall the following inequality (see [26]):

$$u_n(x) \le \frac{C}{\|u_n\|_\infty} \frac{1}{|x|^{N-2}} \qquad \text{if } |x| < \delta \tag{4.13}$$

where C and δ are positive constant which do not depend on n.
From (4.13) we will deduce (4.12). Let us compute

$$\int_\Omega |u_n|^{p_n+1} = \int_{|x|<\delta} |u_n|^{p_n+1} + \int_{\{|x|>\delta\}\cap\Omega} |u_n|^{p_n+1} = I_{n,1} + I_{n.2} \tag{4.14}$$

Let us prove that

$$I_{n,1} \to S_N^{N/2} \tag{4.15}$$

and

(4.16) $$I_{n,2} \to 0$$

We have

$$I_{n,1} = \int_{|x|<\delta} |u_n(x)|^{p_n+1} = \|u_n\|_\infty^{p_n+1-\frac{p_n-1}{2}N} \int_{|x|<\delta\|u_n\|_\infty^{\frac{p_n-1}{2}}} |\widetilde{u}_n(x)|^{p_n+1}$$

(4.17) $$= \|u_n\|_\infty^{\frac{N-2}{2}\left(\frac{N+2}{N-2}-p_n\right)} \int_{\mathcal{R}^N} \widetilde{u}_n(x)|^{p_n+1} \chi_{\{|x|<\delta\|u_n\|_\infty^{\frac{p_n-1}{2}}\}}.$$

From (4.13) we get

(4.18) $$\widetilde{u}_n(x) \le C \frac{1}{\|u_n\|_\infty^{2-\frac{p_n-1}{2}(N-2)}} \frac{1}{|x|^{N-2}} \le \frac{C}{|x|^{N-2}} \qquad \text{if } |x| < \delta\|u_n\|_\infty^{\frac{p_n-1}{2}}$$

Now since $\widetilde{u}_n$ is bounded near the origin and $\widetilde{u}_n \to U$ pointwise in $\mathcal{R}^N$, from (4.18) and dominate convergence theorem we get

(4.19) $$\int_{\mathcal{R}^N} \widetilde{u}_n(x)^{p_n+1} \chi_{\{|x|<\delta\|u_n\|_\infty^{\frac{p_n-1}{2}}\}} \to \int_{\mathcal{R}^N} U(x)^{\frac{2N}{N-2}}$$

Moreover, again by [26], it is possible to deduce

(4.20) $$\|u_n\|_\infty^{\frac{N-2}{2}\left(\frac{N+2}{N-2}-p_n\right)} \to \text{ as } n \to \infty$$

and then

(4.21) $$I_{n,1} = \|u_n\|_\infty^{\frac{N-2}{2}\left(\frac{N+2}{N-2}-p_n\right)} \int_{\mathcal{R}^N} \widetilde{u}_n(x)^{p_n+1} \chi_{\{|x|<\delta\|u_n\|_\infty^{\frac{p_n-1}{2}}\}} \to S_N^{N/2}$$

and this proves (4.15). In order to prove (4.16), we remark that from (4.13) it follows

(4.22) $$u_n(x) \le \frac{C}{\|u_n\|_\infty} \frac{1}{\delta^{N-2}} \qquad \text{if } |x| = \delta$$

From this we deduce that

(4.23) $$u_n(x) \le \frac{C}{\|u_n\|_\infty} \frac{1}{\delta^{N-2}} \qquad \text{in } \{|x| > \} \cap \Omega$$

Indeed, if by contradiction there exists a point $x_n \in \{|x| > \delta\} \cap \Omega$ such that $u_n(x_n) > \frac{C}{\|u_n\|_\infty} \frac{1}{\delta^{N-2}}$, we would get the existence of a maximum point for u_n in $\{|x| > \delta\} \cap \Omega$. But this is not possible by (4.3) and (4.4) and Gidas-Ni-Nirenberg theorem. Hence (4.23) holds and then we get (4.16) and so (4.12).
Finally, since u_n is a solution of (4.2) we get

(4.24) $$\frac{\int_\Omega |\nabla u_n|^2}{\left(\int_\Omega |u_n|^{p_n+1}\right)^{\frac{2}{p_n+1}}} = N(N-2)\left(\int_\Omega |u_n|^{p_n+1}\right)^{1-\frac{2}{p_n+1}} \to S_N \text{ as } n \to \infty$$

and this proves the claim. □
Now we recall some results due to Han (see [23], Theorem 1, Lemma 3 and proposition 1).

