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ASYMPTOTIC PROFILES OF NONSTATIONARY INCOMPRESSIBLE NAVIER-STOKES FLOWS IN THE HALF-SPACE*

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Dedicated to Professor Seiji Ukai

1. Introduction. This paper studies the behavior as $t \to \infty$ of weak and strong solutions of the nonstationary incompressible Navier-Stokes system in the half-space $D^n = \mathbb{R}^n_+$, $n \geq 2$:

$$\text{(NS)} \qquad \begin{aligned} \partial_t u + u \cdot \nabla u &= \Delta u - \nabla p \quad (x \in D^n,\ t > 0) \\ \nabla \cdot u &= 0 \qquad\qquad\quad (x \in D^n,\ t \geq 0) \\ u|_\Gamma = 0, &\qquad u|_{t=0} = a. \end{aligned}$$

Here,

$$D^n = \mathbb{R}^n_+ = \{x = (x', x_n) = (x_1, \cdots, x_n) \in \mathbb{R}^n \ :\ x_n > 0\}$$

is the upper half-space of $\mathbb{R}^n$, the boundary of which will be denoted by Γ. The functions $u = (u^1, \cdots, u^n)$ and p denote, respectively, unknown velocity and pressure; a is a given initial velocity; and

$$\partial_t = \partial/\partial t, \qquad \nabla = (\partial_1, \cdots, \partial_n), \quad \partial_j = \partial/\partial x_j \quad (j = 1, \cdots, n),$$
$$\Delta u = \sum_{j=1}^n \partial_j^2 u, \qquad u \cdot \nabla u = \sum_{j=1}^n u^j \partial_j u, \qquad \nabla \cdot u = \sum_{j=1}^n \partial_j u^j.$$

Our aim is to find asymptotic profiles of Navier-Stokes flows under some specific conditions on the initial velocities. In the previous work [4] we studied the case of flows in $\mathbb{R}^n$ under some integrability assumption on the initial velocities and deduced large-time asymptotic profiles of solutions which are described in terms of the first-order spatial derivatives (in all coordinate directions) of Gaussian-like functions, thereby extending a result of Carpio [2] with slight improvement. The result of [4] was then applied in [14] to find a characterization of flows which admit lower bounds of rates of energy decay in time. The present work extends the results of [4, 14] to the case of flows in the half-space. As will be shown below, our asymptotic expansion (given in Theorem 3.5) involves only the normal derivatives of Gaussian-like functions in contrast to the case of flows in $\mathbb{R}^n$; but the essential feature is the same. Namely, in both cases the functions describing the profiles possess the form $t^{-\frac{n+1}{2}} K(xt^{-\frac{1}{2}})$, where K stands for some specific functions which are bounded and L^q-integrable for all $1 < q < \infty$. However, it should be emphasized here that in the case of flows in $\mathbb{R}^n$ the functions K are all in L^1, while this is not always true for flows in the half-space. This suggests that the Stokes semigroup on the half-space would never be bounded in L^1. We then apply our expansion result to the analysis of the modes of energy decay of

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Navier-Stokes flows in the half-space and prove a result similar to one of [14] regarding the existence of a lower bound of rates of decay. A remarkable difference of our result (given in Corollary 3.6) from one given in [14] is that in the case of the half-space our characterization of flows admitting the lower bound involves interaction of initial velocities with solutions, which did not appear in characterizing such flows in $\mathbb{R}^n$. We should also mention that in the case of flows in $\mathbb{R}^n$, one can deduce higher-order asymptotics for a specific class of solutions. This result was deduced in [4] with the aid of the estimates for L^2-moments of velocities ([6, 21]), the L^1-estimate ([11, 12]) and the pointwise estimates ([13]) with respect to space-time variables for solutions on $\mathbb{R}^n$. We here mention that a similar expansion of higher order can be deduced also for flows in the half-space, once we establish some boundedness and decay results on L^2-moments of velocities. We further note that our main result (Theorem 3.5) exhibits no boundary effects. This is probably because we are dealing only with flows which decay very rapidly as $|x| \to \infty$.

The paper is organized as follows: In Section 2 we collect basic results regarding the Stokes semigroup in general L^p-spaces, which will be applied in the subsequent sections. We then define a class of initial velocities for which the corresponding Stokes flows decay in L^q-norm, $1 < q < \infty$, like $t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}$. We also give the first-order asymptotic expansion of Stokes flows under the same assumption on the initial data; see Theorem 2.3. All the arguments in Section 2 are based on Ukai's representation formula ([24]) for the Stokes semigroup over the half-space (see (2.1)). In fact, this formula is indispensable and will be systematically applied to obtain the results of this paper. Our main results are stated in Section 3. First we establish, in Theorems 3.1 and 3.3, the existence of weak and strong solutions of the Navier-Stokes system which decay in time like $t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}$ ($1 < q \le 2$ for weak solutions), and then assert in Proposition 3.4 that these solutions admit the first-order asymptotic expansion of the form as described above. We next assert in Theorem 3.5 that the expansion given in Proposition 3.4 can be simplified; and in Corollary 3.6 we apply our expansion result to characterizing weak solutions which admit the lower bound of rates of energy decay of the form $\|u(t)\|_2 \ge ct^{-\frac{n+2}{4}}$, extending a result of [14] to flows in the half-space. Theorem 3.5 and Corollary 3.6 will be deduced in Section 4 from Proposition 3.4, and Proposition 3.4 will be proved in Section 5.

In this paper we could deal with weak solutions only when $n \le 4$. The reason is discussed in detail at the end of Section 5.

2. Preliminaries and asymptotics for the Stokes semigroup. We first deduce a few specific properties of solutions $v = (v', v^n)$, $v' = (v^1, \cdots, v^{n-1})$, of the Stokes system

$$\text{(S)} \qquad \begin{aligned} &\partial_t v = \Delta v - \nabla p && (x \in D^n,\ t > 0) \\ &\nabla \cdot v = 0 && (x \in D^n,\ t \ge 0) \\ &v|_{\Gamma} = 0, \qquad v|_{t=0} = a. \end{aligned}$$

Consider the Helmholtz decomposition ([1]):

$$\boldsymbol{L}^r(D^n) \equiv (L^r(D^n))^n = \boldsymbol{L}^r_\sigma \oplus \boldsymbol{L}^r_\pi, \qquad 1 < r < \infty,$$

with

$$\boldsymbol{L}^r_\sigma = \{u \in \boldsymbol{L}^r(D^n) \,:\, \nabla \cdot u = 0, \quad u^n|_\Gamma = 0\},$$

$$\boldsymbol{L}^r_\pi = \{\nabla p \in \boldsymbol{L}^r(D^n) \,:\, p \in L^r_{\mathrm{loc}}(\overline{D}^n)\},$$

and let $P = P_r$ be the associated bounded projection onto $\boldsymbol{L}^r_\sigma$. Then problem (S) is written in the form

$$\partial_t v + A_r v = 0 \quad (t > 0), \qquad v(0) = a \in \boldsymbol{L}^r_\sigma, \tag{S$'$}$$

in terms of the Stokes operator

$$A = A_r = -P\Delta, \quad D(A_r) = \boldsymbol{L}^r_\sigma \cap \{u \in \boldsymbol{W}^{2,r}(D^n) \,:\, u|_\Gamma = 0\}.$$

We know (see [1]) that $-A_r$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in $\boldsymbol{L}^r_\sigma$ so that for each $a \in \boldsymbol{L}^r_\sigma$, the function $v(t) = (v', v^n) = e^{-tA}a$ gives a unique solution of (S$'$) in $\boldsymbol{L}^r_\sigma$. Ukai [24] gave the following concrete representation of the solution v:

$$v^n(t) = Ue^{-tB}[a^n - S\cdot a']\,; \qquad v'(t) = e^{-tB}[a' + Sa^n] - Sv^n. \tag{2.1}$$

Hereafter, $B = -\Delta$ denotes the Dirichlet-Laplacian on D^n ; $\{e^{-tB}\}_{t\geq 0}$ is the bounded analytic semigroup in L^p-spaces generated by $-B$; $S = (S_1, \cdots, S_{n-1})$ are the Riesz transforms on $\mathbb{R}^{n-1}$; and U is the bounded linear operator from $\boldsymbol{L}^r(D^n)$ to itself, $1 < r < \infty$, which is defined in terms of the Fourier transform on $\mathbb{R}^{n-1}$ as

$$\widehat{(Uf)}(\xi', x_n) = |\xi'| \int_0^{x_n} e^{-|\xi'|(x_n - y)} \hat{f}(\xi', y)dy. \tag{2.2}$$

For basic properties of the Riesz transforms, the reader is referred to [23]. In this paper we need only the fact that each S_j defines a bounded linear operator in $L^r(D^n)$, $1 < r < \infty$. As is well known, we have

$$e^{-tB}f = E_t * f^*\big|_{D^n}\,, \tag{2.3}$$

for a function f defined on D^n, where

$$E_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$$

is the heat kernel on $\mathbb{R}^n$ and f^* is the odd extension to $\mathbb{R}^n$ of the function f defined on D^n:

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & (x_n > 0), \\ -f(x', -x_n) & (x_n < 0). \end{cases} \tag{2.4}$$

Let $\|\cdot\|_q$, $1 \leq q \leq \infty$, denote the norm of $\boldsymbol{L}^q(D^n)$. The following are the standard L^r-L^q estimates for the Stokes semigroup.

Proposition 2.1. *There hold the estimates*

$$\|\nabla^k e^{-tA}a\|_q \leq Ct^{-\frac{k}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})}\|a\|_r \tag{2.5}$$

with $k = 0, 1, 2\ldots$, *provided either* $1 \leq r < q \leq \infty$, *or* $1 < r \leq q < \infty$.

Furthermore,

$$\|\nabla e^{-tA}a\|_r \leq Ct^{-\frac{1}{2}}\|a\|_r \qquad (r = 1, \infty). \tag{2.6}$$

Note that in (2.5) the exponents r and q may take on values 1 and ∞, respectively, although the Stokes semigroup itself seems not bounded in $\boldsymbol{L}^1$, nor in $\boldsymbol{L}^\infty$. Estimates (2.5) are proved in [1]; and estimates (2.6) are proved in [5] for $r = 1$ and in [22] for $r = \infty$, respectively.

In this paper we further need the following estimates:

PROPOSITION 2.2. *Let $a \in \boldsymbol{L}^q_\sigma$ for some $1 < q < \infty$ and*

$$\int_{D^n} (1 + y_n)|a(y)|dy < \infty. \tag{2.7}$$

Then,

$$\begin{aligned}
&\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\left(\|a\|_q + \int_{D^n} y_n|a(y)|dy\right)\\
&\|e^{-tA}a\|_r \le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\int_{D^n} y_n|a(y)|dy \qquad (1 < r \le \infty)\\
&\|\nabla e^{-tA}a\|_r \le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_{D^n} y_n|a(y)|dy \qquad (1 < r \le \infty).
\end{aligned} \tag{2.8}$$

More generally,

$$\begin{aligned}
&\|e^{-tA}a\|_r \le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\left(\int_{D^n} (y_n|a(y)|)^q dy\right)^{1/q}\\
&\|\nabla e^{-tA}a\|_r \le Ct^{-1-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\left(\int_{D^n} (y_n|a(y)|)^q dy\right)^{1/q}
\end{aligned} \tag{2.8$'$}$$

for $a \in \boldsymbol{C}^\infty_c(D^n)$ with $\nabla \cdot a = 0$, whenever $1 \le q < r \le \infty$ or $1 < q \le r < \infty$.

Proof. We use representation (2.1) for $e^{-tA}a$. It is easy to see that

$$e^{-tB}S \cdot a' = e^{t\Delta}S \cdot (a')^*,$$

where $e^{t\Delta}$ means convolution with the heat kernel on $\mathbb{R}^n$. The Fourier image of the kernel function of the convolution operator $e^{t\Delta}S$ is given by

$$e^{-t|\xi|^2}\frac{i\xi'}{|\xi'|}, \qquad \xi' = (\xi_1, \ldots, \xi_{n-1}). \tag{2.9}$$

Inserting $|\xi'|^{-1} = \pi^{-1/2}\int_0^\infty \eta^{-\frac{1}{2}}e^{-\eta|\xi'|^2}d\eta$ gives

$$e^{-t|\xi|^2}\frac{i\xi'}{|\xi'|} = \pi^{-\frac{1}{2}}i\xi'e^{-t|\xi|^2}\int_0^\infty \eta^{-\frac{1}{2}}e^{-\eta|\xi'|^2}d\eta = \pi^{-\frac{1}{2}}e^{-t\xi_n^2}\int_0^\infty \eta^{-\frac{1}{2}}i\xi'e^{-(\eta+t)|\xi'|^2}d\eta.$$

Applying the Fourier inversion formula, we get

$$e^{t\Delta}S = \pi^{-\frac{1}{2}}e^{t\partial_n^2}\int_0^\infty \eta^{-\frac{1}{2}}\nabla' e^{(\eta+t)\Delta'}d\eta$$

with $\nabla' = (\partial_1, \cdots, \partial_{n-1})$ and $\Delta' = \sum_{j=1}^{n-1}\partial_j^2$. Thus, the kernel function $F_t = (F_t^1, \cdots, F_t^{n-1})$ of $e^{t\Delta}S$ is given by

$$F_t(x) \equiv (e^{t\Delta}S)(x) = \pi^{-\frac{1}{2}}E_t(x_n)\int_0^\infty \eta^{-\frac{1}{2}}\nabla' E_{\eta+t}(x')d\eta \tag{2.10}$$

Hereafter, we use the same notation E_t to denote the heat kernels in various dimensions. Thus, for example, for $x = (x', x_n) = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we will write

$$E_t(x) = E_t(x')E_t(x_n) = E_t(x_1) \cdots E_t(x_n).$$

We have

$$\begin{aligned} |F_t(x)| &\leq CE_t(x_n) \int_0^\infty \eta^{-\frac{1}{2}}(\eta + t)^{-\frac{n}{2}} e^{-c|x'|^2/(\eta+t)} d\eta \\ &\leq Ct^{-\frac{1}{2}} \int_0^\infty \eta^{-\frac{1}{2}}(\eta + t)^{-\frac{n}{2}} d\eta \leq Ct^{-\frac{n}{2}}. \end{aligned}$$

Furthermore, for $1 < p \leq \infty$,

$$\begin{aligned} \|F_t\|_p &\leq \pi^{-\frac{1}{2}} \|E_t(\cdot)\|_p \left\| \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(\cdot) d\eta \right\|_p \\ &\leq Ct^{-\frac{1}{2}(1-\frac{1}{p})} \int_0^\infty \eta^{-\frac{1}{2}} \|\nabla' E_{\eta+t}(\cdot)\|_p d\eta \\ &\leq Ct^{-\frac{1}{2}(1-\frac{1}{p})} \int_0^\infty \eta^{-\frac{1}{2}} (\eta + t)^{-\frac{1}{2} - \frac{n-1}{2}(1-\frac{1}{p})} d\eta = Ct^{-\frac{n}{2}(1-\frac{1}{p})}. \end{aligned}$$

