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A VARIATIONAL RESULT IN A DOMAIN WITH BOUNDARY*

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When looking for critical points of a function, which are neither minima nor maxima, the function is sometimes defined in a domain with boundary. For example, M. Struwe (see section 11 in [2]) treats functions in convex bodies. P. Majer proved a number of very nice results in [1] on Finsler manifolds with boundary. In this paper we present a rather simple result for a real function F in a bounded domain. Under some conditions on the boundary values ϕ of F we prove that F has a critical point in the domain.

For simplicity we always assume that F is in C^2. We start first with domain in $\mathbb{R}^n$, Theorem 1, while Theorem 1′ considers a domain in a Hilbert space.

THEOREM 1. *Let F be a real C^2 function in the closure $\overline{\Omega}$ of a smooth bounded domain in $\mathbb{R}^n$. Assume that*

$$\phi := F\big|_{\partial\Omega} : \partial\Omega \to \mathbb{R}$$

has only two critical values, max *and* min. *Denote by m the set where ϕ takes its minimum. Assume:*

(i) m is contractible to a point in $\overline{\Omega}$.
(ii) In some α-neighborhood on $\partial\Omega$ of m, m is not contractible to a point.

Then F has a critical point in Ω.

As one would expect, the proof relies on a deformation lemma. It is a modification of the usual one, since we are working in a bounded domain; it is close to several in [1].

LEMMA 1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let F be a real C^2 function on $\overline{\Omega}$ having no critical value in Ω in the interval $[a,b]$, $a < b$. Assume also that*

$$\phi := F\big|_{\partial\Omega} : \partial\Omega \to \mathbb{R}$$

has no critical value in $[a,b]$. Set

$$\widetilde{m} = \{\text{critical points of } F \text{ in } \Omega \text{ where } F < a\} \cup \{\text{critical points of } \phi \text{ where } \phi < a\}$$

Consider the sets

$$W = \{x \in \overline{\Omega}; F(x) \le b\}$$
$$V = \{x \in \overline{\Omega}; F(x) < a\}.$$

Then, there is a continuous deformation of W to V, keeping $\widetilde{m}$ pointwise fixed, that is, for some $T > 0$ there is a continuous map $\eta : W \times [0,T] \to \overline{\Omega}$ such that

$$\begin{cases} \eta(x,0) = x & \forall x \in W \\ \eta(x,T) \in V & \forall x \in W \\ \eta(x,t) = x & \forall x \in \widetilde{m}, \quad \forall t \in [0,T]. \end{cases} \tag{1}$$

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Proof. Near $\partial\Omega$ we make an orthogonal decomposition of ∇F:

$$\nabla F = v_1 + v_2 \tag{2}$$

where v_1 is the component of ∇F pointing normal to $\partial\Omega$. By our hypotheses, there is a neighborhood N of $\partial\Omega$ and there exists $\delta > 0$ such that

$$|v_2(x)| \geq \delta \quad \text{for} \quad x \in (W \setminus V) \cap N. \tag{3}$$

In addition, by possible reducing δ, we have

$$|\nabla F(x)| \geq \delta \quad \text{for} \quad x \in W \setminus V. \tag{4}$$

Let ζ be a real function in $C_0^\infty(\Omega)$ with $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on $\Omega \setminus N$. Consider the flow $\eta(x,t)$ for $x \in W$, $t \geq 0$ defined by

$$\frac{d\eta}{dt} = -\zeta(\eta)\,\nabla F(\eta) - (1-\zeta(\eta))\,v_2(x), \quad \eta(x,0) = x.$$

Clearly

$$\eta(z,t) = z \quad \text{for} \quad z \in \widetilde{m}, \;\; t \geq 0.$$

Note that on $\partial\Omega$, $d\eta/dt$ is tangential to $\partial\Omega$, so $\eta(x,t)$ always lies in $\overline{\Omega}$.

Next,

$$\frac{d}{dt}\,F(\eta(t)) = -\zeta(\eta)\,|\nabla F(\eta)|^2 - (1-\zeta(\eta))\,|v_2(\eta)|^2 \leq 0.$$

Thus, where $\zeta(\eta(t)) = 1$, we have by (4),

$$\frac{d}{dt}\,F(\eta) \leq -\delta^2.$$

On the other hand, if $\zeta(\eta(t)) < 1$ and if $\eta(t)$ lies in $W \setminus V$, then (3) holds and so, again,

$$\frac{d}{dt}\,F(\eta(t)) \leq -\delta^2.$$

It follows that if $x \in W$, then $\eta(x,t) \in W$ and, after time $t = T = (b-a)\,\delta^{-2}$,

$$\eta(x,t) \;\text{ lies in } V.$$

□

Proof. [Proof of Theorem 1] We may assume that the set M where ϕ achieves its maximum is different from m. Otherwise there is nothing to prove. In addition we may suppose $\phi = 1$ on M, $\phi = -1$ on m, and that

$$|F| < 1 \text{ in } \Omega.$$

For otherwise F would have a maximum or minimum in Ω and again there is nothing to prove.

