METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 7, No. 2, pp. 407–416, June 2000

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L^∞ ESTIMATES FOR CONSERVATION LAWS WITH HYPERVISCOUS PARABOLIC TERMS*

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1. Introduction. This paper considers the question of L^∞ *a priori* estimates for hyperbolic conservation laws which are regularized with higher order parabolic terms. These are in the form

$$\begin{aligned} \partial_t u + \partial_x f(u) &= (-1)^{s-1}\partial_x^{2s} u \\ u_0(x) &= u(x,0) \in L^\infty. \end{aligned} \tag{1}$$

The boundary conditions are that $u(x,t) = u(x+2\pi,t)$. The analysis will involve L^∞ estimates of the linear parabolic equation

$$\begin{aligned} \partial_t v &= (-1)^{s-1}\partial_x^{2s} v \\ v_0(x) &= v(x,0) \in L^\infty, \end{aligned} \tag{2}$$

which is studied both on the circle; $v(x,t) = v(x+2\pi,t)$ and on the line $x \in \mathbb{R}$. Such questions are motivated by numerical spectral methods for solutions of hyperbolic conservation laws [2]. In this light it is fitting to contribute this paper to a special volume in honor of Cathleen Morawetz, who has had an interest throughout her career in nonlinear partial differential equations and their numerical solution.

Standard parabolic theory implies that solutions to (1) and (2) are smooth, given L^∞ initial data, at least locally in time. The purpose of the analysis in this paper is to give an *a priori* L^∞ estimate on the solutions, and to understand the behavior of this estimate as $s \to +\infty$. Incidentally, $s > 1$ can be taken to be any real, but for simplicity we will let $s \geq 1$ be an integer. For $s = 1$ the classical parabolic maximum principle holds for both (1) and (2), implying the simple *a priori* estimates

$$\|u(x,t)\|_{L^\infty} \leq \|u_0(x)\|_{L^\infty} \quad , \tag{3}$$

with constant $C = 1$, and furthermore the strong maximum principle holds. In contrast to this, for $s \geq 2$ there is an estimate for (1) of similar form

$$\|u(x,t)\|_{L^\infty} \leq C(s)\|u_0(x)\|_{L_\infty} \quad , \tag{4}$$

but in general $C(s) > 1$. Furthermore there is more structure to the question, in that any constant which is to be taken uniformly in $t \in [0,+\infty)$ satisfies $C(s) \sim O(s)$ for large s. However the solutions actually exhibit an initial layer in time; that is to say there is an initial time interval $[0,T(s)]$ in which $C(s)$ gives the *a priori* bound. After this short time interval the L^∞ estimate becomes better behaved, and we have

$$\sup_{T(s)<t<+\infty} \|u(x,t)\|_{L^\infty} \leq C_0\|u_0(x)\|_{L^\infty} \tag{5}$$

for C_0 uniformly bounded as $s \to \infty$. The size of this initial layer is exponentially small in s; indeed $T(s) \sim 2^{-s}$.

*Received January 7, 2000.
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There is a related initial layer for the linear parabolic equation (2), posed over the circle $x \in [0, 2\pi)$. It is well known that for $s > 1$ the parabolic equation does not satisfy a maximum principle. The convolution kernel for the solution may change signs, unlike the case $s = 1$, and this may cause constructive interference for highly oscillatory initial data $v_0(x)$. However solutions of the equation do obey a maximum principle after an inital layer in time. Indeed there is a time $T^{\text{pos.}}(s)$ after which the solution kernel becomes positive, and the strong maximum principle will hold.

ACKNOWLEDGEMENTS. I would like to thank Heping Ma and David Gottlieb of the Division of Applied Mathematics at Brown University for drawing my attention to this problem, and Jochen Denzler of the Technische Universität - München for his advice.

