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ASYMPTOTICS FOR POLYNOMIALS SATISFYING A CERTAIN TWIN ASYMPTOTIC PERIODIC RECURRENCE RELATION*

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Abstract. In this paper polynomials satisfying a certain twin asymptotic periodic recurrence relation are considered. The asymptotic behaviour of the ratio of contiguous polynomials and their limiting zero distributions are analyzed. Finally, the L-orthogonality relation associated with the twin periodic recurrence relation is given.

1. Introduction. Polynomials B_n, $n \geq 0$, satisfying the recurrence relation

$$B_{n+1}(z) = (z - \beta_{n+1})B_n(z) - \alpha_{n+1} z B_{n-1}(z), \qquad n \geq 1, \tag{1.1}$$

with $B_0(z) = 1$ and $B_1(z) = z - \beta_1$ where $\beta_n > 0$, $\alpha_{n+1} > 0$, $n \geq 1$, are of considerable interest for two point Padé approximants and related quadrature rules. It is well known that the zeros of B_n are all positive, distinct and interlace with those of B_{n-1}. This result is due to Jones et al. [5]. Let $z_1^{(n)} < z_2^{(n)} < \cdots < z_n^{(n)}$ be the n zeros of B_n, then if $\beta_n = \beta$, $n \geq 1$, it follows from [6] that $z_r^{(n)} = \beta^2 / z_{n+1-r}^{(n)}$. Furthermore, if also $\alpha_2 = 2\alpha$ and $\alpha_{n+2} = \alpha$, $n \geq 1$, then the zeros are given explicitly by

$$z_r^{(n)} = \beta^2 / z_{n+1-r}^{(n)} \quad \text{and} \quad z_{n+1-r}^{(n)} = \beta + \alpha\nu_r^{(n)} + \sqrt{\left(\beta + \alpha\nu_r^{(n)}\right)^2 - \beta^2},$$

for $r = 1, 2, \ldots, \lfloor (n+1)/2 \rfloor$ where $\nu_r^{(n)} = 1 + \cos(\pi(2r-1)/n)$.

Consider the sequence

$$a_n(z) = \frac{\alpha_{n+1} z}{(z - \beta_n)(z - \beta_{n+1})} = \{1 - m_{n-1}(z)\} m_n(z), \qquad n \geq 1,$$

which can be obtained from (1.1) where $m_n(z) = 1 - B_{n+1}(z)/\{(z - \beta_{n+1})B_n(z)\}$, $n \geq 0$. It was shown in [9], using chain sequences, that if

$$\hat{\beta}_N = \sup_{1 \leq n \leq N+1} \beta_n, \qquad \check{\beta}_N = \inf_{1 \leq n \leq N+1} \beta_n \quad \text{and} \quad \hat{\alpha}_N = \sup_{2 \leq n \leq N+1} \alpha_n,$$

for any $N \geq 1$ then all the zeros of the polynomials B_n, $1 \leq n \leq N+1$, lie inside the interval $[\check{d}_{N,1}, \hat{d}_{N,1}]$ where

$$\hat{d}_{N,1} = \hat{\beta}_N + 2\hat{\alpha}_N + \sqrt{(\hat{\beta}_N + 2\hat{\alpha}_N)^2 - \check{\beta}_N^2},$$

$$\check{d}_{N,1} = \left\{ \frac{1}{\check{\beta}_N} + \frac{2\hat{\alpha}_N}{\check{\beta}_N^2} + \sqrt{\left(\frac{1}{\check{\beta}_N} + \frac{2\hat{\alpha}_N}{\check{\beta}_N^2}\right)^2 - \frac{1}{\check{\beta}_N^2}} \right\}^{-1}.$$

The sequences $\{\hat{\beta}_n\}$ and $\{\hat{\alpha}_n\}$ are non-decreasing and the sequence $\{\check{\beta}_n\}$ is non-increasing. Let $\hat{\beta}$, $\hat{\alpha}$ and $\check{\beta}$ be the respective limits of these sequences and let

$$\hat{d} = \hat{\beta} + 2\hat{\alpha} + \sqrt{(\hat{\beta} + 2\hat{\alpha})^2 - \check{\beta}^2}$$

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and

$$\check{d} = \left\{ \frac{1}{\check{\beta}} + \frac{2\hat{\alpha}}{\check{\beta}^2} + \sqrt{\left(\frac{1}{\check{\beta}} + \frac{2\hat{\alpha}}{\check{\beta}^2}\right)^2 - \frac{1}{\check{\beta}^2}} \right\}^{-1},$$

then all the zeros of $B_n(z)$, $n \geq 1$, lie inside the interval $[\check{d}, \hat{d}]$. The idea of using chain sequences to obtain limits for the zeros is due to Chihara (see [1]) who considered the recurrence relations associated with orthogonal polynomials. For a very good use of this idea applied to orthogonal polynomials, see [4].

Returning to our recurrence relation (1.1), the occurrence of the situations

$$\hat{\beta} < \infty \quad \text{and} \quad \hat{\alpha} < \infty \tag{1.2}$$

and/or

$$0 < \check{\beta} \quad \text{and} \quad \hat{\alpha} < \infty, \tag{1.3}$$

is interesting. When (1.2) holds then $\hat{\beta} < \hat{d} < \infty$ and when (1.3) holds then $0 < \check{d} < \check{\beta}$. Let $\tilde{X} = (0, \infty) \setminus [\check{d}, \hat{d}]$ and let $\tilde{Z} = \overline{\mathbb{C}} \setminus [\check{d}, \hat{d}]$. Here, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane. When (1.2) and/or (1.3) hold, then $\tilde{X}$ is not empty.

In this paper, we assume that (1.2) always holds, and we consider the behaviour of the polynomials B_n and their zeros under the asymptotic conditions

(1.4)
$$\lim_{n\to\infty} \beta_{2n} = \beta^{(0)}, \quad \lim_{n\to\infty} \alpha_{2n} = \alpha^{(0)}, \quad \lim_{n\to\infty} \beta_{2n+1} = \beta^{(1)}, \quad \lim_{n\to\infty} \alpha_{2n+1} = \alpha^{(1)}.$$

Specifically, we give information on the limiting behaviour of the sequences $\{B_{2n-1}/B_{2n}\}$, $\{B_{2n}/B_{2n+1}\}$ and $\{B'_n/(nB_n)\}$. A previous study of this nature, with recurrence relation associated with orthogonal polynomials, was done in [10].