Note that the last integral diverges when $p = 1$. Similarly, we can show that

$$\|\partial_t^\ell \nabla^m F_t\|_p \leq Ct^{-\frac{m+2\ell}{2} - \frac{n}{2}(1-\frac{1}{p})} \qquad \text{for } 1 < p \leq \infty \text{ and } \ell,\ m = 0, 1, \ldots. \tag{2.11}$$

We now prove (2.8). In what follows integration with respect to the space variables will be performed on the whole space $\mathbb{R}^n$ unless otherwise specified. Suppose (2.7) holds. Since $\int_{-\infty}^\infty f^*(y', y_n) dy_n = 0$ for a.e. $y' \in \mathbb{R}^{n-1}$ whenever $f \in L^1(D^n)$, direct calculation gives

$$\begin{aligned} e^{t\Delta}(a^n)^* &= \int E_t(x' - y')[E_t(x_n - y_n) - E_t(x_n)](a^n)^*(y) dy \\ &= -\int_0^1 \int y_n E_t(x' - y')(\partial_n E_t)(x_n - y_n\theta)(a^n)^*(y) dy d\theta. \end{aligned} \tag{2.12}$$

So, application of Minkowski's inequality for integrals yields

$$\|e^{-tB}a^n\|_q \leq C\|E_t\|_q \|\partial_n E_t\|_q \int |y_n| \cdot |(a^n)^*(y)| dy \leq Ct^{-\frac{1}{2} - \frac{n}{2}(1-\frac{1}{q})} \int_{D^n} y_n |a(y)| dy.$$

Similarly, from

$$\begin{aligned} e^{t\Delta}S \cdot (a')^* &= \pi^{-\frac{1}{2}} \int [E_t(x_n - y_n) - E_t(x_n)] \\ &\qquad \times \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(x' - y') \cdot (a')^*(y) d\eta dy \\ &= -\pi^{-\frac{1}{2}} \int_0^1 \int \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(x' - y') \\ &\qquad \times y_n (\partial_n E_t)(x_n - y_n\theta) \cdot (a')^*(y) d\eta dy d\theta \end{aligned} \tag{2.13}$$

we get

$$\begin{aligned}\|e^{-tB}S\cdot a'\|_q &\le Ct^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{q})}\int_0^\infty \eta^{-\frac{1}{2}}(\eta+t)^{-\frac{1}{2}-\frac{n-1}{2}(1-\frac{1}{q})}d\eta\int |y_n|\cdot|a^*(y)|dy\\ &= Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_{D^n} y_n|a(y)|dy.\end{aligned}$$

Since U and S are bounded in $\boldsymbol{L}^r(D^n)$ for $1<r<\infty$, these calculations imply that

$$\|e^{-tA}a\|_q \le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_{D^n} y_n|a(y)|dy \qquad \text{for all } 1<q<\infty.$$

On the other hand, we have $\|e^{-tA}a\|_q \le C\|a\|_q$ by Proposition 2.1 ; so we obtain

$$\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\left(\|a\|_q+\int_{D^n} y_n|a(y)|dy\right).$$

The above argument and Proposition 2.1 together yield

$$\|\nabla e^{-tA}a\|_r \le Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_r \le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_{D^n} y_n|a(y)|dy \qquad \text{for all } 1<r<\infty.$$

This proves (2.8) in case $1<r<\infty$. When $r=\infty$, we apply Proposition 2.1 to get

$$\|e^{-tA}a\|_\infty \le Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_n \le Ct^{-\frac{1+n}{2}}\int_{D^n} y_n|a(y)|dy,$$

and

$$\|\nabla e^{-tA}a\|_\infty \le Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_\infty \le Ct^{-1-\frac{n}{2}}\int_{D^n} y_n|a(y)|dy.$$

To prove $(2.8)'$, let $1/q+1/p=1+1/r$. From (2.12) and (2.13), we get

$$\|e^{-tB}a^n\|_r \le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|y_n|a^n|\|_q \quad\text{and}\quad \|e^{-tB}S\cdot a'\|_r \le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|y_n|a'|\|_q,$$

respectively, as in the proof of Young's inequality for convolution. The derivatives $\nabla e^{-tA}a$ are similarly estimated and we obtain $(2.8)'$. The proof of Proposition 2.2 is complete.

We can now prove our main result in this section :

Theorem 2.3. *Let $a\in \boldsymbol{L}^q_\sigma(D^n)$ satisfy (2.7). Then, for $v=(v',v^n)=e^{-tA}a$ we get*

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\left\|v^n(t)+2U\left[(\partial_nE_t)(\cdot)\int_{D^n} y_na^n(y)dy-(\partial_nF_t)(\cdot)\cdot\int_{D^n} y_na'(y)dy\right]\right\|_q \to 0$$

$$\begin{aligned}t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\bigg\|v'(t)&+2\left[(\partial_nE_t)(\cdot)\int_{D^n} y_na'(y)dy+(\partial_nF_t)(\cdot)\int_{D^n} y_na^n(y)dy\right]\\ &-2SU\left[(\partial_nE_t)(\cdot)\int_{D^n} y_na^n(y)dy-(\partial_nF_t)(\cdot)\cdot\int_{D^n} y_na'(y)dy\right]\bigg\|_q \to 0\end{aligned}$$

as $t\to\infty$. Here, F_t is the function given in (2.10).

Proof. We rewrite (2.12) in the form

$$\begin{aligned}
e^{t\Delta}(a^n)^* =& -(\partial_n E_t)(x)\int y_n(a^n)^*(y)dy \\
&- \int_0^1\!\!\int y_n[E_t(x'-y')(\partial_n E_t)(x_n-y_n\theta)-(\partial_n E_t)(x)](a^n)^*(y)dyd\theta \\
=& -2(\partial_n E_t)(x)\int_{D^n} y_n a^n(y)dy \\
&- \int_0^1\!\!\int y_n E_t(x'-y')[(\partial_n E_t)(x_n-y_n\theta)-(\partial_n E_t)(x_n)](a^n)^*(y)dyd\theta \\
&- (\partial_n E_t)(x_n)\int y_n[E_t(x'-y')-E_t(x')](a^n)^*(y)dy.
\end{aligned}$$

Here we have used the fact that $\int_{-\infty}^{\infty} y_n(a^n)^*(y)dy_n = 2\int_0^\infty y_n a^n(y)dy_n$. We thus obtain

$$\begin{aligned}
&\left\| Ue^{-tB}a^n + 2U(\partial_n E_t)(\cdot)\int_{D^n} y_n a^n(y)dy\right\|_q \\
&\quad\le Ct^{-\frac12-\frac n2(1-\frac1q)}\int_0^1\!\!\int \|(\partial_n E_1)(\cdot - y_n t^{-\frac12}\theta)-(\partial_n E_1)(\cdot)\|_q|y_n|\cdot|(a^n)^*(y)|dyd\theta \\
&\quad\quad + Ct^{-\frac12-\frac n2(1-\frac1q)}\int \|E_1(\cdot - y't^{-\frac12})-E_1(\cdot)\|_q|y_n|\cdot|(a^n)^*(y)|dy
\end{aligned}$$

for $1<q<\infty$. But, $\|(\partial_n E_1)(\cdot - y_n t^{-\frac12}\theta)-(\partial_n E_1)(\cdot)\|_q$ and $\|E_1(\cdot - y't^{-\frac12})-E_1(\cdot)\|_q$ are bounded and

$$\lim_{t\to\infty}\|E_1(\cdot - y't^{-\frac12})-E_1(\cdot)\|_q = \lim_{t\to\infty}\|(\partial_n E_1)(\cdot - y_n t^{-\frac12}\theta)-(\partial_n E_1)(\cdot)\|_q = 0$$

for any fixed y and θ. So the dominated convergence theorem gives

$$\lim_{t\to\infty} t^{\frac12+\frac n2(1-\frac1q)}\left\| Ue^{-tB}a^n + 2U(\partial_n E_t)(\cdot)\int_{D^n} y_n a^n(y)dy\right\|_q = 0.$$

We next rewrite (2.13) in the form

$$\begin{aligned}
e^{t\Delta}S\cdot(a')^* =& -2(\partial_n F_t)(x)\cdot\int_{D^n} y_n a'(y)dy \\
&- \pi^{-\frac12}\int_0^1\!\!\int\!\!\int_0^\infty y_n\eta^{-\frac12}[(\partial_n E_t)(x_n-y_n\theta)\nabla' E_{\eta+t}(x'-y') \\
&\qquad - (\partial_n E_t)(x_n)\nabla' E_{\eta+t}(x')]\cdot(a')^*(y)d\eta dyd\theta \\
=& -2(\partial_n F_t)(x)\cdot\int_{D^n} y_n a'(y)dy \\
&- \pi^{-\frac12}\int_0^1\!\!\int\!\!\int_0^\infty y_n\eta^{-\frac12}[(\partial_n E_t)(x_n-y_n\theta)-(\partial_n E_t)(x_n)]
\end{aligned}$$

$$\times \nabla' E_{\eta+t}(x'-y') \cdot (a')^*(y) d\eta dy d\theta$$

$$-\pi^{-\frac{1}{2}}(\partial_n E_t)(x_n) \iint_0^\infty y_n \eta^{-\frac{1}{2}} \nabla' [E_{\eta+t}(x'-y') - E_{\eta+t}(x')] \cdot (a')^*(y) d\eta dy.$$

As in the foregoing calculation, we invoke

$$\lim_{t\to\infty} \|(\partial_n E_1)(\cdot - y_n t^{-\frac{1}{2}}\theta) - (\partial_n E_1)(\cdot)\|_q = 0$$

and

$$\lim_{t\to\infty} \|(\nabla' E_1)(\cdot - y'(\eta+t)^{-\frac{1}{2}}) - (\nabla' E_1)(\cdot)\|_q = 0,$$

to obtain

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| U e^{-tB} S \cdot a' + 2U(\partial_n F_t)(\cdot) \cdot \int_{D^n} y_n a'(y) dy \right\|_q = 0.$$

So

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| v^n(t) + 2U \left[(\partial_n E_t)(\cdot) \int_{D^n} y_n a^n(y) dy - (\partial_n F_t)(\cdot) \cdot \int_{D^n} y_n a'(y) dy \right] \right\|_q \to 0$$

as $t \to \infty$, and therefore

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| S v^n(t) + 2SU \left[(\partial_n E_t)(\cdot) \int_{D^n} y_n a^n(y) dy - (\partial_n F_t)(\cdot) \cdot \int_{D^n} y_n a'(y) dy \right] \right\|_q \to 0$$

as $t \to \infty$. Similarly, we can deduce

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| e^{-tB}[a' + S a^n] + 2 \left[(\partial_n E_t)(\cdot) \int_{D^n} y_n a'(y) dy + (\partial_n F_t)(\cdot) \int_{D^n} y_n a^n(y) dy \right] \right\|_q \to 0$$

as $t \to \infty$, and so

$$\begin{aligned} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| v'(t) &+ 2 \left[(\partial_n E_t)(\cdot) \int_{D^n} y_n a'(y) dy + (\partial_n F_t)(\cdot) \int_{D^n} y_n a^n(y) dy \right] \\ &- 2SU \left[(\partial_n E_t)(\cdot) \int_{D^n} y_n a^n(y) dy - (\partial_n F_t)(\cdot) \cdot \int_{D^n} y_n a'(y) dy \right] \Bigg\|_q \to 0 \end{aligned}$$

as $t \to \infty$. This completes the proof of Theorem 2.3.

Remarks. (i) The divergence-free condition for a is not invoked in the above argument. Indeed, all that we needed is the fact that if $f \in L^1(D^n)$, then

$$\int_{-\infty}^\infty f^*(y', y_n) dy_n = 0 \quad \text{and} \quad \int_{-\infty}^\infty y_n f^*(y', y_n) dy_n = 2 \int_0^\infty y_n f(y', y_n) dy_n$$

for almost every $y' \in \mathbb{R}^{n-1}$. For this reason, we need only assume (3.2) instead of the condition $\int (1+|y|)|a(y)| dy < \infty$ which was imposed in [4] for dealing with flows in $\mathbb{R}^n$.

(ii) The function in (2.9) is discontinuous at $\xi' = 0$, so the convolution operator $e^{-t\Delta} S$ is not bounded in L^1. This suggests that $\{e^{-tA}\}_{t\geq 0}$ would not define a semigroup in $\boldsymbol{L}^1$.

3. Weak and strong solutions of (NS) and statement of the main results. We write problem (NS) in the form of the integral equation

$$\text{(IE)} \qquad \begin{aligned} u(t) &= e^{-tA}a - \int_0^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds \\ &= e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds \end{aligned}$$

with $u\otimes u = (u^j u^k)_{j,k=1}^n$, and discuss the existence of weak and strong solutions with specific decay properties that are needed in proving our main result.

We first deal with the weak solutions, which are known (see [1, 10]) to exist globally in time for all $a \in \boldsymbol{L}^2_\sigma$, satisfying the identity:

$$\langle u(t),\varphi\rangle = \langle e^{-tA}a,\varphi\rangle + \int_0^t \langle u\otimes u, \nabla e^{-(t-s)A}\varphi\rangle ds \tag{3.1}$$

for all $\varphi \in \boldsymbol{C}_0^\infty(D^n)$ with $\nabla\cdot\varphi = 0$ and the energy inequality:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \le \|a\|_2^2 \qquad \text{for all } t\ge 0. \tag{E}$$

THEOREM 3.1. *Suppose $a \in \boldsymbol{L}^2_\sigma$ satisfies*

$$\int_{D^n}(1+y_n)|a(y)|dy < \infty. \tag{3.2}$$

(i) *There exists a weak solution u, which is unique in case $n=2$, such that*

$$\|u(t)\|_2 \le C(1+t)^{-\frac{n+2}{4}}. \tag{3.3}$$

Furthermore, this weak solution satisfies

$$\|u(t)\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad \textit{for all } 1<q\le 2. \tag{3.4}$$

(ii) *When $n=3,4$, the weak solution u given in* (i) *is constructed via approximate solutions $\{u_N\}$ as given in* [1, 7, 15], *which satisfy*

$$\lim_{N\to\infty}\int_0^\infty \|u_N(t)-u(t)\|_2^2 dt = 0.$$

Proof. (i) Proposition 2.2 implies

$$\|e^{-tA}a\|_2 \le C(1+t)^{-\frac{n+2}{4}},$$

so the existence of a weak solution with decay property (3.3) is deduced in exactly the same way as in [1]. Therefore, we here omit the proof of (3.3) and prove only (3.4). The assumption implies $a \in \boldsymbol{L}^q_\sigma$ for all $1<q\le 2$; so Proposition 2.2 and (3.2) together imply

$$\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad \text{for all } 1<q\le 2. \tag{3.5}$$

To estimate the nonlinear term of (IE), suppose first $1 < q < n' = n/(n-1)$. Proposition 2.1 implies

$$|\langle u\otimes u, \nabla e^{-(t-s)A}\varphi\rangle| \le \|u\|_2^2\|\nabla e^{-(t-s)A}\varphi\|_\infty \le C(t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\|u\|_2^2\|\varphi\|_{q'}.$$

Since $1/2+n(1-1/q)/2<1$ because $1<q<n'$, we get by duality, (3.1), (3.3) and (3.5),

$$\begin{aligned}\|u(t)\|_q &\le \|e^{-tA}a\|_q + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\|u\|_2^2 ds\\ &\le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}(1+s)^{-1-\frac{n}{2}}ds\\ &\le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}.\end{aligned}$$

This shows (3.4) in case $1<q<n'$. Since we know (3.3), the result is deduced in general case via interpolation. Assertion (ii) is well known and is deduced as in [15]. The proof is complete.