2. Results. Let's start the analysis with the solution kernels of the linear parabolic equation (2). There are two cases to consider, $x \in [0, 2\pi)$ with periodic boundary conditions, and $x \in \mathbb{R}$. The integral kernels of the respective evolution operators are called

$$K_s^{\text{per}}(x,t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ikx} e^{-k^{2s}t} \tag{6}$$

and

$$K_s(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-k^{2s}t} dk \ . \tag{7}$$

Using (6), the nonlinear equation (1) can be rewritten using Duhamel's principle

$$\begin{aligned} u(x,t) &= \int_0^{2\pi} K_s^{\text{per}}(x-y,t) u_0(y) dy \\ &\quad - \int_0^t \int_0^{2\pi} K_s^{\text{per}}(x-y,t-\tau) \partial_y f(u(y,\tau)) dy d\tau \\ &= (K_s^{\text{per}} * u_0)(x,t) \\ &\quad - \int_0^t \int_0^{2\pi} \partial_x K_s^{\text{per}}(x-y,t-\tau) f(u(y,\tau)) dy d\tau. \end{aligned} \tag{8}$$

The main *a priori* L^∞ estimate of this paper is the following. Suppose that $|f(u)| \le C_0(|u|^d)$.

THEOREM 1. *Let $u(x,t)$ be the solution of the nonlinear parabolic equation (1), with initial data $u_0(x) \in L^\infty$. Then there is a uniform L^∞ estimate in time,*

$$\|u(x,t)\|_{L^\infty} \le \overline{C}(s)(\|u_0(x)\|_{L^\infty} + \|u_0(x)\|_{L^\infty}^d)$$

with $\overline{C}(s) = O(s)$ for large s. When $t > T(s) \sim \exp(-s\log(2))$ is outside of the initial layer, an estimate holds for the solution that is uniform in s as $s \to \infty$. Indeed

$$\|u(x,t)\|_{L^\infty} \le \overline{C}_0(\|u_0(x)\|_{L^\infty} + \|u_0(x)\|_{L^\infty}^d),$$

where $\overline{C}_0$ is independent of s, and independent of t for $\exp(-s\log(2)) \le t < +\infty$.

The following results are some of the ingredients that will go into the proof of the main theorem.

THEOREM 2. *Let $u(x,t)$ be a smooth solution to equation (1), then the following elementary estimates hold,*

$$
\begin{aligned}
&(i) \quad \int_0^{2\pi} u(x,t)dx = \int_0^{2\pi} u_0(x)dx = 2\pi\overline{u_0} \\
(9) \qquad &(ii) \quad \|u(x,t)\|_{L^2} \le \|u_0(x)\|_{L^2} \\
&(iii) \quad \int_0^t \|\partial_x^s u(x,\tau)\|_{L^2}^2 d\tau \le \frac{1}{2}\|u_0(x)\|_{L^2}^2.
\end{aligned}
$$

Proof. These facts depend of course on the form of the nonlinear equation. Certainly

$$\partial_t \int_0^{2\pi} u(x,t)dx = \int_0^{2\pi} (-\partial_x f(u) + (-1)^{s-1}\partial_x^{2s} u)dx = 0$$

for spatially periodic solutions $u(x,t)$. The L^2 estimates follow from the calculation

$$
\begin{aligned}
\frac{1}{2}\partial_t \int_0^{2\pi} u^2(x,t)dx &= \int_0^{2\pi} u(-f'(u)\partial_x u)dx + (-1)^{s-1}\int_0^{2\pi} u\partial_x^{2s} u dx \\
&= \int_0^{2\pi} \partial_x g(u)dx - \int_0^{2\pi} (\partial_x^s u)^2 dx,
\end{aligned}
$$

where $\partial_u g(u) = uf'(u)$. The first term vanishes, and the second term gives a positive contribution, when integrated over a time interval:

$$\frac{1}{2}\|u(x,t)\|_{L^2}^2 + \int_0^t \|\partial_x^s u(x,\tau)\|_{L^2}^2 d\tau = \frac{1}{2}\|u_0(x)\|_{L^2}^2, \qquad (10)$$

which gives statements (9) (ii) (iii). □

Estimates on the integral kernels K_s and K_s^{per}, as well as $\partial_x K_s^{\text{per}}$ will play a role, and we use one of the simplest criteria for boundedness of an integral operator:

PROPOSITION 3. *For $1 \le p \le +\infty$ there are estimates of the solution operators for equation (2) on L^p;*

$$\|v(x,t)\|_{L^p} \le C\|v_0(x)\|_{L^p},$$

with $C = \int_0^{2\pi} |K_s^{\text{per}}(x,t)|dx = \int_{-\infty}^{\infty} |K_s(x,t)|dx$. The constant may depend upon t and/or s.