THEOREM 4.4. *Let Ω be a smooth bounded domain of $\mathcal{R}^N$ and u_n a solution of (4.2). If u_n satisfies*

$$\frac{\int_\Omega |\nabla u_n|^2}{\left(\int_\Omega |u_n|^{p_n+1}\right)^{\frac{2}{p_n+1}}} \to S_N \text{ as } n \to \infty \tag{4.25}$$

where S_N is the best Sobolev constant, then (up to a subsequence)

$$\left(\frac{N+2}{N-2} - p_n\right)\|u_n\|_\infty^2 \to 2\sigma_N^2\left[\frac{N(N-2)}{S_N}\right]^{N/2} g(x_0, x_0) \tag{4.26}$$

where $g(x,y)$ is the regular part of the Green's function $G(x,y)$, i.e.

$$g(x,y) = G(x,y) - \frac{1}{(N-2)\sigma_N |x-y|^{N-2}} \tag{4.27}$$

and σ_N is the area of the unit sphere in $\mathcal{R}^N$ and x_0 is a critical point of g. Moreover

$$u_n(x) \le k \frac{\|u_n\|_\infty}{\left(1 + \|u_n\|_\infty^{\frac{4}{N-2}} |x - x_0|^2\right)^{\frac{N-2}{2}}} \tag{4.28}$$

and

$$\|u_n\|_\infty u_n(x) \to (N-2)\sigma_N G(x, x_0) \tag{4.29}$$

in $C^1(\omega)$ for any neighborhood ω of $\partial\Omega$ not containing x_0.
Finally

$$|\nabla u_n|^2 \to N(N-2)\left[\frac{S_N}{N(N-2)}\right]^{N/2} \delta_{x_0} \quad \text{in the sense of distributions} \tag{4.30}$$

REMARK 4.5. *If Ω is a smooth, bounded domain satisfying (4.3) and (4.4) it is not difficult to deduce by Proposition 4.3 that $x_0 = 0$ in Theorem 4.4.*

At this stage we are in condition to prove Theorem 4.2.

Proof of Theorem 4.2.
We argue by contradiction: let us suppose that there exists a sequence $p_n \nearrow \frac{N+2}{N-2}$ and functions u_n, $v_n \in C^\infty(\Omega)$ which solve (4.2) with p replaced by p_n.
Set

$$\widetilde{w}_n(x) = u_n\Big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\Big) - v_n\Big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\Big), \qquad \widetilde{w}_n : \Omega_n \to \mathcal{R} \tag{4.31}$$

As in Theorem 4.2 we have $\|u_n\|_\infty \to \infty$ and $\Omega_n \to \mathcal{R}^N$. Moreover

$$\begin{cases} -\Delta \widetilde{w}_n = c_n(x)\widetilde{w}_n & \text{in } \Omega \\ w_n = 0 & \text{in } \partial\Omega. \end{cases} \tag{4.32}$$

where

$$c_n(x) = N(N-2)p_n \int_0^1 \Big(\frac{t}{\|u_n\|_\infty} u_n\Big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\Big) + \frac{1-t}{\|u_n\|_\infty} v_n\Big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\Big)\Big)^{p_n-1} dt \tag{4.33}$$

We have

$$c_n(x) \to \frac{N(N+2)}{(1+|x|^2)^2} \quad \text{uniformly on compact sets of } \mathcal{R}^N \tag{4.34}$$

So, for $x \in \Omega_n$, let us define

$$w_n = \frac{\widetilde{w}_n}{\|\widetilde{w}_n\|_\infty} \quad , \quad w_n : \Omega_n \to \mathcal{R} \tag{4.35}$$

Of course w_n solves

$$\begin{cases} -\Delta w_n = c_n(x) w_n & \text{in } \Omega \\ \|w_n\|_\infty = 1 & \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.36}$$

and w_n is a symmetric function.
Since w_n is bounded, using standard elliptic estimates we deduce that w_n converges to a symmetric function w uniformly on compact set of $\mathcal{R}^N$. Moreover w satisfies

$$\begin{cases} -\Delta w = N(N+2)\frac{w}{(1+|x|^2)^2} & \text{in } \Omega \\ \|w_n\|_\infty = 1 & \\ \|w\|_\infty \le 1. & \end{cases} \tag{4.37}$$

