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SOBOLEV ORTHOGONAL POLYNOMIALS: THE DISCRETE-CONTINUOUS CASE*

M. ALFARO†, T. E. PÉREZ‡, M. A. PIÑAR‡, AND M. L. REZOLA†

Abstract. In this paper, we study orthogonal polynomials with respect to the bilinear form

$$\mathcal{B}_S^{(N)}(f,g) = F(c)\mathbf{A}G(c)^T + \langle u, f^{(N)}g^{(N)}\rangle,$$

where u is a quasi-definite (or regular) linear functional on the linear space $\mathbb{P}$ of real polynomials, c is a real number, N is a positive integer number, $\mathbf{A}$ is a symmetric $N \times N$ real matrix such that each of its principal submatrices are regular, and $F(c) = (f(c), f'(c), \ldots, f^{(N-1)}(c))$, $G(c) = (g(c), g'(c), \ldots, g^{(N-1)}(c))$. For these non–standard orthogonal polynomials, algebraic and differential properties are obtained, as well as their representation in terms of the standard orthogonal polynomials associated with u.

1. Introduction. It is well known (see [12]) that the monic generalized Laguerre polynomials $\{L_n^{(\alpha)}\}_n$ satisfy, for any real value of α, the three-term recurrence relation

$$xL_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)}(x) + \beta_n^{(\alpha)}L_n^{(\alpha)}(x) + \gamma_n^{(\alpha)}L_{n-1}^{(\alpha)}(x),$$

$$L_{-1}^{(\alpha)}(x) = 0, \qquad L_0^{(\alpha)}(x) = 1,$$

where

$$\beta_n^{(\alpha)} = 2n + \alpha + 1, \qquad \gamma_n^{(\alpha)} = n(n+\alpha).$$

Whenever α is not a negative integer number, we have $\gamma_n^{(\alpha)} \neq 0$ for all $n \geq 1$ and Favard's theorem (see [2], p. 21) ensures that the sequence $\{L_n^{(\alpha)}\}_n$ is orthogonal with respect to a quasi–definite linear functional. Besides, if $\alpha > -1$ the functional is definite positive and the polynomials are orthogonal with respect to the weight $x^\alpha e^{-x}$ on the interval $(0, +\infty)$. For α a negative integer number, since $\gamma_n^{(\alpha)}$ vanishes for some value of n, no orthogonality results can be deduced from Favard's theorem.

In the last few years, orthogonal polynomials with respect to an inner product involving derivatives (the so–called Sobolev orthogonal polynomials) have been the object of increasing interest and in this context, the case $\{L_n^{(\alpha)}\}_n$ with α a negative integer number has been solved. More precisely, Kwon and Littlejohn, in [5], established the orthogonality of the generalized Laguerre polynomials $\{L_n^{(-k)}\}_n, \quad k \geq 1$, with respect to a Sobolev inner product of the form

$$\langle f, g\rangle = F(0)\mathbf{A}G(0)^T + \int_0^{+\infty} f^{(k)}(x)g^{(k)}(x)e^{-x}dx$$

with $\mathbf{A}$ a symmetric $k \times k$ real matrix, $F(0) = (f(0), f'(0), \ldots, f^{(k-1)}(0))$, and $G(0) = (g(0), g'(0), \ldots, g^{(k-1)}(0))$. The particular case $k = 1$ had been considered by the same authors in a previous paper, (see [6]).

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†Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain (alfaro@posta.unizar.es and rezola@posta.unizar.es).

‡Departamento de Matemática Aplicada, Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, 18071 Granada, Spain (tperez@goliat.ugr.es and mpinar@goliat.ugr.es).

In [10], Pérez and Piñar gave an unified approach to the orthogonality of the generalized Laguerre polynomials, for any real value of the parameter α by proving their orthogonality with respect to a Sobolev non–diagonal inner product. So, they obtained the following result:

THEOREM ([10]). *Let* $(.,.)_S^{(N,\alpha)}$ *be the Sobolev inner product defined by*

$$(f,g)_S^{(N,\alpha)} = \int_0^{+\infty} F(x)\mathbf{A}G(x)^T x^\alpha e^{-x} dx,$$

where the (i,j)*-entry of* $\mathbf{A}$ *is given by*

$$m_{i,j}(N) = \sum_{p=0}^{min\{i,j\}} (-1)^{i+j} \binom{N-p}{i-p}\binom{N-p}{j-p}, \quad 0 \le i,j \le N,$$

$F(x) = (f(x), f'(x), \dots, f^{(N)}(x))$, $G(x) = (g(x), g'(x), \dots, g^{(N)}(x))$. *Then, for every* $\alpha \in \mathbb{R}$, *the monic generalized Laguerre polynomials* $\{L_n^{(\alpha)}\}_n$ *are orthogonal with respect to* $(.,.)_S^{(N,\alpha+N)}$ *with* $N = \max\{0, [-\alpha]\}$, *(* $[\alpha]$ *denotes the greatest integer less than or equal to* α*).*

In the case when α is a negative integer, the inner product $(.,.)_S^{(N,\alpha+N)}$ is the same as the one considered by Kwon and Littlejohn.

The above results justify the interest to consider such a kind of inner products. In a more general setting, our aim is to study polynomials which are orthogonal with respect to a symmetric bilinear form such as

$$(1.1)\quad \mathcal{B}_S^{(N)}(f,g) = \left(f(c), f'(c), \dots, f^{(N-1)}(c)\right) \mathbf{A} \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix} + \langle u, f^{(N)} g^{(N)} \rangle,$$

where u is a quasi–definite (or regular) linear functional on the linear space $\mathbb{P}$ of real polynomials, c is a real number, N is a positive integer number, and $\mathbf{A}$ is a symmetric $N \times N$ real matrix such that each of its principal submatrices is regular. By analogy with the usual terminology, we call it a discrete–continuous Sobolev bilinear form. Recently some properties of the polynomials orthogonal with respect to $\mathcal{B}_S^{(1)}(.,.)$ had been considered in [4].

We will emphasize some cases in which the functional u satisfies some extra conditions, namely, u is a semiclassical or a classical linear functional (see [3], [7] and [9]). A quasi–definite linear functional u is called semiclassical if there exist polynomials ϕ and ψ with $\deg\phi \ge 0$ and $\deg\psi \ge 1$ such that u satisfies the distributional differential equation $\mathcal{D}(\phi u) = \psi u$. Whenever $\deg\phi \le 2$ and $\deg\psi = 1$, the functional u is called classical. It is well known that the only classical functionals correspond to the sequences of Hermite, Laguerre, Jacobi and Bessel polynomials.

In Section 2, we give a description of the monic polynomials $\{Q_n\}_n$ which are orthogonal with respect to $\mathcal{B}_S^{(N)}(.,.)$ in terms of the monic polynomials $\{P_n\}_n$ orthogonal with respect to the functional u. In particular, for $n \ge N$, we have that $Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x)$ and $Q_n^{(k)}(c) = 0$ for $k = 0, 1, ..., N-1$, while $\{Q_n\}_{n=0}^{N-1}$ are orthogonal with respect to the discrete part of the symmetric bilinear form (1.1) and they are determined by the matrix $\mathbf{A}$.

By using these results, in Section 3, we give some examples of polynomials orthogonal with respect to (1.1), with an adequate choice of c, namely, Laguerre polynomials $\{L_n^{(-N)}\}_n$ with $c = 0$, Jacobi polynomials $\{P_n^{(-N,\beta)}\}_n$ with $c = 1$, $\beta + N$ not being a negative integer, and $\{P_n^{(\alpha,-N)}\}_n$ with $c = -1$, $\alpha + N$ not being a negative integer. Note that these sequences of polynomials are not orthogonal with respect to any quasi-definite linear functional.

In Section 4, we give a new characterization of classical polynomials as the only orthogonal polynomials such that, for some positive integer number N, they have a N-th primitive satisfying a three-term recurrence relation. In particular, this result is applied to discrete-continuous Sobolev polynomials which satisfy a three-term recurrence relation and then it follows that u is classical with distributional differential equation $\mathcal{D}(\phi u) = \psi u$, and the point c in (1.1) is such that $\phi(c) = 0$ and $\psi(c) = \phi'(c)$. Hence, the only monic discrete-continuous Sobolev polynomials which satisfy a three-term recurrence relation are the ones described in Section 3.

The link between Sobolev orthogonality and polynomials satisfying a second order differential equation is analyzed in Section 5. It is proved that if the sequence $\{Q_n\}_n$ satisfies the equation

$$\phi(x)Q_n''(x) + \sigma(x)Q_n'(x) = \rho_n Q_n(x),$$

where ϕ and σ are polynomials with degree less than or equal to 2 and 1, respectively, and ρ_n are real numbers, then the functional u is classical with distributional differential equation $\mathcal{D}(\phi u) = \psi u$, $\sigma(x) = \psi(x) - N\phi'(x)$ and the point c in (1.1) verifies $\phi(c) = 0$ and $\psi(c) = \phi'(c)$. Hence, the only monic discrete-continuous Sobolev polynomials which satisfy a second order differential equation are again the described in Section 3.

