

## AOMOTO'S MACHINE AND THE DYSON CONSTANT TERM IDENTITY

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**ABSTRACT.** Aomoto has used the fundamental theorem of calculus to give an elegant proof of an extension of Selberg's integral. A constant term formulation of Aomoto's argument is based upon the fact that for  $1 \leq s \leq n$ , the constant term in  $t_s \partial/\partial t_s f(t_1, \dots, t_n)$  is zero provided that  $f(t_1, \dots, t_n)$  has a Laurent expansion around  $t_1 = \dots = t_n = 0$ . We use this as the engine for a simple proof of an Aomoto-type extension of the Dyson constant term identity. We outline the use of Good's proof to evaluate the coefficients of  $t_1/t_n$ ,  $t_1 t_2/t_{n-1} t_n$ , and  $t_1 t_2/t_n^2$  in  $\prod_{1 \leq i < j \leq n} (1 - t_i/t_j)^{a_i} (1 - t_j/t_i)^{a_j}$ . We give a conjecture with some surprising symmetries and its  $q$ -analogue.

### 1. Introduction and summary

Selberg [14] has given an important multivariable beta integral which is related to constant term identities associated with root systems. Aomoto [2] has extended Selberg's integral to

$$\int_0^1 \dots \int_0^1 \prod_{i=1}^n t_i^{(x-1)+\chi(i \leq m)} (1 - t_i)^{(y-1)} \Delta_n^{2k}(t_1, \dots, t_n) dt_1 \dots dt_n$$

$$= \prod_{i=1}^n \frac{\Gamma(x + (n - i)k + \chi(i \leq m)) \Gamma(y + (n - i)k) \Gamma(1 + ik)}{\Gamma(x + y + (2n - i - 1)k + \chi(i \leq m)) \Gamma(1 + k)} \quad (1.1)$$

where  $n$ ,  $k$ , and  $m$  are integers with  $n \geq 1$ ,  $k \geq 0$ , and  $0 \leq m \leq n$ ,  $\text{Re}(x) > 0$ ,  $\text{Re}(y) > 0$ ,  $\chi(A)$  is one or zero according to whether  $A$  is true or false, respectively, and

$$\Delta_n(t_1, \dots, t_n) = \prod_{1 \leq i < j \leq n} (t_i - t_j) \quad (1.2)$$

denotes the Vandermonde determinant. When  $m = 0$ , (1.2) is Selberg's integral in which the integrand is symmetric in  $t_1, \dots, t_n$ . Observe that the effect of the parameter  $m$  is to introduce the product  $t_1 \dots t_m$  into the integrand.

Aomoto's elegant proof [2] is based upon the fact that if  $f(t_1, \dots, t_n)$  is continuous on the unit cube  $[0, 1]^n$ , then, by the fundamental theorem of calculus, we have

$$0 = \int_0^1 \dots \int_0^1 \frac{\partial}{\partial t_1} (t_1(1 - t_1) f(t_1, \dots, t_n)) dt_1 \dots dt_n. \quad (1.3)$$

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Morris [13] has proven the case  $m = 0$  of the following constant term identity which is equivalent to Aomoto's integral (1.1):

$$\begin{aligned}
 [1] \prod_{i=1}^n (1 - t_i)^{a + \chi(i \leq m)} \left(1 - \frac{1}{t_i}\right)^b \prod_{1 \leq i < j \leq n} \left(1 - \frac{t_i}{t_j}\right)^k \left(1 - \frac{t_j}{t_i}\right)^k \\
 = \prod_{i=1}^n \frac{(a + b + (n - i)k + \chi(i \leq m))!}{(a + (n - i)k + \chi(i \leq m))! (b + (n - i)k)!} \frac{(ik)!}{k!}
 \end{aligned} \tag{1.4}$$

where  $[w]f$  denotes the coefficient of the monomial  $w$  in the Laurent expansion of  $f$ .

Let  $1 \leq s \leq n$ . Aomoto's argument may be applied to constant term identities by observing that if  $f(t_1, \dots, t_n)$  has a Laurent expansion around  $t_1 = \dots = t_n = 0$ , then we have

$$0 = [1] t_s \frac{\partial}{\partial t_s} \left( f(t_1, \dots, t_n) \right). \tag{1.5}$$

Let  $n \geq 2$  and  $a_1, \dots, a_n \geq 0$ . We set

$$f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \prod_{1 \leq i < j \leq n} \left(1 - \frac{t_i}{t_j}\right)^{a_i} \left(1 - \frac{t_j}{t_i}\right)^{a_j} \tag{1.6}$$

and use capital letters to denote the constant term

$$F_n(a_1, \dots, a_n) = [1] f_n(a_1, \dots, a_n; t_1, \dots, t_n). \tag{1.7}$$

Dyson [4] conjectured the constant term identity

$$F_n(a_1, \dots, a_n) = \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}, \tag{1.8}$$

which was proven independently by Gunson [6] and Wilson [16]. Good [5] gave a short proof of (1.8) using the identity

$$1 = \sum_{j=1}^n \prod_{\substack{i=1 \\ i \neq j}}^n \left(1 - \frac{t_j}{t_i}\right)^{-1}, \tag{1.9}$$

which may be obtained [7] by expanding the Vandermonde determinant

$$\Delta_n(t_1, \dots, t_n) = \det |t_j^{n-i}|_{n \times n} \tag{1.10}$$

along the bottom row.