In case $s = 1$ it is a simple fact that $K_1 > 0$ for $t > 0$, so that the absolute value signs are superfluous, and one may take $C = 1$. For $s \ge 2$ the kernel may change sign (in the periodic case it will change sign for t small), and in general the constant is bigger. We will be more precise as to the behavior of the solution kernel.

THEOREM 4. *Consider the linear equation (2) posed on the line $x \in I\!R$. The solution kernel (7) satisfies the estimate*

$$|K_s(x,t)| \le \frac{C_0}{t^{\frac{1}{2s}}} e^{-\frac{1}{C(s)}(\frac{x^{2s}}{t})^{\frac{1}{2s-1}}} ((\frac{x}{t^{\frac{1}{2s}}})^{\frac{1}{2s-1}} + 1) \qquad (11)$$

with constant $C^{-1}(s) = (\tan(\frac{\pi}{8s})/2s)^{\frac{2s}{2s-1}}(2s-1)$. *Through proposition 3 this implies that the linear equation (2), posed on* $I\!R$*, satisfy an* a priori L^∞ *estimate*

$$\|v(x,t)\|_{L^\infty(I\!R)} \le C(s)\|v_0(x)\|_{L^\infty(I\!R)}. \tag{12}$$

Remark that $C(s) = O(s)$, and that as stated in the introductory paragraph, this is an estimate valid uniformly in t. The proof of this theorem will be given in section 4, along with the proofs of the next two statements about L^1 estimates of the solution kernel.

THEOREM 5. *Consider the linear equation (2), posed with periodic boundary conditions on* $0 < x < 2\pi$,

$$v(x,t) = v(x+2\pi, t).$$

Then $v(x,t)$ *satisfies the additional estimates*

$$\|v(x,t)\|_{L^\infty} \le C(s)\|v_0(x)\|_{L^\infty} \tag{13)(i}$$

$$\|v(x,t) - \overline{v}_0\|_{L^\infty} \le \frac{C_0 e^{-t(\frac{2s-1}{2s})}}{t^{\frac{1}{2s}}}\|v_0(x) - \overline{v}_0\|_{L^\infty}. \tag{13)(ii}$$

Recall from (9)(i) that $\overline{v}_0 = \frac{1}{2\pi}\int_0^{2\pi} v_0(x)dx$. For large time t, estimate (13) is stronger, while the estimate (12) provides a uniform estimate for equation (2) over the half line $0 \le t < +\infty$. The time at which there is this exchange of strength of these L^∞ estimates occurs for $\tilde{T}(s) \lesssim \exp(-2s\log(s))$. More relevant 'though is the time $T(s)$ where for $t > T(s)$, $\|v(x,t)\|_{L^\infty} \le 2\|v_0(x)\|_{L^\infty}$. From the exponent of (13) this is clearly for $T(s) \sim e^{-s\log(2)}$, giving an exponentially small initial layer for the linear problem.

One notices in (8) that the L^∞ estimate also may depend upon the derivative of the solution kernel with respect to x. For purposes of the nonlinear theorem we have the following bounds on the L^1 norm of the derivative of K_s.