Let U be an α-neighborhood of m on $\partial\Omega$, in which m is not contractible to a point. For possibly smaller α, let V_1 be the set of points in $\overline{\Omega}$ whose distance to $\partial\Omega$

is less than α and such that for each x in V_1 its projection Px to its closest point on $\partial\Omega$, lies in U. Now,

$$\text{for } 0 < 1 + a \text{ small, } V := \{x \in \overline{\Omega};\quad F(x) < a\} \text{ lies in } V_1. \tag{5}$$

By condition (i), there is a continuous map

$$R : m \times [0,1] \to \overline{\Omega}$$

such that

$$\begin{aligned} R(x,0) &= x & \forall x \in m, \\ R(x,1) &= x_0 \in \overline{\Omega} & \forall x \in m. \end{aligned} \tag{6}$$

Since we may move $R(x,t)$ a bit for $t > 0$, we may also suppose that

$$R(x,t) \in \Omega \qquad \forall x \in m,\ \forall t \in (0,1]. \tag{7}$$

In particular, we can ensure that the closed set

$$\begin{cases} K := \{R(x,t);\ x \in m,\ 0 \le t \le 1\} \\ \text{lies in } \overline{\Omega} \setminus W \text{ where } W := \{x \in \overline{\Omega};\ F(x) \le 1 - \tau\} \quad \text{for some } \tau > 0. \end{cases} \tag{8}$$

We now argue by contradiction. Suppose that F has no critical point in Ω. Using V and W as defined in (5) and (8), we apply Lemma 1 and find a deformation $\eta(\cdot,t)$ of $\overline{\Omega} \setminus W$ into V. Thus for $y \in K$, $\eta(y,T) \in V$, i.e., $\forall x \in m$, $\forall t \in [0,1]$, $(\eta(R(x,t)),T) \in V \subset V_1$. Hence

$$D(x,t) := P(\eta(R(x,t)),T) \in U$$

provides a deformation in U of m to the point $\eta(x_0,T)$—contradicting (ii). □

REMARK 1. *Condition (i) in Theorem 1 may not be dropped, as we see from the example*

$$\Omega = \{1 < |x| < 2\} \quad \textit{in } \mathbb{R}^2; \ \textit{and } F(x) = |x|.$$

REMARK 2. *The function F in Lemma 1, and hence in Theorem 1, need not be in $C^2(\Omega)$. It suffices that it is in $C^1(\overline{\Omega})$. One then uses the pseudo-derivative (see for example [2]).*

We now take up an extension of Theorem 1 and of Lemma 1 to a Hilbert space H. Here Ω is a bounded domain (open connected set) in H with smooth boundary. In $\overline{\Omega}$ we consider a real function F belonging to $C^2(\overline{\Omega})$ and satisfying the *strong* condition on its Frechet derivative F';

$$F' \text{ is uniformly continuous in some } \beta-\text{neighborhood of } \partial\Omega. \tag{9}$$

With

$$\phi := F\big|_{\partial\Omega} \to \mathbb{R},$$

we assume that

$$F \text{ satisfies } PS(\overline{\Omega}) \text{ and } \phi \text{ satisfies } PS(\partial\Omega). \tag{10}$$

Here PS means the Palais-Smale condition holds, i.e., for F, any sequence $\{x_j\} \in \overline{\Omega}$ such that $|F(x_j)|$ is bounded and $\|F'(x_j)\| \to 0$, has a strongly convergent subsequence. (Similarly for ϕ on $\partial\Omega$). See for example [2].

THEOREM 1′. *Let Ω and F be as above. Assume that ϕ has only two different critical values, its* max *and* min*; denote by M and m the sets where these are taken on. Assume*

(i) m is contractible to a point in $\overline{\Omega}$.
(ii) In some α-neighborhood on $\partial\Omega$ of m, m is not contractible to a point.

Then F has a critical point in Ω.

The proof makes use of an extension of Lemma 1:

LEMMA 1′. *Suppose that F and ϕ satisfy the conditions of Lemma 1. Then the conclusion of the lemma holds.*

Proof. The proof is the same as that of Lemma 1. But we have to ensure that (3) and (4) hold. We see first that since ϕ satisfies $PS(\partial\Omega)$, for some $\delta' > 0$,

$$\|\nabla\phi\| \geq \delta' \quad \text{on} \quad \partial\Omega \cap (W \setminus V). \tag{11}$$

From the uniform continuity of F' we infer that there is a β-neighborhood N of $\partial\Omega$ and $\exists\, \delta > 0$ such that (3) holds:

$$\|v_2(x)\| \geq \delta \quad \text{for} \quad x \in (W \setminus V) \cap N. \tag{3′}$$

In addition, by possibly reducing δ, since F satisfies $PS(\overline{\Omega})$, we find

$$\|F'(x)\| \geq \delta \quad \text{for} \quad x \in W \setminus V. \tag{4′}$$

The proof of Lemma 1′ then follows that of Lemma 1 and we consider Lemma 1′ to be proved. □

Proof. [Proof of Theorem 1′] We suppose, as before, that

$$\max\phi = 1, \qquad \min\phi = -1.$$

Then $|F| < 1$ in Ω, otherwise there is nothing to prove.