Observe that Good's identity (1.9) gives

$$f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \sum_{j=1}^n f_n(a_1, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n; t_1, \dots, t_n). \tag{1.11}$$

Extracting the constant term from (1.11), we have

$$F_n(a_1, \dots, a_n) = \sum_{j=1}^n F_n(a_1, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n). \tag{1.12}$$

Good's proof [5] of the Dyson constant term identity (1.8) is concluded by observing that the multinomial coefficient  $(a_1 + \dots + a_n)!/a_1! \dots a_n!$  also satisfies (1.12) and the boundary condition

$$F_n(a_1, \dots, a_{m-1}, 0, a_{m+1}, \dots, a_n) = F_{n-1}(a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_n). \tag{1.13}$$

Let  $[1, n] = \{1, \dots, n\}$  denote the set of positive integers from one to  $n$ . We set

$$k_{n,m}(a_1, \dots, a_n; t_1, \dots, t_n) = \sum_{p=1}^n \sum_{\substack{M \subseteq [1,n] - \{p\} \\ |M|=m}} \left( 1 + \sum_{v \in [1,n] - M} a_v \right) \times \prod_{s \in M} \left( 1 - \frac{t_p}{t_s} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n), \tag{1.14}$$

and use capital letters to denote the constant term

$$K_{n,m}(a_1, \dots, a_n) = [1] k_{n,m}(a_1, \dots, a_n; t_1, \dots, t_n). \tag{1.15}$$

Observe that we may relax the restriction  $M \subseteq [1, n] - \{p\}$  to  $M \subseteq [1, n]$  since  $p \in M$  implies that  $\prod_{s \in M} (1 - t_p/t_s) = 0$ . It will be convenient for the proof not to do so.

We use (1.5) as the engine for a simple proof of an Aomoto-type generalization of the Dyson constant term identity (1.8). This is given by the following theorem which is our main result.

**Theorem 1.**

$$K_{n,m}(a_1, \dots, a_n) = n \binom{n-1}{m} \left( 1 + \sum_{v=1}^n a_v \right) \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}. \tag{1.16}$$

Let  $\pi \in S_n$ . The case where  $\pi$  is a transposition of the symmetry

$$f_n(a_{\pi(1)}, \dots, a_{\pi(n)}; t_{\pi(1)}, \dots, t_{\pi(n)}) = f_n(a_1, \dots, a_n; t_1, \dots, t_n) \tag{1.17}$$

plays a silent role in our proof of Theorem 1. We avoid its use by using the engine (1.5) with  $s$  ranging from one to  $n$  and combining terms so as to remove the denominators from the partial derivatives which arise when we use the engine (1.5) of our machine.

Defining the partial  $q$ -derivative by

$$\frac{\partial_q}{\partial_q t_s} (f(t_1, \dots, t_n)) = \frac{(f(t_1, \dots, t_n) - f(t_1, \dots, t_{s-1}, qt_s, t_{s+1}, \dots, t_n))}{t_s(1-q)}, \tag{1.18}$$

we see that if  $f(t_1, \dots, t_n)$  has a Laurent expansion around  $t_1 = \dots = t_n = 0$ , then we have

$$0 = [1] t_s \frac{\partial_q}{\partial_q t_s} (f(t_1, \dots, t_n)). \tag{1.19}$$

Observe that the  $q$ -engine (1.19) expresses the fact that the constant term is unchanged by the substitution  $t_s \rightarrow qt_s$ .

See Kadell [9–11], Stembridge [15], and Zeilberger [17, 18] for some surprisingly simple proofs of certain constant term identities associated with root systems using this idea.

Let  $|q| < 1$  and set  $(x; q)_m = \prod_{i=1}^m (1 - xq^{i-1})$ . Following [1], we set

$$qf_n(a_1, \dots, a_n; t_1, \dots, t_n) = \prod_{1 \leq i < j \leq n} \left( \frac{t_i}{t_j}; q \right)_{a_i} \left( q \frac{t_j}{t_i}; q \right)_{a_j}. \tag{1.20}$$

Andrews'  $q$ -Dyson conjecture [1],

$$[1]_q f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \frac{(q; q)_{a_1 + \dots + a_n}}{(q; q)_{a_1} \cdots (q; q)_{a_n}}, \quad (1.21)$$

has been established by Zeilberger and Bressoud [19] and extended by Bressoud and Goulden [3].

In Section 2, we use the engine (1.5) of our machine to give a constant term identity which involves the weighted averages  $K_{n,0}(a_1, \dots, a_n)$  and  $K_{n,1}(a_1, \dots, a_n)$ .

In Section 3, we continue with the engine (1.5) of our machine and give a functional equation which gives the behavior of  $K_{n,m}(a_1, \dots, a_n)$  as a function of  $m$ .

In Section 4, we establish the Dyson constant term identity (1.8) and then complete the proof of Theorem 1.

In Section 5, we outline the use of Good's proof [5] to evaluate the non-constant term coefficients  $[t_1/t_n] f_n(a_1, \dots, a_n; t_1, \dots, t_n)$ ,  $[t_1 t_2/t_{n-1} t_n] f_n(a_1, \dots, a_n; t_1, \dots, t_n)$ , and  $[t_1 t_2/t_n^2] f_n(a_1, \dots, a_n; t_1, \dots, t_n)$ . We give a conjecture with some surprising symmetries and its  $q$ -analogue.