THEOREM 6. *The following estimate holds*

$$\int_0^{2\pi} |\partial_x K_s^{\text{per}}(x,t)|dx \le \frac{C_1}{t^{\frac{1}{s}}} e^{-t(\frac{2s-1}{2s})}, \tag{14)(i}$$

where C_1 *is independent of* s *and* t. *Furthermore, for* $t > T(s)$,

$$\|\partial_x K_s^{\text{per}}(x,t)\|_{L^\infty} \le C_0 e^{-t(\frac{2s-1}{2s})}. \tag{14)(ii}$$

3. Proof of Theorem 1. Before giving the details of proof of the linear L^∞ estimates, we will refer to them in order to give the nonlinear result. Using the expression (8) of Duhamel's principle, we see

$$\begin{aligned} \|u(x,t)\|_{L^\infty} \le &\left\| \int_0^{2\pi} K_s^{\text{per}}(x-y,t)u_0(y)dy \right\|_{L^\infty} \\ &+ \int_0^t \left\| \int_0^{2\pi} \partial_x K_s^{\text{per}}(x-y,t-\tau)f(u(y,\tau))dy \right\|_{L^\infty} d\tau. \end{aligned} \tag{15}$$

The first term of the R.H.S. obeys the direct linear estimates, either (12) of Theorem 4 in the initial layer $0 < t \le \exp(-s\log(2))$ or by using (9)(i) and (13),

$$\begin{aligned}(16)\qquad &\Big\| \int_0^{2\pi} K_s^{\rm per}(x-y,t)u_0(y)dy \Big\|_{L^\infty} \\ &\le \Big\| \int_0^{2\pi} K_s^{\rm per}(x-y,t)u_0(y)dy - \overline{u}_0 \Big\|_{L^\infty} + |\overline{u}_0| \\ &\le (1 + 2C_0 e^{-t(\frac{2s-1}{2s})})\|u_0\|_{L^\infty}.\end{aligned}$$

Of course the principal effort of the proof is to also control the nonlinear contributions to the growth of $\|u(x,t)\|_{L^\infty}$, corresponding to the second term of the R.H.S. of (15):

$$\begin{aligned}(17)\qquad &\int_0^t \Big\| \int_0^{2\pi} \partial_x K_s^{\rm per}(x-y,t-\tau)f(u(y,\tau))dy \Big\|_{L^\infty} d\tau \\ &\le \int_0^{t-T(s)} C_0 e^{-(t-\tau)(\frac{2s-1}{2s})}\|f(u)\|_{L^1}d\tau + \int_{t-T(s)}^t \frac{C_1}{(t-\tau)^{\frac{1}{s}}}\|f(u(y,\tau))\|_{L^\infty}d\tau\end{aligned}$$

where the first term used (14)(ii), and the second term used (14)(i) in the estimate of the R.H.S.

To simplify this estimate, let's make the particular choice that $f(u) = u^2$. Then

$$(18)\qquad \|f(u(x,\tau))\|_{L^1} = \|u(x,\tau)\|_{L^2}^2 \le \|u_0(x)\|_{L^2}^2$$

by Theorem 2, (9)(ii), therefore we find that

$$(19)\qquad \int_0^{t-T(s)} C_0 e^{-(t-\tau)(\frac{2s-1}{2s})}\|f(u)\|_{L^1}d\tau \le C_0\|u_0(x)\|_{L^\infty}^2.$$

If $0 < t < T(s)$, we remark that this term does not even appear in (17). Assume then that $T(s) \le t$, we will pursue the second term;

$$\begin{aligned}(20)\qquad &\int_{t-T(s)}^t \frac{1}{(t-\tau)^{\frac{1}{s}}}\|f(u(x,\tau))\|_{L^\infty}d\tau \\ &\le \left(\int_0^{T(s)} \left(\frac{1}{\tau'}\right)^{p'/s} d\tau'\right)^{1/p'} \left(\int_{t-T(s)}^t \|f(u)\|_{L^\infty}^p d\tau\right)^{1/p},\end{aligned}$$

with the usual Hölder inequality pairing $\frac{1}{p} + \frac{1}{p'} = 1$. Setting $p' = s-1$ (close but not equal to s)

$$(21)\qquad \left(\int_0^{T(s)} \left(\frac{1}{\tau'}\right)^{\frac{s-1}{s}} d\tau'\right)^{\frac{1}{s-1}} = s^{\frac{1}{s-1}} T(s)^{\frac{1}{s(s-1)}}.$$

Recall that $T(s) \sim \exp(-s\log(2))$, then it is easy to see that this factor is uniformly bounded in $s \ge 2$.