We follow the proof of Theorem 1. However, we have to ensure that (5) and (8) hold. These both follow from

LEMMA 2. *Under the conditions of Theorem* 1′, *outside of any (relatively) open neighborhood W of M, and V of m, in $\overline{\Omega}$ there exists a constant $\delta' > 0$ such that*

$$-1 + \delta' \leq F(x) \leq 1 - \delta'. \tag{12}$$

This clearly ensures (5) and (8); hence the proof of Theorem 1′ is complete once Lemma 2 is proved.

Proof. [Proof of Lemma 2] Suppose W and V are open neighborhoods of M and m in $\overline{\Omega}$. Since ϕ satisfies PS($\partial\Omega$), M and m are compact. So, for some $\beta > 0$, W and V

contain a $\beta-$neighborhood of M and m respectively. We prove the second inequality in (12) by contradiction argument. Suppose that for a sequence $\{x_j\} \subset \overline{\Omega} \setminus W$,

$$F(x_j) \to 1.$$

After passing to a subsequence (still denoted as $\{x_j\}$), there are two cases:
Case 1. For some $0 < \beta' < \beta/2$, $dist(x_j, \partial\Omega) \geq \beta'$ for all j.
Case 2. $dist(x_j, \partial\Omega) \to 0$.

In Case 1, we consider the flow

$$\frac{d\eta}{dt} = \frac{F'(\eta)}{||F'(\eta)||}, \qquad \eta(0) = x_j.$$

Let T_j be the largest number in $(0, \beta'/2]$ such that $||F'(\eta(t))|| \geq 1/j$ for all $0 \leq t \leq T_j$. Pick $0 \leq t_j \leq T_j$ with

$$||F'(\eta(t_j))|| = \min_{0 \leq t \leq T_j} ||F'(\eta(t))||.$$

Clearly,

$$F(\eta(t_j)) \to 1 \qquad \text{and} \qquad dist(F(\eta(t_j)), M) \geq \beta'/2.$$

We will show that

$$||F'(\eta(t_j))|| \to 0. \tag{13}$$

This would lead to contradiction since from the PS property for F we would find a subsequence of $F(\eta(t_j))$ converging to a point y with $F(y) = 1$ (i.e. $y \in M$) and $dist(y, M) \geq \beta'/2$. Impossible.

If $T_j < \beta'/2$, then $||F'(\eta(T_j))|| = 1/j$ and (13) follows immediately. Otherwise, $T_j = \beta'/2$, then the flow is well defined in $[0, \beta'/2]$, and

$$\int_0^{\beta'/2} ||F'(\eta(t))|| dt = F(\eta(\beta'/2)) - F(x_j) \leq 1 - F(x_j) \to 0.$$

This yields (13) as well.

In case 2, since F' is uniformly continuous near $\partial\Omega$ and ϕ satisfies $PS(\partial\Omega)$, there exist $\delta > 0$ and $0 < r < \beta/2$ such that

$$||v_2(x)|| \geq \delta > 0 \quad \forall\, ||x - x_j|| \leq r, \ \forall\, j.$$

Consider the flow

$$\frac{d\eta}{dt} = \frac{v_2(\eta)}{||v_2(\eta)||}, \qquad \eta(0) = x_j, \qquad 0 \leq t \leq r.$$

Since

$$\frac{d}{dt} F(\eta(t)) = ||v_2(\eta(t))|| \geq \delta \quad \forall\, 0 \leq t \leq r,$$

we have

$$F(\eta(r)) \geq F(x_j) + \delta r = 1 + \delta r + o(1).$$

Impossible for large j.

The right hand inequality of (12) is proved, and the other is similar.

The proofs of Lemma 2 and of Theorem 1′ are complete. □

EXAMPLE. Let Ω be a ball, $||x|| < 1$, in a Hilbert space H. Suppose H has the orthogonal decomposition

$$H = H_1 \oplus H_2 \quad \text{with} \quad \dim H_1 < \infty.$$

Let F be a C^2 function in $\overline{\Omega}$ satisfying (PS) and F' uniformly continuous in $\overline{\Omega}$.

Suppose that

$$\phi := F\big|_{\partial\Omega} = ||x_2||^2 - ||x_1||^2;$$

here $x = x_1 + x_2$, $x_1 \in H_1$, $x_2 \in H_2$. Then F has a critical point in Ω.

Proof. Condition (i) in Theorem 1′ is obvious, while condition (ii) is easily verified, since $m = \{x_1, 0\}; \ ||x_1|| = 1\}$ is a finite dimensional sphere. Finally, it is easy to see that ϕ satisfies PS on $\partial\Omega$. □

QUESTION. In the preceding example suppose that both H_1 and H_2 are infinite dimensional. Does the conclusions hold?

We suspect not.

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