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WEIGHTED ESTIMATES FOR NONSTATIONARY NAVIER–STOKES EQUATIONS IN EXTERIOR DOMAINS*

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1. Introduction. Let Ω be an exterior domain in R^3 with smooth boundary $\partial\Omega$. We consider the nonstationary Navier–Stokes equations on the space- time cylinder $\Omega \times [0, +\infty)$:

$$
\left\{
\begin{array}{lll}
\dfrac{\partial u}{\partial t} - & \nu\Delta u + (u\cdot\nabla)u = -\nabla p, & \text{in } \Omega\times(0,\infty),\\
& \mathrm{div} u = 0, & \text{in } \Omega\times(0,\infty),\\
& u = 0, & \text{on } \partial\Omega\times(0,\infty),\\
& u \longrightarrow 0, & \text{as } |x|\to+\infty,\\
& u(x,0) = a(x), & \text{in } \Omega.
\end{array}
\right.
\tag{1.1}
$$

Here $u = u(x,t) = (u_1, u_2, u_3)$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at point $(x,t) \in \Omega \times (0,\infty)$ respectively, while $\nu > 0$ is the viscosity, $a(x)$ is a given initial velocity vector field. For simplicity, let $\nu = 1$. It is well known that system (1.1) models a viscous incompressible fluid flow. For more details about the physical meaning of (1.1), see [1].

There is an extensive literature on the existence of weak solutions and strong solutions to the nonstationary Navier-Stokes equations. Hopf [12] proved the existence of a square-summable weak solution for an arbitrary square summable initial velocity $a(x)$. Later on, Galdi and Maremonti [5] constructed a class of weak solutions with second order spatial derivatives and one order time derivative with 5/4 power summmability. Other properties of weak solutions were discussed in [7]. Also see [21]. As far as the strong solutions of (1.1) is concerned, Heywood [11] proved the existence of global strong solution under the assumption that $||a||_{H^1(\Omega)}$ is small, by using a variant of the Faedo-Galerkin approximation. Applying the theories of semigroup generated by Stokes operator A and some estimates of the Stokes operator with fractional powers, Miyakawa [20] improved Heywood's result for three dimensional exterior domain only assuming $||A^{1/4}a||_{L^2(\Omega)}$ small. On the other hand, Iwashita [13] extended the results of Kato [14] to $n(\geq 3)$ dimensional exterior domain and established the existence of $L^p(p > 3)$ strong solutions, by making use of the $L^p - L^q$ estimates of semigroup generated by Stokes operator A. By completely different method in [8], he showed the existence of global strong solutions only needing $||a||_{L^2(\Omega)}$ or $||\nabla a||_{L^2(\Omega)}$ small, or ν large. For other works, see [3, 4].

However, the fundamental problems, the uniqueness of weak solutions, up to an additive constant for pressure p, and the global existence of strong solutions without

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any restriction on given initial data, still remain open. To solve such problem, some new a priori estimates, for smooth solutions, such as $\sup_{t\geq 0} \|\nabla u\|_{L^2(\Omega)}$, $\sup_{\Omega\times[0,\infty)} |u|$ or for any one of certain other quantities are needed, to serve in a continuation argument via well known existence theorems. In this paper, we will try to establish a kind of new estimates.

We will establish some new estimates in some weighted space. Then a class of strong solutions are showed in the weighted spaces for problem (1.1). Similar weighted weak solutions and strong solutions were shown in [9] for the Cauchy problem of the Navier-Stokes equations. One of the main ingredients of the analysis in [9] is to derive the weighted estimates about the pressure function p. For the Cauchy problem, the pressure function p satisfies that

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j). \tag{1.2}$$

By the fundamental solution of the Laplace's equation, the pressure function p can be expressed as

$$p(x,t) = \sum_{i,j=1}^{3} \Big(C_{i,j} u_i u_j - \frac{3}{4\pi} \int_{R^3} \frac{\partial^2}{\partial x_i \partial x_j} (\frac{1}{|x-y|}) u_i(y) u_j(y) dy \Big). \tag{1.3}$$

Then employing the singular integral theory (cf. Stein [23]), they deduced the necessary weighted estimates. For details, see [9]. But for the exterior domain, a series of new difficulties appear due to the appearance of the boundary. The main difficulty is the weighted estimates about the pressure function p. In order to overcome these difficulties, we use the singular integral expression of the operator $P : L^2(\Omega) \longrightarrow \overset{o}{J}{}^2(\Omega)$ in [17], then give the integral expressions of the approximate solutions, as do in [17]. Therefore, we obtain the weighted estimates for the solutions of (1.1), and show the existence of a weighted strong solution in the class $(1+|x|^2)^{\alpha/2}u \in L^\infty(0,+\infty; L^p(\Omega))$ and $t^{1/2}(1+|x|^2)^{\alpha/2}\nabla u \in L^\infty(0,+\infty; L^p(\Omega))$ for $7 < p \leq +\infty$ and $\alpha = 3/7 - 3/p$. It seems that our results are completely different from any one previously known for the exterior problem.

It should be to point out that α is smaller than that of the Cauchy problem. This is due to the appearance of $\bar{\theta}$, which come from the boundary. Owing to the same reason, we can not obtain the weighted estimates of weak solutions, as doing for Cauchy problem in [9]. For details, see section 5. However, Farwig and Sohr [5] showed a class of weighted ($|x|^\alpha$) weak solutions with second derivative about the spatial variables and one order derivative about time variable in $L^s(0,\infty; L^q(\Omega))$ for $1 < q < 3/2$, $1 < s < 2$ and $0 \leq 3/q + 2/s - 4 \leq \alpha < \min\{1/2, 3 - 3/q\}$.

Meanwhile, the large time behavior of weak solutions has been studied in details by Galdi and Maremonti [5], Borcher and Miyakawa [2], Maremonti [18] and Kozono, Ogawa and Sohr [15], also see [19, 21]; while that of strong solutions with small initial data has also been studied in details by Heywood [11], Miyakawa [20], Iwashita [13] and Kozono and Ogawa [16] etc. Thus this problem is well understood. But there is no results about the decay properties at large distances. For the Cauchy problem, the decay properties of weak solutions and strong solutions are implied, in some sense, by the corresponding weighted estimates in [9]; that of weak solutions for the Navier-Stokes equations in exterior domains was studied in [4]; while for the steady state Navier-Stokes equations in exterior domains, Galdi and Simader [6] showed a velocity field decaying at large distances as $|x|^{-2}$ under the assumption of small given initial data. Similar decay properties of the Stokes equations in exterior domains were obtained in [22]. For other works, see [3] and the literature in [22] and [6]. In this paper, our weighted estimates imply the decay properties at large distances for the strong solutions of the Navier-Stokes equations in an exterior domain in some sense.

The rest of the paper is organized as follows: In section 2, we introduce the notations and state the main results. The approximate solutions are constructed and some basic estimates are given in section 3. The integral representations of the approximate solutions are given in section 4. Finally, we deduce the main weighted estimates in section 5.