As a consequence of the results in Sections 4 and 5, we have that if u is not a classical linear functional then the sequence $\{Q_n\}_n$ does not satisfy neither a three-term recurrence relation nor a second order differential equation. In order to avoid this lack in our study, in Section 6, we introduce a linear differential operator $\mathcal{F}^{(N)}$ on $\mathbb{P}$ symmetric with respect to the bilinear form (1.1). The basic property of this operator is a relationship between the Sobolev bilinear form and the bilinear form associated with the functional u. Handling with $\mathcal{F}^{(N)}$ we can deduce explicit relations between $\{Q_n\}_n$ and $\{P_n\}_n$ as well as a differential substitute of the algebraic recurrence relations. This is done in Section 7.

2. The Sobolev discrete-continuous bilinear form. Let $\mathbb{P}$ be the linear space of real polynomials, u a quasi-definite linear functional on $\mathbb{P}$ (see [2]), N a positive integer number, and $\mathbf{A}$ a quasi-definite and symmetric real matrix of order N, that is, a symmetric and real matrix such that all the principal minors are different from zero. For a given real number c, the expression

$$(2.1)\quad \mathcal{B}_S^{(N)}(f,g) = \left(f(c), f'(c), \ldots, f^{(N-1)}(c)\right) \mathbf{A} \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix} + \langle u, f^{(N)} g^{(N)} \rangle,$$

defines a symmetric bilinear form on $\mathbb{P}$.

Since expression (2.1) involves derivatives, this bilinear form is non–standard, and by analogy with the usual terminology we will call it a *discrete–continuous Sobolev bilinear form.*

In the linear space of real polynomials, we can consider the basis given by

$$\left\{ \frac{(x-c)^m}{m!} \right\}_{m\geq 0}. \tag{2.2}$$

For $n \leq N-1$, the associated Gram matrix $\mathbf{G}_n$ is given by the n–th order principal submatrix of the matrix $\mathbf{A}$. For $n \geq N$, the associated Gram matrix is given by

$$\mathbf{G}_n = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B}_{n-N} \end{array} \right),$$

where $\mathbf{B}_{n-N}$ is the Gram matrix associated with the quasi–definite linear functional u in the basis (2.2).

In both cases, $\mathbf{G}_n$ is quasi–definite (that is, all the principal minors are different from zero) and therefore, we will say that the discrete–continuous Sobolev bilinear form (2.1) is quasi–definite. Thus, we can assure the existence of a sequence of monic polynomials, denoted by $\{Q_n\}_n$, which are orthogonal with respect to (2.1). These polynomials will be called *Sobolev orthogonal polynomials.* Our first aim is to relate this sequence with the monic orthogonal polynomial sequence (in short MOPS) $\{P_n\}_n$ associated with the quasi–definite linear functional u.

THEOREM 2.1. *Let $\{Q_n\}_n$ be the sequence of monic orthogonal polynomials with respect to the bilinear form $\mathcal{B}_S^{(N)}$.*
i) The polynomials $\{Q_n\}_{n=0}^{N-1}$ are orthogonal with respect to the discrete bilinear form

$$\mathcal{B}_D^{(N)}(f,g) = \left(f(c), f'(c), \dots, f^{(N-1)}(c)\right) \mathbf{A} \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix}. \tag{2.3}$$

ii) If $n \geq N$, then

$$Q_n^{(k)}(c) = 0, \quad k = 0, 1, \dots, N-1, \tag{2.4}$$

$$Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x). \tag{2.5}$$

Proof. i) If $0 \leq m, n < N$, then $Q_n^{(N)}(x) = Q_m^{(N)}(x) = 0$, and the value of the Sobolev bilinear form on (Q_n, Q_m) can be computed by means of the following expression

$$\begin{aligned} \mathcal{B}_S^{(N)}(Q_n, Q_m) &= \mathcal{B}_D^{(N)}(Q_n, Q_m) \\ &= \left(Q_n(c), Q_n'(c), \dots, Q_n^{(N-1)}(c)\right) \mathbf{A} \begin{pmatrix} Q_m(c) \\ Q_m'(c) \\ \vdots \\ Q_m^{(N-1)}(c) \end{pmatrix}, \end{aligned}$$

and therefore they are orthogonal with respect to the discrete bilinear form (2.3).

ii) Let $n \geq N$, then from the orthogonality of the polynomial Q_n, we deduce

$$(2.6)\quad 0 = \mathcal{B}_S^{(N)}(Q_n(x), \frac{1}{k!}(x-c)^k) = \left(Q_n(c), Q_n'(c), \ldots, Q_n^{(N-1)}(c)\right) \mathbf{A} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

for $0 \leq k \leq N-1$. Thus, the vector

$$\left(Q_n(c), Q_n'(c), \ldots, Q_n^{(N-1)}(c)\right)$$

is the only solution of a homogeneous linear system with N equations and N unknowns, whose coefficient matrix $\mathbf{A}$ is regular. Then, we conclude $Q_n^{(k)}(c) = 0, \quad k = 0, 1, \ldots, N-1$, that is, Q_n contains the factor $(x-c)^N$.
In this way, if $n, m \geq N$, the discrete part of the bilinear form $\mathcal{B}_S^{(N)}(Q_n, Q_m)$ vanishes and we get

$$\mathcal{B}_S^{(N)}(Q_n, Q_m) = \langle u, Q_n^{(N)} Q_m^{(N)} \rangle = \tilde{k}_n \delta_{n,m}, \quad \tilde{k}_n \neq 0.$$

That is, the polynomials $\{Q_n^{(N)}\}_{n \geq N}$ are orthogonal with respect to the linear functional u, and equality (2.5) follows from a simple inspection of the leading coefficients.
□

Reciprocally, we are going to show that a system of monic polynomials $\{Q_n\}_n$ satisfying equations (2.4) and (2.5) is orthogonal with respect to some discrete–continuous Sobolev bilinear form. This result could be considered a *Favard–type theorem.*

THEOREM 2.2. *Let $\{P_n\}_n$ be the MOPS associated with a quasi–definite linear functional u, and $N \geq 1$ a given integer number. Let $\{Q_n\}_n$ be a sequence of monic polynomials satisfying*
i) $\deg Q_n = n, \quad n = 0, 1, 2, \ldots,$
ii) $Q_n^{(k)}(c) = 0, \quad 0 \leq k \leq N-1, \quad n \geq N,$
iii) $Q_n^{(N)}(x) = \dfrac{n!}{(n-N)!} P_{n-N}(x), \quad n \geq N.$

Then, there exists a quasi–definite and symmetric real matrix $\mathbb{A}$, of order N, such that $\{Q_n\}_n$ is the monic orthogonal polynomial sequence associated with the Sobolev bilinear form defined by (2.1).

Proof. By using the same reasoning as above it is obvious that every polynomial Q_n, with $n \geq N$, is orthogonal to every polynomial with degree less than or equal to $n-1$ with respect to a Sobolev bilinear form like (2.1) containing an arbitrary matrix $\mathbf{A}$ in the discrete part and the functional u in the second part.

Next, we show that we can recover the matrix $\mathbf{A}$ from the N first polynomials $Q_k, k = 0, 1, \ldots, N-1$.

Let us denote

$$\mathbf{Q} = \begin{pmatrix} Q_0(c) & Q_0'(c) & \cdots & Q_0^{(N-1)}(c) \\ Q_1(c) & Q_1'(c) & \cdots & Q_1^{(N-1)}(c) \\ \vdots & \vdots & & \vdots \\ Q_{N-1}(c) & Q_{N-1}'(c) & \cdots & Q_{N-1}^{(N-1)}(c) \end{pmatrix},$$

then $\mathbf{Q}$ is a lower triangular and invertible matrix. Let $\mathbf{D}$ be a diagonal matrix with non zero elements in its diagonal.

Define

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T.$$

Obviously $\mathbf{A}$ is symmetric and quasi–definite and since

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^T = \mathbf{D},$$

the polynomials $Q_0, \ldots, Q_{N-1}$ are orthogonal with respect to the bilinear form (2.1), with the matrix $\mathbf{A}$ in the discrete part. Besides, the elements in the diagonal of $\mathbf{D}$ are the values $\mathcal{B}_S^{(N)}(Q_k, Q_k)$ for $k = 0, \ldots, N-1$. □

REMARK. Observe that the matrix $\mathbf{A}$ is not unique, because its construction depends on the arbitrary regular diagonal matrix $\mathbf{D}$.