## 2. The machine meets Dyson

In this section, we use the engine (1.5) of our machine to give a constant term identity which involves the weighted averages  $K_{n,0}(a_1, \dots, a_n)$  and  $K_{n,1}(a_1, \dots, a_n)$ .

We apply the engine (1.5) of our machine by using the fact that

$$0 = \sum_{1 \leq p \neq s \leq n} [1] t_s \frac{\partial}{\partial t_s} \left( \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \right). \quad (2.1)$$

Observe that

$$\begin{aligned} s \frac{\partial}{\partial s} \left( \left(1 - \frac{s}{t}\right)^a \left(1 - \frac{t}{s}\right)^b \right) &= \left( \frac{-a s/t + b t/s}{1 - s/t + 1 - t/s} \right) \left(1 - \frac{s}{t}\right)^a \left(1 - \frac{t}{s}\right)^b \\ &= \frac{as + bt}{s - t} \left(1 - \frac{s}{t}\right)^a \left(1 - \frac{t}{s}\right)^b. \end{aligned} \quad (2.2)$$

Interchanging the roles of  $s, t$  and  $a, b$  in (2.2), we have

$$t \frac{\partial}{\partial t} \left( \left(1 - \frac{s}{t}\right)^a \left(1 - \frac{t}{s}\right)^b \right) = \frac{as + bt}{t - s} \left(1 - \frac{s}{t}\right)^a \left(1 - \frac{t}{s}\right)^b. \quad (2.3)$$

Replacing  $a$  by  $a + 1$  in (2.3), we obtain

$$t \frac{\partial}{\partial t} \left( \left(1 - \frac{s}{t}\right)^{a+1} \left(1 - \frac{t}{s}\right)^b \right) = \frac{(a+1)s + bt}{t - s} \left(1 - \frac{s}{t}\right)^{a+1} \left(1 - \frac{t}{s}\right)^b. \quad (2.4)$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned} &t_s \frac{\partial}{\partial t_s} \left( \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \right) \\ &= \left( \frac{(a_p + 1)t_p + a_s t_s}{t_s - t_p} + \sum_{v \in [1, n] - \{s, p\}} \frac{a_s t_s + a_v t_v}{t_s - t_v} \right) \\ &\quad \times \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \quad (2.5)$$

Substituting (2.5) into (2.1) and rearranging the result, we obtain

$$\begin{aligned}
 0 = & \sum_{1 \leq p \neq s \leq n} [1] \frac{(a_p + 1)t_p + a_s t_s}{t_s - t_p} \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 & + \sum_{1 \leq p \neq s \leq n} \sum_{v \in [1, n] - \{p, s\}} [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n).
 \end{aligned} \tag{2.6}$$

We now remove the denominators from the terms in the two sums on the right side of (2.6).

For the first sum on the right side of (2.6), observe that

$$\begin{aligned}
 \frac{(a_p + 1)t_p + a_s t_s}{t_s - t_p} \left(1 - \frac{t_p}{t_s}\right) &= (a_p + 1) \frac{t_p}{t_s} + a_s \\
 &= a_p + a_s + 1 - (a_p + 1) \left(1 - \frac{t_p}{t_s}\right).
 \end{aligned} \tag{2.7}$$

Hence, we have

$$\begin{aligned}
 & \sum_{1 \leq p \neq s \leq n} [1] \frac{(a_p + 1)t_p + a_s t_s}{t_s - t_p} \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 &= \sum_{1 \leq p \neq s \leq n} [1] \left(a_p + a_s + 1 - (a_p + 1) \left(1 - \frac{t_p}{t_s}\right)\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n).
 \end{aligned} \tag{2.8}$$

For the second sum on the right side of (2.6), we have

$$\sum_{1 \leq p \neq s \leq n} \sum_{v \in [1, n] - \{p, s\}} h(p, s, v) = \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} h(p, s, v) + h(p, v, s), \tag{2.9}$$

and we observe that

$$\begin{aligned}
 \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) + \frac{a_v t_v + a_s t_s}{t_v - t_s} \left(1 - \frac{t_p}{t_v}\right) &= \frac{t_p}{t_s t_v} (a_s t_s + a_v t_v) \\
 &= a_s \frac{t_p}{t_v} + a_v \frac{t_p}{t_s} = a_s + a_v - a_s \left(1 - \frac{t_p}{t_v}\right) - a_v \left(1 - \frac{t_p}{t_s}\right).
 \end{aligned} \tag{2.10}$$

Setting

$$h(p, s, v) = [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \tag{2.11}$$

in (2.9) and using (2.10), we obtain

$$\begin{aligned}
 & \sum_{1 \leq p \neq s \leq n} \sum_{v \in [1, n] - \{p, s\}} [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 &= \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} [1] \left(\frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) + \frac{a_v t_v + a_s t_s}{t_v - t_s} \left(1 - \frac{t_p}{t_v}\right)\right) \\
 & \quad \times f_n(a_1, \dots, a_n; t_1, \dots, t_n)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} [1] \left( a_s + a_v - a_s \left( 1 - \frac{t_p}{t_v} \right) - a_v \left( 1 - \frac{t_p}{t_s} \right) \right) \\
 &\quad \times f_n(a_1, \dots, a_n; t_1, \dots, t_n). \tag{2.12}
 \end{aligned}$$