To deal with strong solutions, note first that (IE) can be rewritten as

$$\begin{aligned}u(t) &= e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds\\ &= e^{-tA/2}u(t/2) - \int_{t/2}^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds.\end{aligned}$$

Theorem 3.2. *Let $a\in \boldsymbol{L}^q_\sigma$ for all $1<q<\infty$. Given $1<p\le 2$, there is a number $\eta_p>0$ so that if $\|a\|_n\le\eta_p$, a unique strong solution u exists for all $t\ge 0$, satisfying $u\in BC([0,\infty):\boldsymbol{L}^q_\sigma)$ for all $p\le q<\infty$, and*

$$\begin{aligned}&\|u(t)\|_q \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\\ &\|\nabla u(t)\|_q \le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\end{aligned}\qquad \text{for all } p\le q<\infty. \tag{3.6}$$

Theorem 3.2 is proved by following the argument given in [8, 9, 12]. The proof is lengthy and delicate, and so omitted here. We invoke Theorem 3.2 to deduce

Theorem 3.3. *Let $a\in\boldsymbol{L}^1(D^n)\cap\boldsymbol{L}^q_\sigma$ for all $1<q<\infty$ and satisfy (3.2), i.e.,*

$$\int_{D^n}(1+y_n)|a(y)|dy<\infty.$$

If a is small in $\boldsymbol{L}^n_\sigma$, the strong solution u given in Theorem 3.2 satisfies

$$\begin{aligned}&\|u(t)\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\\ &\|\nabla u(t)\|_q \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}\end{aligned}\qquad \text{for all } 1<q<\infty. \tag{3.7}$$

Proof. We recall that (see [1]) whenever $1 < q < \infty$, we have

$$D(A_q^{\frac{1}{2}}) = \boldsymbol{L}_\sigma^q \cap \boldsymbol{W}_0^{1,q}(D^n) \quad \text{and} \quad \|A^{\frac{1}{2}}u\|_q \cong \|\nabla u\|_q \quad \text{(equivalent norms)}. \tag{3.8}$$

This implies via duality that the operator $A^{-\frac{1}{2}}P\nabla$ defined originally on smooth functions extends uniquely to a bounded linear operator from $\boldsymbol{L}^r(D^n)$ to $\boldsymbol{L}_\sigma^r$ for all $1 < r < \infty$.

Now observe that Proposition 2.2 gives the estimates

$$\left.\begin{aligned} &\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \\ &\|\nabla e^{-tA}a\|_q \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})} \end{aligned}\right. \qquad \text{for all } 1 < q < \infty$$

and that we have already proved (3.7) for $\|u\|_q$ in case $1 < q \le 2$, because u is a unique weak solution with $u(0) = a$ and a satisfies (3.2). So it remains to show that

$$\begin{aligned} &\|u(t)\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \quad && (2 < q < \infty) \\ &\|\nabla u(t)\|_q \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})} && (1 < q < \infty). \end{aligned} \tag{3.9}$$

But, to prove (3.9), we need only show that

$$\begin{aligned} &\|u(t)\|_q \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})} \qquad (2 < q < \infty) \\ &\|\nabla u(t)\|_q \le \begin{cases} Ct^{-1-\frac{n}{2}(1-\frac{1}{q})} & (1 < q \le 2) \\ Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} & (2 < q < \infty). \end{cases} \end{aligned} \tag{3.10}$$

Indeed, assume (3.10). Then for $q > 2$, we see that $\|e^{-tA/2}u(t/2)\|_q \le C\|u(t/2)\|_q \le C$ and

$$\|e^{-tA/2}u(t/2)\|_q \le Ct^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}\|u(t/2)\|_2 \le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}.$$

Therefore,

$$\|e^{-tA/2}u(t/2)\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}. \tag{3.11}$$

Similarly,

$$\|\nabla e^{-tA/2}u(t/2)\|_q \le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}\|u(t/2)\|_2 \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}. \tag{3.12}$$

Writing $e^{-(t-s)A}P\nabla\cdot = A^{\frac{1}{2}}e^{-(t-s)A}A^{-\frac{1}{2}}P\nabla\cdot$ and applying $\|A^{\frac{1}{2}}e^{-tA}v\|_q \le Ct^{-\frac{1}{2}}\|v\|_q$, we get

$$\begin{aligned} \left\|\int_{t/2}^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds\right\|_q &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u\|_{2q}^2 ds \\ \le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}(1+s)^{-n(1-\frac{1}{2q})}ds &\le C(1+t)^{\frac{1-n}{2}-\frac{n}{2}(1-\frac{1}{q})} \\ &\le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \end{aligned}$$

for $t > 0$, by the boundedness of $A^{-\frac{1}{2}}P\nabla\cdot$ and the fact that $n \ge 2$. This, together with (3.11), implies (3.9) for $\|u\|_q$. We further obtain

$$\begin{aligned}\left\|\int_{t/2}^t A^{\frac{1}{2}}e^{-(t-s)A}P(u\cdot\nabla u)(s)ds\right\|_q &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u\|_{2q}\|\nabla u\|_{2q}ds\\ &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{2q})}(1+s)^{-\frac{n}{2}(1-\frac{1}{2q})}ds\\ &\le Ct^{-\frac{n}{2}(1-\frac{1}{2q})}(1+t)^{-\frac{n}{2}(1-\frac{1}{2q})} \le Ct^{-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{q})} \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}.\end{aligned}$$

Combining this with (3.8) and (3.12) proves (3.9) for $\|\nabla u\|_q$ in case $q > 2$. The proof of (3.9) is thus complete.

It therefore suffices to show (3.10). Let $q > 2$. We fix $1 < p < 2$ and apply (3.7) for $\|u\|_p$ to see that, with $k = 0, 1$,

$$\|\nabla^k e^{-tA/2}u(t/2)\|_q \le Ct^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u(t/2)\|_p \le Ct^{-\frac{k+1}{2}-\frac{n}{2}(1-\frac{1}{q})}.$$

Furthermore, we use $e^{-(t-s)A}P\nabla\cdot = A^{\frac{1}{2}}e^{-(t-s)A}A^{-\frac{1}{2}}P\nabla\cdot$, to get

$$\begin{aligned}\left\|\int_{t/2}^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds\right\|_q &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u\|_{2q}^2ds\\ &\le C(1+t)^{\frac{1}{2}-n(\frac{1}{p}-\frac{1}{2q})} = C(1+t)^{\frac{1+n}{2}-\frac{n}{p}-\frac{n}{2}(1-\frac{1}{q})} \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})}\end{aligned}$$

by choosing p so that $1 < p < \frac{2n}{n+1} < 2$; and

$$\begin{aligned}\left\|\int_{t/2}^t A^{\frac{1}{2}}e^{-(t-s)A}P(u\cdot\nabla u)(s)ds\right\|_q &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u\cdot\nabla u\|_q ds\\ &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u\|_{2q}\|\nabla u\|_{2q}ds \le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}-n(\frac{1}{p}-\frac{1}{2q})}ds\\ &\le Ct^{-n(\frac{1}{p}-\frac{1}{2q})} = Ct^{\frac{-2n/p+n}{2}-\frac{n}{2}(1-\frac{1}{q})} = Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\end{aligned}$$

by choosing $1 < p = \frac{2n}{n+1} < 2$. This proves (3.10) in case $q > 2$.

We finally prove (3.10) in case $1 < q \le 2$. Let $t \ge 1$. Since $2q > 2$, we can apply (3.10) with $q > 2$ to get

$$\begin{aligned}\left\|\int_{t/2}^t A^{\frac{1}{2}}e^{-(t-s)A}P(u\cdot\nabla u)(s)ds\right\|_q &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u\|_{2q}\|\nabla u\|_{2q}ds\\ &\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{n}{2}(1-\frac{1}{2q})}s^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{2q})}ds\\ &\le Ct^{-\frac{n}{2}(1-\frac{1}{2q})}(1+t)^{-\frac{n}{2}(1-\frac{1}{2q})} \le Ct^{-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{q})} \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}.\end{aligned}$$

When $0 < t < 1$, we have

$$\|\nabla u(t)\|_q \le Ct^{-\frac{1}{2}} = Ct^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \times t^{-1-\frac{n}{2}(1-\frac{1}{q})} \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}.$$

This completes the proof of (3.10) ; so Theorem 3.3 is proved.

To state and prove our main result, we need some specific functions of (x,t) which will describe the profiles of general solutions as $t \to \infty$. Recall that we are using one and the same notation E_t to denote simultaneously the heat kernel of one space variable and several space variables.

The following is the complete list of necessary functions.

$$F_t(x) = \pi^{-\frac{1}{2}} E_t(x_n) \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(x') d\eta.$$

$$E_{jk}(x,t) = \int_0^\infty \partial_j \partial_k \partial_n E_{\tau+t}(x) d\tau, \qquad j,k = 1,\ldots,n.$$

$$F_{jk}(x,t) = \int_0^\infty \partial_j \partial_k \partial_n F_{\tau+t}(x) d\tau, \qquad j,k = 1,\ldots,n.$$

$$G_{jk}(x,t) = \int_0^\infty \int_{-\infty}^\infty (\partial_j \partial_k \nabla' E_{\tau+t})(x') \operatorname{sgn}(z_n) \times E_t(x_n - z_n) E_\tau(z_n) dz_n d\tau \qquad (j,k \le n-1).$$

$$G_{nn}(x,t) = \int_0^\infty \int_{-\infty}^\infty (\nabla' E_{\tau+t})(x') \operatorname{sgn}(z_n) \times (\partial_n E_t)(x_n - z_n)(\partial_n E_\tau)(z_n) dz_n d\tau. \tag{3.13}$$

$$H_{jk}(x,t) = -\pi^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \eta^{-\frac{1}{2}} (\partial_j \partial_k E_{\eta+\tau+t})(x') \operatorname{sgn}(z_n) \times (\partial_n E_t)(x_n - z_n)(\partial_n E_\tau)(z_n) dz_n d\eta d\tau \quad (j,k \le n-1).$$

$$H_{nn}(x,t) = \pi^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \eta^{-\frac{1}{2}} \Delta' E_{\eta+\tau+t}(x') \operatorname{sgn}(z_n) \times (\partial_n E_t)(x_n - z_n)(\partial_n E_\tau)(z_n) dz_n d\eta d\tau.$$

Here $\operatorname{sgn}(z_n) = z_n/|z_n|$ for $z_n \neq 0$ and $\operatorname{sgn}(0) = 0$. Note that the functions F_t, F_{jk} and G_{jk} are $\mathbb{R}^{n-1}$-valued functions. The expressions above are complicated, but the important fact is that all of the above functions except F_t are written in the form $K_t(x) \equiv t^{-\frac{n+1}{2}} K(xt^{-\frac{1}{2}})$ in terms of some functions K which are bounded and L^p-integrable on $\mathbb{R}^n$ for all $1 < p < \infty$ together with their derivatives. So each K_t satisfies

$$\|\partial_t^\ell \nabla^m K_t\|_q = C_{q\ell} t^{-\frac{1+m+2\ell}{2} - \frac{n}{2}(1-\frac{1}{q})} \qquad (1 < q \le \infty, \quad \ell,\ m = 0,1,2,\ldots).$$

By using the functions listed above, we can prove

Proposition 3.4. (i) *Let $u = (u', u^n)$ denote the strong solution given in Theorem 3.3. Then for all $1 < q < \infty$,*

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| & u^n(t) + 2U \left[(\partial_n E_t)(\cdot) \int_{D^n} y_n a^n(y) dy - (\partial_n F_t)(\cdot) \cdot \int_{D^n} y_n a'(y) dy \right] \\ & + 2U \left[(\partial_n E_t)(\cdot) \int_0^\infty \int_{D^n} |u^n|^2 dy ds - (\partial_n F_t)(\cdot) \cdot \int_0^\infty \int_{D^n} u^n u' dy ds \right] \\ & + 2U \left[\sum_{j,k=1}^{n-1} E_{jk}(\cdot,t) \int_0^\infty \int_{D^n} u^j u^k dy ds + E_{nn}(\cdot,t) \int_0^\infty \int_{D^n} |u^n|^2 dy ds \right] \\ & - 2U \left[\sum_{j,k=1}^{n-1} H_{jk}(\cdot,t) \int_0^\infty \int_{D^n} u^j u^k dy ds + H_{nn}(\cdot,t) \int_0^\infty \int_{D^n} |u^n|^2 dy ds \right] \Bigg\|_q = 0 \end{aligned} \tag{3.14}$$

and

(3.15)

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u'(t) + 2\left[(\partial_n F_t)(\cdot)\int_{D^n} y_n a^n(y)dy + (\partial_n E_t)(\cdot)\int_{D^n} y_n a'(y)dy\right]$$
$$- 2SU\left[(\partial_n E_t)(\cdot)\int_{D^n} y_n a^n(y)dy - (\partial_n F_t)(\cdot)\cdot\int_{D^n} y_n a'(y)dy\right]$$
$$+ 2\left[(\partial_n E_t)(\cdot)\int_0^\infty\int_{D^n} u^n u' dyds + (\partial_n F_t)(\cdot)\int_0^\infty\int_{D^n} |u^n|^2 dyds\right]$$
$$- 2SU\left[(\partial_n E_t)(\cdot)\int_0^\infty\int_{D^n} |u^n|^2 dyds - (\partial_n F_t)(\cdot)\cdot\int_0^\infty\int_{D^n} u^n u' dyds\right]$$
$$+ 2\left[\sum_{j,k=1}^{n-1} G_{jk}(\cdot,t)\int_0^\infty\int_{D^n} u^j u^k dyds + G_{nn}(\cdot,t)\int_0^\infty\int_{D^n} |u^n|^2 dyds\right]$$
$$+ 2\left[\sum_{j,k=1}^{n-1} F_{jk}(\cdot,t)\int_0^\infty\int_{D^n} u^j u^k dyds + F_{nn}(\cdot,t)\int_0^\infty\int_{D^n} |u^n|^2 dyds\right]$$
$$- 2SU\left[\sum_{j,k=1}^{n-1} E_{jk}(\cdot,t)\int_0^\infty\int_{D^n} u^j u^k dyds + E_{nn}(\cdot,t)\int_0^\infty\int_{D^n} |u^n|^2 dyds\right]$$
$$+ 2SU\left[\sum_{j,k=1}^{n-1} H_{jk}(\cdot,t)\int_0^\infty\int_{D^n} u^j u^k dyds + H_{nn}(\cdot,t)\int_0^\infty\int_{D^n} |u^n|^2 dyds\right]\Bigg\|_q = 0.$$

(ii) *The weak solutions* u *given in Theorem* 3.1 (ii) *satisfy* (3.14) *and* (3.15) *for* $1 < q \le 2$.

We prove Proposition 3.4 in Section 5. But, expansions (3.14) and (3.15) are unnecessarily complicated and contain many terms that cancel one another. In fact, they can be simplified into the following form, which is our first main result in this paper.

