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ON A CLASS OF QUASILINEAR DIFFERENTIAL EQUATIONS: THE NEUMANN PROBLEM

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ABSTRACT. In this paper, we consider a quasilinear ordinary differential equation with Neumann boundary conditions. Our formulation is general and incorporates the case of the one-dimensional Laplacian. Using an abstract result on the range of the sum of certain monotone operators, we prove the existence of a solution. Our approach is based on the theory of monotone operators and does not use degree theoretic arguments, which is usually the case in the literature for such problems.

1. Introduction

Let $T = [0, b]$ and consider the following quasilinear Neumann problem:

$$\begin{gathered} -\left(a(|x'(t)|^2)x'(t)\right)' + f(t, x(t)) = v(t) \quad \text{a.e. on } T \\ x'(0) = x'(b) = 0. \end{gathered} \tag{1}$$

Here $a : \mathbf{R} \to \mathbf{R}$ is a continuous map which satisfies certain geometric and growth conditions (see hypothesis $\mathbf{H(a)}$ in Section 3). One possibility is to have $a(r^2) = (r^2)^{(p-2)/2}$, $p \geq 2$, in which case we obtain the differential operator $-(|x'(t)|^{p-2}x'(t))'$ (the one-dimensional p-Laplacian). The version of problem (1) with the p-Laplacian and with homogeneous Dirichlet boundary conditions was studied by Boccardo et al. [1] and Pino et al. [6]. In that case, the differential operator

$$Ax(t) = -(|x'(t)|^{p-2}x'(t))'$$

is invertible and compact, and so a degree theoretic approach based on the Leray-Schauder degree is possible. However, for the Neumann problem, as well as for the periodic problem, the differential operator is no longer invertible, and so a different approach is needed. In [5], we studied the periodic problem using techniques from the perturbation theory of maximal monotone operators. In this paper, again our approach is based on the theory of maximal monotone operators and, more specifically, we use a theorem on the range of the sum of two monotone operators, due to Gupta and Hess [3] (see Section 2).

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2. Preliminaries

Let X be a reflexive Banach space, X^* its topological dual, and $A : X \to 2^{X^*}$. We recall some basic notions, namely: the domain of A is the set $D(A) = \{x \in X : A(x) \neq \emptyset\}$, the range of A is the set $R(A) = \{x^* \in X^* : x^* \in A(x), x \in D(A)\}$, and the graph of A is the subset of $X \times X^*$ defined by $\operatorname{Gr} A = \{(x, x^*) \in X \times X^* : x \in D(A), x^* \in A(x)\}$. We note that we can identify each subset $G \subseteq X \times X^*$ with a map $A : X \to 2^{X^*}$ by setting $A(x) = \{x^* \in X^* : (x, x^*) \in G\}$.

Definition. A map $A : X \to 2^{X^*}$ is said to be *monotone*, if for any x, $y \in D(A)$ and $x^* \in A(x)$, $y^* \in A(y)$, we have

$$\langle x^* - y^*, x - y \rangle \geq 0 \tag{2}$$

where by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). We say that A is *strictly monotone*, if equality in (2) implies $x = y$.

A is *maximal monotone*, if it is monotone and for $(y, y^*) \in X \times X^*$, the inequalities

$$\langle x^* - y^*, x - y \rangle \geq 0 \quad \text{for all } (x, x^*) \in \operatorname{Gr} A,$$

imply $(y, y^*) \in \operatorname{Gr} A$.

Remark. From the above definition, it is clear that $A : X \to 2^{X^*}$ is maximal monotone if and only if its graph is a monotone subset of $X \times X^*$ which is maximal with respect to inclusion.

In our analysis of problem (1), we also will need the following specification of a monotone map.

Definition. A map $A : X \to 2^{X^*}$ is said to be 3-*monotone* if

$$\langle x^* - y^*, z - x \rangle \leq \langle y^* - z^*, y - z \rangle$$

for all (x, x^*), (y, y^*), $(z, z^*) \in \operatorname{Gr} A$.

Remark. It is clear that a 3-monotone map is monotone, but the converse is not true in general.

For a single-valued map $B : X \to X^*$ with $D(B) = X$, we say that $B(\cdot)$ is demicontinuous if $x_n \to x$ in X implies $B(x_n) \xrightarrow{w} B(x)$ in X^* as $n \to \infty$. It is well known that a monotone demicontinuous map $B : X \to X^*$ is maximal monotone. We say that $B : X \to X^*$ is bounded if it maps bounded sets of X into bounded sets of X^*. For further details on these and related issues, we refer to Zeidler [7].

Definition. The map $A : X \to 2^{X^*}$ is said to be *boundedly inversely compact*, if for any pair of bounded sets $C \subseteq X$ and $C^* \subseteq X^*$, we have that $C \cap A^{-1}(C^*)$ is relatively compact in X (here $A^{-1}(C^*) = \{x \in D(A) : (x, x^*) \in \operatorname{Gr} A \text{ for some } x^* \in C^*\}$).

Remark. It is easy to see that if $K : X^* \to X$ maps bounded sets of X^* into relatively compact subsets of X, then $K^{-1} : X \to 2^{X^*}$ is boundedly inversely compact. We recall the following abstract result of Gupta and Hess [3], which is crucial in our approach.