Substituting (2.8) and (2.12) into (2.6) gives

$$\begin{aligned}
 0 &= \sum_{1 \leq p \neq s \leq n} [1] \left( a_p + a_s + 1 - (a_p + 1) \left( 1 - \frac{t_p}{t_s} \right) \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 &+ \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} [1] \left( a_s + a_v - a_s \left( 1 - \frac{t_p}{t_v} \right) - a_v \left( 1 - \frac{t_p}{t_s} \right) \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n), \tag{2.13}
 \end{aligned}$$

which we may rearrange as

$$\begin{aligned}
 &\left( \left( \sum_{1 \leq p \neq s \leq n} a_p + a_s + 1 \right) + \left( \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} a_s + a_v \right) \right) F_n(a_1, \dots, a_n) \\
 &= \sum_{1 \leq p \neq s \leq n} [1] (a_p + 1) \left( 1 - \frac{t_p}{t_s} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 &+ \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} [1] \left( a_s \left( 1 - \frac{t_p}{t_v} \right) + a_v \left( 1 - \frac{t_p}{t_s} \right) \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \tag{2.14}
 \end{aligned}$$

Using (2.9) and interchanging the roles of  $s$  and  $v$  in the last sum, we see that (2.14) gives

$$\begin{aligned}
 &\left( \left( \sum_{1 \leq p \neq s \leq n} a_p + a_s + 1 \right) + \left( \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} a_s + a_v \right) \right) F_n(a_1, \dots, a_n) \\
 &= \sum_{1 \leq p \neq s \leq n} [1] (a_p + 1) \left( 1 - \frac{t_p}{t_s} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 &\quad + \sum_{1 \leq p \neq s \leq n} \sum_{v \in [1, n] - \{p, s\}} [1] a_v \left( 1 - \frac{t_p}{t_s} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \tag{2.15}
 \end{aligned}$$

Observe that

$$\begin{aligned}
 &\left( \sum_{1 \leq p \neq s \leq n} a_p + a_s + 1 \right) + \left( \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} a_s + a_v \right) \\
 &= 2(n-1) \left( \sum_{v=1}^n a_v \right) + n(n-1) + (n-1)(n-2) \left( \sum_{v=1}^n a_v \right) \\
 &= n(n-1) \left( 1 + \sum_{v=1}^n a_v \right). \tag{2.16}
 \end{aligned}$$

Choose  $p$  and  $s$  with  $1 \leq p \neq s \leq n$ . Then we have

$$a_p + 1 + \sum_{v \in [1, n] - \{p, s\}} a_v = 1 + \sum_{v \in [1, n] - \{s\}} a_v. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15), we obtain the functional equation

$$\begin{aligned} n(n-1) \left( 1 + \sum_{v=1}^n a_v \right) F_n(a_1, \dots, a_n) \\ = \sum_{1 \leq p \neq s \leq n} \left( 1 + \sum_{v \in [1, n] - \{s\}} a_v \right) [1] \left( 1 - \frac{t_p}{t_s} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \tag{2.18}$$

Observe that  $|M| = 0$  requires that  $M = \emptyset$ . Setting  $m = 0$  in the definition (1.14), we have

$$K_{n,0}(a_1, \dots, a_n) = n \left( 1 + \sum_{v=1}^n a_v \right) F_n(a_1, \dots, a_n). \tag{2.19}$$

Thus, (2.18) becomes

$$(n-1) K_{n,0}(a_1, \dots, a_n) = K_{n,1}(a_1, \dots, a_n), \tag{2.20}$$

in agreement with Theorem 1.

### 3. A functional equation involving the parameter $m$

In this section, we continue with the engine (1.5) of our machine and give a functional equation which gives the behavior of  $K_{n,m}(a_1, \dots, a_n)$  as a function of  $m$ .

Let  $1 \leq m \leq n-1$ . We apply the engine (1.5) of our machine by using the fact that

$$0 = \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - p, s \\ |M| = m-1}} [1] t_s \frac{\partial}{\partial t_s} \left( \left( 1 - \frac{t_p}{t_s} \right) \prod_{j \in M} \left( 1 - \frac{t_p}{t_j} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \right). \tag{3.1}$$

Observe that the case  $m = 1$  of (3.1) is (2.1) which gave the functional equation (2.20).

Let  $M \subseteq [1, n] - p, s$  with  $|M| = m-1$ . Using (2.3) and (2.4), we obtain

$$\begin{aligned} t_s \frac{\partial}{\partial t_s} \left( \left( 1 - \frac{t_p}{t_s} \right) \prod_{j \in M} \left( 1 - \frac{t_p}{t_j} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \right) \\ = \frac{(a_p + 1)t_p + a_s t_s}{t_s - t_p} + \sum_{v \in M} \frac{a_s t_s + a_v t_v}{t_s - t_v} + \sum_{v \in [1, n] - M - \{p, s\}} \frac{a_s t_s + a_v t_v}{t_s - t_v} \\ \times \left( 1 - \frac{t_p}{t_s} \right) \prod_{j \in M} \left( 1 - \frac{t_p}{t_j} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1) and rearranging the result, we obtain

$$\begin{aligned} 0 = \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - p, s \\ |M| = m-1}} [1] \frac{(a_p + 1)t_p + a_s t_s}{t_s - t_p} \left( 1 - \frac{t_p}{t_s} \right) \\ \times \prod_{j \in M} \left( 1 - \frac{t_p}{t_j} \right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \quad + \end{aligned}$$

$$\begin{aligned}
 & + \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - p, s \\ |M| = m - 1}} \sum_{v \in M} [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) \\
 & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 & + \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - p, s \\ |M| = m - 1}} \sum_{v \in [1, n] - M - p, s} [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) \\
 & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \tag{3.3}
 \end{aligned}$$

We now remove the denominators from the terms in the three sums on the right side of (3.3).