2. Notations and the main results. Let Ω be an exterior domain in R^3 with smooth boundary $\partial\Omega$. Without loss of generality, we assume that the complement of Ω, Ω^c, is contained in $B(0,R)$, the ball centered at 0 with radius R. Let $L^p(\Omega), 1 \leq p \leq +\infty$, represent the usual Lesbegue space of scalar functions as well as that of vector functions which norm denoted by $\|\cdot\|_p$. We will use $\|\cdot\|_{p,D}$ to denote the norm of the function in $L^p(D)$. Let $C^\infty_{0,\sigma}(\Omega)$ denote the set of all C^∞ real vector functions $\phi = (\phi_1, \phi_2, \phi_3)$ with compact support in Ω, such that div$\phi = 0$. $\overset{o}{J}{}^p(\Omega)$, $1 \leq p < \infty$, is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to $\|\cdot\|_p$. Finally, given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X), 1 \leq p \leq +\infty$, the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|_X^p dt < +\infty$. Let P be the Helmholtz projection from $L^p(\Omega))$ to $\overset{o}{J}{}^p(\Omega)$. Then the Stokes operator A is defined by $A = -P\Delta$ with $D(A) = H^2(\Omega) \cap \overset{o}{J}{}^2(\Omega)$. Let $< x >= (1+|x|^2)^{1/2}$ and $B(\cdot,\cdot)$ denotes the beta function. Finally, by symbol C, we denote a generic constant whose value is unessential to our aims, and it may change from line to line.

Before stating our main results, we first give the definition of strong solutions.

DEFINITION. A vector u is called a strong solution of (1.1) if
1) $u, \nabla u \in L^\infty(0,T; L^p(\Omega))$ for $7 \leq p \leq +\infty$ and any $T > 0$,
2) u satisfies the equations (1.1) in distribution sense, i.e.,

$$\int_0^\infty \int_\Omega (-\frac{\partial \phi}{\partial \tau} u + \nabla u \cdot \nabla \phi + (u\cdot\nabla)u \cdot \phi) dx d\tau = \int_\Omega \phi(x,0)a(x)$$

for every $\phi \in C^\infty_{0,\sigma}(R \times \Omega)$.
3) div$u = 0$ in distribution sense, i.e.,

$$\int_\Omega u(x,t)\nabla\psi(x) = 0$$

for every $\psi \in C^\infty_0(\Omega)$.

Then our main results are stated in the following two theorems.

THEOREM 1. Let $a \in \overset{o}{J}{}^1(\Omega)$ and $(1+|x|^2)^{\alpha/2} a \in L^p(\Omega)$ for $7 < p \leq +\infty$ and $\alpha = 3/7 - 3/p$. Then there exists a constant δ such that if $\|a\|_p \leq \delta$, there exists a unique solution u to (1.1) such that

$$(1+|x|^2)^{\alpha/2} u, \quad t^{1/2}(1+|x|^2)^{\alpha/2}\nabla u \in L^\infty(0,+\infty; L^p(\Omega)), \tag{2.1}$$

$$\|u\|_2 \leq CN(r) t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})}. \tag{2.2}$$

where $N(r)$ is a constant depend only on $||a||_2$ and $||a||_r$ (see Lemma 3.2).

THEOREM 2. Let $a \in \overset{o}{J}{}^1(\Omega)$ and $(1+|x|^2)^{\beta/2}a \in L^p(\Omega)$ for $3 < p \leq +\infty$ and $\beta < 1 - 3/p$. Then there exists a constant δ such that if $||a||_p \leq \delta$, there exists a unique solution u to (1.1) such that

$$(1+|x|^2)^{\beta/2}u, \quad t^{1/2}(1+|x|^2)^{\beta/2}\nabla u \in L^\infty_{loc}(0,+\infty; L^p(\Omega)). \tag{2.3}$$

REMARKS.

1. The decay property (2.2) for weak solutions was already obtained by Borchers and Miyikawa [2].

2. A class of weighted weak solutions $(1+|x|^2)^{\alpha/2}u \in L^\infty(0,+\infty; L^2(R^3))$ was obtained for Cauchy problem for $0 \leq \alpha \leq 3$. And with small assumption on initial data, the weighted strong solution $t^{\beta/2}(1+|x|^2)u \in L^\infty(0,+\infty; L^p(R^3))(3 < p \leq \infty)$ also was obtained for Cauchy problem in [9] with $\beta + \alpha = 3 - 3/p$ or $4 - 3/p$.

3. Farwig and Sohr [5] showed a class of weighted ($|x|^\alpha$) weak solutions with second derivative about the spatial variables and one order derivative about time variable in $L^s(0,\infty; L^q(\Omega))$ for $1 < q < 3/2$, $1 < s < 2$ and $0 \leq 3/q + 2/s - 4 \leq \alpha < \min\{1/2, 3 - 3/q\}$.

Applying the weighted estimates obtained in section 5, the proof of Theorem 1 and Theorem 2 is standard. So we will only deduce the necessary weighted estimates, and omit the details of the procedure of the proof.

3. The Construction of approximation solutions and its basic estimates. We first define the approximate solutions by using the linearized Navier-Stokes equations in Ω. Let $a \in \overset{o}{J}{}^p(\Omega) \cap \overset{o}{J}{}^q(\Omega)(1 \leq p, q \leq +\infty)$. By Lemma 1 in (Maremonti [18]), we select $a^k \in C^\infty_{0,\sigma}(\Omega)$, such that

$$a^k \longrightarrow a \quad \text{in } \overset{o}{J}{}^p(\Omega) \cap \overset{o}{J}{}^q(\Omega) \text{ strongly}$$

and

$$||a^k||_p \leq 2||a||_p, \qquad ||a^k||_q \leq 2||a||_q. \tag{3.1}$$

We now consider the exterior problems for the linearized Navier-Stokes equations in Ω

$$\begin{cases} \dfrac{\partial u^0}{\partial t} - \Delta u^0 = -\nabla p^0, & \text{in } \Omega \times (0,\infty), \\ \text{div} u^0 = 0, & \text{in } \Omega \times (0,\infty), \\ u^0 = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ u^0 \longrightarrow 0, & \text{as } |x| \to +\infty, \\ u^0(x,0) = a^0(x), & \text{in } \Omega \end{cases} \tag{3.2}$$

and

$$\begin{cases} \dfrac{\partial u^k}{\partial t} - \Delta u^k + (u^{k-1}\cdot\nabla)u^k = -\nabla p^k, & \text{in } \Omega \times (0,\infty), \\ \text{div} u^k = 0, & \text{in } \Omega \times (0,\infty), \\ u^k = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ u^k \longrightarrow 0, & \text{as } |x| \to +\infty, \\ u^k(x,0) = a^k(x), & \text{in } \Omega \end{cases} \tag{3.3}$$

for $k \geq 1$. It is well known (cf. Ladyzhenskaya [17]) that there exists a unique solution $u^k (k \geq 0)$ to (3.2) and (3.3) satisfying

$$\frac{\partial u^k}{\partial t}, \ \frac{\partial u^k}{\partial x_i}, \ \frac{\partial^2 u^k}{\partial x_i \partial x_j}, \ \frac{\partial p^k}{\partial x_i} \in L^2(0,T;L^2(\Omega)) \tag{3.4}$$

for $i,j = 1,2,3$, $k \geq 0$ and any $T > 0$. An easily computation shows that the following Lemma holds.

LEMMA 3.1. If $a \in \overset{o}{J}{}^2(\Omega)$, then the estimates

$$\|u^k(t)\|_2 \leq 2\|a\|_2 \quad \forall t > 0 \tag{3.5}$$

and

$$\int_0^\infty \|\nabla u^k(s)\|_2^2 ds \leq 4\|a\|_2^2 \tag{3.6}$$

hold uniformly for $k \geq 0$.