Again to simplify the argument, assume that $f(u) = u^2$ as above. From (20) we need to bound the factor

$$(22)\qquad \left(\int_{t-T(s)}^t \|u^2\|_{L^\infty}^p d\tau\right)^{1/p},$$

with $p = (s-1)/(s-2)$. (In case $s = 2$, set $p' = s - 1/2$ and adjust this argument accordingly).

LEMMA 7. *We have estimates of the form*

$$\int_{t-T(s)}^{t} \|u(x,\tau)\|_{L^\infty}^2 d\tau \le C_0\Big(T(s)\|u_0(x)\|_{L^2}^2 + \int_{t-T(s)}^{t} \|\partial_x^s u(x,\tau)\|_{L^2}^2 d\tau\Big) \tag{23}$$
$$\le C_0(T(s)+1)\|u_0(x)\|_{L^\infty}^2.$$

Proof. This is the Sobolev lemma, used in conjunction with Theorem 2, (9)(i) and (ii). For each τ in question,

$$\|u(x,\tau)\|_{L^\infty(x)}^2 \le C_0(\|u(x,\tau)\|_{L^2(x)}^2 + \|\partial_x^s u(x,\tau)\|_{L^2(x)}^2)$$

by the Sobolev lemma, as $s \ge 2$ ($s > 1/2$ will do). Integrating over the time interval $(t - T(s), t)$, we obtain

$$\int_{t-T(s)}^{t} \|u(x,\tau)\|_{L^\infty(x)}^2 d\tau \le \int_{t-T(s)}^{t} C_0\|u(x,\tau)\|_{L^2(x)}^2 d\tau + \int_{t-T(s)}^{t} C_0\|\partial_x^s u(x,\tau)\|_{L^2(x)}^2 d\tau.$$

Using (9)(i) on the first term, and (9)(ii) on the second,

$$\le C_0 T(s)\|u_0(x)\|_{L^2}^2 + C_0\|u_0(x)\|_{L^2}^2.$$

Since the L^∞ norm bounds the L^2 norm over the circle, this finishes the proof. □

Returning to expression (22),

$$\left(\int_{t-T(s)}^{t} \|u^2\|_{L^\infty}^p d\tau\right)^{1/p} \tag{24}$$
$$\le \sup_{t-T(s)\le\tau\le t} \|u(x,\tau)\|_{L^\infty(x)}^Q \times \left(\int_{t-T(s)}^{t} \|u(x,\tau)\|_{L^\infty}^2 d\tau\right)^{1/p},$$

where the exponent is $Q = 2 - 2/p = 2/(s-1)$,

$$\le \left(\sup_{t-T(s)\le\tau\le t} \|u(x,\tau)\|_{L^\infty(x)}\right)^Q \times C_0\|u_0(x)\|_{L^\infty}^{\frac{2s-4}{s-1}}. \tag{25}$$

We first prove the estimate which is uniform in $t > 0$, which however blows up as $s \to \infty$. Define

$$M(t) = \sup_{0\le\tau\le t} \|u(x,\tau)\|_{L^\infty},$$

then combining (12) and (19) (25) to control (15), we find an expression

$$\sup_{0\le\tau\le t} \|u(x,\tau)\|_{L^\infty} \tag{26}$$
$$\le C(s)\|u_0(x)\|_{L^\infty} + C_0\|u_0(x)\|_{L^\infty}^2 + C_0\left(\sup_{0\le\tau\le t} \|u(x,\tau)\|_{L^\infty}\right)^Q \|u_0(x)\|_{L^\infty}^{\frac{2s-4}{s-1}}.$$

When $s > 3$ the exponent $Q < 1$, resulting in the expression

$$M(t) \le C(\|u_0(x)\|_{L^\infty})(1 + M^Q(t)) \ , \tag{27)(i}$$

which implies that

$$M(t) \le (C(s)\|u_0(x)\|_{L^\infty} + C_0\|u_0(x)\|^2_{L^\infty})^{\frac{1}{1-Q}}. \tag{27)(ii}$$

This gives the uniform in time estimate with constant $\overline{C}(s) \simeq s^{(s-1)/(s-3)} \simeq O(s)$.