Using (2.7), the first sum on the right side of (3.3) becomes

$$\begin{aligned}
 & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - p, s \\ |M| = m - 1}} [1] \frac{(a_p + 1)t_p + a_s t_s}{t_s - t_p} \left(1 - \frac{t_p}{t_s}\right) \\
 & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 & = \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - p, s \\ |M| = m - 1}} [1] \left(a_p + a_s + 1 - (a_p + 1) \left(1 - \frac{t_p}{t_s}\right)\right) \\
 & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \tag{3.4}
 \end{aligned}$$

Observe that when  $m = 1$ , the condition  $|M| = m - 1 = 0$  implies that  $M = \emptyset$  and the second sum in (3.3) is zero. Thus, for the second sum on the right side of (3.3), we may assume that  $2 \leq m \leq n - 1$ . We have

$$\begin{aligned}
 & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - p, s \\ |M| = m - 1}} \sum_{v \in M} h_M(p, s, v) \\
 & = \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M| = m - 2}} h_{M \cup \{v\}}(p, s, v) + h_{M \cup \{s\}}(p, v, s). \tag{3.5}
 \end{aligned}$$

Observe that the function

$$h_M(p, s, v) = [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \tag{3.6}$$

satisfies

$$\begin{aligned}
 & M \subseteq [1, n] - \{p, s, v\}, \quad 1 \leq s < v \leq n, \quad s, v \in [1, n] - \{p\}, \\
 & \implies h_{M \cup \{s\}}(p, v, s) = -h_{M \cup \{v\}}(p, s, v), \tag{3.7}
 \end{aligned}$$



since the factor  $(1 - t_p/t_s)(1 - t_p/t_v)$  appears in both  $h_{M \cup \{s\}}(p, v, s)$  and  $h_{M \cup \{v\}}(p, s, v)$ . Thus, (3.5) gives

$$0 = \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M|=m-1}} \sum_{v \in M} [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \tag{3.8}$$

For the third sum on the right side of (3.3), we have

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M|=m-1}} \sum_{v \in [1, n] - M - \{p, s\}} h_M(p, s, v) \\ &= \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M|=m-1}} h_M(p, s, v) + h_M(p, v, s). \end{aligned} \tag{3.9}$$

Using (2.10), we see that (3.9) gives

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M|=m-1}} \sum_{v \in [1, n] - M - \{p, s\}} [1] \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) \\ & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M|=m-1}} [1] \left( \frac{a_s t_s + a_v t_v}{t_s - t_v} \left(1 - \frac{t_p}{t_s}\right) + \frac{a_v t_v + a_s t_s}{t_v - t_s} \left(1 - \frac{t_p}{t_v}\right) \right) \\ & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M|=m-1}} [1] \left( a_s + a_v - a_s \left(1 - \frac{t_p}{t_v}\right) - a_v \left(1 - \frac{t_p}{t_s}\right) \right) \\ & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \tag{3.10}$$

Substituting (3.4), (3.8), and (3.10) into (3.3), we obtain

$$\begin{aligned} 0 &= \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M|=m-1}} [1] \left( a_p + a_s + 1 - (a_p + 1) \left(1 - \frac{t_p}{t_s}\right) \right) \\ & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & \quad + \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M|=m-1}} [1] \left( a_s + a_v - a_s \left(1 - \frac{t_p}{t_v}\right) - a_v \left(1 - \frac{t_p}{t_s}\right) \right) \times \end{aligned}$$

$$\times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n), \quad (3.11)$$

which we may rearrange as

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} [1](a_p + a_s + 1) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & + \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M| = m-1}} [1](a_s + a_v) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & = \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} [1](a_p + 1) \left(1 - \frac{t_p}{t_s}\right) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & \quad + \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M| = m-1}} [1] \left( a_s \left(1 - \frac{t_p}{t_v}\right) + a_v \left(1 - \frac{t_p}{t_s}\right) \right) \\ & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \quad (3.12)$$

Using (3.9) and interchanging the roles of  $s$  and  $v$  in the last sum, we see that (3.12) gives

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} [1](a_p + a_s + 1) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & + \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M| = m-1}} [1](a_s + a_v) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & = \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} [1](a_p + 1) \left(1 - \frac{t_p}{t_s}\right) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & \quad + \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} \sum_{v \in [1, n] - M - \{p, s\}} [1] a_v \left(1 - \frac{t_p}{t_s}\right) \\ & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \quad (3.13)$$

Observe that

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} [1](a_p + a_s + 1) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ & = \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M| = m-1}} \left( (n-m)(a_p + 1) + \sum_{v \in [1, n] - M - \{p\}} a_v \right) \times \end{aligned}$$

$$\times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \tag{3.14}$$

and

$$\begin{aligned} & \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M|=m-1}} [1](a_s + a_v) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= (n - m - 1) \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M|=m-1}} \left( \sum_{v \in [1, n] - M - \{p\}} a_v \right) \\ & \quad \times [1] \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \tag{3.15}$$