2. Preliminaries. Together with the polynomials B_n, $n \geq 0$ we also consider the polynomials A_n, $n \geq 0$, given by

$$A_{n+1}(z) = (z - \beta_{n+1})A_n(z) - \alpha_{n+1} z A_{n-1}(z), \qquad n \geq 1,$$

with initial values $A_0(z) = 0$ and $A_1(z) = 1$. For any $n \geq 1$, A_n is a polynomial of degree $n-1$. As in [5] the following results can be verified.

$$\begin{aligned} T_n(z) &= A_{n+1}(z)B_n(z) - A_n(z)B_{n+1}(z) = \alpha_2 \cdots \alpha_{n+1} z^n = \rho_n z^n, \qquad n \geq 1, \\ K_n(z) &= B'_{n+1}(z)B_n(z) - B'_n(z)B_{n+1}(z) \\ &= B_n^2(z) + \alpha_{n+1}\beta_n B_{n-1}^2(z) + \alpha_{n+1}\alpha_n z^2 K_{n-2}(z), \qquad n \geq 2, \end{aligned} \tag{2.1}$$

where $T_0(z) = 1$, $K_0(z) = B_0^2(z) = 1$ and $K_1(z) = B_1^2(z) + \alpha_2\beta_1 B_0^2(z)$. Hence $T_n(z) > 0$ and $K_n(z) > 0$ for $n \geq 0$ and for any real positive z. From this and $(-1)^n B_n(0) = \beta_1\beta_2 \cdots \beta_n > 0$ one can establish that the zeros of B_n are positive and distinct and different from those of A_n and B_{n-1}. Considering

$$\frac{A_n(z)}{B_n(z)} = \sum_{r=1}^{n} \frac{\lambda_r^{(n)}}{z - z_r^{(n)}}, \quad \frac{B_{n-1}(z)}{B_n(z)} = \sum_{r=1}^{n} \frac{\tau_r^{(n)}}{z - z_r^{(n)}}, \qquad n \geq 1, \tag{2.2}$$

we have

$$\lambda_r^{(n)} = \frac{A_n(z_r^{(n)})}{B_n'(z_r^{(n)})} = \frac{T_{n-1}(z_r^{(n)})}{K_{n-1}(z_r^{(n)})} > 0, \qquad 1 \le r \le n, \quad \sum_{r=1}^{n} \lambda_r^{(n)} = 1$$

and

$$\tau_r^{(n)} = \frac{B_{n-1}(z_r^{(n)})}{B_n'(z_r^{(n)})} = \frac{B_{n-1}^2(z_r^{(n)})}{K_{n-1}(z_r^{(n)})} > 0, \qquad 1 \le r \le n, \quad \sum_{r=1}^{n} \tau_r^{(n)} = 1.$$

Note that $\tau_r^{(n)} = \lambda_r^{(n)} B_{n-1}^2(z_r^{(n)})/\{(z_r^{(n)})^n \rho_n\}$.

Now define the non-decreasing (step) function $\psi_n(t) = \sum_{r=1}^n \lambda_r^{(n)} U(t - z_r^{(n)})$ where

$$U(t) = \begin{cases} 1, & t \ge 0, \\ 0, & t < 0, \end{cases}$$

we can then write

$$\frac{A_n(z)}{B_n(z)} = \int_{\check{d}}^{\hat{d}} \frac{1}{z-t}\, d\psi_n(t), \qquad n \ge 1.$$

The above results lead to

$$\text{(2.3)} \qquad \int_{\check{d}}^{\hat{d}} t^{-m+s} B_m(t)\, d\psi_n(t) = \rho_m \delta_{s,m}\,, \qquad 0 \le s \le m, \quad m \le n.$$

Below we sketch a proof of this last result. For more details of this proof and other related results we again refer to [5]. First, we have from (2.1)

$$\frac{A_{n+1}(z)}{B_{n+1}(z)} - \frac{A_n(z)}{B_n(z)} = \frac{\rho_n z^n}{B_n(z) B_{n+1}(z)}, \qquad n \ge 1.$$

Since $\rho_n \ne 0$, $B_n(0) \ne 0$ and $B_{n+1}(0) \ne 0$, one can conclude that there exist two sequences $\{\mu_r\}_{r=0}^{\infty}$ and $\{\mu_{-r-1}\}_{r=0}^{\infty}$ such that

$$\text{(2.4)} \qquad \frac{A_n(z)}{B_n(z)} = \begin{cases} \displaystyle\sum_{r=0}^{n-1} \frac{\mu_r}{z^{r+1}} + \frac{\mu_n^{(n)}}{z^{n+1}} + \mathcal{O}\left((1/z)^{n+2}\right), & z \to \infty, \\ \displaystyle\sum_{r=0}^{n-1} \mu_{-r-1} z^r + \mu_{-n-1}^{(n)} z^n + \mathcal{O}\left(z^{n+1}\right), & z \to 0. \end{cases}$$

Here $\mu_n^{(n)} = \mu_n - \rho_n$ and $\mu_{-n-1}^{(n)} = \mu_{-n-1} + \rho_n/\{B_n(0)B_{n+1}(0)\}$. Hence for $1 \le m \le n$, the first correspondence property in (2.4) gives

$$\int_{\check{d}}^{\hat{d}} \frac{1}{z-t}\, d\psi_n(t) - \frac{A_m(z)}{B_m(z)} = \frac{G_m^{(n)}(z)}{z^{m+1}}$$

where $G_m^{(n)}(z) = \int_{\check{d}}^{\hat{d}} \frac{z t^m}{z-t} (d\psi_n(t) - d\psi_m(t))$. Observe that

$$|G_m^{(n)}(z)| < \int_{\check{d}}^{\hat{d}} t^m (d\psi_n(t) + d\psi_m(t)) = \mu_m + \mu_m^{(m)},$$

for any $z \in \Lambda \equiv \{z : z = iy, y > \delta > 0\}$. Now in the equation

(2.5)
$$\int_{\check{d}}^{\hat{d}} \frac{B_m(z) - B_m(t)}{z-t}\, d\psi_n(t) - A_m(z) = \frac{1}{z^{m+1}} G_m^{(n)}(z) B_m(z) - \int_{\check{d}}^{\hat{d}} \frac{B_m(t)}{z-t}\, d\psi_n(t),$$

the right-hand side is a bounded function for $z \in \Lambda$, which tends to zero as $z \to \infty$ in Λ. However, the left-hand side is a polynomial of degree less than or equal to $m-1$. Therefore, both sides of this equation must be identically zero. The left-hand side of (2.5) then gives

$$A_m(z) = \int_{\check{d}}^{\hat{d}} \frac{B_m(z) - B_m(t)}{z-t}\, d\psi_n(t), \qquad 1 \le m \le n.$$

If we write this equation in the form

$$\frac{1}{B_m(z)} \int_{\check{d}}^{\hat{d}} \frac{B_m(t)}{z-t}\, d\psi_n(t) = \int_{\check{d}}^{\hat{d}} \frac{1}{z-t}\, d\psi_n(t) - \frac{A_m(z)}{B_m(z)},$$

then the second correspondence property in (2.4) gives the results in (2.3) for $0 \le s \le m-1$. For $s = m$, the required result can be established from the right-hand side of (2.5).