LEMMA 3.2. Let $a \in \overset{o}{J}{}^2(\Omega) \cap \overset{o}{J}{}^r(\Omega)$ for $1 < r < 2$. Then

$$\|u^k(t)\|_2 \leq CN(r)t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})} \tag{3.7}$$

holds uniformly for $k \geq 0$ and $t > 0$. Where

$$N(r) = \begin{cases} \|a\|_r + \|a\|_2 + \|a\|_2^2, & \text{if } 3/2 < r < 2, \\ \|a\|_r + \|a\|_r^2 + \|a\|_2 + \|a\|_2^2 + \|a\|_2^4, & \text{if } 6/5 < r \leq 3/2, \\ \sum_{i=1}^{2} \|a\|_r^{2^i} + \sum_{i=0}^{3} \|a\|_2^{2^i}, & \text{if } 1 < r \leq 6/5. \end{cases}$$

Proof. Similar to the discussion of (5.8) in Borchers and Miyakawa([2],P_{221}), we can deduce that the inequality

$$\begin{aligned} \|u^k(t)\|_2^2 \leq \ & Ct^{-m}\int_0^t ms^{m-1}\|e^{-sA}a^k\|_2^2 ds \\ & + C\|a\|_2^{2(1+2\varepsilon)}t^{-m}\int_0^t s^{m-5/2+2\varepsilon}(\int_0^s \|u^k(\tau)\|_2^2 d\tau)^{1-2\varepsilon} ds \end{aligned} \tag{3.8}$$

holds uniformly for $k \geq 0$. Here m is a sufficiently large number and $\varepsilon \in [0,1/4]$.

The $L^p - L^q$ estimates of semigroup generated by the Stokes operator in the exterior domain Ω (cf. Iwashita[13]) is

$$\|e^{-sA}a\|_q \leq C\|a\|_p s^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}, \quad \forall s > 0 \tag{3.9}$$

for $1 < p \leq q < +\infty$.

If $3/2 < r < 2$, (3.8) implies us, with the help of (3.5) and (3.9), that

$$\|u^k(t)\|_2^2 \leq C\|a\|_r^2 t^{-3(1/r-1/2)} + C\|a\|_2^4 t^{-1/2} \tag{3.10}$$

holds uniformly for $k \geq 0$ and $t > 0$. By inequality

$$(c+d)^{1/2} \leq c^{1/2} + d^{1/2} \quad \text{for any } c,d \geq 0, \tag{3.11}$$

(3.10) implies (3.7) for $t \geq 1$, since $3(1/r - 1/2) < 1/2$ so far as $3/2 < r < 2$. When $t < 1$, (3.5) gives (3.7).

Let $6/5 < r \leq 3/2$, then (3.8) implis us that

$$\|u^k(t)\|_2^2 \leq C\Big(\|a^k\|_2^{\frac{4(3-2r)}{3(2-r)}} \|a^k\|_r^{\frac{2r}{3(2-r)}} + \|a^k\|_2^4\Big)t^{-1/2}.$$

Let $A_1 = \|a\|_2^{\frac{4(3-2r)}{3(2-r)}} \|a\|_r^{\frac{2r}{3(2-r)}} + \|a\|_2^4$. It follows from (3.1), that

$$\|u^k(t)\|_2^2 \leq CA_1t^{-1/2} \quad \text{for } t > 0. \tag{3.12}$$

Substituting (3.12) into the second term at the right hand of (3.8), we deduce that

$$\|u^k(t)\|_2^2 \leq C\|a\|_r^2 t^{-3(1/r-1/2)} + C\|a\|_2^{\frac{1}{2}(1+2\varepsilon)} A_1^{1-2\varepsilon} t^{-1+\varepsilon}. \tag{3.13}$$

By the Young inequality and (3.11), we have (3.7) for $6/5 < r \leq 3/2$.

Let $1 < r \leq 6/5$. We substitute estimate (3.12) into (3.8) to get

$$\|u^k(t)\|_2^2 \leq C\Big(\|a^k\|_2^{\frac{2(6-5r+2r\varepsilon)}{3(2-r)}} \|a^k\|_r^{\frac{4r(1-\varepsilon)}{3(2-r)}} + \|a^k\|_2^{2(1+2\varepsilon)} A_1^{1-2\varepsilon}\Big)t^{-1+\varepsilon}. \tag{3.14}$$

Due to (3.1) and the Young inequality, the constant at right hand side of (3.14) can be estimated as

$$\|a^k\|_2^{\frac{2(6-5r+2r\varepsilon)}{3(2-r)}} \|a^k\|_r^{\frac{4r(1-\varepsilon)}{3(2-r)}} + \|a^k\|_2^{2(1+2\varepsilon)} A_1^{1-2\varepsilon} \leq C\Big(\|a\|_2^2 + \|a\|_r^2 + \|a\|_2^4 + A_1^2\Big) \triangleq CA_2.$$

Substituting (3.14) into (3.8), one show that

$$\begin{aligned}\|u^k(t)\|_2^2 &\leq C\|a^k\|_r^2 t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})} + \|a^k\|_r^{2(1+2\varepsilon)} A_2^{1-2\varepsilon} t^{-\frac{3}{2}+3\varepsilon-2\varepsilon^2} \\ &\leq C\|a^k\|_r^2 t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})} + C(\|a\|_r^4 + A_2^2) t^{-\frac{3}{2}+3\varepsilon-2\varepsilon^2}.\end{aligned}$$

As $1 < r \leq 6/5$ and $0 < \varepsilon < 1/4$, $(3/2)(1/r - 1/2) \leq 3/2 + 2\varepsilon^2 - 3\varepsilon$. So (3.7) follows from the last fact and (3.5). □

LEMMA 3.3. Let $a \in \overset{o}{J}{}^p(\Omega)(3 \leq p < +\infty)$. Then there exists a positive constant δ_1 such that, if $\|a\|_p \leq \delta_1$, then the estimate

$$\begin{cases} \|u^k(t)\|_p \leq C\|a\|_p, \\ t^{1/2}\|\nabla u^k\|_p \leq C\|a\|_p \end{cases} \tag{3.15}$$

holds uniformly for $k \geq 0$ and $t \geq 0$.

The proof is completely similar to that in (Iwashita [13]). Here we omit the details.