Theorem 3.5. (i) *For all* $1 < q < \infty$, *the strong solution* u *given in Theorem* 3.3 *satisfies*

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u^n(t) + 2U\partial_n E_t(\cdot)\int_{D^n} y_n a^n(y)dy \\ -2U\partial_n F_t(\cdot)\cdot\left(\int_{D^n} y_n a'(y)dy + \int_0^\infty\int_{D^n} u^n u'(y,s)dyds\right)\Bigg\|_q = 0 \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u'(t) + 2(\partial_n F_t(\cdot) - SU\partial_n E_t(\cdot))\int_{D^n} y_n a^n(y)dy \\ +2\partial_n E_t(\cdot)\left(\int_{D^n} y_n a'(y)dy + \int_0^\infty\int_{D^n} u^n u'(y,s)dyds\right) \\ +2SU\partial_n F_t(\cdot)\cdot\left(\int_{D^n} y_n a'(y)dy + \int_0^\infty\int_{D^n} u^n u'(y,s)dyds\right)\Bigg\|_q = 0. \end{aligned} \tag{3.17}$$

(ii) *The weak solutions* u *given in Theorem* 3.1 (ii) *satisfy* (3.16) *and* (3.17) *for* $1 < q \le 2$.

Note that (3.16) and (3.17) exhibit no boundary effects. We next apply Theorem 3.5 to a characterization of flows with the lower bound of rates of energy decay. The result below extends a result of [14] to flows in the half-space, and it is our second main result.

COROLLARY 3.6. *The weak solutions* u *given in Theorem* 3.1 (ii) *satisfy*

$$\|u(t)\|_2 \ge ct^{-\frac{n+2}{4}} \qquad \text{for large} \ \ t > 0 \tag{3.18}$$

if and only if

$$\left(\int_{D^n} y_n a'(y)dy + \int_0^\infty \int_{D^n} (u^n u')(y,s)dyds, \ \ \int_{D^n} y_n a^n(y)dy \right) \ne (0,0). \tag{3.19}$$

It should be noticed here that our characterization given in [14] for flows in $\mathbb{R}^n$ involves all of the quantities $\int y_j a^k(y)dy$ and $\int_0^\infty \int (u^j u^k)(y,s)dyds$, and this reflects the fact that no coordinate direction plays a distinguished role in describing the motion of a fluid in $\mathbb{R}^n$ which is at rest at the spatial infinity. For related results on flows in $\mathbb{R}^n$, the reader is referred to [17]–[20]. In contrast to the case of flows in $\mathbb{R}^n$, Corollary 3.6 shows that in describing the behavior of flows in the half-space a distinguished role is played by the normal components a^n and u^n and the normal derivatives $\partial_n E_t$ and $\partial_n F_t$. Moreover, the quantities $\int_0^\infty \int_{D^n} (u^j u^k)(y,s)dyds$, $j,k = 1,\dots,n-1$, and $\int_0^\infty \int_{D^n} (u^n u^n)(y,s)dyds$ do not appear in Corollary 3.6.

In the next section, we shall deduce Theorem 3.5 and Corollary 3.6 from Proposition 3.4. Proposition 3.4 will be proved in Section 5.

4. Proof of Theorem 3.5 and Corollary 3.6. We first deduce Corollary 3.6 from Theorem 3.5, and then Theorem 3.5 from Proposition 3.4.

Proof of Corollary 3.6. In view of (2.2), the functions $U\partial_n E_t$ and $U\partial_n F_t^j$, $j = 1,\cdots,n-1$, have the form $t^{-\frac{n+1}{2}} K(xt^{-\frac{1}{2}})$, so

$$\|U\partial_n E_t\|_2^2 = C_1 t^{-\frac{n+2}{2}} > 0, \qquad \|U\partial_n F_t^j\|_2^2 = C_2 t^{-\frac{n+2}{2}} > 0.$$

We easily see that $U\partial_n E_t$ is an even function of x', and $U\partial_n F_t^j$ is an odd function of x_j. Furthermore, let $j \le n-1$, $k \le n-1$ and $j \ne k$. Then $U\partial_n F_t^j$ is odd in x_j and even in x_k, while $U\partial_n F_t^k$ is odd in x_k and even in x_j. So we easily see that

$$\begin{aligned} &(U\partial_n E_t, U\partial_n F_t^j) = 0, \qquad\qquad j = 1,\dots,n-1, \\ &\|U\partial_n F_t^1\|_2^2 = \ldots = \|U\partial_n F_t^{n-1}\|_2^2, \\ &(U\partial_n F^j, U\partial_n F_t^k) = \delta_{jk} \|U\partial_n F_t^j\|_2^2, \end{aligned} \tag{4.1}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(D^n)$. Using (4.1) we see that if we set

$$\alpha = 2\int_{D^n} y_n a^n(y)dy, \quad \boldsymbol{\beta} = 2\int_{D^n} y_n a'(y)dy, \quad \boldsymbol{\gamma} = 2\int_0^\infty \int_{D^n} (u^n u')(y,s)dyds,$$

then

$$\begin{aligned} \|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2^2 &= \|U\partial_n E_t\|_2^2\alpha^2 + \|U\partial_n F_t^1\|_2^2|\boldsymbol{\beta}+\boldsymbol{\gamma}|^2 \\ &= Ct^{-\frac{n+2}{2}}. \end{aligned} \tag{4.2}$$

We shall apply (4.2) to the proof of Corollary 3.6. Firstly, suppose that $(\boldsymbol{\beta}+\boldsymbol{\gamma},\alpha) \neq (0,0)$. Then (4.2) implies

$$\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2 = Ct^{-\frac{n+2}{4}} > 0 \qquad \text{for all } t>0\,;$$

so (3.16) yields, for large $t>0$,

$$\begin{aligned} \|u^n(t)\|_2 \geq \|U\partial_n E_t(\cdot)\alpha &- U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2 \\ &- \|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2 \\ &= Ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) \geq ct^{-\frac{n+2}{4}}. \end{aligned}$$

Secondly, suppose that $\|u^n(t)\|_2 \geq ct^{-\frac{n+2}{4}}$ for large $t>0$. Then (3.16) implies

$$\begin{aligned} &\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2 \\ &\qquad \geq \|u^n(t)\|_2 - \|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2 \\ &\qquad \geq ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) > 0 \end{aligned}$$

for large $t>0$, and so we conclude that $(\boldsymbol{\beta}+\boldsymbol{\gamma},\alpha) \neq (0,0)$.

Suppose finally that

$$\liminf_{t\to\infty} t^{\frac{n+2}{4}}\|u^n(t)\|_2 = 0 \qquad \text{and} \qquad \|u(t)\|_2 \geq ct^{-\frac{n+2}{4}}. \tag{4.3}$$

In this case we invoke

$$\begin{aligned} C &= t^{\frac{n+2}{4}}\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2 \\ &\leq t^{\frac{n+2}{4}}\|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\boldsymbol{\beta}+\boldsymbol{\gamma})\|_2 + t^{\frac{n+2}{4}}\|u^n(t)\|_2. \end{aligned}$$

Passing to the limit as $t\to\infty$ and applying (3.16) and (4.3) gives $C=0$, since

$$\liminf_{t\to\infty}[f(t)+g(t)] = \lim_{t\to\infty} f(t) + \liminf_{t\to\infty} g(t).$$

This implies that $(\boldsymbol{\beta}+\boldsymbol{\gamma},\alpha) = (0,0)$, so (3.16) and (3.17) together yield

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|u(t)\|_2 = 0,$$

contradicting the assumption (4.3). Hence $\|u(t)\|_2 \geq ct^{-\frac{n+2}{4}}$ implies $\|u^n(t)\|_2 \geq ct^{-\frac{n+2}{4}}$, and so we get $(\boldsymbol{\beta}+\boldsymbol{\gamma},\alpha) \neq (0,0)$. This completes the proof of Corollary 3.6.

Proof of Theorem 3.5. Let

$$c^{jk} = \int_0^\infty\int_{D^n}(u^j u^k)(y,s)\,dy\,ds,$$

where $1 \le j \le n-1$ and $1 \le k \le n-1$, or $j = k = n$. By (3.14) and (3.15), it suffices to show that

$$\partial_n E_t c^{nn} + E_{jk} c^{jk} + E_{nn} c^{nn} - (H_{jk} c^{jk} + H_{nn} c^{nn}) = 0 \tag{4.4}$$

and

$$\partial_n F_t c^{nn} + G_{jk} c^{jk} + G_{nn} c^{nn} + F_{jk} c^{jk} + F_{nn} c^{nn} = 0. \tag{4.5}$$

Here, and in what follows, we will employ the summation convention for repeated indices with respect to $1 \le j \le n-1$ and $1 \le k \le n-1$. Now we apply the Fourier transform with respect to $x' \in \mathbb{R}^{n-1}$ to the left-hand side of (4.4), to get

$$\begin{aligned} & e^{-t|\xi'|^2} \Bigg(c^{nn} \partial_n E_t(x_n) - c^{jk} \xi_j \xi_k \int_0^\infty e^{-\tau|\xi'|^2} \partial_n E_{\tau+t}(x_n) d\tau \\ & \qquad\qquad + c^{nn} \int_0^\infty e^{-\tau|\xi'|^2} \partial_n^3 E_{\tau+t}(x_n) d\tau \Bigg) \\ & - e^{-t|\xi'|^2} \Bigg(c^{jk} \frac{\xi_j \xi_k}{|\xi'|} \int_0^\infty e^{-\tau|\xi'|^2} \int_{-\infty}^\infty \operatorname{sgn}(z_n) \partial_n E_t(x_n - z_n) \partial_n E_\tau(z_n) dz_n d\tau \\ & \qquad - |\xi'| c^{nn} \int_0^\infty e^{-\tau|\xi'|^2} \int_{-\infty}^\infty \operatorname{sgn}(z_n) \partial_n E_t(x_n - z_n) \partial_n E_\tau(z_n) dz_n d\tau \Bigg). \end{aligned}$$

We then multiply the above function by $|\xi'| e^{t|\xi'|^2}$, to obtain

$$\begin{aligned} & |\xi'| \Bigg(c^{nn} \partial_n E_t(x_n) - c^{jk} \xi_j \xi_k \int_0^\infty e^{-\tau|\xi'|^2} \partial_n E_{\tau+t}(x_n) d\tau \\ & \qquad\qquad + c^{nn} \int_0^\infty e^{-\tau|\xi'|^2} \partial_n^3 E_{\tau+t}(x_n) d\tau \Bigg) \\ & \quad - c^{jk} \xi_j \xi_k \int_0^\infty e^{-\tau|\xi'|^2} \int_{-\infty}^\infty \operatorname{sgn}(z_n) \partial_n E_t(x_n - z_n) \partial_n E_\tau(z_n) dz_n d\tau \\ & \quad + c^{nn} |\xi'|^2 \int_0^\infty e^{-\tau|\xi'|^2} \int_{-\infty}^\infty \operatorname{sgn}(z_n) \partial_n E_t(x_n - z_n) \partial_n E_\tau(z_n) dz_n d\tau. \end{aligned}$$

But, since $\partial_\tau E_\tau(x_n) = \partial_n^2 E_\tau(x_n)$, we have

$$\begin{aligned} \int_0^\infty e^{-\tau|\xi'|^2} \partial_n^3 E_{\tau+t}(x_n) d\tau &= \int_0^\infty e^{-\tau|\xi'|^2} \partial_\tau \partial_n E_{\tau+t}(x_n) d\tau \\ &= -\partial_n E_t(x_n) + |\xi'|^2 \int_0^\infty e^{-\tau|\xi'|^2} \partial_n E_{\tau+t}(x_n) d\tau, \end{aligned} \tag{4.6}$$

so the resulting function is written as

$$\begin{aligned} & |\xi'| (c^{nn} |\xi'|^2 - c^{jk} \xi_j \xi_k) \int_0^\infty e^{-\tau|\xi'|^2} \partial_n E_{\tau+t}(x_n) d\tau \\ & + (c^{nn} |\xi'|^2 - c^{jk} \xi_j \xi_k) \int_0^\infty e^{-\tau|\xi'|^2} \int_{-\infty}^\infty \operatorname{sgn}(z_n) \partial_n E_t(x_n - z_n) \partial_n E_\tau(z_n) dz_n d\tau. \end{aligned} \tag{4.7}$$

We regard the above function as an odd function of $x_n \in \mathbb{R}$ and apply the Fourier transform with respect to x_n. The first term of (4.7) is then transformed to

$$i \xi_n e^{-t\xi_n^2} \frac{|\xi'| (c^{nn} |\xi'|^2 - c^{jk} \xi_j \xi_k)}{|\xi'|^2 + \xi_n^2};$$

and the convolution in the second term of (4.7) has the Fourier transform

$$i\xi_n e^{-t\xi_n^2}[\widehat{\operatorname{sgn}(\cdot)\partial_n} E_\tau](\xi_n).$$

Therefore, if we show that

$$\frac{|\xi'|}{|\xi'|^2+\xi_n^2}+\int_0^\infty e^{-\tau|\xi'|^2}[\widehat{\operatorname{sgn}(\cdot)\partial_n} E_\tau](\xi_n)d\tau=0, \tag{4.8}$$

then (4.4) will be deduced irrespective of the values of c^{jk} and c^{nn}. Direct calculation gives

$$\begin{aligned}
[\widehat{\operatorname{sgn}(\cdot)\partial_n} E_\tau](\xi_n) &= \left(\int_0^\infty-\int_{-\infty}^0\right)e^{-iz_n\xi_n}\partial_n E_\tau(z_n)dz_n \\
&= 2\int_0^\infty \cos(z_n\xi_n)\partial_n E_\tau(z_n)dz_n.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^\infty e^{-\tau|\xi'|^2}[\widehat{\operatorname{sgn}(\cdot)\partial_n} E_\tau](\xi_n)d\tau &= 2\int_0^\infty \cos(z_n\xi_n)\partial_n\left[\int_0^\infty e^{-\tau|\xi'|^2}E_\tau(z_n)d\tau\right]dz_n \\
&= 2\int_0^\infty \cos(z_n\xi_n)\partial_n\left[(2\pi)^{-1}\int_{-\infty}^\infty \frac{e^{itz_n}}{|\xi'|^2+t^2}dt\right]dz_n \\
&= |\xi'|^{-1}\int_0^\infty \cos(z_n\xi_n)\partial_n e^{-|\xi'|z_n}dz_n \\
&= -\int_0^\infty \cos(z_n\xi_n)e^{-|\xi'|z_n}dz_n \\
&= -\frac{|\xi'|}{|\xi'|^2+\xi_n^2}.
\end{aligned}$$

This proves (4.8) and so (4.4) is proved.