Suppose that (1.2) holds, then by considering the convergence of the associated continued fraction we can also show that $A_n(z)/B_n(z)$ converges uniformly to a limit $R_1(z)$ on every compact subset of $\tilde{Z}$. Hence the Grommer-Hamburger theorem [10] implies that there exists a non-decreasing function ψ on $E \subset [\check{d}, \hat{d}]$, such that

$$(2.6) \quad R_1(z) = \int_E \frac{1}{z-t}\, d\psi(t) \quad \text{and} \quad \int_E t^{-n+s} B_n(t)\, d\psi(t) = \rho_n \delta_{s,n}\,, \quad 0 \le s \le n.$$

Consequently,

$$\int_E f(t)\, d\psi(t) = \sum_{r=1}^{n} \lambda_r^{(n)} f(z_r^{(n)}), \quad \text{for} \quad t^n f(t) \in \mathbb{P}_{2n-1}.$$

Moreover, for every bounded and continuous function f on $(0, \infty)$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \lambda_r^{(n)} f(z_r^{(n)}) = \int_E f(t)\, d\psi(t).$$

The second equation in (2.6) gives the orthogonality property, or perhaps to be more precise the L-orthogonality property, satisfied by the polynomials B_n, $n \ge 1$.

3. Chain sequences and convergence results. From (1.1) the following recurrence relation can be obtained

$$B_{n+1}(z) = q_{n+1}(z) B_{n-1}(z) - p_{n+1}(z) B_{n-3}(z), \qquad n \ge 3,$$

where

$$q_{n+1}(z) = (z - \beta_n)(z - \beta_{n+1}) - \alpha_n \frac{z - \beta_{n+1}}{z - \beta_{n-1}} z - \alpha_{n+1} z$$

and

$$p_{n+1}(z) = \alpha_{n-1}\alpha_n \frac{z-\beta_{n+1}}{z-\beta_{n-1}} z^2.$$

One can easily verify that for $z \in \tilde{X}$ one has $\frac{B_{n+1}(z)}{B_{n-1}(z)} > 0$, $n \geq 1$ and $p_{n+1}(z) > 0$, $n \geq 3$. Since one can write

$$q_{n+1}(z) = \frac{B_{n+1}(z)}{B_{n-1}(z)} + p_{n+1}(z)\frac{B_{n-3}(z)}{B_{n-1}(z)}, \qquad n \geq 3, \tag{3.1}$$

it follows that $q_{n+1}(z) > 0$, $n \geq 3$, for $z \in \tilde{X}$. Let

$$q_2(z) = (z-\beta_1)(z-\beta_2) - \alpha_2 z = B_2(z)$$

and

$$q_3(z) = (z-\beta_2)(z-\beta_3) - \alpha_2 \frac{z-\beta_3}{z-\beta_1} z - \alpha_3 z = B_3(z)/B_1(z),$$

then also $q_2(z) > 0$ and $q_3(z) > 0$ for $z \in \tilde{X}$. From the above recurrence relation one also gets

$$\frac{B_{n-1}(z)}{q_{n-1}(z)B_{n-3}(z)}\left\{1 - \frac{B_{n+1}(z)}{q_{n+1}(z)B_{n-1}(z)}\right\} = \frac{p_{n+1}(z)}{q_{n-1}(z)q_{n+1}(z)}, \qquad n \geq 3,$$

which for any $z \in \tilde{X}$ gives the two chain sequences $\{a_n^{(0)}(z)\}$ and $\{a_n^{(1)}(z)\}$ where

$$a_n^{(\nu)}(z) = \{1 - m_{n-1}^{(\nu)}(z)\} m_n^{(\nu)}(z), \qquad n \geq 1,$$

with

$$a_n^{(\nu)}(z) = \frac{p_{2n+2+\nu}(z)}{q_{2n+\nu}(z)q_{2n+2+\nu}(z)}, \qquad n \geq 1,$$

and

$$m_n^{(\nu)}(z) = 1 - \frac{B_{2n+2+\nu}(z)}{q_{2n+2+\nu}(z)B_{2n+\nu}(z)}, \qquad n \geq 0.$$

Since $m_0^{(\nu)} = 0$ for $\nu = 0$ and $\nu = 1$, the sequence $\{m_n^{(\nu)}(z)\}$ is the minimal parameter sequence of $\{a_n^{(\nu)}(z)\}$. Now under the asymptotic conditions (1.4) we have

$$\lim_{n\to\infty} q_n(z) = (z-\beta^{(0)})(z-\beta^{(1)}) - (\alpha^{(0)}+\alpha^{(1)})z, \quad \lim_{n\to\infty} p_n(z) = \alpha^{(0)}\alpha^{(1)}z^2$$

and

$$\lim_{n\to\infty} a_n^{(\nu)}(z) = \frac{\alpha^{(0)}\alpha^{(1)}z^2}{\{(z-\beta^{(0)})(z-\beta^{(1)}) - (\alpha^{(0)}+\alpha^{(1)})z\}^2} = a(z),$$

for any $z \in \tilde{X}$. Hence, it follows from [1, Thm 6.4, p. 102] that $0 \leq a(z) \leq 1/4$ and that the corresponding parameter sequences $\{m_n^{(\nu)}(z)\}$ converge to limits that depend only on $a(z)$. This means that they both converge to the same limit and consequently,

$B_{n+1}(z)/B_{n-1}(z)$ converges to a limit $R_2(z)$ for any $z \in \tilde{X}$. To obtain this limit we let $n \to \infty$ in (3.1) and get

$$R_2^2(z) - \left\{(z-\beta^{(0)})(z-\beta^{(1)}) - (\alpha^{(0)}+\alpha^{(1)})z\right\} R_2(z) + \alpha^{(0)}\alpha^{(1)}z^2 = 0.$$

From this we find

$$R_2(z) = \frac{1}{2}\Big\{(z-\beta^{(0)})(z-\beta^{(1)}) - (\alpha^{(0)}+\alpha^{(1)})z \\ \pm \sqrt{[(z-\beta^{(0)})(z-\beta^{(1)}) - (\alpha^{(0)}+\alpha^{(1)})z]^2 - 4\alpha^{(0)}\alpha^{(1)}z^2}\Big\}.$$

Since $B_{n+1}(0)/B_{n-1}(0) = \beta_{n+1}\beta_n$ and $B_{n+1}(z)/B_{n-1}(z) \to \infty$ as $z \to \infty$, the positive sign gives the desired limit.