3. Classical examples.

3.1. The Laguerre case. Let $\alpha \in \mathbb{R}$, the *n–th monic generalized Laguerre polynomial* is defined in [12], p. 102, by means of its explicit representation

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n+\alpha}{n-j} x^j, \qquad n \geq 0, \tag{3.1}$$

where $\binom{a}{k}$ denotes the *generalized binomial coefficient*

$$\binom{a}{k} = \frac{(a-k+1)_k}{k!}, \tag{3.2}$$

and $(a-k+1)_k$ stands for the so-called *Pochhammer's symbol* defined by

$$(b)_0 = 1, \quad (b)_n = b(b+1)\ldots(b+n-1), \qquad \text{for } b \in \mathbb{R}, \quad n \geq 0. \tag{3.3}$$

In this way, we have

$$(L_n^{(\alpha)})^{(k)}(0) = (-1)^{n+k} n! \binom{n+\alpha}{n-k}, \quad n \geq k.$$

If α is a negative integer number, say $\alpha = -N$, for $n \geq N$, we have

$$(L_n^{(-N)})^{(k)}(0) = 0, \quad k = 0, 1, \ldots, N-1,$$

and, for $n < N$, we get

$$(L_n^{(-N)})^{(k)}(0) = n! \binom{N-k-1}{n-k}, \quad k = 0, 1, \ldots, n.$$

On the other hand, since the derivatives of Laguerre polynomials are again Laguerre polynomials, we have

$$(L_n^{(-N)})^{(N)}(x) = \frac{n!}{(n-N)!} L_{n-N}^{(0)}(x), \qquad \text{for } n \geq N.$$

Therefore, from the previous Section, we conclude that Laguerre polynomials $L_n^{(-N)}$ are orthogonal with respect to the Sobolev bilinear form

$$\mathcal{B}_S^{(N)}(f,g) = F(0)\mathbf{A}G(0)^T + \int_0^{+\infty} f^{(N)}(x)g^{(N)}(x)e^{-x}\,dx,$$

with $F(0) = (f(0), f'(0), \ldots, f^{(N-1)}(0))$, $G(0) = (g(0), g'(0), \ldots, g^{(N-1)}(0))$, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T,$$

$\mathbf{Q}$ is the matrix of the derivatives of Laguerre polynomials $L_n^{(-N)}$ evaluated at zero

$$\mathbf{Q} = \begin{pmatrix} 0!\binom{N-1}{0} & 0 & \ldots & 0 \\ 1!\binom{N-1}{1} & 1!\binom{N-2}{0} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ (N-1)!\binom{N-1}{N-1} & (N-1)!\binom{N-2}{N-2} & \ldots & (N-1)!\binom{0}{0} \end{pmatrix}$$

and $\mathbf{D}$ is an arbitrary regular diagonal matrix. Similar results have been obtained with different techniques in [5] and [10]. We recover the results in [5] by using a diagonal matrix $\mathbf{D}$ whose elements are $(0!)^2$, $(1!)^2$, ..., $((N-1)!)^2$.

3.2. The Jacobi case. For α and β arbitrary real numbers, the *generalized Jacobi polynomials* can be defined (see [12], p. 62) by means of their explicit representation

$$\mathcal{P}_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{m}\binom{n+\beta}{n-m}\left(\frac{x-1}{2}\right)^{n-m}\left(\frac{x+1}{2}\right)^{m}, \qquad n \geq 0.$$

When α, β and $\alpha+\beta+1$ are not a negative integer, Jacobi polynomials are orthogonal with respect to the quasi–definite linear functional $u^{(\alpha,\beta)}$. This linear functional is positive definite for $\alpha > -1$ and $\beta > -1$.

For $\alpha = -N$, with N a positive integer, and β being not a negative integer, the *n–th monic generalized Jacobi polynomial* is given by

$$P_n^{(-N,\beta)}(x) = \binom{2n-N+\beta}{n}^{-1} \sum_{m=0}^{n} \binom{n-N}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m. \tag{3.4}$$

In this case, for $n \geq N$, $x = 1$ will be a zero of multiplicity N ([12], p. 65).

On the other hand, since the derivatives of Jacobi polynomials are again Jacobi polynomials, we have

$$(P_n^{(-N,\beta)})^{(N)}(x) = \frac{n!}{(n-N)!}P_{n-N}^{(0,\beta+N)}(x), \qquad \text{for } n \geq N.$$

Therefore, from the previous Section, we conclude that Jacobi polynomials $P_n^{(-N,\beta)}$, when $\beta + N$ is not a negative integer, are orthogonal with respect to the Sobolev bilinear form

$$\mathcal{B}_S^{(N)}(f,g) = F(1)\mathbf{A}G(1)^T + \langle u^{(0,\beta+N)}, f^{(N)}g^{(N)}\rangle,$$

where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T,$$

$\mathbf{Q}$ is the matrix of the derivatives of Jacobi polynomials

$$\mathbf{Q} = \left((P_n^{(-N,\beta)})^{(k)}(1)\right)_{n,k=0,\ldots,N-1}$$

which are given by

$$(P_n^{(-N,\beta)})^{(k)}(1) = 2^{n-k}\frac{n!}{(n-k)!}\frac{(-N+k+1)_{n-k}}{(n-N+\beta+k+1)_{n-k}},$$

and $\mathbf{D}$ is an arbitrary regular diagonal matrix.

Of course, a similar result can be stated in the case when $\alpha+N$ is not a negative integer, $\beta=-N$, and $c=-1$.

4. Sobolev orthogonal polynomials and three-term recurrence relations. Laguerre and Jacobi polynomials satisfy a three–term recurrence relation even for negative integer values of their respective parameters (see [12]). In the previous Section, we have seen that Laguerre polynomials with α a negative integer and Jacobi polynomials with either α or β a negative integer are Sobolev orthogonal polynomials. In this way a natural question arises: do the Sobolev orthogonal polynomials satisfy a three–term recurrence relation? As we are going to show, the answer is very restrictive, the existence of a three–term recurrence relation for the Sobolev orthogonal polynomials implies the classical character of the linear functional u associated with the bilinear form (2.1).

DEFINITION 4.1. *We will say that a family of polynomials $\{Q_n\}_{n\geq 0}$ is a monic polynomial system (MPS) if*

i) $\deg(Q_n) = n, \quad n \geq 0,$

ii) $Q_0(x) = 1, \quad Q_n(x) = x^n + \text{lower degree terms}, \quad n \geq 1.$

Obviously, every MPS is a basis of the linear space $\mathbb{P}$ and every MOPS is a MPS.

DEFINITION 4.2. *A monic polynomial system $\{Q_n\}_{n\geq 0}$ satisfies a three–term recurrence relation if there exist two sequences of real numbers $\{b_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$, such that*

$$xQ_n(x) = Q_{n+1}(x) + b_nQ_n(x) + g_nQ_{n-1}(x), \quad n \geq 0,$$

$$Q_{-1}(x) = 0, \quad Q_0(x) = 1.$$

From Favard's theorem (see [2], p. 21) we can deduce the existence of monic polynomial systems satisfying a three–term recurrence relation which are not orthogonal with respect to any linear functional. This case appears when some of the coefficients g_n are zero. For instance, Laguerre polynomials with parameter α a negative integer and Jacobi polynomials with parameters either α or β or $\alpha+\beta+1$ a negative integer.

PROPOSITION 4.3. *Let $\{Q_n\}_{n\geq 0}$ be a monic polynomial system satisfying a three–term recurrence relation and let N be a positive integer number such that the system of monic N-th order derivatives*

$$P_n(x) := \frac{n!}{(n+N)!}Q_{n+N}^{(N)}(x), \quad n \geq 0,$$

constitutes a monic orthogonal polynomial sequence. Then, the polynomials $\{P_n\}_n$ *are classical.*

Proof. Since $\{P_n\}_{n\geq 0}$ is a MOPS, it satisfies a three–term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 0,$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1,$$

with $\gamma_n \neq 0, n \geq 0$.

In this way,

$$(4.1)\ xQ^{(N)}_{n+N}(x) = \frac{n+1}{n+N+1}Q^{(N)}_{n+N+1}(x) + \beta_n Q^{(N)}_{n+N}(x) + \gamma_n \frac{n+N}{n}Q^{(N)}_{n+N-1}(x).$$

On the other hand, the monic polynomial sequence $\{Q_n\}_{n\geq 0}$ satisfies a three–term recurrence relation

$$xQ_{n+N}(x) = Q_{n+N+1}(x) + b_{n+N}Q_{n+N}(x) + g_{n+N}Q_{n+N-1}(x).$$

Taking N–th order derivatives in this relation, we get

$$(4.2)\ xQ^{(N)}_{n+N}(x) + NQ^{(N-1)}_{n+N}(x) = Q^{(N)}_{n+N+1}(x) + b_{n+N}Q^{(N)}_{n+N}(x) + g_{n+N}Q^{(N)}_{n+N-1}(x).$$

By eliminating the term $xQ^{(N)}_{n+N}(x)$ between (4.1) and (4.2), we obtain

$$\begin{aligned} NQ^{(N-1)}_{n+N}(x) &= \frac{N}{n+N+1}Q^{(N)}_{n+N+1}(x)+ \\ &+ (b_{n+N} - \beta_n)\, Q^{(N)}_{n+N}(x) + \left(g_{n+N} - \frac{n+N}{n}\gamma_n\right) Q^{(N)}_{n+N-1}(x). \end{aligned} \tag{4.3}$$

Taking again derivatives in this relation we deduce that each polynomial P_n can be expressed as a linear combination of the derivatives of three consecutive polynomials in the sequence $\{P_n\}_n$ and, therefore, we conclude that they are classical by using the characterization of classical orthogonal polynomials obtained by Marcellán et al. in [7]. □

REMARK. This result characterizes the classical orthogonal polynomials as the only system of orthogonal polynomials having a N–th order primitive ($N \geq 1$) which satisfies a three–term recurrence relation.