Now we need the following estimate:

$$\int_{\Omega_n} |\nabla w_n|^2 \le C \tag{4.38}$$

In order to prove (4.38) let us multiply (4.36) by w_n. Then by Sobolev inequality and for some $0 < \delta < \frac{4}{N-2}$

$$S_N\Big(\int_{\Omega_n} |w_n|^{2^*}\Big)^{\frac{2}{2^*}} \le \int_{\Omega_n} |\nabla w_n|^2 = \int_{\Omega_n} |c_n(x)| w_n^2 \le \int_{\Omega_n} |c_n(x)| w_n^{2-\delta} \tag{4.39}$$

since $\|w_n\|_\infty = 1$. Then using Hölder inequality and (4.32)

$$S_N\Big(\int_{\Omega_n} |w_n|^{2^*}\Big)^{\frac{2}{2^*}} \le \Big(\int_{\Omega_n} |w_n|^{2^*}\Big)^{\frac{2-\delta}{2^*}} \le \Big(\int_{\Omega_n} c_n(x)^{\frac{2^*}{2^*-2+\delta}}\Big)^{\frac{2^*-2+\delta}{2^*}} \le$$

$$\le C\Big(\int_{\Omega_n} |w_n|^{2^*}\Big)^{\frac{2-\delta}{2^*}} \Big(\int_{\Omega_n} \frac{1}{(1+|x|^2)^{(p_n-1)\frac{N-2}{2}\frac{2^*}{2^*-2+\delta}}}\Big)^{\frac{2^*-2+\delta}{2^*}} \tag{4.40}$$

and then

$$C\Big(\int_{\Omega_n} |w_n|^{2^*}\Big)^{\frac{\delta}{2^*}} \le \Big(\int_{\Omega_n} \frac{1}{(1+|x|^2)^{(p_n-1)\frac{N-2}{2}\frac{2^*}{2^*-2+\delta}}}\Big)^{\frac{2^*-2+\delta}{2^*}} \le$$

$$\le \Big(\int_{\mathcal{R}^N} \frac{1}{(1+|x|^2)^{(p_n-1)\frac{N-2}{2}\frac{2^*}{2^*-2+\delta}}}\Big)^{\frac{2^*-2+\delta}{2^*}} < \infty \tag{4.41}$$

since $0 < \delta < \frac{4}{N-2}$. So $\int_{\Omega_n} |w_n|^{2^*}$ is bounded and then again by (4.36) the claim follows.

The structure of the solutions of (4.37) satisfying ((4.38)) was described in [4], where was proved that the following cases occur:
i) $w = 0$,
ii) $w = \frac{\partial U}{\partial x_i}$, $\quad i = 1,..,N$ where $U(x) = \big(\frac{1}{1+|x|^2}\big)^{\frac{N-2}{2}}$.
or
iii) $w = \frac{1-|x|^2}{(1+|x|^2)^{N/2}}$

So we have that $w(x) = \sum_{i=1}^{N} \alpha_i \frac{x_i}{(1+|x|^2)^{N/2}} + \beta \frac{1-|x|^2}{(1+|x|^2)^{N/2}}$. It is easily seen that the study of w is equivalent to consider i), ii), iii) separately.

Now we will prove that in any case a contradiction arises.

Case i) $w = 0$

From (4.38) it is possible to deduce the following crucial estimates, (see [22], Appendix 2):

$$|w_n(x)| \le \frac{C}{(1+|x|^2)^{\frac{N-2}{2}}} \qquad \text{for any } x \in \mathcal{R}^N \tag{4.42}$$

From (4.42) a contradiction follows easily. Indeed since $\|w_n\|_\infty = 1$ we can assume that $\exists x_n \in \Omega_n$ such that $\max_{x\in\Omega_n} w_n(x) = w_n(x_n) = 1$ and $\|x_n\|_{\mathcal{R}^N} \to \infty$ because $w_n \to 0$ uniformly on any compact set of $\mathcal{R}^N$. But this is not possible by (4.42).

Case ii) $w = \frac{\partial U}{\partial x_i}$, $\quad i = 1,..,N$

In this case we have a contradiction since $w = \frac{\partial U}{\partial x_i}$ is not a symmetric function, i.e. does not satisfy $w(x_1, \dots, -x_{i-1}, x_i, x_{i+1}, \dots, x_N) = w(x)$ for any $x \in \mathcal{R}^N$.