Observe that the sum of (3.14) and (3.15) is given by

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M|=m-1}} [1](a_p + a_s + 1) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &+ \sum_{p=1}^n \sum_{\substack{1 \leq s < v \leq n \\ s, v \in [1, n] - \{p\}}} \sum_{\substack{M \subseteq [1, n] - \{p, s, v\} \\ |M|=m-1}} [1](a_s + a_v) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= (n - m) \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M|=m-1}} \left(1 + \sum_{v \in [1, n] - M} a_v\right) \\ & \quad \times [1] \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \tag{3.16}$$

We also have

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M|=m-1}} [1](a_p + 1) \left(1 - \frac{t_p}{t_s}\right) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= m \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M|=m}} (a_p + 1) [1] \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M|=m-1}} \sum_{v \in [1, n] - M - \{p, s\}} [1] a_v \left(1 - \frac{t_p}{t_s}\right) \\ & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= m \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M|=m}} \left( \sum_{v \in [1, n] - M - \{p\}} a_v \right) [1] \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n). \end{aligned} \tag{3.18}$$

Observe that the sum of (3.17) and (3.18) is given by

$$\begin{aligned}
 & \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} [1] (a_p + 1) \left(1 - \frac{t_p}{t_s}\right) \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 & + \sum_{1 \leq p \neq s \leq n} \sum_{\substack{M \subseteq [1, n] - \{p, s\} \\ |M| = m-1}} \sum_{v \in [1, n] - M - \{p, s\}} [1] a_v \left(1 - \frac{t_p}{t_s}\right) \\
 & \quad \times \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 & = m \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M| = m}} \left(1 + \sum_{v \in [1, n] - M} a_v\right) [1] \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n).
 \end{aligned} \tag{3.19}$$

Combining our results (3.14)–(3.19), we see that (3.13) becomes

$$\begin{aligned}
 & (n - m) \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M| = m-1}} \left(1 + \sum_{v \in [1, n] - M} a_v\right) [1] \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 & = m \sum_{p=1}^n \sum_{\substack{M \subseteq [1, n] - \{p\} \\ |M| = m}} \left(1 + \sum_{v \in [1, n] - M} a_v\right) [1] \prod_{j \in M} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n).
 \end{aligned} \tag{3.20}$$

Comparing with the definition (1.14), we see that the functional equation (3.20) is given by

$$(n - m) K_{n, m-1}(a_1, \dots, a_n) = m K_{n, m}(a_1, \dots, a_n), \quad 1 \leq m \leq n - 1. \tag{3.21}$$

Observe that this agrees with (2.20) when  $m = 1$ .

#### 4. A proof of Theorem 1

In this section, we establish the Dyson constant term identity (1.8) and then complete the proof of Theorem 1.

Observe that Theorem 1 is trivially true for  $m = n$  since

$$K_{n, n}(a_1, \dots, a_n) = k_{n, n}(a_1, \dots, a_n; t_1, \dots, t_n) = n \binom{n-1}{n} = 0. \tag{4.1}$$

We may rearrange (3.21) as

$$K_{n, m}(a_1, \dots, a_n) = \frac{(n - m)}{m} K_{n, m-1}(a_1, \dots, a_n). \tag{4.2}$$

Using (4.2), we see by induction on  $m$  that

$$\begin{aligned}
 K_{n, m}(a_1, \dots, a_n) &= \prod_{j=1}^m \frac{(n - j)}{j} K_{n, 0}(a_1, \dots, a_n) \\
 &= \binom{n-1}{m} K_{n, 0}(a_1, \dots, a_n),
 \end{aligned} \tag{4.3}$$

in agreement with Theorem 1.

Setting  $m = n - 1$  in (4.3) gives

$$K_{n,n-1}(a_1, \dots, a_n) = K_{n,0}(a_1, \dots, a_n). \tag{4.4}$$

Observe from the definition (1.14) that

$$\begin{aligned} K_{n,n-1}(a_1, \dots, a_n) &= [1] k_{n,n-1}(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= \sum_{p=1}^n (1 + a_p) [1] \prod_{j \in [1,n] - \{p\}} \left(1 - \frac{t_p}{t_j}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= \sum_{p=1}^n (1 + a_p) F_n(a_1, \dots, a_{p-1}, a_p + 1, a_{p+1}, \dots, a_n). \end{aligned} \tag{4.5}$$

Substituting (4.5) and (2.19) into (4.4) gives

$$\sum_{p=1}^n (1 + a_p) F_n(a_1, \dots, a_{p-1}, a_p + 1, a_{p+1}, \dots, a_n) = n \left(1 + \sum_{v=1}^n a_v\right) F_n(a_1, \dots, a_n). \tag{4.6}$$

We now use the functional equation (3.21) to prove the Dyson constant term identity (1.8) where  $n$  is a positive integer and  $a_1, \dots, a_n$  are nonnegative integers.

We proceed by induction on  $n$  and the minimum

$$z = \min(a_1, \dots, a_n) \tag{4.7}$$

of the parameters  $a_1, \dots, a_n$ .