THEOREM 4.4. *Let* $\{Q_n\}_n$ *be the monic orthogonal polynomial sequence associated with the Sobolev bilinear form (2.1). If the polynomials* $\{Q_n\}_n$ *satisfy a three-term recurrence relation, then the linear functional* u *is classical and the point* c *in (2.1) satisfies*

$$\phi(c) = 0, \tag{4.4}$$

$$\psi(c) - \phi'(c) = 0, \tag{4.5}$$

where ϕ *and* ψ *are the polynomials in the distributional differential equation* $\mathcal{D}(\phi u) = \psi u$ *satisfied by* u.

Proof. Let $\{P_n\}_n$ be the monic orthogonal polynomial sequence associated with the linear functional u. From Theorem 2.1, we have

$$Q_n^{(k)}(c) = 0, \quad k = 0, 1, \ldots, N-1, \tag{4.6}$$

$$Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x), \tag{4.7}$$

for all $n \geq N$. Therefore, using Proposition 4.3, we deduce the classical character of the polynomials $\{P_n\}_n$ and then the classical linear functional u satisfies a distributional differential equation

$$\mathcal{D}(\phi u) = \psi u,$$

where ϕ and ψ are polynomials with $\deg \phi \leq 2$ and $\deg \psi = 1$. From Bochner's characterization of the classical orthogonal polynomials, (see [2]), we deduce that the polynomials $\{P_n\}_n$ satisfy the second order differential equation

$$\phi(x) P_n''(x) + \psi(x) P_n'(x) = \lambda_n P_n(x),$$

for all $n \geq 0$.

Thus the polynomials $\{Q_n\}_n$ satisfy the differential equation

$$\phi(x) Q_{n+N}^{(N+2)}(x) + \psi(x) Q_{n+N}^{(N+1)}(x) = \lambda_n Q_{n+N}^{(N)}(x),$$

for $n \geq 0$. This differential equation can be written in a more convenient way

$$\left(\phi(x) Q_{n+N}^{(N+1)}(x)\right)' + \left((\psi(x) - \phi'(x)) Q_{n+N}^{(N)}(x)\right)' = \kappa_n Q_{n+N}^{(N)}(x), \tag{4.8}$$

where $\kappa_n = \lambda_n + \psi'(x) - \phi''(x)$. Integrating (4.8) we get

$$\phi(x) Q_{n+N}^{(N+1)}(x) + (\psi(x) - \phi'(x)) Q_{n+N}^{(N)}(x) = \kappa_n Q_{n+N}^{(N-1)}(x) + \mu_n,$$

where μ_n is a constant.

For $n \geq 2$, let p be a polynomial with $\deg p \leq n-2$, then

$$\begin{aligned}\langle u, p\left[\phi Q_{n+N}^{(N+1)} + (\psi - \phi') Q_{n+N}^{(N)}\right]\rangle &= \langle u, p\phi Q_{n+N}^{(N+1)}\rangle - \langle u, (\phi p Q_{n+N}^{(N)})'\rangle \\ &= -\langle u, (p\phi)' Q_{n+N}^{(N)}\rangle \\ &= -\frac{(n+N)!}{n!}\langle u, (p\phi)' P_n\rangle = 0.\end{aligned}$$

Thus, the polynomial $\phi Q_{n+N}^{(N+1)} + (\psi - \phi') Q_{n+N}^{(N)} = \kappa_n Q_{n+N}^{(N-1)} + \mu_n$ is orthogonal, with respect to u, to every polynomial of degree less than or equal to $n-2$, and then it can be written as a linear combination of three consecutive polynomials P_n

$$\kappa_n Q_{n+N}^{(N-1)} + \mu_n = \kappa_n P_{n+1} + s_n P_n + t_n P_{n-1}.$$

From (4.3) we have that the polynomial $Q_{n+N}^{(N-1)}$ is a linear combination of the three polynomials P_{n+1}, P_n and P_{n-1}, and, since the sequence $\{P_n\}_n$ constitutes a basis of the linear space of the polynomials, we conclude that $\mu_n = 0$, for $n \geq 2$.

In this way, the polynomials $\{Q_{n+N}\}_n$ satisfy the differential equation

$$\phi(x)Q_{n+N}^{(N+1)}(x) + (\psi(x) - \phi'(x))Q_{n+N}^{(N)}(x) = \kappa_n Q_{n+N}^{(N-1)}(x), \tag{4.9}$$

for $n \geq 2$.

Replacing $x = c$ in (4.9), from (4.6) we conclude

$$\phi(c)P_n'(c) + (\psi(c) - \phi'(c))P_n(c) = 0, \tag{4.10}$$

for $n \geq 2$.

From recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

satisfied by $\{P_n\}_n$ and (4.10) written for $n+1$, n and $n-1$, we obtain

$$c\phi(c)P_n'(c) + [\phi(c) + c(\psi(c) - \phi'(c))]\, P_n(c) = 0, \qquad n \geq 3, \tag{4.11}$$

and subtracting (4.10) from (4.11), we get

$$\phi(c)P_n(c) = 0, \qquad n \geq 3.$$

Therefore, we conclude $\phi(c) = 0$ and using again (4.10), $\psi(c) - \phi'(c) = 0$. □

COROLLARY 4.5. *The only sequences of monic polynomials which are orthogonal with respect to a Sobolev bilinear form (2.1) and satisfy a three–term recurrence relation are*

a) The generalized Laguerre polynomials $L_n^{(-N)}$,

b) The generalized Jacobi polynomials $P_n^{(-N,\beta)}$, *with* $\beta+N$ *not a negative integer,*

c) The generalized Jacobi polynomials $P_n^{(\alpha,-N)}$, *with* $\alpha+N$ *not a negative integer.*

Proof. Suppose that the monic polynomials $\{Q_n\}_n$ orthogonal with respect to (2.1) satisfy a three–term recurrence relation. Theorem 4.4 assures that u is a classical linear functional. If $\mathcal{D}(\phi u) = \psi u$ is its distributional differential equation, the polynomials ϕ and ψ are given by the following table

Name	ϕ	ψ	Restrictions
Hermite	1	$-2x$	
Laguerre	x	$(\alpha+1)-x$	$\alpha \neq -n, n \geq 1$
Jacobi	$1-x^2$	$(\beta-\alpha)-(\alpha+\beta+2)x$	$\alpha \neq -n, \beta \neq -n,$ $\alpha+\beta+1 \neq -n, n \geq 1$
Bessel	x^2	$(\alpha+2)x+2$	$\alpha \neq -n, n \geq 2$

Then, conditions (4.4) and (4.5) exclude Hermite and Bessel cases. Moreover, in Laguerre and Jacobi cases the only possibilities are the following:

- Laguerre case with $\alpha = 0$ and $c = 0$.
- Jacobi case with $\alpha = 0$, $\beta \neq -m$, $m \geq 1$ and $c = 1$.
- Jacobi case with $\beta = 0$, $\alpha \neq -m$, $m \geq 1$ and $c = -1$.

Taking into account the results of Section 3, we conclude. □

Note that a reduction of the degree of $P_n^{(\alpha,\beta)}$ could occur when either both α and $\beta+n$ or $\alpha+\beta+1$ are negative integers (see [12], p.64). An interesting open problem is to give some kind of orthogonality relations valid for these polynomials. For the particular case $\alpha = \beta$ negative integers, this problem has been solved in [1]: the corresponding Gegenbauer polynomials are orthogonal with respect to a discrete-continuous Sobolev bilinear form, where the discrete part is concentrated in two points, namely, 1 and -1.

COROLLARY 4.6. *The monic orthogonal polynomials associated to the Sobolev bilinear form (2.1) satisfy a three term recurrence relation if and only if the linear functional u is classical and the point c in (2.1) satisfies $\phi(c) = 0$ and $\psi(c) = \phi'(c)$.*

Proof. It follows from Theorem 4.4 and Corollary 4.5 taking in mind that Laguerre and Jacobi polynomials satisfy a three–term recurrence relation for all values of their parameters. □

5. Sobolev orthogonal polynomials and second order differential equations. As it is well known (see [12]) Laguerre and Jacobi polynomials satisfy a second order differential equation for every value of their respective parameters.

In this Section, our aim is to characterize the sequences of monic Sobolev orthogonal polynomials satisfying a second order differential equation. We can observe that if, for every n, the polynomials $\{Q_n\}_n$ satisfy a second order differential equation

$$\phi(x)Q_n''(x) + \sigma(x)Q_n'(x) = \rho_n Q_n(x),$$

where ϕ and σ are fixed polynomials and $\rho_n \in \mathbb{R}$, then

$$\deg \phi \leq 2, \quad \deg \sigma \leq 1.$$

Moreover, if $\rho_1 \neq 0$ then $\deg \sigma = 1$.