Now consider the case where $t > T(s)$ is outside of the initial layer. Reiterating (15), we have the expression that

$$\begin{aligned} \|u(x,t)\|_{L^\infty} \le & \left\| \int_0^{2\pi} K_s^{\text{per}}(x-y,\tau)u_0(y)dy \right\|_{L^\infty} \\ & + \int_0^{t-T(s)} C_0 e^{-(t-\tau)(\frac{2s-1}{2s})} \|f(u)\|_{L^1} d\tau \\ & + \int_{t-T(s)}^t \frac{C_0}{(t-\tau)^{\frac{1}{s}}} \|f(u(y,\tau))\|_{L^\infty} d\tau. \end{aligned} \tag{28}$$

Assuming again that $f(u) = u^2$; the first two terms are bounded by

$$(1 + 2C_0 e^{-t(\frac{2s-1}{2s})})\|u_0\|_{L^\infty} + C_0\|u_0\|^2_{L^\infty}.$$

The estimate for the last integral again uses the Hölder inequality in time:

$$\begin{aligned} & \int_{t-T(s)}^t \frac{C_0}{(t-\tau)^{\frac{1}{s}}} \|u^2(y,\tau)\|_{L^\infty} d\tau \\ & \le \left(\int_0^{T(s)} \frac{C_0}{(\tau')^{p'/s}} d\tau' \right)^{1/p'} \left(\int_{t-T(s)}^t \|u(y,\tau)\|^{2p}_{L^\infty} d\tau \right)^{1/p}. \end{aligned} \tag{29}$$

In this case make a different choice for p, p', namely $p' = 2$ will do (or if $s = 2, p' = 3/2$). The R.H.S. is bounded by

$$C_0 T(s)^{\frac{s-2}{2s}} \left(\frac{s}{s-2} \right)^{1/2} \left(\int_{t-T(s)}^t d\tau \right) M^2(\tau). \tag{30}$$

Use the estimate (27) of $M(t)$ which is globally valid, we find that

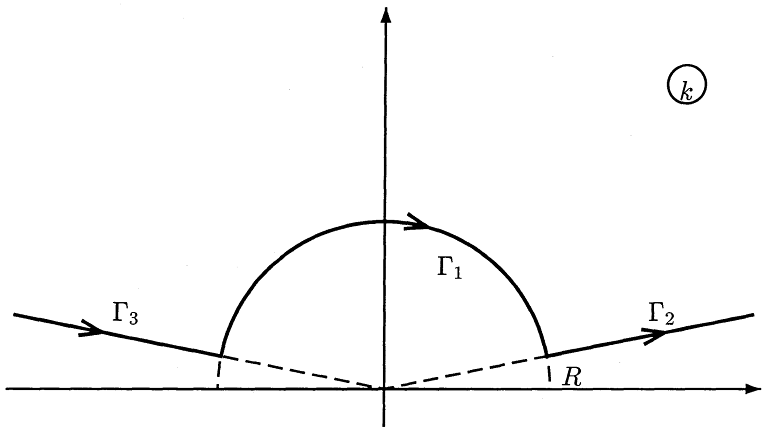
$$(29) \le C_0 T^{1+\frac{s-2}{2s}}(s) C(s)^{\frac{2}{(1-Q)}} \left(\|u_0(x)\|_{L^\infty} + \|u_0(x)\|^2_{L^\infty} \right)^{\frac{2}{1-Q}}.$$

Since $T(s) \sim \exp(-s\log(2))$ this term is clearly uniformly bounded as $s \to \infty$. This finishes the second estimate of Theorem 1.