Now for $z \in \tilde{Z}$,

$$\left|\frac{B_{n-1}(z)}{B_{n+1}(z)}\right| = \left|\frac{B_{n-1}(z)}{B_n(z)}\right| \left|\frac{B_n(z)}{B_{n+1}(z)}\right| \le \frac{1}{\delta^2}\left|\sum_{r=1}^{n}\tau_r^{(n)}\right|\left|\sum_{r=1}^{n+1}\tau_r^{(n+1)}\right| = \frac{1}{\delta^2},$$

where $\delta = \operatorname{dist}(z, [\check{d}, \hat{d}])$. This means that the sequence $B_{n-1}(z)/B_{n+1}(z)$, $n \ge 1$, is uniformly bounded on every compact subset of $\tilde{Z}$. Therefore, from the Stieltjes-Vitali theorem (see for example [3]) one gets

THEOREM 3.1. *Assuming that* (1.2) *holds then under the asymptotic conditions of* (1.4)

$$\lim_{n\to\infty} \frac{B_{n+1}(z)}{B_{n-1}(z)} = R_2(z) = \frac{1}{2}\left\{z^2 - u_1 z + u_2 + \sqrt{(z^2 - u_1 z + u_2)^2 - 4u_3 z^2}\right\},$$

uniformly on every compact subset of $\tilde{Z}$. *Here,* $u_1 = \beta^{(0)} + \beta^{(1)} + \alpha^{(0)} + \alpha^{(1)}$, $u_2 = \beta^{(0)}\beta^{(1)}$ *and* $u_3 = \alpha^{(0)}\alpha^{(1)}$.

From (1.1) we deduce

$$\frac{B_{n+1}(z)}{B_{n-1}(z)} = (z - \beta_{n+1})\frac{B_n(z)}{B_{n-1}(z)} - \alpha_{n+1}z,$$

hence one can also conclude

COROLLARY 1. *Suppose* (1.2) *and* (1.4) *hold, then*

$$\lim_{n\to\infty} \frac{B_{2n-1}(z)}{B_{2n}(z)} = R_3^{(1)}(z) = \frac{z-\beta^{(1)}}{R_2(z) + \alpha^{(1)}z},$$
$$\lim_{n\to\infty} \frac{B_{2n}(z)}{B_{2n+1}(z)} = R_3^{(0)}(z) = \frac{z-\beta^{(0)}}{R_2(z) + \alpha^{(0)}z},$$

uniformly on every compact subset of $\tilde{Z}$.

Since $B_{n+1}(z)/B_{n-1}(z)$ is analytic in $\tilde{Z}$, we can take derivatives in these asymptotic formulas to find

$$\frac{d}{dz}\left(\frac{B_{n+1}(z)}{B_{n-1}(z)}\right) = \frac{B'_{n+1}(z)}{B_{n-1}(z)} - \frac{B'_{n-1}(z)}{B_{n-1}(z)}\frac{B_{n+1}(z)}{B_{n-1}(z)} \to R'_2(z),$$

for any $z \in \tilde{Z}$. Thus, from

$$\frac{d}{dz}\left(\frac{B_{n+1}(z)}{B_{n-1}(z)}\right) \Big/ \left(\frac{B_{n+1}(z)}{B_{n-1}(z)}\right) = \frac{B'_{n+1}(z)}{B_{n+1}(z)} - \frac{B'_{n-1}(z)}{B_{n-1}(z)},$$

we obtain

$$\lim_{n\to\infty}\left(\frac{B'_{n+1}(z)}{B_{n+1}(z)} - \frac{B'_{n-1}(z)}{B_{n-1}(z)}\right) = \frac{R'_2(z)}{R_2(z)}, \tag{3.2}$$

for any $z \in \tilde{Z}$.

THEOREM 3.2. *Assuming that* (1.2) *holds, then under the asymptotic conditions* (1.4) *we have*

$$\lim_{n\to\infty}\frac{B'_n(z)}{nB_n(z)} = R_4(z) = \frac{R'_2(z)}{2R_2(z)} = \frac{1}{2z}\frac{z^2-u_2}{\sqrt{(z^2-u_1z+u_2)^2-4u_3z^2}} + \frac{1}{2z},$$

uniformly on every compact subset of $\tilde{Z}$.

Proof. Consider the Cesàro summation of the odd and even combinations of the sequence given by (3.2), then it follows that $B'_n(z)/[nB_n(z)] \to R_4(z)$ for any $z \in \tilde{Z}$. Since $B'_n(z)/[nB_n(z)]$ is uniformly bounded on any compact subset of $\tilde{Z}$, the convergence is also uniform on compact subsets. To obtain the value of the limit function we take

$$2R_2(z) = z^2 - u_1z + u_2 + \left\{(z^2-u_1z+u_2)^2 - 4u_3z^2\right\}^{1/2}$$

and a straightforward differentiation of this gives

$$2R'_2(z) = \{(2z-u_1)2R_2(z) - 4u_3z\} \,/\, \left\{(z^2-u_1z+u_2)^2-4u_3z^2\right\}^{1/2}.$$

Hence, the observation

$$\frac{2u_3z^2}{R_2(z)} = z^2 - u_1z + u_2 - \left\{(z^2-u_1z+u_2)^2-4u_3z^2\right\}^{1/2}$$

immediately leads to the required result of the theorem. □

Note that

$$\frac{B'_n(z)}{nB_n(z)} = \sum_{r=1}^{n}\frac{1/n}{z-z_r^{(n)}} = \int_{\check{d}}^{\hat{d}}\frac{1}{z-t}\,dF_n(t) \tag{3.3}$$

where the step function $nF_n(t) = \sum_{r=1}^{n} U(t-z_r^{(n)})$ represents the number of zeros of B_n less than or equal to t. Hence one can extract information regarding the asymptotic behaviour of the zeros of B_n from the limit function R_4. First of all, we can write

$$R_4(z) = \frac{1}{2z}\frac{z^2-\beta^{(0)}\beta^{(1)}}{\sqrt{z-a}\sqrt{z-\tilde{a}}\sqrt{z-\tilde{b}}\sqrt{z-b}} + \frac{1}{2z}, \tag{3.4}$$

where $0 \le a \le \tilde{a} \le b$ satisfy $ab = \tilde{a}\tilde{b} = u_2$, $a+b = u_1 + 2\sqrt{u_3}$ and $\tilde{a}+\tilde{b} = u_1 - 2\sqrt{u_3}$. Furthermore, since $ab = \tilde{a}\tilde{b} = \beta^{(0)}\beta^{(1)}$, one also has the decomposition

$$\begin{aligned} R_4(z) &= \frac{-\gamma_0}{z}\frac{\sqrt{z-b}\sqrt{z-\tilde{b}}}{\sqrt{z-\tilde{a}}\sqrt{z-a}} + \frac{\gamma_1}{z}\frac{\sqrt{z-a}\sqrt{z-\tilde{a}}}{\sqrt{z-b}\sqrt{z-\tilde{b}}} + \frac{1}{2z} \\ &= \frac{\gamma_0}{z}\left[\sqrt{\frac{b\tilde{b}}{a\tilde{a}}} - \frac{\sqrt{z-b}\sqrt{z-\tilde{b}}}{\sqrt{z-\tilde{a}}\sqrt{z-a}}\right] + \frac{\gamma_1}{z}\left[\frac{\sqrt{z-a}\sqrt{z-\tilde{a}}}{\sqrt{z-b}\sqrt{z-\tilde{b}}} - \sqrt{\frac{a\tilde{a}}{b\tilde{b}}}\right] \end{aligned}$$

where

$$\gamma_0 = \frac{\beta^{(0)}\beta^{(1)} + a\tilde{a}}{2(b\tilde{b} - a\tilde{a})} \quad \text{and} \quad \gamma_1 = \frac{\beta^{(0)}\beta^{(1)} + b\tilde{b}}{2(b\tilde{b} - a\tilde{a})}.$$

The above relations for R_4 indicate that $R_4(z)$ is finite and real if and only if z does not belong to the union $[a, \tilde{a}] \cup [\tilde{b}, b]$.