Observe that when  $a_m = 0$  for  $1 \leq m \leq n$ , the variable  $t_m$  does not occur to a positive power in any of the terms in the expansion of  $f_n(a_1, \dots, a_n; t_1, \dots, t_n)$ . Thus, we have the boundary condition

$$\begin{aligned} F_n(a_1, \dots, a_{m-1}, 0, a_{m+1}, \dots, a_n) \\ = F_{n-1}(a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_n), \quad 1 \leq m \leq n, \end{aligned} \tag{4.8}$$

which using our induction assumption on  $n$  establishes the Dyson constant term identity (1.8) when  $a_m = 0$  for some  $m$  with  $1 \leq m \leq n$ .

We now assume that  $a_1, \dots, a_n$  are positive integers. We let  $a_u = z$  where  $1 \leq u \leq n$  and we have  $z \geq 1$ .

Replace  $a_1, \dots, a_n$  in (4.6) by  $\alpha_1, \dots, \alpha_n$  where  $\alpha_u = z - 1$  and  $\alpha_i = a_i$  for  $1 \leq i \leq n$ ,  $i \neq u$ . We may rearrange the result as

$$\begin{aligned} a_u F_n(a_1, \dots, a_n) &= n \left(\sum_{v=1}^n a_v\right) F_n(\alpha_1, \dots, \alpha_n) \\ &\quad - \sum_{\substack{m=1 \\ m \neq u}}^n (1 + a_m) F_n(\alpha_1, \dots, \alpha_{m-1}, \alpha_m + 1, \alpha_{m+1}, \dots, \alpha_n). \end{aligned} \tag{4.9}$$

Observe that the multinomial coefficient  $(a_1 + \dots + a_n)! / a_1! \dots a_n!$  satisfies (4.6). Each term in the sum on the left side of (4.6) equals  $1/n$  times the right side of (4.6). Thus, the multinomial coefficient also satisfies (4.9).

Observe that

$$\min(\alpha_1, \dots, \alpha_n) = z - 1 \tag{4.10}$$

and

$$1 \leq m \leq n \text{ and } m \neq u \implies \min(\alpha_1, \dots, \alpha_{m-1}, \alpha_m + 1, \alpha_{m+1}, \dots, \alpha_n) = z - 1. \tag{4.11}$$

Thus, we see by our induction assumption on  $z$  that all of the constant terms on the right side of (4.9) are given by the Dyson constant term identity (1.8). This completes our induction on  $z$  and establishes the Dyson constant term identity (1.8).

Theorem 1 then follows using (2.19) and (4.3).

### 5. A conjecture and its $q$ -analogue

In this section, we outline the use of Good’s proof [5] to evaluate the non constant term coefficients  $[t_1/t_n]f_n(a_1, \dots, a_n; t_1, \dots, t_n)$ ,  $[t_1 t_2/t_{n-1} t_n]f_n(a_1, \dots, a_n; t_1, \dots, t_n)$  and  $[t_1 t_2/t_n^2]f_n(a_1, \dots, a_n; t_1, \dots, t_n)$ . We give a conjecture with some surprising symmetries and its  $q$ -analogue.

Extracting the coefficient of  $t_1/t_n$  from (1.11), we see that the coefficient

$$\mathcal{F}_n(a_1, \dots, a_n) = [t_1/t_n] f_n(a_1, \dots, a_n; t_1, \dots, t_n) \tag{5.1}$$

satisfies the functional equation (1.12). Observe that it also satisfies the boundary conditions

$$\mathcal{F}_n(0, a_2, \dots, a_n) = 0, \tag{5.2}$$

$$\mathcal{F}_n(a_1, \dots, a_{r-1}, 0, a_{r+1}, \dots, a_n) = \mathcal{F}_{n-1}(a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_n), \tag{5.3}$$

$2 \leq r \leq n - 1,$

and

$$\begin{aligned} \mathcal{F}_n(a_1, \dots, a_{n-1}, 0) &= -a_1 \mathcal{F}_{n-1}(a_1, \dots, a_{n-1}) \\ &\quad - (a_2 + \dots + a_{n-1}) \mathcal{F}_{n-1}(a_1, \dots, a_{n-1}). \end{aligned} \tag{5.4}$$

The reader may readily use Good’s proof [5] to obtain

$$\left(1 + \sum_{v=2}^n a_v\right) \mathcal{F}_n(a_1, \dots, a_n) = -a_1 \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}. \tag{5.5}$$

Observe that the coefficient  $\mathcal{F}_n(a_1, \dots, a_n)$  is symmetric in  $a_2, \dots, a_n$ .

Let  $1 \leq p \neq s \leq n$ . Taking the Dyson constant term identity (1.9) minus (5.1) and using the symmetry (1.17) with the transpositions  $\pi = (p, n)$  and  $\pi = (1, s)$ , we obtain

$$\begin{aligned} &\left(1 + \sum_{v \in [1, n] - \{s\}} a_v\right) [1] \left(1 - \frac{t_p}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= \left(1 + \sum_{v=1}^n a_v\right) \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}. \end{aligned} \tag{5.6}$$

Observe that (5.6) states that the contribution of each term on the right side of (2.18) to the constant term  $K_{n,1}(a_1, \dots, a_n)$  is independent of  $p$  and  $s$ .