We next prove (4.5). The Fourier transform, with respect to $x'\in\mathbb{R}^{n-1}$, of the ℓ-th component of the left-hand side is written, after dividing by $e^{-t|\xi'|^2}$, as

$$\begin{aligned}
c^{nn}\frac{i\xi_\ell}{|\xi'|}\partial_n E_t(x_n) &- i\xi_\ell c^{jk}\xi_j\xi_k\int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\operatorname{sgn}(z_n)E_t(x_n-z_n)E_\tau(z_n)dz_n d\tau \\
&+ i\xi_\ell c^{nn}\int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\operatorname{sgn}(z_n)\partial_n E_t(x_n-z_n)\partial_n E_\tau(z_n)dz_n d\tau \\
&- \frac{i\xi_\ell}{|\xi'|}c^{jk}\xi_j\xi_k\int_0^\infty e^{-\tau|\xi'|^2}\partial_n E_{\tau+t}(x_n)d\tau \\
&+ \frac{i\xi_\ell}{|\xi'|}c^{nn}\int_0^\infty e^{-\tau|\xi'|^2}\partial_n^3 E_{\tau+t}(x_n)d\tau, \qquad \ell=1,\dots,n-1.
\end{aligned}$$

We then apply (4.6) to see that the above function is written, after dividing by $i\xi_\ell$, as

$$
\begin{aligned}
-c^{jk}\xi_j\xi_k & \int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\mathrm{sgn}\,(z_n)E_t(x_n-z_n)E_\tau(z_n)dz_nd\tau \\
& + c^{nn}\int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\mathrm{sgn}\,(z_n)\partial_n E_t(x_n-z_n)\partial_n E_\tau(z_n)dz_nd\tau \\
& + |\xi'|^{-1}(c^{nn}|\xi'|^2-c^{jk}\xi_j\xi_k)\int_0^\infty e^{-\tau|\xi'|^2}\partial_n E_{\tau+t}(x_n)d\tau.
\end{aligned}
\tag{4.9}
$$

But, the first term of (4.9) is computed via integration by parts as

$$
\begin{aligned}
&= |\xi'|^{-2}c^{jk}\xi_j\xi_k\int_0^\infty \partial_\tau e^{-\tau|\xi'|^2}\left(\int_{-\infty}^\infty \mathrm{sgn}\,(z_n)E_t(x_n-z_n)E_\tau(z_n)dz_n\right)d\tau \\
&= -\,|\xi'|^{-2}c^{jk}\xi_j\xi_k\int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\mathrm{sgn}\,(z_n)E_t(x_n-z_n)\partial_\tau E_\tau(z_n)dz_nd\tau \\
&= -\,|\xi'|^{-2}c^{jk}\xi_j\xi_k\int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\mathrm{sgn}\,(z_n)E_t(x_n-z_n)\partial_n^2 E_\tau(z_n)dz_nd\tau \\
&= -\,|\xi|^{-2}c^{jk}\xi_j\xi_k\int_0^\infty e^{-\tau|\xi'|^2}\left(\left\{\int_0^\infty-\int_{-\infty}^0\right\}E_t(x_n-z_n)\partial_n^2E_\tau(z_n)dz_n\right)d\tau \\
&= -\,|\xi'|^{-2}c^{jk}\xi_j\xi_k\int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\mathrm{sgn}\,(z_n)\partial_n E_t(x_n-z_n)\partial_n E_\tau(z_n)dz_nd\tau.
\end{aligned}
$$

Here we have used $(\partial_n E_\tau)(0)=0$. So (4.9) is rewritten in the form

$$
\begin{aligned}
&|\xi'|^{-2}(c^{nn}|\xi'|^2-c^{jk}\xi_j\xi_k)\int_0^\infty\int_{-\infty}^\infty e^{-\tau|\xi'|^2}\mathrm{sgn}\,(z_n)\partial_n E_t(x_n-z_n)\partial_n E_\tau(z_n)dz_nd\tau \\
&+|\xi'|^{-1}(c^{nn}|\xi'|^2-c^{jk}\xi_j\xi_k)\int_0^\infty e^{-\tau|\xi'|^2}\partial_n E_{\tau+t}(x_n)d\tau,
\end{aligned}
$$

which is regarded as an odd function of $x_n\in\mathbb{R}$. We take the Fourier transform with respect to x_n, divide the resulting function by $i\xi_n e^{-t\xi_n^2}$ and then apply (4.8), to obtain

$$
(c^{nn}|\xi'|^2-c^{jk}\xi_j\xi_k)\left[-\,|\xi'|^{-2}\frac{|\xi'|}{|\xi'|^2+\xi_n^2}+|\xi'|^{-1}\frac{1}{|\xi'|^2+\xi_n^2}\right]=0.
$$

This proves (4.5) and the proof of Theorem 3.5 is now complete.

5. Proof of Proposition 3.4. Let u be a strong solution given in Theorem 3.3. We write the nonlinear term of (IE) as

$$
-\left(\int_0^{t/2}+\int_{t/2}^t\right)e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds.
$$

By (3.7) and the boundedness of $A^{-\frac12}P\nabla\cdot$, the second term is estimated in $\boldsymbol{L}^q_\sigma$ as

$$
\le C\int_{t/2}^t(t-s)^{-\frac12}\|u(s)\|_{2q}^2ds\le C\int_{t/2}^t(t-s)^{-\frac12}(1+s)^{-1-n(1-\frac{1}{2q})}ds=o(t^{-\frac12-\frac{n}{2}(1-\frac1q)})
$$

as $t \to \infty$. Therefore, in view of Theorem 2.3, we need only estimate the function

$$\begin{aligned} w(t) &= -\int_0^{t/2} e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s)ds \\ &= w_1(t) + w_2(t) = (w_1'(t), w_1^n(t)) + (w_2'(t), w_2^n(t)) \end{aligned}$$

where $u \otimes u = (u^j u^k)_{j,k=1}^n$ and

(5.1)

$$\begin{aligned} w_1^n(t) &= -\int_0^{t/2} Ue^{-(t-s)B}[\nabla \cdot (uu^n) - S \cdot (\nabla \cdot (uu'))](s)ds \\ w_1'(t) &= -\int_0^{t/2} e^{-(t-s)B}[(\nabla \cdot (uu') + S\nabla \cdot (uu^n))](s)ds - Sw_1^n(t) \\ w_2^n(t) &= -\int_0^{t/2} Ue^{-(t-s)B}[\partial_n N(\partial_j\partial_k(u^j u^k)) - S \cdot (\nabla' N(\partial_j\partial_k(u^j u^k))](s)ds \\ w_2'(t) &= -\int_0^{t/2} e^{-(t-s)B}[(\nabla' N(\partial_j\partial_k(u^j u^k)) + S\partial_n N(\partial_j\partial_k(u^j u^k))](s)ds - Sw_2^n(t). \end{aligned}$$

Here and in what follows, $g = Nf$ will denote the solution of the Neumann problem

$$-\Delta g = f \quad \text{in } D^n\,; \qquad \partial_n g|_\Gamma = 0.$$

Observe that since $u = 0$ on ∂D^n, we have

$$P\nabla \cdot (u \otimes u) = \partial_j(u^j u) + \nabla N(\partial_j\partial_k(u^j u^k)) \tag{5.2}$$

by using the summation convention. This explains why the operator N appears in (5.1). We note that if Q_n is the fundamental solution of $-\Delta$, then Nf equals the restriction to D^n of the function $Q_n * f_*$, with f_* the even extension of f with respect to x_n:

$$Nf = Q_n f_*|_{D^n} \equiv Q_n * f_*|_{D^n}\,, \qquad f_*(x', x_n) = \begin{cases} f(x', x_n) & (x_n > 0) \\ f(x', -x_n) & (x_n < 0). \end{cases}$$

The lemma below plays the fundamental role in proving (3.14) and (3.15).

LEMMA 5.1. *Let $x \in D^n$, $y \in \mathbb{R}^n$, $t > 0$, and consider the function*

$$K(x, y, t) = t^{-\frac{n+1}{2}} K^0(xt^{-\frac12}, yt^{-\frac12}),$$

where $K^0(\xi, \eta)$ is smooth and satisfies

$$\|\nabla_{\xi,\eta}^m K^0(\cdot, \eta)\|_q \le C_{q,m}$$

for all $m = 0, 1, 2, \ldots$, all $\eta \in \mathbb{R}^n$ and for some $1 < q \le \infty$. Then

$$\|\partial_t^\ell \nabla_{x,y}^m K(\cdot, y, t)\|_q \le C_{q,\ell,m} t^{-\frac{1+2\ell+m}{2} - \frac{n}{2}(1-\frac1q)}, \qquad \ell, m = 0, 1, 2, \ldots, \tag{5.3}$$

for all $y \in \mathbb{R}^n$. Moreover, if we set

$$\begin{aligned} I_*(x, t) &= \int_0^{t/2}\!\!\int K(x, y, t-s)(u \otimes u)_*(y, s)dyds, \\ I^*(x, t) &= \int_0^{t/2}\!\!\int K(x, y, t-s)(u \otimes u)^*(y, s)dyds, \end{aligned}$$

with u the strong solutions given in Theorem 3.3, then

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| I_*(t) - 2K(\cdot,0,t)\int_0^\infty\int_{D^n}(u\otimes u)(y,s)dyds\right\|_q = 0 \tag{5.4}$$

and

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I^*(t)\|_q = 0. \tag{5.5}$$

Proof. We here prove only (5.4), since (5.5) is proved similarly and (5.3) is directly verified. We write

$$\begin{aligned}
I_*(x,t) = {} & 2K(x,0,t)\int_0^{t/2}\int_{D^n}(u\otimes u)(y,s)dyds \\
& + \int_0^{t/2}\int [K(x,y,t-s)-K(x,0,t)](u\otimes u)_*(y,s)dyds \\
= {} & 2K(x,0,t)\int_0^{\infty}\int_{D^n}(u\otimes u)(y,s)dyds - K(x,0,t)\int_{t/2}^{\infty}\int (u\otimes u)_*(y,s)dyds \\
& + \int_0^{t/2}\int [K(x,y,t-s)-K(x,0,t-s)](u\otimes u)_*(y,s)dyds \\
& + \int_0^{t/2}\int [K(x,0,t-s)-K(x,0,t)](u\otimes u)_*(y,s)dyds \\
\equiv {} & 2K(x,0,t)\int_0^{\infty}\int_{D^n}(u\otimes u)(y,s)dyds + I_1 + I_2 + I_3.
\end{aligned}$$

We easily see that $\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I_1\|_q = 0$. Since $t-s \geq t/2$ if $0\leq s\leq t/2$, application of Minkowski's inequality for integrals and a change of variables gives

$$\begin{aligned}
\|I_2\|_q &\leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_0^{t/2}\int \|K^0(\cdot,y(t-s)^{-\frac{1}{2}}) - K^0(\cdot,0)\|_q|u_*(y,s)|^2dyds \\
&\equiv Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_0^{t/2}\int \varphi_t(y,s)dyds \equiv Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_0^{t/2}\psi_t(s)ds.
\end{aligned}$$

By the assumption on K^0, the function

$$\varphi_t(y,s) = \|K^0(\cdot,y(t-s)^{-\frac{1}{2}}) - K^0(\cdot,0)\|_q|u_*(y,s)|^2$$

satisfies $0\leq \varphi_t(y,s)\leq C|u_*(y,s)|^2$. Furthermore, $\lim_{t\to\infty}\varphi_t(y,s) = 0$. Indeed, from

$$K^0(x,y(t-s)^{-\frac{1}{2}}) - K^0(x,0) = y(t-s)^{-\frac{1}{2}}\cdot\int_0^1(\nabla_y K^0)(x,y(t-s)^{-\frac{1}{2}}\theta)d\theta$$

and the assumption on K^0, we obtain

$$\varphi_t(y,s)\leq |y|(t-s)^{-\frac{1}{2}}\sup_z\|\nabla_z K^0(\cdot,z)\|_q|u_*(y,s)|^2 \leq C|y|(t-s)^{-\frac{1}{2}}|u_*(y,s)|^2 \to 0$$

as $t \to \infty$, for each fixed y and s. Since $|u_*(y,s)|^2$ in integrable in $y \in \mathbb{R}^n$ for each fixed s, the dominated convergence theorem gives

$$\lim_{t\to\infty} \psi_t(s) = \lim_{t\to\infty} \int \varphi_t(y,s)dy = 0.$$

Since $\psi_t(s) \le C\|u(s)\|_2^2$, the bounded convergence theorem yields

$$\lim_{t\to\infty} \int_0^T \psi_t(s)ds = 0 \qquad \text{for each fixed } T > 0.$$

Now, given an $\varepsilon > 0$, choose $T > 0$ so that $\int_T^\infty \|u(s)\|_2^2 ds < \varepsilon$. For $t > 2T$, we have

$$\int_0^{t/2} \psi_t(s)ds \le \int_0^T \psi_t(s)ds + C\int_T^\infty \|u(s)\|_2^2 ds \le \int_0^T \psi_t(s)ds + C\varepsilon.$$

Hence, $\limsup_{t\to\infty} \int_0^{t/2} \psi_t(s)ds \le C\varepsilon$, and this proves $\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I_2\|_q = 0$. To estimate I_3, note that $K(x,0,t) - K(x,0,t-s) = -\int_0^1 s(\partial_t K)(x,0,t-s\theta)d\theta$, and so, for $0 \le s \le t/2$,

$$\|K(\cdot,0,t) - K(\cdot,0,t-s)\|_q \le C\int_0^1 s(t-s\theta)^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{q})}d\theta \le Cst^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{q})}.$$

Since $\|u(s)\|_2^2 \le C(1+s)^{-1-\frac{n}{2}}$ and $n \ge 2$, it follows that

$$\begin{aligned}\|I_3\|_q &\le Ct^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_0^{t/2} s\|u(s)\|_2^2 ds \le Ct^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_0^{t/2}(1+s)^{-\frac{n}{2}}ds \\ &\le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \times t^{-1}\int_0^t (1+s)^{-1}ds.\end{aligned}$$

We thus conclude that $\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I_3\|_q = 0$. This proves Lemma 5.1.