4. The Integral Representations of the Approximate Solutions. In this section, we deduce the integral expressions of the approximate solutions. At first, the singular integral expression of the projection operator $P : L^2(R^3) \longrightarrow \overset{o}{J}{}^2(R^3)$ is:

$$P\phi = \phi + \frac{1}{4\pi}\nabla \text{div} \int_{R^3} \frac{\phi(y)}{|x-y|} dy \tag{4.1}$$

for any $\phi \in L^2(R^3)$(cf. Ladyzhenskaya[17]). By the fundamental solution of the heat equation, the solution for the Cauchy problem of the Stokes equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -\nabla p + f, \\ \text{div} v = 0, \\ v(x,0) = 0 \end{cases}$$

can be written as

$$v_i = \int_0^t \int_{R^3} V^i(x-y,t-\tau)\cdot f(y,\tau)dyd\tau, \quad i=1,2,3, \tag{4.2}$$

where

$$\begin{cases} V^i(x,t) = \Gamma(x,t)e^i + \dfrac{1}{4\pi}\nabla\dfrac{\partial}{\partial x_i}\displaystyle\int_{R^3}\dfrac{\Gamma(x-z,t)}{|z|}dz \\ \Gamma(x,t) = (4\pi t)^{-3/2}e^{-|x|^2/4t} \end{cases} \tag{4.3}$$

and e^i is the unite vector along x_i- axis. It is easy to see that

$$V^i(x,t) = \text{curl}(\text{curl}\omega^i) = -\Delta\omega^i + \nabla\text{div}\omega^i, \quad i=1,2,3$$

with

$$\begin{cases} \omega^i(x,t) = \dfrac{1}{4\pi}\displaystyle\int_{R^3}\dfrac{\Gamma(x-z,t)}{|z|}dze^i = \dfrac{1}{2|x|\pi^{3/2}}\theta(\dfrac{|x|}{2\sqrt{t}})e^i, \\ \theta(\rho) = \displaystyle\int_0^\rho e^{-\eta^2}d\eta,\ \bar{\theta}(x,t) = \dfrac{1}{4\pi}\int_{R^3}\dfrac{\Gamma(x-z,t)}{|z|}dz. \end{cases} \tag{4.4}$$

For the details about the deducement of (4.3) and (4.4), see Ladyzhenskaya[17].

Now we deduce the integral expressions of the approximate solutions u^k, as do in Ladyzhenskaya[17]. For this purpose, let $\zeta \in C_0^\infty(\Omega)$ such that $\zeta \equiv 0$ for $x \in \{x|0 \le dist(x,\partial\Omega) \le \lambda\}$ and $\zeta \equiv 1$ for $x \in \Omega_\lambda = \{x|dist(x,\partial\Omega) \ge 2\lambda\}$ with some given positive constant λ, here $dist(x,\partial\Omega)$ denotes the distance function from x to $\partial\Omega$. Then

$$\begin{cases} \text{curl}_y(\text{curl}_y[\omega^i(x-y,t-\tau)\zeta(y)]) = \zeta(y)V^i(x-y,t-\tau) + R_1^i(x,y,t,\tau), \\ R_1^i(x,y,t,\tau) = \nabla\zeta\times\text{curl}_y\omega^i + \nabla\zeta\cdot\text{div}\omega^i - \omega^i\Delta\zeta + (\omega^i\cdot\nabla)\nabla\zeta - (\nabla\zeta\cdot\nabla)\omega^i. \end{cases}$$

Let y and τ denote the variables in equations (3.3). In the following, we drop the right upper label k of the solution u^k of (3.3) and use b to denote u^{k-1} for convenience' sake of writing. We multiply both sides of (3.3) by $\text{curl}_y\text{curl}_y[\zeta(y)\omega^i(x-y,t-\tau)]$, then integrate for $y \in R^3$ and $\tau \in [0, t-\varepsilon]$ for arbitrary $0 < \varepsilon < t$, to get that

$$\begin{aligned} &\int_0^{t-\varepsilon}\int_{R^3}(\frac{\partial u}{\partial t} - \Delta u)(y,\tau)\Big(\zeta(y)V^i(x-y,t-\tau) + R_1^i(x,y,t,\tau)\Big)dyd\tau \\ &= \int_0^{t-\varepsilon}\int_{R^3}(-\nabla_y p - (b\cdot\nabla)u)(y,\tau)\text{curl}_y\text{curl}_y[\omega^i(x-y,t-\tau)\zeta(y)]dyd\tau. \end{aligned}$$

Let $R_2^i(x,y,t,\tau) = -2(\nabla\zeta\cdot\nabla)V^i - \Delta\zeta\cdot V^i$. Since $(-\partial/\partial\tau - \Delta_y)V^i = 0$, it follows, with the help of the integration by parts, that

(4.5)

$$\begin{aligned}\int_0^{t-\varepsilon}\int_{R^3} u(y,\tau)R_2^i(x,y,t,\tau)dyd\tau - \int_0^{t-\varepsilon}\int_{R^3} u(y,\tau)(\frac{\partial}{\partial\tau} + \Delta_y)R_1^i(x,y,t,\tau)dyd\tau \\ + \int_{R^3} u(y,t-\varepsilon)\zeta(y)V^i(x-y,\varepsilon)dy + \int_{R^3} u(y,t-\varepsilon)R_1^i(x,y,t,t-\varepsilon)dy \\ - \int_{R^3} a(y)\Big(\zeta(y)V^i(x-y,t) + R_1^i(x,y,t,0)\Big)dy \\ = -\int_0^{t-\varepsilon}\int_{R^3}(b\cdot\nabla)u(y,\tau)\Big(\zeta(y)V^i(x-y,t-\tau) + R_1^i(x,y,t,\tau)\Big)dyd\tau.\end{aligned}$$

Since $\zeta V^i(x-y,t-\tau)$ is continuous, then one obtains

$$\lim_{\varepsilon\to 0}\int_{R^3} u(y,t-\varepsilon)\zeta(y)V^i(x-y,\varepsilon)dy = (u\zeta)_i(x,t) + \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{R^3}\frac{\operatorname{div}(\zeta(y)u_i(x-y,t))}{|y|}dy,$$

here $(u\zeta)_i$ denotes the i-th component of vector $u\zeta$. Thus, (4.5) implies that

$$\begin{aligned}(u\zeta)_i = -\int_0^t\int_{R^3}(b\cdot\nabla)u(y,\tau)\zeta(y)V^i(x-y,t-\tau)dyd\tau \\ -\int_0^t\int_{R^3}(b\cdot\nabla)u(y,\tau)R_1^i(x,y,t,\tau)dyd\tau + \int_{R^3}a(y)\zeta(y)V^i(x-y,t)dy \\ + \int_{R^3}a(y)R_1^i(x,y,t,0)dy - \int_0^t\int_{R^3}u(y,\tau)R_2^i(x,y,t,\tau)dyd\tau \\ + \int_0^t\int_{R^3}u(y,\tau)(\frac{\partial}{\partial\tau}+\Delta_y)R_1^i(x,y,t,\tau)dyd\tau - \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{R^3}\frac{\operatorname{div}(\zeta(y)u_i(x-y,t))}{|y|}dy.\end{aligned}$$

Substituting (4.3) into above equation, we get that

$$\begin{aligned}(u\zeta)_i = -\int_0^t\int_{R^3}(b\cdot\nabla)u(y,\tau)\zeta(y)\Gamma(x-y,t-\tau)e^i dyd\tau \\ -\int_0^t\int_{R^3}(b\cdot\nabla)u(y,\tau)\zeta(y)\nabla\frac{\partial}{\partial y_i}\bar\theta(x-y,t-\tau)dyd\tau \\ +\int_0^t\int_{R^3}\sum_{l,k=1}^3 b_l u_k(y,\tau)\frac{\partial}{\partial y_j}(R_1^i(x,y;t,\tau))_k dyd\tau \\ +\int_{R^3}a(y)\zeta(y)\Gamma(x-y,t)e^i dy + \int_{R^3}a(y)\zeta(y)\nabla\frac{\partial}{\partial y_i}\bar\theta(x-y,t)dy \\ +\int_{R^3}a(y)R_1^i(x,y;t,0)dy + \int_0^t\int_{R^3}u(y,\tau)(\frac{\partial}{\partial\tau}+\Delta_y)R_1^i(x,y;t\tau)dyd\tau \\ -\int_0^t\int_{R^3}u(y,\tau)R_2^i(x,y;t,\tau)dyd\tau - \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{R^3}\frac{\operatorname{div}(\zeta(y)u_i(x-y,t))}{|y|}dy.\end{aligned} \tag{4.6}$$