The reader may consult [12] for the details of using Good’s proof [5] to establish the non-constant term coefficient identity

$$[t_1 t_2/t_{n-1} t_n] f_n(a_1, \dots, a_n; t_1, \dots, t_n) = [t_1 t_2/t_n^2] f_n(a_1, \dots, a_n; t_1, \dots, t_n)$$

$$\begin{aligned}
 &= \left( \frac{a_1}{\left(1 + \sum_{v=2}^n a_v\right)} \frac{a_2}{\left(1 + \sum_{v=3}^n a_v\right)} + \frac{a_2}{\left(1 + a_1 + \sum_{v=3}^n a_v\right)} \frac{a_1}{\left(1 + \sum_{v=3}^n a_v\right)} \right) \\
 &\quad \times \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}. \tag{5.7}
 \end{aligned}$$

Observe that this common coefficient is symmetric in  $a_1, a_2$  and in  $a_3, \dots, a_n$ .

Let  $3 \leq p_1, p_2 \leq n$ . Using the constant term identities (5.1) and (5.7), their symmetries, and some tedious computation, we obtain

$$\begin{aligned}
 &\left(1 + \sum_{v=3}^n a_v\right) [1] \left(1 - \frac{t_{p_1}}{t_1}\right) \left(1 - \frac{t_{p_2}}{t_2}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 &= \left(1 + \sum_{v=1}^n a_v\right) \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}. \tag{5.8}
 \end{aligned}$$

Observe that (5.8) states that the contribution of each term on the right side of the definition (1.14) to the constant term  $K_{n,2}(a_1, \dots, a_n)$  is independent of  $p$  and the subset  $M$ . Moreover, we may let  $p$  be a function whose range is a subset of  $[1, n] - M$ .

This leads us to make the following conjecture which provides a refinement of Theorem 1.

**Conjecture 2.** *Let  $M \subseteq [1, n]$ ,  $|M| = m$ ,  $0 \leq m \leq n - 1$ , and  $\{p_s \mid s \in M\} \cap M = \emptyset$ . Then we have*

$$\begin{aligned}
 &\left(1 + \sum_{v \in [1, n] - M} a_v\right) [1] \prod_{s \in M} \left(1 - \frac{t_{p_s}}{t_s}\right) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \\
 &= \left(1 + \sum_{v=1}^n a_v\right) \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}. \tag{5.9}
 \end{aligned}$$

Conjecture 2 states that the contribution of each term on the right side of the definition (1.14) to the constant term  $K_{n,m}(a_1, \dots, a_n)$  is independent of  $p$  and the subset  $M$ . Moreover, we may let  $p$  be a function whose range is a subset of  $[1, n] - M$ . That is, we may replace the variable  $t_p$  in the numerator of the extra factors in the definition (1.14) by  $t_{p_s}$  where  $t_{p_s} \in [1, n] - M$  for all  $s \in M$ .

It is well known (see [1, 7]) that the case  $n = 2$  of (1.21)

$$[1] \left(\frac{s}{t}; q\right)_a \left(q \frac{t}{s}; q\right)_b = \frac{(q; q)_{a+b}}{(q; q)_a (q; q)_b} \tag{5.10}$$

is equivalent to the  $q$ -binomial theorem. Observe that we may interpret the symmetry of (5.10) in  $a$  and  $b$  as stating that if we add one to the subscript  $a$  or  $b$  and multiply by  $(1 - q^{a+1})$  or  $(1 - q^{b+1})$ , respectively, then we obtain the same constant term. In particular, the presence or absence of  $q$  in the argument  $s/t$  or  $qt/s$  does not affect the result.

This leads us to make the following conjecture which provides a  $q$ -analogue of Conjecture 2.

**Conjecture 3.** *Let  $M \subseteq [1, n]$ ,  $|M| = m$ ,  $0 \leq m \leq n - 1$ , and  $\{p_s \mid s \in M\} \cap M = \emptyset$ . Then we have*

$$\left(1 - q^{1 + \sum_{v \in [1, n] - M} a_v}\right) [1] \prod_{1 \leq i < j \leq n} \left(\frac{t_i}{t_j}; q\right)_{a_i + \chi(j \in M \text{ and } i = p_j)} \left(q \frac{t_j}{t_i}; q\right)_{a_j + \chi(i \in M \text{ and } j = p_i)}$$

$$= \left(1 - q^{1+\sum_{v=1}^n a_v}\right) \frac{(q; q)_{a_1+\dots+a_n}}{(q; q)_{a_1} \cdots (q; q)_{a_n}}. \quad (5.11)$$

Observe that the right side of (5.11) is independent of the function  $p$  and the subset  $M$ .

In [7], we observed the elementary symmetry

$${}_q f_n(a_2, \dots, a_n, a_1; t_2, \dots, t_n, qt_1) = {}_q f_n(a_1, \dots, a_n; t_1, \dots, t_n), \quad (5.12)$$

which extends (1.17) for long cycles in  $S_n$ . As happened in [8], (5.12) allows us to compute the partial  $q$ -derivatives required by the  $q$ -engine (1.19). Since the case  $a_1 = \dots = a_n = k$  of (5.12) is central to the simple proof in [10], we may ask if there is a simple proof of the Zeilberger-Bressoud theorem (1.21) using the symmetry (5.12) and the fact that the constant term is unchanged by the substitution  $t_s \rightarrow qt_s$  where  $1 \leq s \leq n$ .

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