4. Linear estimates. This section finishes the analysis of the paper, by verifying the estimates of the linear equation (2) that are stated in theorems 4, 5, and 6. The Fourier representation of the solution kernel for (2) posed on the line $x \in \mathbb{R}$, is that

$$K_s(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-k^{2s}t} dk = \frac{1}{2\pi t^{\frac{1}{2s}}} \int_{-\infty}^{\infty} e^{iky} e^{-k^{2s}} dk, \quad y = x/t^{\frac{1}{2s}}. \tag{31}$$

Assume without loss of generality that $x \geq 0$ (the kernel is symmetric), we will deform the contour of integration in (31) into the upper half-plane, to an appropriately chosen position. The one that we choose consists of an arc Γ_1 of radius R and two rays, Γ_2, Γ_3, at angles $\pi/8s$ and $\pi - \pi/8s$, as pictured here:



In the exponent of (31), $iyk - k^{2s}$ considered along the arc Γ_1, the real part is bounded by $|\mathrm{re}\,(iyk - k^{2s})| \leq -y\tan(\pi/8s)R + R^{2s}$. This is minimized over R by the choice $y\tan(\pi/8s) = 2sR^{2s-1}$, hence $R = (\tan(\pi/8s)/2s)^{\frac{1}{2s-1}} y^{\frac{1}{2s-1}}$. Notice that the constant $(\tan(\pi/8s)/2s)^{\frac{1}{2s-1}} = O(1)$ is bounded for large s. The integral over Γ_1 therefore has the bound

$$(32) \qquad \left| \int_{\Gamma_1} e^{iky} e^{-k^{2s}} dk \right| \leq \int_{\pi/8s}^{\pi-\pi/8s} e^{-y\tan(\frac{\pi}{8s})R + R^{2s}} R d\theta$$
$$\leq e^{-(2s-1)(\tan(\pi/8s)/2s)^{\frac{2s}{2s-1}} y^{\frac{2s}{2s-1}}} \left(\frac{\tan(\frac{\pi}{8s})}{2s} \right)^{\frac{1}{2s}} y^{\frac{1}{2s-1}}.$$

The constant in the exponent is $C^{-1}(s) = (2s-1)(\tan(\frac{\pi}{8s})/2s)^{\frac{2s}{2s-1}} = O(s^{-1})$.

The estimates over the contours Γ_2 and Γ_3 are similar; one notices that

$$(33) \qquad \left| \int_{\Gamma_2} e^{iky} e^{-k^{2s}} dk \right| \leq \int_R^{+\infty} e^{-\rho y \tan(\pi/8s)} e^{-\rho^{2s}/\sqrt{2}} d\rho$$
$$\leq e^{-Ry\tan(\pi/8s)} \int_R^{+\infty} e^{-\rho^{2s}/\sqrt{2}} d\rho$$
$$\leq e^{-C^{-1}(s) y^{\frac{2s}{2s-1}}}.$$

This finishes the proof of (11). To estimate the norm of the evolution operation from $L^\infty(I\!R)$ to $L^\infty(I\!R)$, it suffices by proposition 3 to estimate the integral

$$(34) \qquad \int_{-\infty}^{\infty} |K_s(x,t)| dx \leq C_0 \int_{-\infty}^{\infty} e^{-C^{-1}(s)(\frac{x}{t^{\frac{1}{2s}}})^{\frac{2s}{2s-1}}} \left(\left(\frac{x}{t^{\frac{1}{2s}}} \right)^{\frac{1}{2s-1}} + 1 \right) \frac{dx}{t^{\frac{1}{2s}}}$$
$$\leq 2C_0 \int_0^\infty e^{-C^{-1}(s) y^{\frac{2s}{2s-1}}} \left(y^{\frac{1}{2s-1}} + 1 \right) dy$$
$$= 2C_0 \left(C(s) \left(\frac{2s}{2s-1} \right) + C(s) \right),$$

which demonstrates the estimate (12).