Integration by parts several times yield
(4.7)

$$\begin{aligned}(u\zeta)_i = &\int_0^t \int_{R^3} \sum_j b_j u_i(y,\tau)\zeta(y)\frac{\partial}{\partial y_j}\Gamma(x-y,t-\tau)dyd\tau \\ &+ \int_0^t \int_{R^3} \sum_j b_j u_i(y,\tau)\frac{\partial \zeta(y)}{\partial y_j}\Gamma(x-y,t-\tau)dyd\tau \\ &+ \int_0^t \int_{R^3} \sum_{l,k=1}^{3} b_l u_k(y,\tau)\zeta(y)\frac{\partial^3}{\partial y_i \partial y_l \partial y_k}\bar{\theta}(x-y,t-\tau)dyd\tau \\ &+ \int_0^t \int_{R^3} \sum_{l,k=1}^{3} b_l u_k(y,\tau)\frac{\partial \zeta}{\partial y_l}\frac{\partial^2}{\partial y_i \partial y_k}\bar{\theta}(x-y,t-\tau)dyd\tau \\ &+ \int_0^t \int_{R^3} \sum_{l,k=1}^{3} b_l u_k(y,\tau)\frac{\partial}{\partial y_l}(R_1^i(x,y,t,\tau))_k dyd\tau \\ &+ \int_{R^3} a(y)\zeta(y)\Gamma(x-y,t)e^i dy + \int_{R^3} a(y)\zeta(y)\nabla\frac{\partial}{\partial y_i}\bar{\theta}(x-y,t)dy \\ &+ \int_{R^3} a(y)R_1^i(x,y,t,0)dy + \int_0^t \int_{R^3} u(y,\tau)(\frac{\partial}{\partial \tau}+\Delta_y)R_1^i(x,y,t,\tau)dyd\tau \\ &- \int_0^t \int_{R^3} u(y,\tau)R_2^i(x,y,t,\tau)dyd\tau - \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{R^3}\frac{\operatorname{div}(\zeta(y)u_i(x-y,t))}{|y|}dy \triangleq \sum_{k=1}^{11} I_k^i.\end{aligned}$$

It is obvious that

$$\text{supp}R_1^i \text{ and } \text{supp}R_2^i \subset \{y|\lambda \le \text{dist}(y,\partial\Omega) \le 2\lambda\} \triangleq D. \tag{4.8}$$

Thus, the actual integral region of $I_2^i, I_4^i, T_5^i, I_8^i - I_{11}^i$ is D. Moreover,

$$\begin{cases} |R_1^i| \le C(|\bar{\theta} + |\nabla\bar{\theta}|), \\ |\nabla R_1^i| \le C(|\bar{\theta}| + |\nabla\bar{\theta}| + |\partial^2\bar{\theta}|), \\ |R_2^i| \le C(|\nabla\Gamma| + |\Gamma| + |\partial^2\bar{\theta}| + |\partial^3\bar{\theta}|), \\ |(\partial/\partial\tau + \Delta_y)R_1^i| \le C(|\bar{\theta}| + |\nabla\bar{\theta}| + |\partial^2\bar{\theta}| + |\partial^3\bar{\theta}| + |\partial_t\bar{\theta}| + |\partial_t\nabla\bar{\theta}|), \end{cases} \tag{4.9}$$

where $|\partial^2\bar{\theta}| = \sum_{i,j=1}^2 |\partial^2\bar{\theta}/\partial x_i \partial x_j|$ and $|\partial^3\bar{\theta}| = \sum_{i,j,k=1}^3 |\partial^3\bar{\theta}/\partial x_i \partial x_j \partial x_k|$.

5. Weighted Estimates for the Approximate Solutions. By simple calculation, we have, with the help of the inequality

$$\tau^\alpha e^{-C\tau} \le C \qquad \forall \alpha > 0$$

that

$$\begin{cases} \||x|^\alpha \Gamma\|_p \le Ct^{\frac{\alpha}{2}-\frac{3}{2}(1-\frac{1}{p})}, \\ \||x|^\alpha \nabla\Gamma\|_p \le Ct^{\frac{\alpha-1}{2}-\frac{3}{2}(1-\frac{1}{p})} \end{cases} \tag{5.1}$$

for $1 \le p \le +\infty$ and $\alpha \ge 0$. It follows from the weighted estimates about the singular integral (cf. Stein [23-25]), that

(5.2)
$$\begin{cases} \||x|^\alpha \bar{\theta}\|_p \le C\||x|^\alpha \Gamma\|_r \le Ct^{\frac{\alpha}{2}-\frac{3}{2}(1-\frac{1}{r})}, \ 1/p=1/r-2/3, 1<r<3/2, 0\le\alpha<1-3/p, \\ \||x|^\alpha \nabla\bar{\theta}\|_p \le C\||x|^\alpha \Gamma\|_r \le Ct^{\frac{\alpha}{2}-\frac{3}{2}(1-\frac{1}{r})}, \ 1/p=1/r-1/3, 1<r<3, 0\le\alpha<2-3/p, \\ \||x|^\alpha \partial^2\bar{\theta}\|_p \le C\||x|^\alpha \Gamma\|_p \le Ct^{\frac{\alpha}{2}-\frac{3}{2}(1-\frac{1}{p})}, \ 1<p<+\infty, -1/p<\alpha<3-3/p. \end{cases}$$

In the following, let $\alpha = 3/7 - 3/p$ for $7 < p \le +\infty$. Let $x \in \Omega_{3\lambda}$, then $|x-y| \ge \lambda$ for $y \in D$. According to (4.9) and the expression of $\bar{\theta}$, it is obvious that

$$|R_1^i| + |\nabla_y R_1^i| + |R_2^i| + |(\partial/\partial\tau + \Delta_y)R_1^i| \le C(|\bar{\theta}| + t\Gamma)$$

for $x \in \Omega_{3\lambda}$ and $y \in D$. Thus the theory on singular integral operator (cf. Stein [23-25]) implies that

LEMMA 5.1. Let $\alpha = 3/7 - 3/p$ for $7 < p \le +\infty$. Then for $3 < r < +\infty$, we have

$$\|R_1^i\|_{r,\Omega_{3\lambda}} + \|\nabla_y R_1^i\|_{r,\Omega_{3\lambda}} + \|R_2^i\|_{r,\Omega_{3\lambda}} + \|(\partial/\partial\tau + \Delta_y)R_1^i\|_{r,\Omega_{3\lambda}} \le Ct^{-\frac{1}{2}+\frac{3}{2r}}, \tag{5.3}$$

$$\begin{aligned} &\||x-y|^\alpha R_1^i\|_{r,\Omega_{3\lambda}} + \||x-y|^\alpha \nabla_y R_1^i\|_{r,\Omega_{3\lambda}} + \||x-y|^\alpha R_2^i\|_{r,\Omega_{3\lambda}} \\ &+\||x-y|^\alpha (\partial/\partial\tau + \Delta_y)R_1^i\|_{r,\Omega_{3\lambda}} \le Ct^{-\frac{2}{7}-\frac{3}{2p}+\frac{3}{2r}}. \end{aligned} \tag{5.4}$$