Case iii) $w = \frac{1-|x|^2}{(1+|x|^2)^{N/2}}$

In what follows we will use Theorem 4.4, where the point $x_0 = O$ (see Remark 4.5). First of all we notice that, for any neighborhood $\emptyset$ of $\partial\Omega$ not containing O it holds that

$$\|u_n\|_\infty^2 \frac{u_n(x) - v_n(x)}{\|u_n - v_n\|_\infty} \to -(N+2)\sigma_N G(x,0) \text{ in } C^1(\emptyset) \tag{4.43}$$

where, as in Theorem 4.4, G is the Green function.

Let us prove (4.2). We have

$$-\Delta\Big(\|u_n\|_\infty^2 \frac{u_n(x) - v_n(x)}{\|u_n - v_n\|_\infty}\Big) = \|u_n\|_\infty^2 d_n(x) \frac{u_n(x) - v_n(x)}{\|u_n - v_n\|_\infty} \tag{4.44}$$

where $d_n(x) = N(N-2)p_n \int_0^1 (tu_n(x) + (1-t)v_n(x))^{p_n-1} dt$.

So, using (4.28), (4.42) and the dominate convergence theorem we obtain

$$\|u_n\|_\infty^2 \int\limits_\Omega d_n(x)\frac{u_n(x)-v_n(x)}{\|u_n-v_n\|_\infty} = \|u_n\|_\infty^{2-\frac{p_n-1}{2}N} \int\limits_{\Omega_n} d_n\Big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\Big)w_n(x) =$$

$$(4.45)= \|u_n\|_\infty^{p_n+1-\frac{p_n-1}{2}N} \int\limits_{\Omega_n} c_n(x)w_n(x) \to -(N+2)\sigma_N$$

(the last integral can easily be computed by recalling that $-\Delta w = \frac{1}{(1+|x|^2)^2}w$). Moreover, again by (4.28), (4.42) and for any $x \neq 0$

$$\|u_n\|_\infty^2 d_n(x)\frac{u_n(x)-v_n(x)}{\|u_n-v_n\|_\infty}$$

$$\leq C\frac{\|u_n\|_\infty^2}{\|u_n\|_\infty^{(p_n-1)(\frac{N-2}{2}(p_n-1)-1)}}\frac{w_n(\|u_n\|_\infty^{\frac{p_n-1}{2}}x)}{|x|^{(N-2)(p_n-1)}} \leq$$

$$\leq \frac{C}{\|u_n\|_\infty^{(p_n-1)(\frac{N-2}{2}p_n-1)-2}}\frac{1}{|x|^{(N-2)p_n}} \leq \frac{C}{|x|^{(N-2)p_n}} \tag{4.46}$$

Finally since, in the sense of distributions

$$\|u_n\|_\infty^2 \int\limits_\Omega d_n(x)\frac{u_n(x)-v_n(x)}{\|u_n-v_n\|_\infty}\phi(x) \to -(N+2)\sigma_N\delta_0 \tag{4.47}$$

by Lemma 2 of [23] we get (4.43).
Now let us write down the Pohozaev identity for u_n and v_n (see [31])

$$\frac{1}{2}\int\limits_{\partial\Omega}(x\cdot\nu)\Big(\frac{\partial u_n}{\partial\nu}\Big)^2 = \Big(\frac{N}{p_n+1}-\frac{N-2}{2}\Big)\int\limits_\Omega u_n^{p_n+1} \tag{4.48}$$

$$\frac{1}{2}\int\limits_{\partial\Omega}(x\cdot\nu)\Big(\frac{\partial v_n}{\partial\nu}\Big)^2 = \Big(\frac{N}{p_n+1}-\frac{N-2}{2}\Big)\int\limits_\Omega v_n^{p_n+1} \tag{4.49}$$