THEOREM 5.1. *Let $\{Q_n\}_n$ be the monic orthogonal polynomial sequence associated with the Sobolev bilinear form (2.1). If, for $n \geq N$, every polynomial Q_n satisfies a second order differential equation*

$$\phi(x)Q_n''(x) + \sigma(x)Q_n'(x) = \rho_n Q_n(x), \tag{5.1}$$

where ϕ and σ are fixed polynomials with degree less than or equal to 2 and 1 respectively, and $\rho_n \in \mathbb{R}$, then the linear functional u is classical with distributional differential equation $\mathcal{D}(\phi u) = \psi u$, $\sigma(x) = \psi(x) - N\phi'(x)$ and the point c in (2.1) satisfies

$$\phi(c) = 0, \tag{5.2}$$

$$(N-1)\phi'(c) + \sigma(c) = 0. \tag{5.3}$$

Proof. Taking k–th order derivatives in (5.1), from Leibniz rule, we get

$$\begin{aligned} &\phi(x)Q_n^{(k+2)}(x) + (k\phi'(x) + \sigma(x))\, Q_n^{(k+1)}(x) \\ &= \left(\rho_n - \frac{k(k-1)}{2}\phi''(x) - k\sigma'(x)\right) Q_n^{(k)}(x). \end{aligned} \tag{5.4}$$

Let $k = N$ in (5.4), then we deduce that the polynomials P_n orthogonal with respect to the linear functional u satisfy the second order differential equation

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) = \lambda_n P_n(x), \quad n \geq 0,$$

where

$$\psi(x) = N\phi'(x) + \sigma(x),$$
$$\lambda_n = \rho_{n+N} - \frac{N(N-1)}{2}\phi''(x) - N\sigma'(x).$$

Therefore u satisfies $\mathcal{D}(\phi u) = \psi u$ and $\deg \psi \geq 1$. Now, since $\deg \phi \leq 2$ and $\deg \sigma \leq 1$, from Bochner's characterization of the classical orthogonal polynomials, (see [2]), we deduce the classical character of the linear functional u.

Writing equation (5.4) for $n = N$ and $k = N-1$ we get

$$\begin{aligned}&((N-1)\phi'(x) + \sigma(x))\, Q_N^{(N)}(x)\\ &= \left(\rho_N - \frac{(N-1)(N-2)}{2}\phi''(x) - (N-1)\sigma'(x)\right) Q_N^{(N-1)}(x),\end{aligned}$$

and by substitution in c, since $Q_N^{(N-1)}(c) = 0$, we deduce

$$(N-1)\phi'(c) + \sigma(c) = 0.$$

Finally, if $N = 1$ writing (5.1) for $n = 2$, we get $\phi(c) = 0$ and when $N \geq 2$, writing equation (5.4) for $n = N$, $k = N-2$, we get

$$\begin{aligned}&\phi(x)Q_N^{(N)}(x) + ((N-2)\phi'(x) + \sigma(x))Q_N^{(N-1)}(x)\\ &= \left(\rho_N - \frac{(N-2)(N-3)}{2}\phi''(x) - (N-2)\sigma'(x)\right) Q_N^{(N-2)}(x)\end{aligned}$$

and by substitution in c we deduce $\phi(c) = 0$. □

Using the same reasoning as in Corollary (4.5), we obtain

COROLLARY 5.2. *The only sequences of monic polynomials which are orthogonal with respect to a Sobolev bilinear form (2.1) and satisfy a second order differential equation (5.1) are*

a) The generalized Laguerre polynomials $L_n^{(-N)}$,

b) The generalized Jacobi polynomials $P_n^{(-N,\beta)}$, *with* $\beta + N$ *not a negative integer,*

c) The generalized Jacobi polynomials $P_n^{(\alpha,-N)}$, *with* $\alpha + N$ *not a negative integer.*

COROLLARY 5.3. *The monic orthogonal polynomials associated to the Sobolev bilinear form (2.1) satisfy a second order differential equation like (5.1) if and only if the linear functional* u *is classical and the point* c *in (2.1) satisfies* $\phi(c) = 0$ *and* $\psi(c) = \phi'(c)$.

Proof. It follows from Theorem 5.1 and Corollary 5.2 taking in mind that Laguerre and Jacobi polynomials satisfy a second order differential equation for all values of their parameters. □

6. A symmetric differential operator: Properties. In order to obtain explicit relations between the sequences $\{Q_n\}_n$ and $\{P_n\}_n$ associated with $\mathcal{B}_S^{(N)}$ and u, respectively, we introduce a linear differential operator $\mathcal{F}^{(N)}$ closely related to u. To do this, u must satisfy an extra condition. This is why, from now on, the functional u in (2.1) will be a semiclassical one.

DEFINITION 6.1 ([3], [9]). *A linear functional u on $\mathbb{P}$ is called semiclassical, if there exist two polynomials ϕ and ψ, with $\deg\phi = p \geq 0$ and $\deg\psi = q \geq 1$, such that u satisfies the following distributional differential equation*

$$\mathcal{D}(\phi u) = \psi u, \tag{6.1}$$

or equivalently

$$\phi\mathcal{D}u = (\psi - \phi')u. \tag{6.2}$$

Equation (6.2) can be generalized in the following way:

LEMMA 6.2 ([8]). *Let u be a semiclassical linear functional, then for every n we have*

$$\phi^n(x)\mathcal{D}^n u = \psi(x;n)u, \tag{6.3}$$

where the polynomials $\psi(x;n)$ are recursively defined by

$$\psi(x;0) = 1,$$
$$\psi(x;n) = \phi(x)\psi'(x;n-1) + \psi(x;n-1)[\psi(x) - n\phi'(x)], \quad n \geq 1. \tag{6.4}$$

Observe that, now, (6.2) adopts the form $\phi(x)\mathcal{D}u = \psi(x;1)u$. Taking derivatives $(n-1)$–times in this formula, we get

$$\phi(x)\mathcal{D}^n u = \sum_{i=0}^{n-1}\left[\binom{n-1}{i}\mathcal{D}^{n-1-i}\psi(x;1) - \binom{n-1}{i-1}\mathcal{D}^{n-i}\phi(x)\right]\mathcal{D}^i u, \tag{6.5}$$

for $n \geq 1$, where $\binom{n}{m} = 0$ whenever $m < 0$. Multiplying by ϕ^{n-1} and using (6.3), we obtain another recursive expression for $\psi(x;n)$:

$$\psi(x;n) = \sum_{i=0}^{n-1}\left[\binom{n-1}{i}\mathcal{D}^{n-1-i}\psi(x;1) - \binom{n-1}{i-1}\mathcal{D}^{n-i}\phi(x)\right]\phi^{n-1-i}(x)\psi(x;i) \tag{6.6}$$

valid for $n \geq 1$.

LEMMA 6.3. *In the above conditions, we have*

$$\phi^{n-j}(x)\psi(x;j)\mathcal{D}^n u = \psi(x;n)\mathcal{D}^j u, \quad n \geq j, \tag{6.7}$$

and consequently

$$\phi^i(x)\psi(x;N-i)\mathcal{D}^{N-j}u = \phi^j(x)\psi(x;N-j)\mathcal{D}^{N-i}u, \quad 0 \leq i,j \leq N. \tag{6.8}$$

Proof. We will show the result by induction on n. The case $n = 1, j = 0$ comes from (6.3), while the case $n = j = 1$ is trivial. Suppose that

$$\phi^{n-1-j}(x)\psi(x;j)\mathcal{D}^{n-1}u = \psi(x;n-1)\mathcal{D}^j u,$$

holds for all j, $0 \leq j \leq n-1$. Then,
i) For $0 \leq j < n-1$, taking derivatives, multiplying by ϕ, using (6.4), and the induction hypothesis, we have

$$\begin{aligned}
\phi^{n-j}(x)\psi(x;j)\mathcal{D}^n u &= \phi(x)\psi'(x,n-1)\mathcal{D}^j u + \phi(x)\psi(x;n-1)\mathcal{D}^{j+1}u - \\
&\quad -\{(n-1-j)\phi'(x)\psi(x;j) + \psi(x;j+1) - \\
&\quad -\psi(x;j)[\psi(x) - (j+1)\phi'(x)]\}\phi^{n-1-j}(x)\mathcal{D}^{n-1}u \\
&= \phi(x)\psi'(x,n-1)\mathcal{D}^j u + \phi(x)\psi(x;n-1)\mathcal{D}^{j+1}u - \\
&\quad -\{\psi(x;j+1) - \psi(x;j)[\psi(x) - n\phi'(x)]\}\phi^{n-1-j}(x)\mathcal{D}^{n-1}u \\
&= \{\phi(x)\psi'(x,n-1) + \psi(x;n-1)[\psi(x) - n\phi'(x)]\}\mathcal{D}^j u + \\
&\quad +\phi(x)\psi(x;n-1)\mathcal{D}^{j+1}u - \psi(x;j+1)\phi^{n-1-j}(x)\mathcal{D}^{n-1}u \\
&= \psi(x;n)\mathcal{D}^j u + \phi(x)\psi(x;n-1)\mathcal{D}^{j+1}u - \phi(x)\psi(x;n-1)\mathcal{D}^{j+1}u \\
&= \psi(x;n)\mathcal{D}^j u.
\end{aligned}$$

ii) If $j = n-1$, multiplying (6.5) by $\psi(x;n-1)$, and using the induction hypothesis, we obtain the result taking into account (6.6).