Since

$$|\nabla_x R_1^i| + |\nabla_x|\nabla_y R_1^i|| + |\nabla_x R_2^i|| + |\nabla_x(\partial/\partial\tau + \Delta_y)R_1^i| \le C(|\bar{\theta}|/|x-y| + t\Gamma/|x-y|)$$

for $x \in \Omega_{3\lambda}$ and $y \in D$. Thus the theory on singular integral operator (cf. Stein [23-25]) implies that

LEMMA 5.2. Let $\alpha = 3/7 - 3/p$ for $7 < p \le +\infty$. Then for $3/2 < r < +\infty$, we have

(5.5)
$$\|\nabla_x R_1^i\|_{r,\Omega_{3\lambda}} + \|\nabla_x|\nabla_y R_1^i|\|_{r,\Omega_{3\lambda}} + \|\nabla_x R_2^i\|_{r,\Omega_{3\lambda}} + \|\nabla_x(\partial/\partial\tau+\Delta_y)R_1^i\|_{r,\Omega_{3\lambda}} \le Ct^{-1+\frac{3}{2r}},$$

$$\begin{aligned} &\||x-y|^\alpha \nabla_x R_1^i\|_{r,\Omega_{3\lambda}} + \||x-y|^\alpha \nabla_x|\nabla_y R_1^i|\|_{r,\Omega_{3\lambda}} + \||x-y|^\alpha \nabla_x R_2^i\|_{r,\Omega_{3\lambda}} \\ &+\||x-y|^\alpha \nabla_x(\partial/\partial\tau + \Delta_y)R_1^i\|_{r,\Omega_{3\lambda}} \le Ct^{-\frac{11}{14}-\frac{3}{2p}+\frac{3}{2r}}. \end{aligned} \tag{5.6}$$

Now we can give our main weighted estimates as follows.

LEMMA 5.3. Let $a \in \overset{o}{J}{}^1 \cap \overset{o}{J}{}^p$ and $(1+|x|^2)^{\alpha/2} a \in L^p(\Omega)$ for $7 < p \le +\infty$, and $\alpha = 3/7 - 3/p$. Then there exists a constant δ such that, if $\|a\|_p \le \delta$, the following weighted estimate is valid for $k \ge 0$ and $t \ge 0$,

$$\|(1+|x|^2)^{\alpha/2} u^k\|_p \le CA, \tag{5.7}$$

where A only depend on $\|a\|_1$ and $\|(1+|x|^2)^{\alpha/2} a\|_p$.

Proof. For convenience's of presentation, we drop the right upper label of the approximation solutions u^k and use b to denote u^{k-1}. By the interpolation inequality, $a \in L^q(\Omega)$ for every $q \in [1,p]$ and

$$\|a\|_q \le \|a\|_1^{\frac{p-q}{q(p-1)}} \|a\|_p^{\frac{p(q-1)}{q(p-1)}}. \tag{5.8}$$

First let $\|a\|_p \le \delta_1$, then Lemma 3.3 hold. It is obvious that

$$< x >^\alpha \le 2^{\alpha/2}(< y >^\alpha + |x-y|^\alpha).$$

Let $J_u = \| < x >^\alpha u\|_{p,\Omega}$. Applying (5.1), The Young inequality, Minkowski inequality and Hölder inequality, we can estimate I_1^i in (4.7) as follows,

(5.9)

$$\begin{aligned}
\| < x >^\alpha I_1^i\|_{p,\Omega_{3\lambda}} &\le C\| \int_0^t \int_\Omega |b||u|(y) < y >^\alpha |\nabla\Gamma|(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\
&\quad + C\| \int_0^t \int_\Omega |b||u|(y)|x-y|^\alpha|\nabla\Gamma|(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\
&\le C\int_0^t \| < y >^\alpha |b||u|\|_{2m}(t-\tau)^{\frac{3}{2p}-\frac{1}{2}-\frac{3}{4m}}d\tau + C\int_0^t \||b||u|\|_{2n}(t-\tau)^{-\frac{2}{7}-\frac{3}{4n}}d\tau \\
&\le C\int_0^t J_b\|u\|_p^{\frac{mp-p+2m}{m(p-2)}}\|u\|_2^{\frac{p-4m}{m(p-2)}}(t-\tau)^{\frac{3}{2p}-\frac{1}{2}-\frac{3}{4m}}d\tau \\
&\quad + C\int_0^t \|b\|_p\|u\|_p^{\frac{np-p+2n}{n(p-2)}}\|u\|_2^{\frac{p-4n}{n(p-2)}}(t-\tau)^{-\frac{2}{7}-\frac{3}{4m}}d\tau \\
&\le C\int_0^t J_b\|a\|_p^{\frac{mp-p+2m}{m(p-2)}}\|u\|_2^{\frac{p-4m}{m(p-2)}}(t-\tau)^{\frac{3}{2p}-\frac{1}{2}-\frac{3}{4m}}d\tau \\
&\quad + C\|a\|_p^{\frac{p(2n-1)}{n(p-2)}}N(p_0)^{\frac{p-4n}{n(p-2)}}B(\tfrac{2}{7}-\tfrac{3}{4n},\tfrac{5}{7}+\tfrac{3}{4n}) \\
&\le C\|a\|_p^{\frac{mp-p+2m}{m(p-2}}N(p_0)^{\frac{p-4m}{m(p-2)}}\int_0^t J_b\tau^{-\frac{5(p-4m)}{7m(p-2)}}(t-\tau)^{\frac{3}{2p}-\frac{1}{2}-\frac{3}{4m}}d\tau \\
&\quad + C\|a\|_p^{\frac{p(2n-1)}{n(p-2)}}N(p_0)^{\frac{p-4n}{n(p-2)}}.
\end{aligned}$$

Here we have used Lemma 3.3, $p_0 = 42/41$, $m = 41p^2 - 42p/(14p^2+66p-28)$ and $n = 41/20 - 31/(5(p-2))$. For I_2^i, it holds that

(5.10)

$$\begin{aligned}
\| < x >^\alpha I_2^i\|_{p,\Omega_{3\lambda}} &\le C\| \int_0^t \int_D |b||u|(y) < y >^\alpha \Gamma(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\
&\quad + C\| \int_0^t \int_D |b||u|(y)|x-y|^\alpha\Gamma(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\
&\le C\int_0^t \|b\|_p\|u\|\|_2(t-\tau)^{-\frac{3}{4}}d\tau + C\int_0^t \|b\|_p\|u\|\|_2(t-\tau)^{-\frac{15}{28}-\frac{3}{2p}}d\tau \\
&\le C\|a\|_p\|a\|_2^{\frac{13}{20}}N^{\frac{7}{20}}(p_0)B(\frac{3}{4},\frac{1}{4})d\tau \\
&\quad + C\|a\|_p\|a\|_2^{7/20+21/10p}N^{13/20-21/10p}(p_0)B(\frac{15}{28}-\frac{3}{2p},\frac{13}{28}+\frac{3}{2p})d\tau.
\end{aligned}$$

Note the fact

$$\nabla_y \frac{\partial^2}{\partial y_i \partial y_j}\bar{\theta}(x-y,t-\tau) = -\frac{\partial^2}{\partial y_i \partial y_j}\int_{R^3}\frac{\nabla\Gamma(x-y-z,t-\tau)}{|z|}dz.$$