Expansion of $\boldsymbol{w}_1(t)$. We first deal with $w_1^n(t)$ in (5.1). Direct calculation gives

$$\begin{aligned}&\sum_{j=1}^{n-1}\int_0^{t/2} Ue^{-(t-s)B}[\partial_j((u^j u^n) - S\cdot(u^j u'))](s)ds \\ &\qquad = \sum_{j=1}^{n-1} U\int_0^{t/2}\!\!\int [(\partial_j E_{t-s})(x-y) - (\partial_j F_{t-s})(x-y)](u^j u^n)^*(y,s)dyds.\end{aligned}$$

These integrals are of the form I^* treated in Lemma 5.1, and so $o(t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})})$ as $t \to \infty$. We next estimate

$$I_1 = -\int_0^{t/2} Ue^{-(t-s)B}[\partial_n(u^n u^n) - S\cdot\partial_n(u^n u')](s)ds \equiv I_{11} + I_{12}.$$

Since $[\partial_n(u^n u^n)]^* = \partial_n(|u^n|^2)_*$ and $[\partial_n(u^n u')]^* = \partial_n(u^n u')_*$, we easily see that

$$\begin{aligned}I_{11} &= -U\int_0^{t/2}\!\!\int (\partial_n E_{t-s})(x-y)(|u^n|^2)_*(y,s)dyds, \\ I_{12} &= U\int_0^{t/2}\!\!\int (\partial_n F_{t-s})(x-y)\cdot(u^n u')_*(y,s)dyds.\end{aligned}$$

Lemma 5.1 implies

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| I_{11} + 2U(\partial_n E_t)(\cdot)\int_0^\infty\!\!\int_{D^n} |u^n|^2(y,s)dyds \right\|_q = 0,$$
$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| I_{12} - 2U(\partial_n F_t)(\cdot)\cdot\int_0^\infty\!\!\int_{D^n} (u^n u')(y,s)dyds \right\|_q = 0,$$

and therefore,

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Big\| w_1^n(t) + 2U(\partial_n E_t)(\cdot)\int_0^\infty\!\!\int_{D^n} |u^n|^2(y,s)dyds$$
$$- 2U(\partial_n F_t)(\cdot)\cdot\int_0^\infty\!\!\int_{D^n} (u^n u')(y,s)dyds \Big\|_q = 0.$$

We next deal with $w_1'(t)$ in (5.1). Consider first the integral

$$\begin{aligned} J &= -\int_0^{t/2} e^{-(t-s)B}[\nabla\cdot(uu') + S\nabla\cdot(uu^n)](s)ds \\ &= -\sum_{j=1}^{n-1}\int_0^{t/2} e^{-(t-s)B}[\partial_j(u^j u') + S\partial_j(u^j u^n)](s)ds \\ &\quad - \int_0^{t/2} e^{-(t-s)B}[\partial_n(u^n u') + S\partial_n(u^n u^n)](s)ds \\ &\equiv J_1 + J_2. \end{aligned}$$

Direct calculation gives

$$\begin{aligned} J_1 = &-\sum_{j=1}^{n-1}\int_0^{t/2}\!\!\int (\partial_j E_{t-s})(x-y)(u^j u')^*(y,s)dyds \\ &- \sum_{j=1}^{n-1}\int_0^{t/2}\!\!\int (\partial_j F_{t-s})(x-y)(u^j u^n)^*(y,s)dyds \end{aligned}$$

and

$$\begin{aligned} J_2 = &-\int_0^{t/2}\!\!\int (\partial_n E_{t-s})(x-y)(u^n u')_*(y,s)dyds \\ &- \int_0^{t/2}\!\!\int (\partial_n F_{t-s})(x-y)(u^n u^n)_*(y,s)dyds. \end{aligned}$$

Thus, Lemma 5.1 implies $\lim\limits_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|J_1\|_q = 0$ and

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Big\| J_2 + 2(\partial_n E_t)(\cdot)\int_0^\infty\int_{D^n} (u^n u')(y,s)dyds$$
$$+ 2(\partial_n F_t)(\cdot)\int_0^\infty\int_{D^n} |u^n|^2(y,s)dyds \Big\|_q \to 0.$$

Therefore,

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Big\|w_1'(t)+2(\partial_n E_t)(\cdot)\int_0^\infty\!\!\int_{D^n}(u^n u')(y,s)dyds \\ +2[(\partial_n F_t)(\cdot)-SU(\partial_n E_t)(\cdot)]\int_0^\infty\!\!\int_{D^n}|u^n|^2(y,s)dyds \\ +2SU(\partial_n F_t)(\cdot)\cdot\int_0^\infty\!\!\int_{D^n}(u^n u')(y,s)dyds\Big\|_q\to 0.$$

We have thus proved

PROPOSITION 5.2. *The function* $(w_1'(t), w_1^n(t))$ *given in* (5.1) *satisfies*

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Big\|w_1^n(t)+2U(\partial_n E_t)(\cdot)\int_0^\infty\!\!\int_{D^n}|u^n|^2(y,s)dyds \\ -2U(\partial_n F_t)(\cdot)\cdot\int_0^\infty\!\!\int(u^n u')(y,s)dyds\Big\|_q=0$$

and

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Big\|w_1'(t)+2(\partial_n E_t)(\cdot)\int_0^\infty\!\!\int_{D^n}(u^n u')(y,s)dyds \\ +2[(\partial_n F_t)(\cdot)-SU(\partial_n E_t)(\cdot)]\int_0^\infty\!\!\int_{D^n}|u^n|^2(y,s)dyds \\ +\ 2SU(\partial_n F_t)(\cdot)\cdot\int_0^\infty\!\!\int_{D^n}(u^n u')(y,s)dyds\Big\|_q=0$$

for all $1<q<\infty$.

Expansion of $\boldsymbol{w}_2(t)$. Let Q_n be the fundamental solution of $-\Delta$. We first consider

$$K_1=-\sum_{j,k=1}^n\int_0^{t/2}Ue^{-(t-s)B}[\partial_n N(\partial_j\partial_k(u^j u^k))](s)ds.$$

Direct calculation gives

$$\begin{aligned}K_1&=-\sum_{j,k=1}^n U\int_0^{t/2}\partial_n e^{(t-s)\Delta}[N(\partial_j\partial_k(u^j u^k))]_*(s)ds\\ &=-\sum_{j,k=1}^n U\int_0^{t/2}\partial_n e^{(t-s)\Delta}Q_n[\partial_j\partial_k(u^j u^k)]_*(s)ds\\ &=-\sum_{j,k=1}^{n-1} U\int_0^{t/2}\partial_n e^{(t-s)\Delta}Q_n\partial_j\partial_k(u^j u^k)_*(s)ds\\ &\quad-2\sum_{k=1}^{n-1}U\int_0^{t/2}\partial_n e^{(t-s)\Delta}Q_n\partial_k(\partial_n(u^k u^n))_*(s)ds\end{aligned}$$

$$
\begin{aligned}
&- U\int_0^{t/2} \partial_n e^{(t-s)\Delta} Q_n[\partial_n^2(u^n u^n)]_*(s)ds \\
&= -\sum_{j,k=1}^{n-1} U\int_0^{t/2} \partial_n e^{(t-s)\Delta}(\partial_j\partial_k Q_n)(u^j u^k)_*(s)ds \\
&\quad - 2\sum_{k=1}^{n-1} U\int_0^{t/2} \partial_n e^{(t-s)\Delta}(\partial_k\partial_n Q_n)(u^k u^n)^*(s)ds \\
&\quad - U\int_0^{t/2} \partial_n e^{(t-s)\Delta}\partial_n^2 Q_n(u^n u^n)_*(s)ds.
\end{aligned}
$$

The kernel function

$$E_{jk}(x,t) = [(\partial_n E_t) * (\partial_j\partial_k Q_n)](x)$$

of the operator $\partial_n e^{t\Delta}(\partial_j\partial_k Q_n)$ has the Fourier transform

$$\frac{i\xi_n i\xi_j i\xi_k}{|\xi|^2} e^{-t|\xi|^2} = i\xi_n i\xi_j i\xi_j \int_0^\infty e^{-(\tau+t)|\xi|^2} d\tau$$

and so

$$E_{jk}(x,t) = \int_0^\infty \partial_n\partial_j\partial_k E_{\tau+t}(x)d\tau.$$

We thus obtain

$$
\begin{aligned}
K_1 = &-\sum_{j,k=1}^{n-1} U\int_0^{t/2}\!\!\int E_{jk}(x-y,t-s)(u^j u^k)_*(y,s)dyds \\
&- 2\sum_{k=1}^{n-1} U\int_0^{t/2}\!\!\int E_{kn}(x-y,t-s)(u^k u^n)^*(y,s)dyds \\
&- U\int_0^{t/2}\!\!\int E_{nn}(x-y,t-s)(u^n u^n)_*(y,s)dyds,
\end{aligned}
$$

and Lemma 5.1 gives

$$
\begin{aligned}
\lim_{t\to\infty} t^{\frac12+\frac n2(1-\frac1q)} \Bigg\| K_1 + 2U\sum_{j,k=1}^{n-1} E_{jk}(\cdot,t)\int_0^\infty\!\!\int_{D^n}(u^j u^k)(y,s)dyds \\
+ 2UE_{nn}(\cdot,t)\int_0^\infty\!\!\int_{D^n}(u^n u^n)(y,s)dyds\Bigg\|_q = 0.
\end{aligned}
\tag{5.6}
$$

We next consider

$$
\begin{aligned}
K_2 &= -\int_0^{t/2} e^{-(t-s)B} S\partial_n[N(\partial_j\partial_k(u^j u^k))](s)ds \\
&= -\int_0^{t/2} e^{(t-s)\Delta} S\partial_n[Q_n(\partial_j\partial_k(u^j u^k))]_*(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^{t/2}\!\!\int (\partial_n F_{t-s})(x-y) Q_n(\partial_j\partial_k(u^j u^k))_*(y,s)\,dyds \\
&= -\sum_{j,k=1}^{n-1}\int_0^{t/2}\!\!\int (\partial_n F_{t-s})(x-y)(\partial_j\partial_k Q_n)(u^j u^k)_*(y,s)\,dyds \\
&\quad - 2\sum_{k=1}^{n-1}\int_0^{t/2}\!\!\int (\partial_n F_{t-s})(x-y)(\partial_k\partial_n Q_n)(u^k u^n)^*(y,s)\,dyds \\
&\quad - \int_0^{t/2}\!\!\int (\partial_n F_{t-s})(x-y)(\partial_n^2 Q_n)(u^n u^n)_*(y,s)\,dyds.
\end{aligned}
$$

Denoting

$$
F_{jk}(x,t) = [\partial_n F_t * (\partial_j\partial_k Q_n)](x) = \left(\int_0^\infty \partial_n\partial_j\partial_k E_{\tau+t}\,d\tau\right) * S = \int_0^\infty \partial_n\partial_j\partial_k F_{\tau+t}\,d\tau,
$$

we can apply Lemma 5.1 to $F_{jk}(x-y,t)$ to obtain

$$
\begin{aligned}
\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| K_2 &+ 2\sum_{j,k=1}^{n-1} F_{jk}(\cdot,t)\int_0^\infty\!\!\int_{D^n}(u^j u^k)(y,s)\,dyds \\
&+ 2F_{nn}(\cdot,t)\int_0^\infty\!\!\int_{D^n}(u^n u^n)(y,s)\,dyds\Bigg\|_q = 0.
\end{aligned} \tag{5.7}
$$

Consider next the function

$$
\begin{aligned}
K_3 &= U\int_0^{t/2} e^{-(t-s)B} S\cdot\nabla'[N(\partial_j\partial_k(u^j u^k))](s)\,ds \\
&= U\int_0^{t/2} \nabla'\cdot e^{-(t-s)B} S[N(\partial_j\partial_k(u^j u^k))](s)\,ds \\
&= U\int_0^{t/2} \nabla'\cdot e^{-(t-s)B} S Q_n(\partial_j\partial_k(u^j u^k))_*(s)\,ds \\
&= U\sum_{j,k=1}^{n-1}\int_0^{t/2} \nabla'\cdot e^{-(t-s)B} S(\partial_j\partial_k Q_n)(u^j u^k)_*(s)\,ds \\
&\quad + U\int_0^{t/2} \nabla'\cdot e^{-(t-s)B} S(\partial_n^2 Q_n)(u^n u^n)_*(s)\,ds \\
&\quad + 2U\sum_{k=1}^{n-1}\int_0^{t/2} \nabla'\cdot e^{-(t-s)B} S(\partial_k\partial_n Q_n)(u^k u^n)^*(s)\,ds.
\end{aligned}
$$

The kernel function of the operator $\nabla'\cdot e^{-tB}S(\partial_j\partial_k Q_n)$ is written as

$$
\int_{z_n>0} \nabla'\cdot[F_t(x-z) - F_t(x-\hat{z})](\partial_j\partial_k Q_n)(z-y)\,dz \equiv I_1 + I_2
$$

where $x \in D^n$, $y \in \mathbb{R}^n$, and $\hat{z} = (z', -z_n)$ for $z = (z', z_n)$. Direct calculation gives

$$I_2 = -\int_{z_n<0} \nabla' \cdot F_t(x-z)(\partial_j \partial_k Q_n)(\hat{z}-y)dz.$$

Hence

$$\begin{aligned} I_1 + I_2 &= \int_{z_n>0} \nabla' \cdot F_t(x-z)(\partial_j \partial_k Q_n)(z-y)dz \\ &\quad - \int_{z_n<0} \nabla' \cdot F_t(x-z)(\partial_j \partial_k Q_n)(\hat{z}-y)dz \\ &= \int_0^\infty \int_{z_n>0} \nabla' \cdot F_t(x-z)(\partial_j \partial_k E_\tau)(z-y)dzd\tau \\ &\quad - \int_0^\infty \int_{z_n<0} \nabla' \cdot F_t(x-z)(\partial_j \partial_k E_\tau)(\hat{z}-y)dzd\tau \\ &\equiv H_{jk}^+(x,y,t) + H_{jk}^-(x,y,t). \end{aligned}$$

By scaling argument, we see that $H_{jk}^\pm$ are of the form $t^{-\frac{n+1}{2}} K^0(xt^{-\frac{1}{2}}, yt^{-\frac{1}{2}})$. To find more concrete expressions, recall that

$$F_t(x-z) = \pi^{-\frac{1}{2}} E_t(x_n - z_n) \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(x'-z')d\eta$$

and so

$$\nabla' \cdot F_t(x-z) = \pi^{-\frac{1}{2}} E_t(x_n - z_n) \int_0^\infty \eta^{-\frac{1}{2}} \Delta' E_{\eta+t}(x'-z')d\eta.$$

Let $j < n$ and $k < n$. Since $\Delta' E_{\eta+\tau+t}(x') = \partial_\tau E_{\eta+\tau+t}(x')$, and since $E_\tau(z_n) \to \delta(z_n)$ as $\tau \to 0$, integration by parts gives

$$\begin{aligned} H_{jk}^+(x,y,t) &= \pi^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \int_0^\infty \eta^{-\frac{1}{2}} \Delta' \partial_j \partial_k E_{\eta+\tau+t}(x'-y') \\ &\qquad\qquad \times E_t(x_n - z_n) E_\tau(z_n - y_n) dz_n d\eta d\tau \\ &= \pi^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \eta^{-\frac{1}{2}} \partial_\tau \partial_j \partial_k E_{\eta+\tau+t}(x'-y') \\ &\qquad\qquad \times \left(\int_0^\infty E_t(x_n - z_n) E_\tau(z_n - y_n) dz_n \right) d\tau d\eta \\ &= -\pi^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \eta^{-\frac{1}{2}} \partial_j \partial_k E_{\eta+\tau+t}(x'-y') \\ &\qquad\qquad \times \left(\int_0^\infty E_t(x_n - z_n)(\partial_n^2 E_\tau)(z_n - y_n) dz_n \right) d\tau d\eta \\ &\quad - \partial_j F_t^k(x-y) Y(y_n) \\ &= \pi^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \eta^{-\frac{1}{2}} \partial_j \partial_k E_{\eta+\tau+t}(x'-y') E_t(x_n)(\partial_n E_\tau)(-y_n) d\eta d\tau \end{aligned}$$

$$
\begin{aligned}
&-\pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty\int_0^\infty \eta^{-\frac{1}{2}} \partial_j\partial_k E_{\eta+\tau+t}(x'-y') \\
&\qquad\qquad \times (\partial_n E_t)(x_n - z_n)(\partial_n E_\tau)(z_n - y_n)\,dz_n d\eta d\tau \\
&-\partial_j F_t^k(x-y)Y(y_n) \\
&\equiv M_1^+(x,y,t) + M_2^+(x,y,t) - \partial_j F_t^k(x-y)Y(y_n),
\end{aligned}
$$

where Y is the Heaviside function. Similarly,

$$
\begin{aligned}
H_{jk}^-(x,y,t) &= \pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty \eta^{-\frac{1}{2}} \partial_j\partial_k E_{\eta+\tau+t}(x'-y')E_t(x_n)(\partial_n E_\tau)(y_n)\,d\eta d\tau \\
&\quad + \pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty\int_{-\infty}^0 \eta^{-\frac{1}{2}} \partial_j\partial_k E_{\eta+\tau+t}(x'-y') \\
&\qquad\qquad \times (\partial_n E_t)(x_n - z_n)(\partial_n E_\tau)(z_n + y_n)\,dz_n d\eta d\tau \\
&\quad + \partial_j F_t^k(x-\hat{y})Y(y_n) \\
&\equiv M_1^-(x,y,t) + M_2^-(x,y,t) + \partial_j F_t^k(x-\hat{y})Y(y_n).
\end{aligned}
$$

The functions $M_2^\pm$ have the form $t^{-\frac{n+1}{2}}K^0(xt^{-\frac{1}{2}}, yt^{-\frac{1}{2}})$ and $M_2 = M_2^+ + M_2^-$ satisfies

$$
\|\partial_t^\ell \nabla_{x,y}^m M_2(\cdot,y,t)\|_q \le C_{q,\ell,m} t^{-\frac{1+m+2\ell}{2} - \frac{n}{2}(1-\frac{1}{q})} \qquad (1<q<\infty,\ \ell,\, m = 0,1,\ldots).
$$

Furthermore, since $(u^j u^k)_*(\hat{y}) = (u^j u^k)_*(y)$,

$$
\begin{aligned}
-\int_0^{t/2}\!\!\int [\partial_j F_{t-s}^k(x-y) - \partial_j F_{t-s}^k(x-\hat{y})]Y(y_n)(u^j u^k)_*(y)\,dyds& \\
= -\int_0^{t/2}\!\!\int \partial_j F_{t-s}^k(x-y)\,\mathrm{sgn}\,(y_n)(u^j u^k)_*(y)\,dyds& \\
= -\int_0^{t/2}\!\!\int \partial_j F_{t-s}^k(x-y)(u^j u^k)^*(y)\,dyds.&
\end{aligned}
$$

Therefore, by Lemma 5.1 this term behaves in L^q as $o(t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})})$.