Now, from *i)* and *ii)*, we conclude, since the case $j = n$ is trivial. □

We define a linear differential operator $\mathcal{F}^{(N)}$ on the linear space of real polynomials $\mathbb{P}$ in the following way

$$\mathcal{F}^{(N)} = (-1)^N (x-c)^N \sum_{i=0}^{N} \binom{N}{i} \phi^i(x)\psi(x;N-i)\mathcal{D}^{N+i}, \tag{6.9}$$

where $\mathcal{D}$ denotes the derivative operator and the polynomials $\psi(x;n)$ are defined as in Lemma 6.2.

REMARK. In the particular case of a semiclassical linear functional u defined from a weight function, expression (6.9) can be written in a very compact form

$$\mathcal{F}^{(N)} = (-1)^N (x-c)^N \frac{\phi^N(x)}{\rho(x)} {}^N \left(\rho(x)\mathcal{D}^N\right)$$

where ρ denotes the weight function associated with the semiclassical linear functional u.

In the next Lemma, we recall a very useful formula involving derivatives.

LEMMA 6.4 ([8]). *Let f and g be n–times and $2n$–times differentiable functions, respectively. Then,*

$$f^{(n)}g^{(n)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(fg^{(n+i)}\right)^{(n-i)}.$$

In the following Proposition, we show how the linear operator $\mathcal{F}^{(N)}$ allows us to obtain a representation for the Sobolev bilinear form (2.1), in terms of the consecutive derivatives of the semiclassical linear functional u.

PROPOSITION 6.5. *Let $\mathcal{B}_S^{(N)}$ be a Sobolev bilinear form with u semiclassical and f, g arbitrary polynomials. Then, for $0 \le i \le N$, we have*

$$\mathcal{B}_S^{(N)}\left((x-c)^N\phi^i(x)\psi(x;N-i)f,g\right) = \langle \mathcal{D}^{N-i}u, f\mathcal{F}^{(N)}g\rangle.$$

Proof. From Lemmas 6.2, 6.3 and 6.4, we get

$$\begin{aligned}
&\mathcal{B}_S^{(N)}\left((x-c)^N\phi^i(x)\psi(x;N-i)f,g\right)\\
&= \langle u, \left((x-c)^N\phi^i(x)\psi(x;N-i)f\right)^{(N)} g^{(N)}\rangle\\
&= \sum_{j=0}^{N}(-1)^j\binom{N}{j}\langle u, \left((x-c)^N\phi^i(x)\psi(x;N-i)fg^{(N+j)}\right)^{(N-j)}\rangle\\
&= \sum_{j=0}^{N}(-1)^N\binom{N}{j}\langle \mathcal{D}^{N-j}u, (x-c)^N\phi^i(x)\psi(x;N-i)fg^{(N+j)}\rangle\\
&= \sum_{j=0}^{N}(-1)^N\binom{N}{j}\langle \phi^i(x)\psi(x;N-i)\mathcal{D}^{N-j}u, (x-c)^N fg^{(N+j)}\rangle\\
&= \sum_{j=0}^{N}(-1)^N\binom{N}{j}\langle \phi^j(x)\psi(x;N-j)\mathcal{D}^{N-i}u, (x-c)^N fg^{(N+j)}\rangle\\
&= \langle \mathcal{D}^{N-i}u, f[(-1)^N(x-c)^N\sum_{j=0}^{N}\binom{N}{j}\phi^j(x)\psi(x;N-j)g^{(N+j)}]\rangle\\
&= \langle \mathcal{D}^{N-i}u, f\mathcal{F}^{(N)}g\rangle. \quad \square
\end{aligned}$$

THEOREM 6.6. *The linear operator $\mathcal{F}^{(N)}$ is symmetric with respect to the Sobolev bilinear form (2.1), that is*

$$\mathcal{B}_S^{(N)}(\mathcal{F}^{(N)}f,g) = \mathcal{B}_S^{(N)}(f,\mathcal{F}^{(N)}g).$$

Proof. From Proposition 6.5 and Lemma 6.4, we can deduce

$$\begin{aligned}
\mathcal{B}_S^{(N)}(\mathcal{F}^{(N)}f,g) &= \sum_{i=0}^{N}(-1)^N\binom{N}{i}\mathcal{B}_S^{(N)}\left((x-c)^N\phi^i(x)\psi(x;N-i)f^{(N+i)},g\right)\\
&= \sum_{i=0}^{N}(-1)^N\binom{N}{i}\langle \mathcal{D}^{N-i}u, f^{(N+i)}\mathcal{F}^{(N)}g\rangle\\
&= \sum_{i=0}^{N}(-1)^i\binom{N}{i}\langle u, \left(f^{(N+i)}\mathcal{F}^{(N)}g\right)^{(N-i)}\rangle\\
&= \langle u, f^{(N)}\left(\mathcal{F}^{(N)}g\right)^{(N)}\rangle = \mathcal{B}_S^{(N)}(f,\mathcal{F}^{(N)}g). \quad \square
\end{aligned}$$

Now, we study the degree of the polynomial $\mathcal{F}^{(N)}x^n$ for every n. Observe that $\mathcal{F}^{(N)}$ vanishes on every polynomial with degree less than N.

PROPOSITION 6.7. *For every $n \geq 0$, we have*

$$\deg\left(\mathcal{F}^{(N)}x^n\right) \leq n + N\max\{p-1, q\},$$

where $p = \deg\phi$ and $q = \deg\psi$.

Proof. By using the induction method it is very easy to see that $\deg\psi(x;n) \leq n + \max\{p-1, q\}$ for all $n \geq 0$. Taking into account the definition of the linear operator $\mathcal{F}^{(N)}$, the conclusion follows. □

On the other hand, the linear operator $\mathcal{F}^{(N)}$ never reduces the degree for all polynomials.

PROPOSITION 6.8. *There exists $n_0 \geq N$ such that*

$$\deg\mathcal{F}^{(N)}x^{n_0} \geq n_0.$$

Proof. Suppose that $\deg\mathcal{F}^{(N)}x^n < n$, for all $n \geq N$. Then, we can expand

$$\mathcal{F}^{(N)}Q_n = \sum_{i=0}^{n-1} a_{n,i}Q_i.$$

Thus, since $\mathcal{F}^{(N)}$ vanishes on every polynomial of degree less than N, using its symmetry property, we have

$$a_{n,i} = \frac{\mathcal{B}_S^{(N)}\left(\mathcal{F}^{(N)}Q_n, Q_i\right)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = \frac{\mathcal{B}_S^{(N)}\left(Q_n, \mathcal{F}^{(N)}Q_i\right)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = 0, \quad i = 0, 1, \ldots, n-1,$$

and the result follows. □

To study the degree of $\mathcal{F}^{(N)}x^n$, we need to know the degree of the polynomials $\psi(x; N-i)$, $i = 0, \ldots, N$ in formula (6.9). The following Lemma provides us some combinatorial identities, that will be useful for our purpose.

LEMMA 6.9. *Let a and b arbitrary real numbers. Then for every non negative integer n, we have*

i) $(a)_n = (-1)^n(-a-n+1)_n$,

ii) $\displaystyle\sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$,

iii) $\displaystyle\sum_{i=0}^{n}(-1)^i\binom{a}{i}\binom{b-i}{n-i} = \binom{b-a}{n}$,

where $\binom{a}{k}$ and $(a)_k$ are given by (3.2) and (3.3).

Proof. i) It is a direct consequence of the definition of the Pochhammer's symbol.
ii) It can be derived from the power series expansion of the identity

$$(1+x)^a(1+x)^b = (1+x)^{a+b},$$

comparing the coefficients.

iii) This formula can be deduced from *i)* and *ii)*. □

REMARK. For a and b positive integer numbers, formulas *ii)* and *iii)* can be found on page 8 in [11].

Let assume that the explicit representation for the polynomials ϕ and ψ is given by

$$\phi(x) = \sum_{i=0}^{p} a_i x^i, \quad a_p \neq 0, \quad p \geq 0, \quad \psi(x) = \sum_{i=0}^{q} b_i x^i, \quad b_q \neq 0, \quad q \geq 1,$$

and without loss of generality, we can suppose that $a_p = 1$.

Next Lemma gives us the degree of the polynomial $\psi(x;n)$ and its leading coefficient in terms of p, q and b_q.