Using (5.2) instead of (5.1), the estimates of I_3^i and I_4^i are completely similar to that of I_1^i and I_2^i. Then we have

$$\begin{aligned} & \| < x >^\alpha I_3^i\|_{p,\Omega_{3\lambda}} + \| < x >^\alpha I_4^i\|_{p,\Omega_{3\lambda}} \\ \leq\ & C\|a\|_p^{\frac{mp-p+2m}{m(p-2)}} N(p_0)^{\frac{p-4m}{m(p-2)}} \int_0^t J_b \tau^{-\frac{5(p-4m)}{7m(p-2)}}(t-\tau)^{\frac{3}{2p}-\frac{1}{2}-\frac{3}{4m}} d\tau \\ & + C\|a\|_p^{\frac{p(2n-1)}{n(p-2)}} N(p_0)^{\frac{p-4n}{n(p-2)}} + C\|a\|_p\|a\|_2^{\frac{13}{20}} N^{\frac{7}{20}}(p_0) \\ & + C\|a\|_p\|a\|_2^{7/20+21/10p} N^{13/20-21/10p}(p_0). \end{aligned} \tag{5.11}$$

Let χ_D be the characteristic function of D, i.e., $\chi_D = 1$ for $y \in D$, 0 for $y \notin D$. Similar to the above discussion, we have that

(5.12)

$$\begin{aligned} \| < x >^\alpha I_5^i\|_{p,\Omega_{3\lambda}} & \leq C\| \int_0^t \int_\Omega \chi_D|b||u|(y) < y >^\alpha |\nabla_y R_1^i|(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\ & + C\| \int_0^t \int_\Omega \chi_D|b||u|(y)|x-y|^\alpha|\nabla_y R_1^i|(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\ & \leq C\int_0^t \|b\|_p\|\chi_D u\|_{7/6}(t-\tau)^{-\frac{2}{7}}d\tau + C\int_0^t \|b\|_2\|u\|_3(t-\tau)^{-\frac{1}{28}}d\tau \\ & \leq C\|a\|_p N(p_0) + C\|a\|_p^{\frac{p}{6(p-2)}}\|a\|_2^{\frac{5p-12}{6(p-2)}-\frac{7}{20}} N(p_0)^{\frac{27}{20}} \end{aligned}$$

$$\begin{aligned} \| < x >^\alpha I_6^i\|_{p,\Omega_{3\lambda}} & \leq C\| \int_\Omega |a|(y) < y >^\alpha \Gamma(x-y,t)dy\|_{p,\Omega_{3\lambda}} \\ & + C\| \int_\Omega |a|(y)|x-y|^\alpha \Gamma(x-y,t)dy\|_{p,\Omega_{3\lambda}} \\ & \leq C\| < y >^\alpha a\|_p + C\|a\|_7 \\ & \leq C\| < y >^\alpha a\|_p + C\|a\|_1^{\frac{p-7}{7(p-1)}}\|a\|_p^{\frac{6p}{7(p-1)}}. \end{aligned}$$

Similarly,

$$\| < x >^\alpha I_7^i\|_{p,\Omega_{3\lambda}} \leq C\| < y >^\alpha a\|_p + C\|a\|_1^{\frac{p-7}{7(p-1)}}\|a\|_p^{\frac{6p}{7(p-1)}}.$$

By the fact that $\mathrm{supp}_y R_1^i \subset D$ and $|R_1^i(x,y,t,0)| \leq C/|x-y|$ for $x \in \Omega_{3\lambda}$. We estimate I_8^i as follows

$$\begin{aligned} \| < x >^\alpha I_8^i\|_{p,\Omega_{3\lambda}} & \leq C\| < x >^\alpha \int_\Omega \frac{\chi_D a(y)}{|x-y|}dy\|_{p,\Omega_{3\lambda}} \\ & \leq C\| < x >^\alpha \chi_D a\|_{3p/(3+2p)} \leq C\|a\|_p. \end{aligned}$$

Here we have used the weighted inequality of singular integral operator. By Lemma 5.1 and 5.2, we have, for I_9^i, that

$$\| < x >^\alpha I_9^i\|_{p,\Omega_{3\lambda}} \leq C\| \int_0^t \int_\Omega \chi_D|u| < y >^\alpha |(\partial/\partial\tau + \Delta_y)R_1^i|dyd\tau\|_{p,\Omega_{3\lambda}}$$

$$+C\|\int_0^t\int_\Omega \chi_D|u||x-y|^\alpha|(\partial/\partial\tau+\Delta_y)R_1^i|dyd\tau\|_{p,\Omega_{3\lambda}}$$
$$\le C\int_0^t\|\chi_D u\|_{7p/(7+6p)}(t-\tau)^{-2/7}d\tau+C\int_0^t\|\chi_D u\|_r(t-\tau)^{-2/7+3/2r'}$$
$$\le CN(p_0)+N(\frac{100}{99}),$$

where $r'=r/(r-1)$ is a sufficient number. Similarly,

$$\|<x>^\alpha I_{10}^i\|_{P,\Omega_{3\lambda}}\le CN(p_0)+N(\frac{100}{99}).$$

Note that

$$|\frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{R^3}\frac{\operatorname{div}(\zeta u)}{|x-y|}dy|\le C\int_{R^3}\frac{\chi_D|u|(y)}{|x-y|^2}dy.$$

By the weighted inequality of singular integral operator, we get

$$\|<x>^\alpha I_{11}^i\|_{p,\Omega_{3\lambda}}\le C\|<x>^\alpha\chi_D u\|_{3p/(3+p)}\le C\|a\|_p.$$

Applying Lemma 3.3 and the Minkowski inequality, we obtain, from above discussion, that

$$\begin{aligned}(5.13)\qquad J_u\le\ & C\|a\|_pN(p_0)^{\frac{p-4m}{m(p-2)}}\int_0^t J_b\tau^{-\frac{5(p-4m)}{7m(p-2)}}(t-\tau)^{\frac{3}{2p}-\frac{1}{2}-\frac{3}{4m}}d\tau\\ &+C\Big(\|a\|_p^{\frac{p(2n-1)}{n(p-2)}}N(p_0)^{\frac{p-4n}{n(p-2)}}+\|a\|_p\|a\|_2^{\frac{13}{20}}N^{\frac{7}{20}}(p_0)\\ &+\|a\|_p\|a\|_2^{7/20+21/10p}N^{13/20-21/10p}(p_0)+\|a\|_pN(p_0)\\ &+\|a\|_p^{\frac{p}{6(p-2)}}\|a\|_2^{\frac{5p-12}{6(p-2)}-\frac{7}{20}}N(p_0)^{\frac{27}{20}}\\ &+\|<x>^\alpha a\|_p+\|a\|_1^{\frac{p-7}{7(p-1)}}\|a\|_p^{\frac{6p}{7(p-1)}}+N(p_0)+N(\frac{100}{99})\Big).\end{aligned}$$

Let A be the terms in bracket. By the above discussion, it is obvious that $J_{u_0}\le CA$. By induction, we have $J_{u^k}\in L^\infty(0,T;L^p(\Omega))$ for arbitrary $T>0$. Thus (5.13) give us that

$$(5.14)\qquad J_u\le C\|a\|_pN(p_0)^{\frac{p-4m}{m(p-2)}}J_b+CA.$$

Therefore, there exists δ_2 such that if $\|a\|_p\le\delta_2$, $C\|a\|_pN(p_0)^{p-4m/(m(p-2))}<1$. Taking $\delta=\min\{\delta_1,\delta_2\}$. Then (5.14) implies our result if $\|a\|_p\le\delta$. □