4. Integral representations. From the convergence of (3.3), which we have already established, and the Grommer-Hamburger theorem, it follows that there exists a non-decreasing function F on $E \subset [\check{d}, \hat{d}]$, such that

$$\lim_{n\to\infty} \int_{\check{d}}^{\hat{d}} \frac{1}{z-t}\, dF_n(t) = R_4(z) = \int_E \frac{1}{z-t}\, dF(t),$$

uniformly on every compact subsets of $\tilde{Z} = \overline{\mathbb{C}}\backslash[\check{d}, \hat{d}]$ and as a consequence F_n converges (weakly) to F. This result also means that for every bounded continuous function f on $(0, \infty)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(z_r^{(n)}) = \int_E f(t)\, dF(t).$$

THEOREM 4.1. *Suppose that* (1.2) *and* (1.4) *hold, and that* $0 < \beta^{(0)}\beta^{(1)}$ *and* $0 < \alpha^{(0)}\alpha^{(1)}$. *Then*

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{t} \frac{|x-\beta|\ (x+\beta)/x}{\sqrt{\gamma^2 x - (x-\beta)^2}\sqrt{(x-\beta)^2 - \tilde{\gamma}^2 x}}\, I_E(x)\, dx$$

where $I_E(x) = U(x-a) - U(x-\tilde{a}) + U(x-\tilde{b}) - U(x-b)$ *is the indicator function of the set* $E = B = [a, \tilde{a}] \cup [\tilde{b}, b]$, $\gamma^2 = (\sqrt{\beta^{(0)}} - \sqrt{\beta^{(1)}})^2 + (\sqrt{\alpha^{(0)}} + \sqrt{\alpha^{(1)}})^2 = (\sqrt{b} - \sqrt{a})^2$, $\tilde{\gamma}^2 = (\sqrt{\beta^{(0)}} - \sqrt{\beta^{(1)}})^2 + (\sqrt{\alpha^{(0)}} - \sqrt{\alpha^{(1)}})^2 = (\sqrt{\tilde{b}} - \sqrt{\tilde{a}})^2$ *and* $\beta = \sqrt{\beta^{(0)}\beta^{(1)}}$.

Proof. Since $0 < \beta^{(0)}\beta^{(1)} < \infty$ and $0 < \alpha^{(0)}\alpha^{(1)} < \infty$ we have $0 < a < \tilde{a} \le \tilde{b} < b < \infty$. Now observe that we can also write

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{t} \frac{|T'(x)|}{\sqrt{4u_3 - \{T(x)\}^2}} I_B(x)\, dx$$

where $T(x) = x - u_1 + u_2/x$. Hence we need to prove that

$$R(z) = \frac{1}{2\pi} \int_B \frac{1}{z-t} \frac{|T'(t)|}{\sqrt{4u_3 - \{T(t)\}^2}}\, dt$$

is equal to $R_4(z)$. Since $z\sqrt{\{T(z)\}^2 - 4u_3} = \sqrt{(z-a)(z-\tilde{a})(z-b)(z-\tilde{b})}$, we obtain after the substitution $y = T(t)$

$$R(z) = \frac{1}{2\pi} \int_{-2\sqrt{u_3}}^{2\sqrt{u_3}} \frac{2}{2z - \left\{(u_1+y) - \sqrt{(u_1+y)^2 - 4u_2}\right\}} \frac{dy}{\sqrt{4u_3 - y^2}}$$
$$+ \frac{1}{2\pi} \int_{-2\sqrt{u_3}}^{2\sqrt{u_3}} \frac{2}{2z - \left\{(u_1+y) + \sqrt{(u_1+y)^2 - 4u_2}\right\}} \frac{dy}{\sqrt{4u_3 - y^2}}.$$

Thus

$$R(z) = \frac{1}{2\pi}\int_{-2\sqrt{u_3}}^{2\sqrt{u_3}} \frac{2z-(u_1+y)}{z^2-(u_1+y)z+u_2}\frac{dy}{\sqrt{4u_3-y^2}}$$
$$= \frac{1}{\pi}\int_{-2\sqrt{u_3}}^{2\sqrt{u_3}} \frac{1-\frac{1}{2z}(u_1+y)}{T(z)-y}\frac{dy}{\sqrt{4u_3-y^2}}.$$

Since,

$$\frac{1}{\pi}\int_{-\lambda}^{\lambda}\frac{1}{z-y}\frac{dy}{\sqrt{\lambda^2-y^2}} = \frac{1}{\sqrt{z^2-\lambda^2}} \quad \text{and} \quad \frac{1}{\pi}\int_{-\lambda}^{\lambda}\frac{dy}{\sqrt{\lambda^2-y^2}} = 1,$$

for any $\lambda > 0$, we then have

$$R(z) = \frac{1}{\pi}\int_{-2\sqrt{u_3}}^{2\sqrt{u_3}} \frac{1-\frac{1}{2z}\{u_1+T(z)\}+\frac{1}{2z}\{T(z)-y\}}{T(z)-y}\frac{dy}{\sqrt{4u_3-y^2}},$$
$$= \left\{1-\frac{u_1}{2z}-\frac{T(z)}{2z}\right\}\frac{1}{\sqrt{\{T(z)\}^2-4u_3}}+\frac{1}{2z}.$$

Substituting the expression for $T(z)$ then gives the required result $R(z) = R_4(z)$, which completes the proof. □

Note that one can also write

$$R_4(z) = \frac{1}{2z}\frac{z^2-\beta^2}{\sqrt{(z-\beta)^2-\gamma^2 z}\sqrt{(z-\beta)^2-\tilde{\gamma}^2 z}}+\frac{1}{2z}$$

and

$$(4.1) \qquad R_2(z) = \frac{1}{2}\left\{(z-\beta)^2-\hat{\gamma}^2 z+\sqrt{(z-\beta)^2-\tilde{\gamma}^2 z}\sqrt{(z-\beta)^2-\gamma^2 z}\right\}$$

where $\hat{\gamma}^2 = (\sqrt{\beta^{(0)}}-\sqrt{\beta^{(1)}})^2+\alpha^{(0)}+\alpha^{(1)}$.

Now we consider the convergence of the ratio $B_{n-1}(z)/B_n(z)$. First we give the following theorem.