LEMMA 6.10. *The following assertions are true:*
i) If $p-1 < q$, *then*

$$\psi(x;n) = b_q^n x^{nq} + \text{lower degree terms},$$

and $\deg \psi(x;n) = nq, \quad n \geq 0$.
ii) If $p-1 > q$, *then*

$$\psi(x;n) = (-1)^n (p)_n x^{n(p-1)} + \text{lower degree terms},$$

and $\deg \psi(x;n) = n(p-1), \quad n \geq 0$.
iii) If $p-1 = q$, *then*

$$\psi(x;n) = (-1)^n (p-b_q)_n x^{n(p-1)} + \text{lower degree terms}.$$

Therefore,
iii.1) If $p - b_q \neq 0, -1, \dots, -(n-1)$ *then* $\deg \psi(x;n) = n(p-1), \quad n \geq 0$,
iii.2) If $p - b_q = -k$, $k \geq 0$, *then* $\deg \psi(x;n) = n(p-1)$, $0 \leq n \leq k$, *and* $\deg \psi(x;n) < n(p-1)$, $n > k$.

Proof. These results can be obtained by induction on n. □

As we are going to see, the equality in Proposition 6.7 is true for almost all n, that is, the action of $\mathcal{F}^{(N)}$ on a polynomial of degree bigger than or equal to N increases its degree exactly in Nt being $t = \max\{p-1, q\}$. Therefore, we can write

$$\mathcal{F}^{(N)} x^n = F(n; N, t) x^{n+Nt} + \dots,$$

where $F(n;N,t)$ denotes the leading coefficient of the polynomial $\mathcal{F}^{(N)} x^n$. We want to notice that this coefficient can be zero for some specific values of n and in particular for every $n < N$.

To prove this, we decompose the operator $\mathcal{F}^{(N)}$ in $N+1$ differential operators defined by

$$\mathcal{F}_i^{(N)} = (-1)^N (x-c)^N \binom{N}{i} \phi^i(x) \psi(x; N-i) \mathcal{D}^{N+i}, \quad i = 0, 1, \dots, N. \tag{6.10}$$

Thus,

$$\mathcal{F}^{(N)} = \sum_{i=0}^{N} \mathcal{F}_i^{(N)}, \quad \text{and} \quad \mathcal{F}^{(N)} x^n = \sum_{i=0}^{\min\{N,n-N\}} \mathcal{F}_i^{(N)} x^n, n \geq N,$$

where, for $i = 0, \dots, \min\{N, n-N\}$,

$$\mathcal{F}_i^{(N)} x^n = (-1)^N (x-c)^N \binom{N}{i} \phi^i(x) \psi(x; N-i) \frac{n!}{(n-N-i)!} x^{n-N-i}.$$

Let us denote by $F_i(n)$ the leading coefficient of the polynomial $\mathcal{F}_i^{(N)} x^n$ and, for the sake of simplicity, we will put $F(n; N, t) = F(n)$.

THEOREM 6.11. *Let $t = \max\{p-1, q\}$. Except for finitely many values of $n \geq N$, we have*

$$\deg \mathcal{F}^{(N)} x^n = n + Nt,$$

that is,

$$\mathcal{F}^{(N)} x^n = F(n) x^{n+Nt} + \text{lower terms degree},$$

with $F(n) \neq 0$. *More precisely*
i) If $p - 1 < q$, then

$$F(n) = (-b_q)^N \frac{n!}{(n-N)!},$$

and $\deg \mathcal{F}^{(N)} x^n = n + Nt, \quad n \geq N.$
ii) If $p - 1 > q$, then

$$F(n) = (p - n + N)_N \frac{n!}{(n-N)!},$$

and

$$\begin{aligned} \deg \mathcal{F}^{(N)} x^n &< n + Nt, \quad N + p \leq n \leq 2N - 1 + p, \\ \deg \mathcal{F}^{(N)} x^n &= n + Nt, \quad N \leq n < N + p, \quad n \geq 2N + p. \end{aligned}$$

iii) If $p - 1 = q$, then

$$F(n) = (p - b_q - n + N)_N \frac{n!}{(n-N)!},$$

and
iii.1) if $p - b_q = -k$, $\quad k = 0, 1, \dots, N-1$, then

$$\begin{aligned} \deg \mathcal{F}^{(N)} x^n &< n + Nt, \quad N \leq n \leq 2N - 1 - k, \\ \deg \mathcal{F}^{(N)} x^n &= n + Nt, \quad n \geq 2N - k, \end{aligned}$$

iii.2) if $p - b_q$ is a positive integer, then

$$\begin{aligned} \deg \mathcal{F}^{(N)} x^n &< n + Nt, \quad N + p - b_q \leq n \leq 2N - 1 + p - b_q, \\ \deg \mathcal{F}^{(N)} x^n &= n + Nt, \quad N \leq n < N + p - b_q, \quad n \geq 2N + p - b_q. \end{aligned}$$

iii.3) in another case,

$$\deg \mathcal{F}^{(N)} x^n = n + Nt, \quad n \geq N.$$

Proof. To prove the theorem a basic tool will be Lemma 6.10. For this reason, we distinguish three different cases.
i) Case $p-1 < q$. In this situation, we have

$$\deg \mathcal{F}_i^{(N)} x^n = n + Nq - i(q-(p-1)), \quad i = 0, 1, \ldots, \min\{N, n-N\},$$

and then, $\deg \mathcal{F}^{(N)} x^n = \deg \mathcal{F}_0^{(N)} x^n = n + Nq$, for all $n \geq N$.
The explicit expression for $\mathcal{F}_0^{(N)} x^n$ is

$$\begin{aligned}\mathcal{F}_0^{(N)} x^n &= (-1)^N (x-c)^N \psi(x; N) \mathcal{D}^N x^n \\ &= (-b_q)^N \frac{n!}{(n-N)!} x^{n+Nq} + \text{lower terms degree,}\end{aligned}$$

and the leading coefficient for $\mathcal{F}^{(N)} x^n$ is

$$F(n) = F_0(n) = (-b_q)^N \frac{n!}{(n-N)!}, \quad n \geq N.$$

ii) Case $p-1 > q$. In this case, as $p > 2$, we get

$$\deg \mathcal{F}_i^{(N)} x^n = n + N(p-1), \quad i = 0, \ldots, \min\{N, n-N\},$$

and then $\deg \mathcal{F}^{(N)} x^n \leq n + N(p-1)$, for all $n \geq N$. The leading coefficient of $\mathcal{F}_i^{(N)} x^n$ is

$$F_i(n) = (-1)^i \binom{N}{i} (p)_{N-i} \frac{n!}{(n-N-i)!}, \quad i = 0, 1, \ldots, \min\{N, n-N\}.$$

Taking into account that

$$F(n) = \sum_{i=0}^{\min\{N, n-N\}} F_i(n), \tag{6.11}$$

and using Lemma 6.9 *iii)*, we can show that:
If $N \leq n < 2N$, since $\min\{N, n-N\} = n-N$, we obtain

$$\begin{aligned}F(n) &= \sum_{i=0}^{n-N} F_i(n) = n!\ (p)_{2N-n} \sum_{i=0}^{n-N} (-1)^i \binom{N}{i} \binom{p-1+N-i}{n-N-i} \\ &= n!\ (p)_{2N-n} \binom{p-1}{n-N} = (p-n+N)_N \frac{n!}{(n-N)!}.\end{aligned}$$

On the other hand, if $n \geq 2N$,

$$\begin{aligned}F(n) &= \sum_{i=0}^{N} F_i(n) = \frac{n!\ N!}{(n-N)!} \sum_{i=0}^{N} (-1)^i \binom{n-N}{i} \binom{p-1+N-i}{N-i} \\ &= \frac{n!\ N!}{(n-N)!} \binom{p-1-n+2N}{N} = (p-n+N)_N \frac{n!}{(n-N)!}.\end{aligned}$$

Observe that, since p is an integer bigger than 2, from the definition of the Pochhammer's symbol $(p-n+N)_N$, we have $F(n)=0$ if and only if $N+p\le n\le 2N-1+p$. Then, there exist exactly N values of n such that $\deg \mathcal{F}^{(N)}x^n < n+N(p-1)$. In another case, we have $\deg \mathcal{F}^{(N)}x^n = n+N(p-1)$.

iii) Case $p-1=q$.

First, we assume that $p-b_q=-k$, $k=0,1,\dots,N-1$. In this case, by Lemma 6.10, $\deg \psi(x;N-i)=(N-i)(p-1)$, if $0\le N-i\le k$ and $\deg\psi(x;N-i)<(N-i)(p-1)$ for $k+1\le N-i\le N$.

Therefore, for $n\ge N$, we have $\deg \mathcal{F}_i^{(N)}x^n < n+N(p-1)$ when $i=0,1,\dots,N-k-1$, and $\deg \mathcal{F}_i^{(N)}x^n = n+N(p-1)$ if $N-k\le i\le N$. In this way, $\deg \mathcal{F}^{(N)}x^n < n+N(p-1)$ when $N\le n\le 2N-k-1$.