LEMMA 5.4. Assume the conditions of Lemma 5.3 hold. Then we hold uniform estimate as follows

$$(5.15)\qquad t^{1/2}\|<x>^\alpha\nabla u\|_p\le CA.$$

Proof. Let $\bar{I}_1^i$ and $\bar{I}_2^i$ denote the first two terms at the right hand of (4.6). The other terms at the right hand side of (4.6) are same as $I_5^i--I_{11}^i$. Differentiating in

x at both side of (4.6), we have, by the Minkowski inequality, that

$$(5.16)\quad \begin{aligned} \| < x >^\alpha \nabla u\|_{p,\Omega_{3\lambda}} \le\ & C\sum_{i=1}^{3}(\| < x >^\alpha \nabla \bar{I}_1^i\|_{p,\Omega_{3\lambda}} + \| < x >^\alpha \nabla \bar{I}_2^i\|_{p,\Omega_{3\lambda}} \\ & + \sum_{j=5}^{11} \| < x >^\alpha \nabla I_j^i\|_{p,\Omega_{3\lambda}}). \end{aligned}$$

Employing Lemma 5.2 instead of Lemma 5.1 in the above discussion, we can obtain that

$$(5.17)\quad \sum_{i=1}^{3}\sum_{j=5}^{11} \| < x >^\alpha I_j^i\|_{p,\Omega_{3\lambda}} \le CAt^{-1/2}.$$

So we only need to estimate the first two terms at right hand side of (5.16). Due to (4.9) and the Young inequality, we have that

$$\begin{aligned} \| < x >^\alpha \nabla \bar{I}_1^i\|_{p,\Omega_{3\lambda}} \le\ & C\| \int_0^t \int_\Omega |b||\nabla u| < y >^\alpha |\nabla\Gamma|(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\ & + C\| \int_0^t \int_\Omega |b||\nabla u||x-y|^\alpha|\nabla\Gamma|(x-y,t-\tau)dyd\tau\|_{p,\Omega_{3\lambda}} \\ \le\ & C\int_0^t K_u\|b\|_4(t-\tau)^{-\frac{7}{8}}d\tau + C\int_0^t \|b\|_{7/2}\|\nabla u\|_p(t-\tau)^{-\frac{5}{7}-\frac{3}{2p}}d\tau \\ \le\ & C\int_0^t K_u\|b\|_2^{\frac{p-4}{2(p-2)}}\|b\|_p^{\frac{p}{2(p-2)}}(t-\tau)^{-\frac{7}{8}}d\tau \\ & + C\int_0^t \|\nabla u\|_p\|b\|_2^{\frac{2(2p-7)}{7(p-2)}}\|b\|_p^{\frac{3p}{7(p-2)}}(t-\tau)^{-\frac{5}{7}-\frac{3}{2p}}d\tau \\ \le\ & C\|a\|_{p_0}^{\frac{7}{40}}\|a\|_2^{\frac{p-4}{2(p-2)}-\frac{7}{40}}\|a\|_p^{\frac{p}{2(p-2)}}\int_0^t K_u\tau^{-\frac{1}{8}}(t-\tau)^{-\frac{7}{8}}d\tau \\ & + C\|a\|_{p_0}^{\frac{3}{5}-\frac{21}{10p}}\|a\|_2^{\frac{2(2p-7)}{7(p-2)}-\frac{3}{5}+\frac{21}{10p}}\|a\|_p^{\frac{10p-14}{7(p-2)}}. \end{aligned}$$

Here $K_u = \| < x >^\alpha \nabla u\|_p$. Similarly,

$$\begin{aligned} \| < x >^\alpha \nabla \bar{I}_2^i\|_{p,\Omega_{3\lambda}} \le\ & C\|a\|_{p_0}^{\frac{7}{40}}\|a\|_2^{\frac{p-4}{2(p-2)}-\frac{7}{40}}\|a\|_p^{\frac{p}{2(p-2)}}\int_0^t K_u\tau^{-\frac{1}{8}}(t-\tau)^{-\frac{7}{8}}d\tau \\ & + C\|a\|_{p_0}^{\frac{3}{5}-\frac{21}{10p}}\|a\|_2^{\frac{2(2p-7)}{7(p-2)}-\frac{3}{5}+\frac{21}{10p}}\|a\|_p^{\frac{10p-14}{7(p-2)}}. \end{aligned}$$

Thus by Lemma 3.3, we have

$$t^{\frac{1}{2}}K_u \le CA + C\|a\|_{p_0}^{\frac{7}{40}}\|a\|_2^{\frac{p-4}{2(p-2)}-\frac{7}{40}}\|a\|_p^{\frac{p}{2(p-2)}}t^{\frac{1}{2}}\int_0^t K_u\tau^{-\frac{1}{8}}(t-\tau)^{-\frac{7}{8}}d\tau.$$

By the Gronwall inequality, we obtain our estimate. □

Because of the appearance of the function $\bar{\theta}$ in the expression of u^k, we only choose α not to exceed $1-3/p$ for $3 < p$, according the theory on singular integral operator (cf. Stein [24]). However, in order to obtain the uniform estimates about time variable, we must restrict α no to exceed $1/2-3/p$ for $p > 6$. Thus, the estimates (5.7) and (5.15) are valid for any $0 < \alpha < 1/2 - 3/p$ and $p > 6$. In this paper, for sake of simplificity, we take $\alpha = 3/7 - 3/p$ for $7 < p \le \infty$. In view of the procedure of the proof of Lemma 5.3 and 5.4, the value of α don't affect on the uniform estimates about the term depending on J_b, it only change the estimates about terms containing factor $|x-y|^\alpha$ and $I_5^i - I_{11}^i$. Thus, similar to the above discussion, we can obtain, for $\beta < 1 - 3/p$, that

(5.18)

$$\begin{cases} \| < x >^\beta u^k\|_p \le C\|a\|_p N(p_0)^{\frac{p-4m}{m(p-2)}} \| < x >^\beta u^{k-1}\|_p + C(T)A \\ t^{\frac{1}{2}} \| < x >^\beta \nabla u^k\|_p \le C(T)A + C(T)A \int_0^t \| < x >^\beta \nabla u^k\|_p \tau^{-\frac{1}{8}} (t-\tau)^{-\frac{7}{8}} d\tau \end{cases}$$

for $3 < p \le +\infty$ and any $T > 0$. Therefore, by Gronwall inequality and small assumption on $\|a\|_p$, we have

LEMMA 5.5. Let $a \in \overset{o}{J}{}^1 \cap \overset{o}{J}{}^p$ and $(1+|x|^2)^{\beta/2} a \in L^p(\Omega)$ for $3 < p \le +\infty$, and $\beta < 1 - 3/p$. Then there exists a constant δ such that, if $\|a\|_p \le \delta$, the following weighted estimate is valid for $k \ge 0$ and $0 < t < T$,

$$\begin{cases} \|(1+|x|^2)^{\beta/2} u^k\|_p \le C(T)A, \\ t^{1/2} \|(1+|x|^2)^{\beta/2} \nabla u^k\|_p \le C(T)A. \end{cases} \tag{5.19}$$

where A only depend on $\|a\|_1$ and $\|(1+|x|^2)^{\beta/2} a\|_p$.

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