THEOREM 4.2. *Let a, b, $\tilde{a}$, $\tilde{b}$ be such that $0 < a < \tilde{a} \le \tilde{b} < b < \infty$ and $ab = \tilde{a}\tilde{b} = \beta^2$. Then for $d_2 = (\sqrt{b}-\sqrt{a})$, $d_1 = (\sqrt{\tilde{b}}-\sqrt{\tilde{a}})$, $d_0^2 \le d_1^2$ and $B = [a,\tilde{a}]\cup[\tilde{b},b]$,*

$$D\int_B \frac{1}{z-t}V(t;d_1,d_2,d_0,\delta)\,dt = L(z;d_1,d_2,d_0,\delta)$$

where

$$D = \frac{2}{\pi}\frac{\left[(d_2^2-d_0^2)^{1/2}+(d_1^2-d_0^2)^{1/2}\right]^2}{(d_2^2-d_1^2)^2},$$
$$V(t;d_1,d_2,d_0,\delta) = \frac{\sqrt{d_2^2 t-(t-\beta)^2}\sqrt{(t-\beta)^2-d_1^2 t}}{t\left[(t-\beta)^2-d_0^2 t\right]}\frac{(t-\beta)(t-\delta)}{|t-\beta|}$$

and

$$L(z;d_1,d_2,d_0,\delta)$$
$$= \frac{2(z-\delta)}{(z-\beta)^2-d_0^2 z-\left[(d_2^2-d_0^2)(d_1^2-d_0^2)\right]^{1/2} z+\sqrt{(z-\beta)^2-d_1^2 z}\sqrt{(z-\beta)^2-d_2^2 z}}.$$

Proof. For $|d_0| \le d_1 < d_2$ and $\hat{B} = [-d_2, -d_1] \cup [d_1, d_2]$ we have

$$D \int_{\hat{B}} \frac{1}{w-x} \frac{\sqrt{d_2^2 - x^2}\sqrt{x^2 - d_1^2}}{[x^2 - d_0^2]} |x|\, dx = \frac{2w}{w^2 - d_0^2 - [(d_2^2 - d_0^2)(d_1^2 - d_0^2)]^{1/2} + \sqrt{w^2 - d_1^2}\sqrt{w^2 - d_2^2}}.$$

This result can be verified from [10, eq.(3.6)] where one considers $S(G(x; d_1, d_2, d_0); z) + S(G(x; d_1, d_2, -d_0); z)$. Applying the change of variables $w = (z-\beta)/\sqrt{z}$ and $x = (t-\beta)/\sqrt{t}$ (see [7]) then leads to

$$D \int_B \frac{1}{z-t} V(t; d_1, d_2, d_0, \beta)\, dt = L(z; d_1, d_2, d_0, \beta).$$

From this the result of the theorem follows since

$$(\beta - \delta) \int_B \frac{1}{z-t} \frac{V(t; d_1, d_2, d_0, \beta)}{t - \beta}\, dt = \frac{(\beta - \delta)}{z - \beta} \int_B \frac{1}{z-t} V(t; d_1, d_2, d_0, \beta)\, dt.$$

□

Note that the function $V(t; d_1, d_2, d_0, \delta)$ is a (positive) weight function on B only if $\tilde{a} \le \delta \le \tilde{b}$.

THEOREM 4.3. *Suppose that* (1.2) *and* (1.4) *hold, together with* $0 < \beta^{(0)}\beta^{(1)}$ *and* $0 < \alpha^{(0)}\alpha^{(1)}$. *Then*

$$R_3^{(0)}(z) = \int_{E^{(0)}} \frac{1}{z-t}\, d\phi^{(0)}(t) = \frac{1 - \alpha_{\min}/\alpha^{(0)}}{z - \beta^{(1)}} + \frac{1}{2\pi\alpha^{(0)}} \int_B \frac{1}{z-t} V(t; \tilde{\gamma}, \gamma, \lambda, \beta^{(0)})\, dt$$

and

$$R_3^{(1)}(z) = \int_{E^{(1)}} \frac{1}{z-t}\, d\phi^{(1)}(t) = \frac{1 - \alpha_{\min}/\alpha^{(1)}}{z - \beta^{(0)}} + \frac{1}{2\pi\alpha^{(1)}} \int_B \frac{1}{z-t} V(t; \tilde{\gamma}, \gamma, \lambda, \beta^{(1)})\, dt$$

where B, γ *and* $\tilde{\gamma}$ *are as in Theorem* 4.1, $\lambda^2 = (\sqrt{\beta^{(0)}} - \sqrt{\beta^{(1)}})^2$ *and* $\alpha_{\min} = \min\{\alpha^{(0)}, \alpha^{(1)}\}$.

Proof. From (4.1) and Corollary 1

(4.2)
$$R_3^{(0)}(z) = \frac{2(z - \beta^{(0)})}{(z-\beta)^2 - \lambda^2 z - (\alpha^{(1)} - \alpha^{(0)})z + \sqrt{(z-\beta)^2 - \tilde{\gamma}^2 z}\sqrt{(z-\beta)^2 - \gamma^2 z}}.$$

Since this limit can be written as

$$\frac{|\alpha^{(1)} - \alpha^{(0)}| - (\alpha^{(1)} - \alpha^{(0)})}{2\alpha^{(0)}(z - \beta^{(1)})} + \frac{\{(\alpha^{(1)} + \alpha^{(0)}) - |\alpha^{(1)} - \alpha^{(0)}|\}}{2\alpha^{(0)}} L(z; \tilde{\gamma}, \gamma, \lambda, \beta^{(0)}),$$

the first result of this theorem follows from Theorem 4.2. Similarly, since

(4.3)
$$R_3^{(1)}(z) = \frac{2(z - \beta^{(1)})}{(z-\beta)^2 - \lambda^2 z - (\alpha^{(0)} - \alpha^{(1)})z + \sqrt{(z-\beta)^2 - \tilde{\gamma}^2 z}\sqrt{(z-\beta)^2 - \gamma^2 z}},$$

the other result of the theorem is obtained by interchanging $(\alpha^{(1)}, \beta^{(1)})$ and $(\alpha^{(0)}, \beta^{(0)})$. □

The results of Theorem 4.3 and equation (2.2) also means that for every bounded and continuous function f on $(0, \infty)$

$$\lim_{n\to\infty} \sum_{r=1}^{2n} \tau_r^{(2n)} f(z_r^{(2n)}) = \int_{E^{(0)}} f(t)\, d\phi^{(0)}(t)$$

and

$$\lim_{n\to\infty} \sum_{r=1}^{2n+1} \tau_r^{(2n+1)} f(z_r^{(2n+1)}) = \int_{E^{(1)}} f(t)\, d\phi^{(1)}(t).$$

We now consider some special cases. Note that, even though the coefficients β_n, α_{n+1} $(n \geq 1)$ are positive, any of their limits, that is any of $\beta^{(0)}$, $\alpha^{(0)}$, $\beta^{(1)}$ and $\alpha^{(1)}$ is allowed to take the value zero.