By (4.48) and (4.49) we obtain

$$\frac{1}{2}\int\limits_{\partial\Omega}(x\cdot\nu)\frac{\partial}{\partial\nu}\Big(\|u_n\|_\infty^2\frac{u_n(x)-v_n(x)}{\|u_n-v_n\|_\infty}\Big)\frac{\partial}{\partial\nu}[\|u_n\|_\infty(u_n+v_n)] =$$

$$= \frac{\frac{N+2}{N-2}-p_n}{2(N-2)(p_n+1)}\|u_n\|_\infty^2\int\limits_\Omega h_n(x)\|u_n\|_\infty\frac{u_n(x)-v_n(x)}{\|u_n-v_n\|_\infty} \tag{4.50}$$

with $h_n(x) = (p_n+1)\int\limits_0^1(tu_n(x)+(1-t)v_n(x))^{p_n}dx$. Then from (4.26), (4.29) and (4.42) we get

$$C\int\limits_{\partial\Omega}(x\cdot\nu)\Big(\frac{\partial G(x,0)}{\partial\nu}\Big)^2 + o(1) = C\int\limits_\Omega h_n(x)\|u_n\|_\infty\frac{u_n(x)-v_n(x)}{\|u_n-v_n\|_\infty} + o(1)$$

$$= \frac{C}{\|u_n\|_\infty^{\frac{p_n-1}{2}N-(p_n+1)}} \int_{\|u_n\|_\infty^{\frac{p_n-1}{2}}\Omega} \int_0^1 \Big(\frac{t}{\|u_n\|_\infty} u_n\big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\big)$$

$$+\frac{(1-t)}{\|u_n\|_\infty} v_n\big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\big)\Big)^{p_n} w_n(x) + o(1)$$

$$= C\int_{\mathcal{R}^N} \frac{1-|x|^2}{(1+|x|^2)^{N+1}} + o(1) = C\int_0^1 \frac{1-\rho^2}{(1+\rho^2)^{N+1}}\rho^{N-1}$$

$$+C\int_1^\infty \frac{1-\rho^2}{(1+\rho^2)^{N+1}}\rho^{N-1} + o(1) = C\int_1^\infty \frac{1-\frac{1}{\rho^2}}{(1+\frac{1}{\rho^2})^{N+1}}\frac{1}{\rho^{N+1}}$$

$$+C\int_1^\infty \frac{1-\rho^2}{(1+\rho^2)^{N+1}}\rho^{N-1} + o(1) = o(1) \tag{4.51}$$

again by the dominate convergence theorem. Finally, since (see [8] or [23])

$$\int_{\partial\Omega} (x\cdot\nu)\big(\frac{\partial G(x,0)}{\partial\nu}\big)^2 = (2-N)g(0,0) > 0 \tag{4.52}$$

we have a contradiction. This proves iii) and hence the uniqueness result .

5. The Nonradial Setting: The Critical Case.. In this section we consider uniqueness and nondegeneracy results for a classical elliptic equation involving critical Sobolev exponent, namely

$$\begin{cases} -\Delta u = N(N-2)u^p + \varepsilon u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where $p = \frac{N+2}{N-2}$. It is well known by the theorem of Brezis and Nirenberg (see [7]) that if $N \geq 4$ and for $0 < \varepsilon < \lambda_1$, there exists a solution of (5.1) while, if $\varepsilon = 0$ and the domain is starshaped, the Pohozaev identity shows that there is not any solution. The asymptotic behaviour of the solution u_ε was studied in [33] where it was proved that, for $\varepsilon \to 0$, u_ε concentrates around a critical point of the Robin function. We recall that, if $g(x,y)$ is the regular part of the Greeen function for the laplacian with zero boundary condition, then the Robin function $\psi(x)$ is defined by

$$\psi(x) = g(x,x). \tag{5.2}$$

Converesely, in [21] it is shown that if $N \geq 5$, for any nondegenerate critical point of the function $\psi(x)$ there is only one solution u_ε of (5.1) with the property that u_ε concentrates at x_0. In the next theorem we state a more general uniqueness result

THEOREM 5.1. *Let Ω be a smooth and bounded domain of $\mathcal{R}^N$ with $N \geq 5$, such that it is symmetric with respect to the coordinate hyperplanes and convex in the x_k- directions. Let us suppose that u_ε and v_ε are two solutions of (5.1). Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,*

$$u_\varepsilon = v_\varepsilon. \tag{5.3}$$

Moreover this solution is nondegenerate.