When $j<n$, we have

$$
\begin{aligned}
H_{jn}^+(x,y,t) &= \pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty\int_{z_n>0} \eta^{-\frac{1}{2}} \partial_j \Delta' E_{\eta+t}(x'-z')E_\tau(z'-y') \\
&\qquad\qquad \times E_t(x_n - z_n)(\partial_n E_\tau)(z_n - y_n)\,dz d\eta d\tau \\
&= \pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty\int_0^\infty \eta^{-\frac{1}{2}} \partial_j \Delta' E_{\eta+\tau+t}(x'-y') \times \\
&\qquad\qquad \times E_t(x_n - z_n)(\partial_n E_\tau)(z_n - y_n)\,dz_n d\eta d\tau.
\end{aligned}
$$

This observation implies that $H_{jn}^\pm(x,y,t)$ have the form $t^{-\frac{n+1}{2}}K^0(xt^{-\frac{1}{2}}, yt^{-\frac{1}{2}})$, and

$$
\|\partial_t^\ell \nabla_{x,y}^m H_{jn}(\cdot,y,t)\|_q \le C_{q,\ell,m} t^{-\frac{1+m+2\ell}{2} - \frac{n}{2}(1-\frac{1}{q})} \quad (1<q<\infty,\ j<n,\ \ell,\, m = 0,1,\ldots).
$$

Therefore, the contribution of $H_{jn}(x,y,t-s)$, $j<n$, is $o(t^{-\frac12-\frac n2(1-\frac1q)})$ by Lemma 5.1, since these functions are integrated against the measure $(u^ju^n)^*(y,s)dyds$.

To treat the function $H_{nn}=H_{nn}^+ + H_{nn}^-$, observe that

$$\begin{aligned}
H_{nn}^+(x,y,t) &= \pi^{-\frac12}\int_0^\infty\int_0^\infty\int_0^\infty \eta^{-\frac12}\Delta' E_{\eta+\tau+t}(x'-y') \\
&\qquad\times E_t(x_n-z_n)(\partial_n^2 E_\tau)(z_n-y_n)dz_nd\tau d\eta \\
&= -\pi^{-\frac12}\int_0^\infty\int_0^\infty \eta^{-\frac12}\Delta' E_{\eta+\tau+t}(x'-y')E_t(x_n)(\partial_n E_\tau)(-y_n)d\tau d\eta \\
&\quad+\pi^{-\frac12}\int_0^\infty\int_0^\infty\int_0^\infty \eta^{-\frac12}\Delta' E_{\eta+\tau+t}(x'-y') \\
&\qquad\times(\partial_n E_t)(x_n-z_n)(\partial_n E_\tau)(z_n-y_n)dz_nd\tau d\eta \\
&\equiv N_1^+(x,y,t)+N_2^+(x,y,t)
\end{aligned}$$

and

$$\begin{aligned}
H_{nn}^-(x,y,t) &= -\pi^{-\frac12}\int_0^\infty\int_0^\infty\int_{-\infty}^0 \eta^{-\frac12}\Delta' E_{\eta+\tau+t}(x'-y') \\
&\qquad\times E_t(x_n-z_n)(\partial_n^2 E_\tau)(z_n+y_n)dz_nd\eta d\tau \\
&= -\pi^{-\frac12}\int_0^\infty\int_0^\infty \eta^{-\frac12}\Delta' E_{\eta+\tau+t}(x'-y')E_t(x_n)(\partial_n E_\tau)(y_n)d\eta d\tau \\
&\quad-\pi^{-\frac12}\int_0^\infty\int_0^\infty\int_{-\infty}^0 \eta^{-\frac12}\Delta' E_{\eta+\tau+t}(x'-y') \\
&\qquad\times(\partial_n E_t)(x_n-z_n)(\partial_n E_\tau)(z_n+y_n)dz_nd\eta d\tau \\
&\equiv N_1^-(x,y,t)+N_2^-(x,y,t).
\end{aligned}$$

The functions $N_2^\pm$ have the form $t^{-\frac{n+1}{2}}K^0(xt^{-\frac12},yt^{-\frac12})$ and $N_2=N_2^+ + N_2^-$ satisfies

$$\|\partial_t^\ell\nabla_{x,y}^m N_2(\cdot,y,t)\|_q \le C_{q,\ell,m}t^{-\frac{1+m+2\ell}{2}-\frac n2(1-\frac1q)} \qquad (1<q<\infty,\ \ell,\, m=0,1,\ldots).$$

The functions $M_1=M_1^+ + M_1^-$ and $N_1=N_1^+ + N_1^-$ make no contribution. Indeed, we have

Lemma 5.3. *For fixed $t>0$, $x\in D^n$ and $0<s<t$, we have*

$$\int M_1(x,y,t-s)(u^nu^n)_*(y,s)dy=\int N_1(x,y,t-s)(u^nu^n)_*(y,s)dy=0.$$

Proof. For simplicity we write $t-s=\tau$. It suffices to prove

$$\int |M_1^\pm(x,y,\tau)(u^nu^n)_*(y,s)|dy<\infty, \qquad \int |N_1^\pm(x,y,\tau)(u^nu^n)_*(y,s)|dy<\infty.$$

The result then follows via Fubini's theorem, since $M_1^\pm$ and $N_1^\pm$ are odd in y_n while $(u^n u^n)_*$ is even in y_n. We estimate only N_1^+; the others are estimated similarly. Direct calculation gives

$$\begin{aligned}
|N_1^+(x,y,\tau)| &\le C\tau^{-\frac{1}{2}} \int_0^\infty\!\!\int_0^\infty \eta^{-\frac{1}{2}}\sigma^{-1}e^{-y_n^2/c\sigma}(\sigma+\eta+\tau)^{-\frac{n+1}{2}}\,d\eta d\sigma \\
&= \tau^{-\frac{n+1}{2}} \int_0^\infty\!\!\int_0^\infty \eta^{-\frac{1}{2}}\sigma^{-1}e^{-y_n^2/c\tau\sigma}(\sigma+\eta+1)^{-\frac{n+1}{2}}\,d\eta d\sigma \\
&= \tau^{-\frac{n+1}{2}} \int_0^\infty \eta^{-\frac{1}{2}} \left(\left\{ \int_0^1 + \int_1^\infty \right\} \sigma^{-1}e^{-y_n^2/c\tau\sigma}(\sigma+\eta+1)^{-\frac{n+1}{2}}\,d\sigma \right) d\eta.
\end{aligned}$$

Observe first that

$$\int_1^\infty \sigma^{-1}e^{-y_n^2/c\tau\sigma}(\sigma+\eta+1)^{-\frac{n+1}{2}}\,d\sigma \le \int_1^\infty \sigma^{-1-\varepsilon}(\eta+1)^{\varepsilon-\frac{n+1}{2}}\,d\sigma = C(\eta+1)^{\varepsilon-\frac{n+1}{2}}$$

irrespective of the size of $|y_n|$. Secondly, if $|y_n| > 1$, then

$$\begin{aligned}
\int_0^1 \sigma^{-1}e^{-y_n^2/c\tau\sigma}(\sigma+\eta+1)^{-\frac{n+1}{2}}\,d\sigma &\le \int_0^1 \sigma^{-1}e^{-1/c\tau\sigma}(\sigma+\eta+1)^{-\frac{n+1}{2}}\,d\sigma \\
&\le C\tau^\varepsilon(\eta+1)^{\varepsilon-\frac{n+1}{2}} \int_0^\infty \sigma^{-1-\varepsilon}e^{-1/c\sigma}\,d\sigma \\
&= C\tau^\varepsilon(\eta+1)^{\varepsilon-\frac{n+1}{2}}.
\end{aligned}$$

Therefore,

$$|N_1^+(x,y,\tau)| \le C(\tau^{\varepsilon-\frac{n+1}{2}} + \tau^{-\frac{n+1}{2}}) \qquad \text{whenever } |y_n| > 1.$$

When $|y_n| < 1$, we see that

$$\int_0^1 \sigma^{-1}e^{-y_n^2/c\tau\sigma}(\sigma+\eta+1)^{-\frac{n+1}{2}}\,d\sigma \le (\eta+1)^{-\frac{n+1}{2}} \int_0^1 \sigma^{-1}e^{-y_n^2/c\tau\sigma}\,d\sigma,$$

and

$$\int_0^1 \sigma^{-1}e^{-y_n^2/c\tau\sigma}d\sigma = \int_{y_n^2/c\tau}^\infty \zeta^{-1}e^{-\zeta}d\zeta = \left(\int_{y_n^2/c\tau}^{1/c\tau} + \int_{1/c\tau}^\infty \right) \zeta^{-1}e^{-\zeta}d\zeta \le C(|\log y_n| + \tau).$$

Hence,

$$|N_1^+(x,y,\tau)| \le C\tau^{-\frac{n+1}{2}}(1+\tau+|\log y_n|) \qquad \text{whenever } |y_n| < 1.$$

Since $(u^n u^n)_*$ is in L^q for all $1 < q < \infty$, and since $|\log y_n|$ is in an arbitrary L^p in $|y_n| < 1$, we conclude the desired assertion. The proof of Lemma 5.3 is complete.

We can now apply Lemma 5.1 to $M_2 + N_2$ to conclude that

$$\begin{aligned}
\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| K_3 - 2U \sum_{j,k=1}^{n-1} H_{jk}(\cdot,t) \int_0^\infty\!\!\int_{D^n} u^j u^k\,dyds \qquad \\
- 2U H_{nn}(\cdot,t) \int_0^\infty\!\!\int_{D^n} |u^n|^2\,dyds \Bigg\|_q = 0,
\end{aligned} \tag{5.8}$$

where the functions $H_{jk}(x,t)$ are those listed in (3.13).

Finally, consider

$$
\begin{aligned}
K_4 &= -\int_0^{t/2} e^{-(t-s)B}[(\nabla' N)(\partial_j\partial_k(u^j u^k))](s)ds \\
&= -\int_0^{t/2} \nabla' e^{-(t-s)B}[N(\partial_j\partial_k(u^j u^k))](s)ds \\
&= -\sum_{j,k=1}^{n-1}\int_0^{t/2} \nabla' e^{-(t-s)B}(\partial_j\partial_k Q_n)(u^j u^k)_*(s)ds \\
&\quad -2\sum_{j=1}^{n-1}\int_0^{t/2} \nabla' e^{-(t-s)B}(\partial_n\partial_j Q_n)(u^n u^j)^*(s)ds \\
&\quad -\int_0^{t/2} \nabla' e^{-(t-s)B}(\partial_n^2 Q_n)(u^n u^n)_*(s)ds.
\end{aligned}
$$

These integrals are treated in the same way as those of K_3 by using the kernel functions

$$
\begin{aligned}
G_{jk}(x,y,t) &= \int_0^\infty\int_{z_n>0} \nabla' E_t(x-z)(\partial_j\partial_k E_\tau)(z-y)dzd\tau \\
&\quad -\int_0^\infty\int_{z_n<0} \nabla' E_t(x-z)(\partial_j\partial_k E_\tau)(\hat{z}-y)dzd\tau,
\end{aligned}
$$

and we can apply Lemma 5.1 to obtain, with $G_{jk}(x,t)$ as listed in (3.13),

$$
\begin{aligned}
\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Bigg\| K_4 &+ 2\sum_{j,k=1}^{n-1} G_{jk}(\cdot,t)\int_0^\infty\int_{D^n} r i^j u^k dyds \\
&+ 2G_{nn}(\cdot,t)\int_0^\infty\int_{D^n} |u^n|^2 dyds\Bigg\|_q = 0.
\end{aligned} \tag{5.9}
$$

Combining (5.6) – (5.9), Proposition 5.2 and Theorem 2.3 proves Proposition 3.4 (i).