CASE 1. First we consider the case

$$\alpha^{(0)} = \alpha^{(1)} = \alpha > 0 \quad \text{and} \quad \beta^{(0)} = \beta^{(1)} = \beta > 0.$$

Then $\tilde{a} = \tilde{b} = \beta$, $a = \beta + 2\alpha - \sqrt{(\beta + 2\alpha)^2 - \beta^2}$ and $b = \beta + 2\alpha + \sqrt{(\beta + 2\alpha)^2 - \beta^2}$. Substitution of these results in (3.4) and Theorem 4.1 gives

$$R_4(z) = \frac{1}{2z} + \frac{1}{2z} \frac{z + \beta}{\sqrt{z - a}\sqrt{z - b}} = \frac{1}{2\pi} \int_a^b \frac{1}{z - t} \frac{1 + \beta/t}{\sqrt{b - t}\sqrt{t - a}}\, dt. \tag{4.4}$$

From Theorem 4.3 it follows that $R_3^{(0)}(z) = R_3^{(1)}(z) = R_3(z)$ where

$$R_3(z) = \frac{2}{z - \beta + \sqrt{(z - \beta)^2 - 4\alpha z}} = \frac{1}{2\pi\alpha} \int_a^b \frac{1}{z - t} V(t; 0, 2\sqrt{\alpha}, 0, \beta)\, dt.$$

Here, $(z - a)(z - b) = (z - \beta)^2 - 4\alpha z$.

CASE 2. Now we consider the case where one of $\alpha^{(0)}$ or $\alpha^{(1)}$ takes the value zero. We consider (for ν equal to 0 or 1),

$$\alpha^{(\nu)} = 0, \quad \alpha^{(1-\nu)} = \alpha \geq 0, \quad \beta^{(\nu)} > 0 \quad \text{and} \quad \beta^{(1-\nu)} > 0.$$

Then, from (3.4), it is seen that $a = \tilde{a}$ and $b = \tilde{b}$. With this we can write

$$R_4(z) = \frac{1}{2z} + \frac{1}{2z} \frac{z^2 - \beta^{(0)}\beta^{(1)}}{(z - a)(z - b)} = \frac{1/2}{z - a} + \frac{1/2}{z - b}.$$

Therefore, $F(t) = \frac{1}{2}U(t - a) + \frac{1}{2}U(t - b)$. Note that if $\alpha = 0$ then $a = \beta_{\min} = \min\{\beta^{(0)}, \beta^{(1)}\}$ and $b = \beta_{\max} = \max\{\beta^{(0)}, \beta^{(1)}\}$. From (4.2) and (4.3) the following also hold. If $\alpha^{(0)} = \alpha$ then

$$R_3^{(0)}(z) = \frac{1}{z - \beta^{(1)}} \quad \text{and} \quad R_3^{(1)}(z) = \frac{z - \beta^{(1)}}{(z - a)(z - b)} = \frac{\beta^{(1)} - a}{b - a} \frac{1}{z - a} + \frac{b - \beta^{(1)}}{b - a} \frac{1}{z - b}.$$

If $\alpha^{(1)} = \alpha$ then

$$R_3^{(0)}(z) = \frac{z-\beta^{(0)}}{(z-a)(z-b)} = \frac{\beta^{(0)}-a}{b-a}\frac{1}{z-a} + \frac{b-\beta^{(0)}}{b-a}\frac{1}{z-b} \quad \text{and} \quad R_3^{(1)}(z) = \frac{1}{z-\beta^{(0)}}.$$

CASE 3. Now we consider the case where

$$\alpha^{(\nu)} \geq 0, \quad \alpha^{(1-\nu)} \geq 0, \quad \beta^{(\nu)} = 0 \quad \text{and} \quad \beta^{(1-\nu)} = \beta \geq 0.$$

It follows that $u_2 = 0$, and hence,

$$a = \tilde{a} = 0, \quad \tilde{b} = \beta + (\sqrt{\alpha^{(0)}} - \sqrt{\alpha^{(1)}})^2, \quad \text{and} \quad b = \beta + (\sqrt{\alpha^{(0)}} + \sqrt{\alpha^{(1)}})^2.$$

Substitution of these values in (3.4) gives

$$R_4(z) = \frac{1/2}{z} + \frac{1/2}{\sqrt{z-\tilde{b}}\sqrt{z-b}} = \frac{1}{2z} + \frac{1}{2\pi}\int_{\tilde{b}}^{b} \frac{1}{z-t}\frac{1}{\sqrt{b-t}\sqrt{t-\tilde{b}}}\,dt$$

and

$$F(t) = \frac{1}{2}U(t) + \frac{1}{2\pi}\int_{-\infty}^{t} \frac{U(x-\tilde{b}) - U(x-b)}{\sqrt{b-x}\sqrt{x-\tilde{b}}}\,dx.$$

If $\alpha^{(0)} = \alpha^{(1)}$ and $\beta = 0$ then $\tilde{b}$ is also zero.

CASE 4. Finally, we assume that

$$\alpha^{(\nu)} = 0, \quad \alpha^{(1-\nu)} = \alpha \geq 0, \quad \beta^{(\nu)} = 0 \quad \text{and} \quad \beta^{(1-\nu)} = \beta \geq 0.$$

Then $a = \tilde{a} = 0$, $b = \tilde{b} = \beta + \alpha$ and

$$R_4(z) = \frac{1/2}{z} + \frac{1/2}{z-b}.$$

This means that $F(t) = \frac{1}{2}U(t) + \frac{1}{2}U(t-b)$. If $\beta = \alpha = 0$ then $F(t) = U(t)$.

5. Examples. We now give some examples of polynomials that satisfy the recurrence relation (1.1) for which the coefficients have the properties (1.2) and (1.4).

EXAMPLE 1. For $\lambda > 0$ and $0 < a < b < \infty$ the polynomials B_n defined by

$$\int_a^b t^{-n+s} B_n(t) t^{-\lambda}(b-t)^{\lambda-1/2}(t-a)^{\lambda-1/2}\,dt = 0, \qquad 0 \leq s \leq n-1,$$

satisfy the recurrence relation (1.1) with

$$\beta_n = \beta, \quad \alpha_{n+1} = \frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}\alpha, \qquad n \geq 1,$$

where $\beta = \sqrt{ab}$ and $\alpha = (\sqrt{b} - \sqrt{a})^2/4$. For a proof of this result, see [7]. This result is also valid for $\lambda = 0$ if we take $\alpha_2 = 2\alpha$. This case has been worked out in detail by

Cooper and Gustafson [2] and their polynomials R_{2n} are related to our polynomials B_n by $B_n(x^2) = x^n R_{2n}(x)$.

Since $\beta^{(0)} = \beta^{(1)} = \beta$ and $\alpha^{(0)} = \alpha^{(1)} = \alpha$, we are dealing with case 1 of the last section.