The kernel of the problem on the line is now used to estimate the problem on the circle $0 \leq x < 2\pi$. By the method of images,

$$K_s^{\mathrm{per}}(x,t) = \sum_{j=-\infty}^{\infty} K_s(x+2\pi j, t),$$

therefore

$$\int_0^{2\pi} |K_s^{\mathrm{per}}(x,t)|dx = \sum_{j=-\infty}^{\infty} \int_0^{2\pi} |K_s^{\mathrm{per}}(x+2\pi j,t)|dx \tag{35}$$
$$= \int_{-\infty}^{\infty} |K_s(x,t)|dx \leq C(s),$$

which through proposition 3 gives the first estimate (13)(i). To analyse the second estimate, we use the Fourier series expression of the solution kernel

$$K_s^{\mathrm{per}}(x,t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} e^{ikx} e^{-k^{2s}t} = \frac{1}{2\pi} + \sum_{k\geq 1} \frac{1}{\pi}\cos(kx) e^{-k^{2s}t}. \tag{36}$$

When one considers $v(x,t) - \overline{v}_0$ it is tantamount to considering the sum over $k \geq 1$, which has an L^1 estimate

$$\int_0^{2\pi} \left| \sum_{k\geq 1} \frac{1}{\pi}\cos(kx)e^{-k^{2s}t} \right| dx \leq \sum_{k\geq 1} \frac{4}{\pi} e^{-k^{2s}t} \leq \frac{4}{\pi}e^{-t} + \frac{4}{\pi}\sum_{k\geq 2} e^{-k^{2s}t}.$$

The last sum is bounded by an integral, as $e^{-k^{2s}t}$ is decreasing in k,

$$\frac{4}{\pi}\sum_{k\geq 2} e^{-k^{2s}t} \leq \frac{4}{\pi}\int_1^{+\infty} e^{-k^{2s}t}dk = \frac{4}{\pi}\int_t^{\infty} \frac{1}{2st^{\frac{1}{2s}}} \xi^{\frac{1}{2s}-1} e^{-\xi} d\xi \tag{37}$$
$$\leq \left(\frac{2s}{2s-1}\right) t^{-\frac{1}{2s}} e^{-t(\frac{2s-1}{2s})}$$

Estimate (13)(ii) follows from this statement, which finishes the proof of Theorem 5.

The final result that is needed occurs in theorem 6, where we must estimate derivatives of the kernel. It is sufficient to study the sum

$$\sum_{k\geq 1} ke^{-k^{2s}t} = e^{-t} + \sum_{k\geq 2} ke^{-k^{2s}t}. \tag{38}$$

The function $\log(k) - k^{2s}t$ has a maximum at $k = (2st)^{-\frac{1}{2s}}$, therefore

$$\sum_{k\geq 2} ke^{-k^{2s}t} \leq \int_1^{(\frac{1}{2st})^{\frac{1}{2s}}} \left(\frac{1}{2st}\right)^{\frac{1}{2s}} e^{-k^{2s}t}dk + \int_{(\frac{1}{2st})^{\frac{1}{2s}}}^{+\infty} ke^{-k^{2s}t}dk, \tag{39}$$

where the first term is taken to be zero if $(1/2st) \leq 1$. As above, the first term can be estimated by the quantity $(2s)^{-\frac{1}{2s}} t^{-\frac{1}{s}} e^{-t(\frac{2s-1}{2s})}$, and the second term looks like this

$$\int_{(\frac{1}{2st})^{\frac{1}{2s}}}^{+\infty} ke^{-k^{2s}t}dk = \frac{1}{2}\int_{(\frac{1}{2st})^{\frac{1}{2s}}} e^{-k^{2s}t}d(k^2) \tag{40}$$
$$= \frac{1}{2}\int_{1/2s}^{+\infty} e^{-\xi} d\left(\frac{\xi}{t}\right)^{\frac{1}{s}} = \left(\frac{1}{2st}\right)^{\frac{1}{s}} e^{-\frac{1}{2s}},$$

which has coefficients of $t^{-1/s}$ bounded for large s. The L^∞ estimates of the kernel work in the same way, so that (14)(i) and (ii) follow from this.

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