Proof. In [9] this result was firstly proved by assuming that

$$\lim_{\varepsilon\to 0}\frac{\int_\Omega |\nabla u_\varepsilon|^2dx}{(\int_\Omega |u_\varepsilon|^{p+1})^{2/p+1}dx}=S_N, \tag{5.4}$$

Assumption (5.4) was removed in [12]. The proof of this results follow the lines of the theorem of the previous section and so we omit it. We only show the nondegeneracity. First of all... Let us denote by $\lambda_{2,n}$ the second eigenvalue of the operator $\Delta+N(N-2)p_nu_n^{p_n-1}$. Then

$$\lambda_{2,n}>0 \tag{5.5}$$

Proof. Since the solution of (2.1) is unique for n large, we can obtain it as a mountain pass solution of the following functional

$$F(u)=\int_\Omega |\nabla u_n|^2-\frac{1}{p_n+1}\int_\Omega (u_n^+)^{p_n+1} \tag{5.6}$$

Then, by a Hofer's result (see [24]), we get that $\lambda_{2,n}\geq 0$.
Now we suppose , by contradiction, that $\lambda_{2,n}=0$. Again let us set

$$\widetilde{v}_{2,n}(x)=\frac{1}{\|u_n\|_\infty}v_{2,n}\Big(\frac{x}{\|u_n\|_\infty^{\frac{p_n-1}{2}}}\Big)\quad,\quad \widetilde{v}_{2,n}:\Omega_n\to\mathcal{R} \tag{5.7}$$

and

$$w_{2,n}=\frac{\widetilde{v}_{2,n}}{\|\widetilde{v}_{2,n}\|_\infty} \tag{5.8}$$

We get that $w_{2,n}$ satisfies

$$\begin{cases} -\Delta w_{2,n}=N(N-2)p_n\widetilde{u}_n^{p_n-1}w_{2,n} & \text{in } \Omega_n\\ w_{2,n}=0 & \text{on } \partial\Omega.\end{cases} \tag{5.9}$$

So w_n converges to function w uniformly on compact set of $\mathcal{R}^N$, where
i) $w=0$,
ii) $w=\frac{\partial U}{\partial x_i}$, $i=1,..,N$ where $U(x)=\big(\frac{1}{1+|x|^2}\big)^{\frac{N-2}{2}}$,
or
iii) $w=\frac{1-|x|^2}{(1+|x|^2)^{N/2}}$
As in the previous section the study of w can be reduced to the study of i), ii), iii). Case i) is treated analogously to case i) of Section 3, i.e. using the estimate

$$|w_n(x)|\leq\frac{C}{|x|^{N-2}}\qquad \text{for } x\in\Omega_n\cap\{|x|>1\} \tag{5.10}$$

whose proof is the same as that (4.42).
Case ii) cannot occur because of Theorem (3.5).
In order to avoid case iii) we follow an idea of Zhang (see [38]). Since $v_{2,n}$ is the second eigenfunction of the operator $\Delta+N(N-2)p_nu_n^{p_n-1}$ then it has two nodal zones.

Now we remark that, since $w_{2,n} \to \frac{1-|x|^2}{(1+|x|^2)^{N/2}}$ uniformly on $B(0,2)$ we get for n large and ¡1

$$\begin{cases} w_{2,n}(x) > 0 & \text{on } |x| = \frac{\delta}{2} \\ w_n(x) < 0 & \text{on } |x| = \frac{3\delta}{2}. \end{cases} \tag{5.11}$$

This implies that

$$\begin{cases} v_{2,n} > 0 & \text{on } |x| = \frac{\delta}{2\|u_n\|_\infty^{\frac{p_n-1}{2}}} \\ v_{2,n} < 0 & \text{on } |x| = \frac{3\delta}{2\|u_n\|_\infty^{\frac{p_n-1}{2}}} \end{cases} \tag{5.12}$$

Hence, since $v_{2,n}$ has two nodal zone, $\frac{\partial v_{2,n}}{\partial \nu}$ does not change sign on the boundary of Ω. Finally a contradiction follows by the identity

$$\int_{\partial\Omega} (x \cdot \nu) \frac{\partial v_{2,n}}{\partial \nu} \frac{\partial u_n}{\partial \nu} = 0 \tag{5.13}$$

which can be obtained by considering the function $\eta = x \cdot \nabla u_n$ which satisfies

$$-\Delta \eta = N(N-2) p_n u_n^{p_n - 1} \eta + 2 u_n^{p_n} \tag{5.14}$$

Then multiplying (4.1) by η and (4.30) by $v_{2,n}$ we obtain (5.13). □

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