EXAMPLE 2. For $0 < a < b < \infty$, we consider the polynomials B_n defined by

$$\int_a^b t^{-n+s} B_n(t) \frac{1+\sqrt{ab}/t}{\sqrt{b-t}\sqrt{t-a}}\, dt = 0, \qquad 0 \le s \le n-1 .$$

The distribution function here is the one that appears in the limit in (4.4). In [8], it was proved that these polynomials satisfy the recurrence relation (1.1) with

$$\beta_n = \beta \ell_{n-1}/\ell_n, \quad \alpha_{n+1} = (\ell_n^2 - 1)\beta_n, \qquad n \ge 1,$$

where $\ell_n = [(1+\ell)^n - (1-\ell)^n]\ell/[(1+\ell)^n + (1-\ell)^n]$, $\ell = \sqrt{1+\alpha/\beta}$, $\beta = \sqrt{ab}$ and $\alpha = (\sqrt{b} - \sqrt{a})^2/4$. Since $\beta^{(0)} = \beta^{(1)} = \beta$ and $\alpha^{(0)} = \alpha^{(1)} = \alpha$, we are again in case 1 of the last section.

EXAMPLE 3. For $\lambda > 0$ we now consider the polynomials B_n defined by

$$\int_0^\infty t^{-n+s} B_n(t)\, d\psi^{(\lambda)}(t) = 0, \qquad 0 \le s \le n-1,$$

where $\psi^{(\lambda)}$ is a step function with jumps

$$d\psi^{(\lambda)}(t) = \frac{2t}{t+\beta} \frac{(k+\lambda)^{-1} e^{-k}}{k!},$$

at the points

$$t = t_{k+1} = \frac{1 + 2\beta(k+\lambda) + \sqrt{1+4\beta(k+\lambda)}}{2(k+\lambda)} \quad \text{and} \quad t = t_{-k-1} = \beta^2/t_{k+1},$$

for $k = 0, 1, \ldots$. These polynomials are related to the Tricomi-Carlitz polynomials through the transformation considered in [7]. The coefficients of the associated recurrence relation satisfy

$$\beta_n = \beta, \quad \alpha_{n+1} = \frac{n}{(n+\lambda)(n+\lambda-1)}, \qquad n \ge 1.$$

Since $\beta^{(0)} = \beta^{(1)} = \beta$ and $\alpha^{(0)} = \alpha^{(1)} = 0$, we now are in case 2 of the last section, and we have

$$R_4(z) = \frac{1}{z-\beta} \quad \text{and} \quad R_3(z) = \frac{1}{z-\beta}.$$

EXAMPLE 4. As a final example, we consider the polynomials B_n that satisfy the recurrence relation (1.1) with

$$\alpha_{2n} = \alpha^{(0)}, \quad \alpha_{2n+1} = \alpha^{(1)}, \quad \beta_{2n} = \beta^{(0)} \text{ for } n \ge 1 \quad \text{and} \quad \beta_{2n+1} = \beta^{(1)} \text{ for } n \ge 0,$$

where $0 < \beta^{(0)}\beta^{(1)} < \infty$ and $0 < \alpha^{(0)}\alpha^{(1)} < \infty$. Hence the results of Theorems 3.1, 3.2, 4.3 and 4.3 are valid. Since $A_n(z)/B_n(z) \to R_1(z)$ where in this case

$$R_1(z) = \frac{1}{z-\beta^{(1)}} - \frac{\alpha^{(0)}z}{z-\beta^{(0)}} - \frac{\alpha^{(1)}z}{z-\beta^{(1)}} - \frac{\alpha^{(0)}z}{z-\beta^{(0)}} - \frac{\alpha^{(1)}z}{z-\beta^{(1)}} - \cdots,$$

we obtain from the theory of continued fractions

$$R_1(z) = \frac{2(z-\beta^{(0)})}{(z-\beta)^2 - \lambda^2 z - (\alpha^{(0)} - \alpha^{(1)})z + \sqrt{(z-\beta)^2 - \tilde{\gamma}^2 z}\sqrt{(z-\beta)^2 - \gamma^2 z}}.$$

Hence, in the same way as in Theorem 4.3, we obtain

$$R_1(z) = \int_E \frac{1}{z-t}\, d\psi(t) = \frac{1 - \alpha_{\min}/\alpha^{(1)}}{z-\beta^{(1)}} + \frac{1}{2\pi\alpha^{(1)}}\int_B \frac{1}{z-t} V(t;\tilde{\gamma},\gamma,\lambda,\beta^{(0)})\, dt$$

where B, γ, $\tilde{\gamma}$ and λ are as in Theorem 4.3. From this we conclude that, with the above distribution function ψ, the polynomials B_n satisfy

$$\int_E t^{-n+s} B_n(t)\, d\psi(t) = 0, \qquad 0 \le s \le n-1.$$

Observe that if $\alpha^{(0)} = \alpha^{(1)} = \alpha$ then $\lambda = \tilde{\gamma}$ and

$$d\psi(t) = \frac{1}{2\pi\alpha} \frac{\sqrt{b-t}\sqrt{t-a}}{t} \sqrt{\frac{t-\beta^{(0)}}{t-\beta^{(1)}}}\, dt$$

on $B = [a, \beta_{\min}] \cup [\beta_{\max}, b]$ where $ab = \beta^{(0)}\beta^{(1)}$ and $a + b = \beta^{(0)} + \beta^{(1)} + 4\alpha$.

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REFERENCES

[1] T. S. CHIHARA, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[2] S. CLEMENT COOPER AND P. E. GUSTAFSON, *The strong Chebyshev distribution and orthogonal Laurent polynomials*, J. Approx. Theory, 92 (1998), pp. 361–378.

[3] E. HILLE, *Analytic Function Theory, vol. 2*, Blaisdell Publ. Co., Waltham, 1962.

[4] M. E. H. ISMAIL AND X. LI, *Bounds on the extreme zeros of orthogonal polynomials*, Proc. Amer. Math. Soc., 115 (1992), pp. 131–140.

[5] W. B. JONES, W. J. THRON, AND H. WAADELAND, *A strong Stieltjes moment problem*, Trans. Amer. Math. Soc., 261 (1980), pp. 503–528.

[6] A. SRI RANGA, *Another quadrature rule of highest algebraic degree of precision*, Numer. Math., 68 (1994), pp. 283–294.

[7] ———, *Symmetric orthogonal polynomials and the associated orthogonal L-polynomials*, Proc. Amer. Math. Soc., 123 (1995), pp. 3135–3141.

[8] A. SRI RANGA AND C. F. BRACCIALI, *A continued fraction associated with a special Stieltjes function*, Commun. Anal. Theory Continued Fractions, 3 (1994), pp. 60–64.

[9] A. SRI RANGA AND L. C. MATIOLI, *Bounds for the extreme zeros of polynomials generated by a certain recurrence relation*, Rev. Mat. Estat., 14 (1996), pp. 113–120.

[10] W. VAN ASSCHE, *Asymptotic properties of orthogonal polynomials from their recurrence formula, I*, J. Approx. Theory, 44 (1985), pp. 258–276.