For $n\ge 2N-k$, we can observe that

$$F(n)=\sum_{i=N-k}^{\min\{N,n-N\}} F_i(n),$$

where, by Lemma 6.9 *i)*,

$$\begin{aligned}F_i(n)&=(-1)^N\binom{N}{i}(-1)^{N-i}(-k)_{N-i}\frac{n!}{(n-N-i)!}\\&=(-1)^N\binom{N}{i}(i+1-(N-k))_{N-i}\frac{n!}{(n-N-i)!}.\end{aligned}$$

Now, we give an explicit expression of $F(n)$. Suppose $2N-k\le n<2N$, then $\min\{N,n-N\}=n-N$, and using Lemma 6.9 *ii)*

$$\begin{aligned}F(n)&=(-1)^N\sum_{i=N-k}^{n-N}\binom{N}{i}(i+1-(N-k))_{N-i}\frac{n!}{(n-N-i)!}\\&=(-1)^N\frac{n!\,N!}{(n-N)!}\sum_{i=0}^{n-(2N-k)}\binom{k}{i}\binom{n-N}{n-(2N-k)-i}\\&=(-1)^N\frac{n!\,N!}{(n-N)!}\binom{n+k-N}{n-(2N-k)}\\&=(-1)^N(n+1-(2N-k))_N\frac{n!}{(n-N)!}.\end{aligned}$$

If $n\ge 2N$, again by Lemma 6.9 *ii)*, we have

$$\begin{aligned}F(n)&=(-1)^N\sum_{i=N-k}^{N}\binom{N}{i}(i+1-(N-k))_{N-i}\frac{n!}{(n-N-i)!}\\&=(-1)^N\frac{n!\,k!}{(n-(2N-k))!}\sum_{i=0}^{k}\binom{n-(2N-k)}{i}\binom{N}{k-i}\\&=(-1)^N\frac{n!\,k!}{(n-(2N-k))!}\binom{n+k-N}{k}\\&=(-1)^N(n+1-(2N-k))_N\frac{n!}{(n-N)!}.\end{aligned}$$

Hence, if $n \geq 2N - k$,

$$F(n) = (-1)^N(n+1-(2N-k))_N \frac{n!}{(n-N)!} = (-k-n+N)_N \frac{n!}{(n-N)!} \neq 0.$$

Now, we assume that $p - b_q \neq 0, -1, \ldots, -(N-1)$ and thus, from Lemma 6.10, $\deg \psi(x; N-i) = (N-i)(p-1)$, $i = 0, 1, \ldots, N$. As in the case $p - 1 > q$, we have $\deg \mathcal{F}_i^{(N)} x^n = n + N(p-1)$, $i = 0, 1, \ldots, \min\{N, n-N\}$, $\deg \mathcal{F}^{(N)} x^n \leq n + N(p-1)$, and also (6.11), where

$$F_i(n) = (-1)^i \binom{N}{i} (p - b_q)_{N-i} \frac{n!}{(n-N-i)!}, \quad i = 0, 1, \ldots, \min\{N, n-N\}.$$

Using the same technique, we obtain

$$F(n) = (p - b_q - n + N)_N \frac{n!}{(n-N)!}, \quad n \geq N.$$

If $p-b_q$ is a positive integer, then $F(n) = 0$ if and only if $N+p-b_q \leq n \leq 2N-1+p-b_q$, that is, there exist precisely N values of n such that $\deg \mathcal{F}^{(N)} x^n < n+N(p-1)$, and for the other values of $n \geq N$, $F(n) \neq 0$ and hence $\deg \mathcal{F}^{(N)} x^n = n + N(p-1)$.

In another case, $\deg \mathcal{F}^{(N)} x^n = n + N(p-1)$, for all $n \geq N$. □

7. Recurrence relations and differential operators. As a direct consequence of Proposition 6.5, which relates the discrete–continuous Sobolev bilinear form $\mathcal{B}_S^{(N)}$ and the one defined from a semiclassical linear functional u, we can establish some relations between the monic Sobolev orthogonal polynomials $\{Q_n\}_n$ and the monic orthogonal polynomials $\{P_n\}_n$, associated with the semiclassical linear functional u. In the sequel, for the sake of simplicity, we will denote

$$k_n = \langle u, P_n^2 \rangle \neq 0, \quad \tilde{k}_n = \mathcal{B}_S^{(N)}(Q_n, Q_n) \neq 0, \quad \forall n \geq 0.$$

PROPOSITION 7.1. *The following formulas hold:*

$$i) \quad (x-c)^N \phi^N(x) P_n(x) = \sum_{i=r}^{n+N(p+1)} \alpha_i^{(n)} Q_i(x), \quad n \geq 0, \tag{7.1}$$

where $r = \max\{N, n - Nt\}$, $\alpha_{n+N(p+1)}^{(n)} = 1$ *and* $\alpha_r^{(n)} = \dfrac{\langle u, P_n \mathcal{F}^{(N)} Q_r \rangle}{\tilde{k}_r}$.

$$ii) \quad \mathcal{F}^{(N)} Q_n(x) = \sum_{i=n-N(p+1)}^{n+Nt} \beta_i^{(n)} P_i(x), \quad n \geq N(p+1), \tag{7.2}$$

where $\beta_{n+Nt}^{(n)} = F(n)$, $\beta_{n-N(p+1)}^{(n)} = \dfrac{\tilde{k}_n}{k_{n-N(p+1)}}$.

Proof. i) Expanding the polynomial $(x-c)^N \phi^N P_n$ in terms of the Sobolev polynomials Q_n, we have

$$(x-c)^N \phi^N(x) P_n(x) = \sum_{i=0}^{n+N(p+1)} \alpha_i^{(n)} Q_i(x),$$

where, taking into account Proposition 6.5,

$$\alpha_i^{(n)} = \frac{\mathcal{B}_S^{(N)}\left((x-c)^N \phi^N P_n, Q_i\right)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = \frac{\langle u, P_n \mathcal{F}^{(N)} Q_i \rangle}{\tilde{k}_i}.$$

From the orthogonality of $\{P_n\}_n$ and since $\mathcal{F}^{(N)} Q_i = 0$ for $i < N$, we deduce that $\alpha_i^{(n)} = 0$ when $0 \le i < r = \max\{N, n - Nt\}$.
ii) Because of Proposition 6.7, the expansion of the polynomial $\mathcal{F}^{(N)} Q_n$ in terms of P_n is

$$\mathcal{F}^{(N)} Q_n(x) = \sum_{i=0}^{n+Nt} \beta_i^{(n)} P_i(x).$$

The coefficients $\beta_i^{(n)}$ can be computed using again Proposition 6.5, and therefore

$$\beta_i^{(n)} = \frac{\langle u, P_i \mathcal{F}^{(N)} Q_n \rangle}{\langle u, P_i^2 \rangle} = \frac{\mathcal{B}_S^{(N)}\left((x-c)^N \phi^N P_i, Q_n\right)}{k_i}.$$

Finally, from the orthogonality of $\{Q_n\}_n$ it follows $\beta_i^{(n)} = 0$ for $0 \le i < n - N(p+1)$.
□

From the symmetry of the linear operator $\mathcal{F}^{(N)}$, we can obtain a difference–differential relation satisfied by the Sobolev orthogonal polynomials with respect to the Sobolev bilinear form (2.1), where u is a semiclassical linear functional.

PROPOSITION 7.2 (Difference–Differential Relation). *For every $n \ge N$, the following relation holds*

$$\mathcal{F}^{(N)} Q_n(x) = \sum_{i=r}^{n+Nt} \gamma_i^{(n)} Q_i(x), \tag{7.3}$$

where $r = \max\{N, n - Nt\}$, $\gamma_{n+Nt}^{(n)} = F(n)$ and $\gamma_r^{(n)} = \dfrac{\mathcal{B}_S^{(N)}\left(Q_n, \mathcal{F}^{(N)} Q_r\right)}{\tilde{k}_r}$.

Proof. Consider the Fourier expansion of the polynomial $\mathcal{F}^{(N)} Q_n$ in terms of Q_n which, by Proposition 6.7, is

$$\mathcal{F}^{(N)} Q_n(x) = \sum_{i=0}^{n+Nt} \gamma_i^{(n)} Q_i(x).$$

Then

$$\gamma_i^{(n)} = \frac{\mathcal{B}_S^{(N)}\left(\mathcal{F}^{(N)} Q_n, Q_i\right)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = \frac{\mathcal{B}_S^{(N)}\left(Q_n, \mathcal{F}^{(N)} Q_i\right)}{\tilde{k}_i},$$

where we have used Theorem 6.6. Notice that $\gamma_i^{(n)} = 0$ for $0 \le i < N$ and that the orthogonality of the polynomials $\{Q_n\}_n$ leads to $\gamma_i^{(n)} = 0$ for $0 \le i < n - Nt$. So the result follows. □

REMARK. In formulas (7.1) and (7.3), when $r = n - Nt$, the coefficients $\alpha_r^{(n)}$ and $\gamma_r^{(n)}$ can explicitly be given by

$$\alpha_r^{(n)} = F(r)\frac{k_n}{\tilde{k}_r}, \quad \gamma_r^{(n)} = F(r)\frac{\tilde{k}_n}{\tilde{k}_r}.$$

Recall that the values of $F(n)$ had been calculated in Theorem 6.11.

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