To establish Proposition 3.4 (ii), we again invoke the splitting

$$
-\left(\int_0^{t/2} + \int_{t/2}^t\right) e^{-(t-s)A} P\nabla\cdot(u\otimes u)(s)ds. \tag{5.10}
$$

To deal with the integral over $[t/2,t]$ we need another idea, since in this case $u\in \boldsymbol{L}^{2q}$ only for $1\le q\le n/(n-2)$ and we know nothing about the explicit decay rates of $\|u(t)\|_{2q}$ except when $q=1$. Therefore, we first describe how to deal with the the integral over $[t/2,t]$. In doing so, we replace u in (5.10) by the approximate solutions u_N which are obtained as in [1, 15] by solving

$$
\begin{aligned}
&u_N(t) = e^{-tA}a_N - \int_0^t e^{-(t-s)A}P\nabla\cdot(\overline{u}_N\otimes u_N)(s)ds, \\
&a_N = (I+N^{-1}A)^{-1-[\frac{n}{2}]}a, \qquad \overline{u}_N = (I+N^{-1}A)^{-1-[\frac{n}{2}]}u_N,
\end{aligned} \tag{5.11}
$$

where $[c]$ is the greatest integer in $c \in \mathbb{R}$. Proposition 2.2 implies

$$\|e^{-tA}a_N\|_2 \le \|e^{-tA}a\|_2 \le C(1+t)^{-\frac{n+2}{4}} \qquad \text{uniformly in } N,$$

and so the spectral method as developed in [1, 7, 16, 25] yields

$$\|\overline{u}_N(t)\|_2 \le \|u_N(t)\|_2 \le C(1+t)^{-\frac{n+2}{4}} \qquad \text{uniformly in } N. \tag{5.12}$$

We shall apply the same spectral method to deduce the desired convergence result:

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| \int_{t/2}^{t} e^{-(t-s)A} P(\overline{u}_N \cdot \nabla u_N)(s)ds \right\|_q = 0 \qquad (1 < q \le 2) \tag{5.13}$$

which has to be uniform in N. To this end, we define

$$v_N(t,\tau) = -\int_\tau^t e^{-(t-s)A} P(\overline{u}_N \cdot \nabla u_N)(s)ds = u_N(t) - e^{-(t-\tau)A}u_N(\tau)$$

for $0 < \tau \le t$. Since

$$\partial_t v_N + Av_N = -P(\overline{u}_N \cdot \nabla u_N) \quad (t > \tau), \qquad v_N(\tau) = 0,$$

and since $(\overline{u}_N \cdot \nabla v_N, v_N) = 0$, the standard energy method gives

$$\partial_t \|v_N\|_2^2 + 2\|A^{\frac{1}{2}}v_N\|_2^2 = -2(\overline{u}_N \cdot \nabla u_N, v_N) = -2(\overline{u}_N \cdot \nabla u_N^0, v_N) = 2(\overline{u}_N \cdot \nabla v_N, u_N^0)$$

with $u_N^0(t) = e^{-(t-\tau)A}u_N(\tau) = u_N(t) - v_N(t)$. Proposition 2.1 implies

$$\|u_N^0(t)\|_\infty \le C(t-\tau)^{-\frac{n}{4}}\|u_N(\tau)\|_2 \le C(t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{n+2}{4}}$$

and (5.12) implies $\|u_N(t)\|_2 \le C(t-\tau)^{-\frac{n+2}{4}}$, with both $C > 0$ independent of N. Thus,

$$\begin{aligned}\partial_t \|v_N\|_2^2 + 2\|A^{\frac{1}{2}}v_N\|_2^2 &\le C\|u_N\|_2 \|A^{\frac{1}{2}}v_N\|_2 \|u_N^0\|_\infty \\ &\le C\|A^{\frac{1}{2}}v_N\|_2 (t-\tau)^{-\frac{n+1}{2}}(1+\tau)^{-\frac{n+2}{4}} \\ &\le \|A^{\frac{1}{2}}v_N\|_2^2 + C(t-\tau)^{-n-1}(1+\tau)^{-1-\frac{n}{2}},\end{aligned}$$

and therefore

$$\partial_t \|v_N\|_2^2 + \|A^{\frac{1}{2}}v_N\|_2^2 \le C(t-\tau)^{-n-1}(1+\tau)^{-1-\frac{n}{2}}.$$

We apply $\|A^{\frac{1}{2}}v_N\|_2^2 \ge \varrho(\|v_N\|_2^2 - \|E_\varrho v_N\|_2^2)$, with $\{E_\varrho\}_{\varrho\ge 0}$ the spectral measure associated with the positive self-adjoint operator A in $\boldsymbol{L}^2_\sigma$, to get

$$\partial_t \|v_N\|_2^2 + \varrho\|v_N\|_2^2 \le \varrho\|E_\varrho v_N\|_2^2 + C(t-\tau)^{-n-1}(1+\tau)^{-1-\frac{n}{2}}.$$

But, $\|E_\varrho v_N\|_2^2 \le C\varrho^{\frac{n+2}{2}}(\int_\tau^t \|\overline{u}_N\|_2\|u_N\|_2 ds)^2 \le C\varrho^{\frac{n+2}{2}}(\int_\tau^t \|u_N\|_2^2 ds)^2$ as shown in [1], and so

$$\partial_t \|v_N\|_2^2 + \varrho\|v_N\|_2^2 \le C\varrho^{1+\frac{n+2}{2}}\left(\int_\tau^t \|u_N\|_2^2 ds\right)^2 + C(t-\tau)^{-n-1}(1+\tau)^{-1-\frac{n}{2}}.$$

Here we set $\varrho = m/(t-\tau)$, with large $m > 0$ and multiply both sides by $(t-\tau)^m$. Then

$$\partial_t[(t-\tau)^m \|v_N\|_2^2] \le C(t-\tau)^{m-2-\frac{n}{2}} \left(\int_\tau^t \|u_N\|_2^2 ds \right)^2 + C(t-\tau)^{m-n-1}(1+\tau)^{-1-\frac{n}{2}}.$$

Fixing $m > n+1$ and then integrating the above inequality over $[\tau, t]$, we see that

$$\begin{aligned}\|v_N(t)\|_2^2 &\le C(t-\tau)^{-m} \int_\tau^t (s-\tau)^{m-2-\frac{n}{2}} \left(\int_\tau^\infty \|u_N\|_2^2 d\sigma \right)^2 ds \\ &\quad + C(1+\tau)^{-1-\frac{n}{2}}(t-\tau)^{-n} \\ &\le C(t-\tau)^{-1-\frac{n}{2}}\tau^{-n} + C(t-\tau)^{-n}(1+\tau)^{-1-\frac{n}{2}}.\end{aligned}$$

Here we have used (5.12). Fixing $\tau = t/2$, we obtain

$$\left\| \int_{t/2}^t e^{-(t-s)A} P(\overline{u}_N \cdot \nabla u_N)(s) ds \right\|_2^2 \le C t^{-1-\frac{3n}{2}}$$

with $C > 0$ independent of N, and so

$$t^{\frac{n+2}{4}} \left\| \int_{t/2}^t e^{-(t-s)A} P(\overline{u}_N \cdot \nabla u_N)(s) ds \right\|_2 \le C t^{-\frac{n}{2}} \to 0 \qquad \text{uniformly in } N$$

as $t \to \infty$. On the other hand, as shown in the proof of Theorem 3.1, we know that

$$\|v_N(t)\|_q \le C \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \|u_N(s)\|_2^2 ds.$$

Let $1 < q < n'$; then $1/2 + n(1-1/q)/2 < 1$, and so by (5.12)

$$\begin{aligned}\|v_N(t)\|_q &\le C \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}(1+s)^{-1-\frac{n}{2}} ds \\ &= C \left(\int_\tau^{(t+\tau)/2} + \int_{(t+\tau)/2}^t \right) (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}(1+s)^{-1-\frac{n}{2}} ds \\ &\le C(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \int_\tau^t (1+s)^{-1-\frac{n}{2}} ds + C(1+\tau)^{-1-\frac{n}{2}}(t-\tau)^{\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \\ &\le C(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\tau^{-\frac{n}{2}} + C(t-\tau)^{\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\tau^{-1-\frac{n}{2}}.\end{aligned}$$

Here we fix $\tau = t/2$ to get

$$\left\| \int_{t/2}^t e^{-(t-s)A} P(\overline{u}_N \cdot \nabla u_N)(s) ds \right\|_q \le C t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})-\frac{n}{2}}$$

and therefore

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| \int_{t/2}^t e^{-(t-s)A} P(\overline{u}_N \cdot \nabla u_N)(s) ds \right\|_q \le C t^{-\frac{n}{2}} \to 0 \qquad \text{uniformly in } N$$

as $t \to \infty$, whenever $1 < q < n'$. The case of $n' \le q < 2$ is now treated via interpolation, and this proves (5.13).

It remains to deduce the uniform (in N) asymptotic expansion for the function

$$w_N(t) = -\int_0^{t/2} e^{-(t-s)A} P\nabla \cdot (\overline{u}_N \otimes u_N)(s)ds,$$

starting from (5.1) with $u \otimes u$ replaced by $\overline{u}_N \otimes u_N$. To do so, observe the following: Firstly, $(\overline{u}_N^n u_N^n)_*$ is in $L^{n/(n-2)}$ when $n \ge 3$, due to the Sobolev inequality $\|\overline{u}_N\|_{2n/(n-2)} \le C\|u_N\|_{2n/(n-2)} \le C\|\nabla u_N\|_2$; and $(u^n u^n)_*$ is in an arbitrary L^q, $1 < q < \infty$, when $n = 2$, since in this case u is also a unique strong solution. Since $|\log z_n|$ is in an arbitrary L^p in $|z_n| < 1$, $1 \le p < \infty$, the proof of Lemma 5.3 applies to our present case with no change. Secondly, the cut-off argument as given in [15] applies to our case and ensures that if $n = 3, 4$, then for each $\varepsilon > 0$ and $T > 0$, there exists $M = M_{\varepsilon,T} > 0$ satisfying

$$\int_0^T \int_{|y| \ge M} |u_N(y,s)|^2 dyds < \varepsilon \qquad \text{for all } N.$$

It thus follows from (5.12) that for each $\varepsilon > 0$ there is a constant $M' > 0$ satisfying

$$\int_0^\infty \int_{|y| \ge M'} |u_N(y,s)|^2 dyds < \varepsilon \qquad \text{for all } N. \tag{5.14}$$

Applying the standard compactness argument then implies that when $n = 3, 4$, there is a subsequence, denoted also $\{u_N\}$, such that

$$\lim_{N\to\infty} \|\overline{u}_N(t) - u(t)\|_2 = \lim_{N\to\infty} \|u_N(t) - u(t)\|_2 = 0 \qquad \text{for a.e. } t > 0, \tag{5.15}$$

and so

$$\lim_{N\to\infty} \|\overline{u}_N(t)\|_2 = \lim_{N\to\infty} \|u_N(t)\|_2 = \|u(t)\|_2 \qquad \text{for a.e. } t > 0. \tag{5.16}$$

It should be emphasized that (5.14) – (5.16) are not yet proved when $n \ge 5$, even in the case of Navier-Stokes flows in $\mathbb{R}^n$; see [8]. From (5.12), (5.14) – (5.16) and the dominated convergence theorem, we obtain

$$\lim_{N\to\infty} \int_0^\infty \int_{D^n} (\overline{u}_N^j u_N^k)(y,s)dyds = \int_0^\infty \int_{D^n} (u^j u^k)(y,s)dyds.$$

Therefore, if Lemma 5.1 holds uniformly in N with $u \otimes u$ replaced by $\overline{u}_N \otimes u_N$, then in view of Theorem 2.3, the proof of Proposition 3.4 (ii) will be complete.

It thus suffices to establish the above-mentioned uniform version of Lemma 5.1. To do so, we need only show that the functions

$$I_{2,N} = \int_0^{t/2} \int [K(x,y,t-s) - K(x,0,t-s)](\overline{u}_N \otimes u_N)_*(y,s)dyds$$

satisfy

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \|I_{2,N}\|_q = 0 \qquad \text{uniformly in } N \tag{5.17}$$

for all $1 < q \leq \infty$. As in the proof of Lemma 5.1, we have

$$\|I_{2,N}\|_q \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \int_0^{t/2}\!\!\int \varphi_{t,N}(y,s)dyds,$$

with $C > 0$ independent of N, where

$$\varphi_{t,N}(y,s) = \|K^0(\cdot, y(t-s)^{-\frac{1}{2}}) - K^0(\cdot,0)\|_q |(\overline{u}_N \otimes u_N)_*(y,s)|.$$

Since $\|K^0(\cdot, y(t-s)^{-\frac{1}{2}}) - K^0(\cdot,0)\|_q \leq C_q$ by assumption, we see from (5.14) that given an $\varepsilon > 0$, there is a constant $M > 0$ such that

$$\int_0^{t/2}\!\!\int_{|y|\geq M} \varphi_{t,N}(y,s)dyds < \varepsilon \qquad \text{for all } N \text{ and } t$$

and so

$$\limsup_{t\to\infty} \int_0^{t/2}\!\!\int_{|y|\geq M} \varphi_{t,N}(y,s)dyds \leq \varepsilon \qquad \text{uniformly in } N. \tag{5.18}$$

On the other hand, if $T > 0$, $|y| \leq M$ and $s \in [0,T]$, then the proof of Lemma 5.1 shows

$$\begin{aligned}\varphi_{t,N}(y,s) &\leq |y|(t-s)^{-\frac{1}{2}} \sup_z \|\nabla_z K^0(\cdot,z)\|_q |(\overline{u}_N \otimes u_N)_*(y,s)| \\ &\leq CM(t-T)^{-\frac{1}{2}} |(\overline{u}_N \otimes u_N)_*(y,s)|\end{aligned}$$

for $t > T$; so we see from (5.12) that, as $t \to \infty$,

$$\begin{aligned}\int_0^T\!\!\int_{|y|\leq M} \varphi_{t,N}(y,s)dyds &\leq CM(t-T)^{-\frac{1}{2}} \int_0^T\!\!\int_{|y|\leq M} |(\overline{u}_N \otimes u_N)_*(y,s)|dyds \\ &\leq CM(t-T)^{-\frac{1}{2}} \int_0^\infty \|u_N(s)\|_2^2 ds \leq C'M(t-T)^{-\frac{1}{2}} \to 0\end{aligned} \tag{5.19}$$

uniformly in N. Here we choose $T > 0$ satisfying $\int_T^\infty \|u_N(s)\|_2^2 ds < \varepsilon$ for all N, which is possible by (5.12), to see that if $t > 2T$, then

$$\begin{aligned}\int_0^{t/2}\!\!\int_{|y|\leq M} \varphi_{t,N}(y,s)dyds &\leq \int_0^T\!\!\int_{|y|\leq M} \varphi_{t,N}(y,s)dyds + C\int_T^\infty \|u_N(s)\|_2^2 ds \\ &\leq \int_0^T\!\!\int_{|y|\leq M} \varphi_{t,N}(y,s)dyds + C\varepsilon\end{aligned}$$

with $C > 0$ independent of N. This, together with (5.19), implies

$$\limsup_{t\to\infty} \int_0^{t/2}\!\!\int_{|y|\leq M} \varphi_{t,N}(y,s)dyds \leq C\varepsilon \qquad \text{uniformly in } N. \tag{5.20}$$

Since $\varepsilon > 0$ was arbitrary, combining (5.18) and (5.20) gives

$$\lim_{t\to\infty} \int_0^{t/2}\!\!\int \varphi_{t,N}(y,s)dyds = 0 \qquad \text{uniformly in } N.$$

This proves (5.17), and so the desired uniform version of Lemma 5.1 is obtained.

By the above argument and Theorem 2.3, the proof of Proposition 3.4 (ii) is complete.

Remark. In dealing with weak solutions over the half-space, we have employed (5.2) for representing the projection P, which involves the Green's function N of the homogeneous Neumann problem for the Poisson equation. This representation is valid for a strong solution u, since u is smooth up to the boundary and so satisfies $u \cdot \nabla u^n|_\Gamma = 0$. However, for a weak solution u, we do not know in general if the boundary value $u \cdot \nabla u^n|_\Gamma$ exists even in some weak sense. Therefore, we cannot directly apply (5.2) to weak solutions. To avoid this difficulty, we have employed in this paper the approximate solutions $\{u_N\}$ as introduced in [1, 15] for which (5.2) holds with $u \otimes u$ replaced by $\overline{u}_N \otimes u_N$. However, when $n \geq 5$, we do not know whether (a subsequence of) $\{u_N\}$ satisfies (5.14) – (5.16), although it is readily verified that (a subsequence of) $\{u_N\}$ converges to a weak solution in some weaker sense. For this reason, we are forced to restrict our consideration to the case when $n \leq 4$.

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