Theorem 1. *Let* $A : X \to 2^{X^*}$ *be monotone, let* $B_1 : X \to X^*$ *be 3-monotone such that* (i) $D(A) \subseteq D(B_1)$, (ii) $0 \in (A + B_1)(0)$, *and* (iii) $A + B_1 : X \to 2^{X^*}$ *is maximal monotone and boundedly inversely compact, and let* $B_2 : X \to X^*$ *be a demicontinuous* *map satisfying the condition that for every $k \geq 0$, we can find a constant $c(k)$ such that*

$$\langle B_2(x), x\rangle \geq k\|B_2(x)\| - c(k) \quad \textit{for all } x \in X.$$

Then $u \in \operatorname{int}(R(A) + R(B_1))$ implies that $u \in R(A + B_1 + B_2)$.

Remark. If $B_2 \equiv 0$, then we do not need to assume that the mapping is boundedly inversely compact. Also note that because of the condition we imposed, B_2 is bounded. Finally, in our subsequent considerations, we will need the following basic inequality:

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)(a-b) \geq 2^{p-2}|a-b|^p.$$

3. Auxiliary results

First, let us introduce our hypotheses on the data of problem (1).

H(a) $a : \mathbf{R} \to \mathbf{R}$ is a continuous function with $a(0) = 0$ such that:

- (i) $t \to h(t) = \frac{1}{2}\int_0^{t^2} a(s)ds$ is strictly convex, and
- (ii) $c_1|t|^{p-2} \leq a(t^2) \leq c_2 + c_3|t|^{p-2}$ for all $t \in \mathbf{R}$ and with c_1, $c_3 > 0$, $c_2 \geq 0$, and $p \geq 2$.

Remark. If $a(t)$ is a polynomial with nonnegative coefficients, $a(t) = 1 + 1/(t+1)^2$, or $a(t) = t^{\frac{p-2}{2}}$, $p \geq 2$, then $a(\cdot)$ satisfies hypotheses **H(a)**. Of special interest is the last case because it corresponds to the p-Laplacian.

H(f) $f : T \times \mathbf{R} \to \mathbf{R}$ is a function such that:

- (i) for every $x \in \mathbf{R}$, $t \to f(t, x)$ is measurable,
- (ii) for almost all $t \in T$, $x \to f(t, x)$ is continuous,
- (iii) $|f(t,x)| \leq \beta(t) + \gamma|x|^{p-1}$ a.e. on T with $\beta \in L^q(T)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\gamma > 0$, and
- (iv) there exists $u \in L^p(T)_+$ such that for almost all $t \in T$ and all $|x| \geq u(t)$, we have $f(t,x)x \geq 0$ (generalized sign condition).

We introduce the operator $A : W^{1,p}(T) \to W^{1,p}(T)^*$ defined by

$$\langle A(x), y\rangle = \int_0^b a\left(|x'(t)|^{p-2}\right) x'(t)y'(t)\, dt.$$

Henceforth, we will denote the duality brackets for the pair $(W^{1,p}(T),\ W^{1,p}(T)^*)$ by $\langle\cdot,\cdot\rangle$.

Proposition 2. *$A : W^{1,p}(T) \to W^{1,p}(T)^*$ is maximal monotone.*

Proof. We will show that A is monotone and demicontinuous. Then this will imply that A is maximal monotone (see Section 2). First, we show the monotonicity of A. By hypothesis, $h(\cdot)$ is strictly convex $r \to h'(r) = a(r^2)r$ is strictly monotone. For all $x \in W^{1,p}(T)$, we have

$$\begin{aligned}\langle A(x) - A(y), x - y\rangle &= \int_0^b \left(a\left(|x'(t)|^2\right)x'(t) - a\left(|y'(t)|^2\right)y'(t)\right)(x'(t) - y'(t))\, dt \\ &= \int_0^b \left(h'(x'(t)) - h'(y'(t))\right)(x'(t) - y'(t))\, dt \geq 0,\end{aligned}$$

and equality holds only when $x' = y'$. This proves the (strict) monotonicity of A.

Next we will show the demicontinuity of A. To this end, let $x_n \overset{n\to\infty}{\longrightarrow} x$ in $W^{1,p}(T)$. By passing to a subsequence if necessary, we may assume that $x_n'(t) \to x'(t)$ a.e. on T. As $a(\cdot)$ is continuous by hypothesis, we have that $a(|x_n'(t)|^2)x_n'(t) \to a(|x'(t)|^2)x'(t)$ a.e. on T. Moreover, from hypothesis $\mathbf{H(a)}$(ii), we see that $\{a(|x_n'|^2)x_n'\}_{n\geq 1}$ is bounded in $L^q(T)$ (recall that $p \geq 2$, hence $q \leq 2$). Thus we deduce that $a(|x_n'|^2)x_n' \overset{w}{\to} a(|x'|^2)x'$ in $L^q(T)$ (see, for example, Hewitt and Stromberg [4], Theorem 13.44, p. 207). From this, it follows that for every $y \in W^{1,p}(T)$, we have

$$\langle A(x_n) - A(x), y\rangle = \int_0^b \left(a\left(|x_n'(t)|^2\right)x_n'(t) - a\left(|x'(t)|^2\right)x'(t)\right)\,dt \overset{n\to\infty}{\longrightarrow} 0,$$

therefore $A(x_n) \overset{w}{\to} A(x)$ in $W^{1,p}(T)^*$, i.e., A is demicontinuous.

Because A is monotone and demicontinuous, it is maximal monotone (see, for example, Zeidler [7], Proposition 32.7, p. 854). □

Now let $A_1 : D_1 \subseteq L^p(T) \to L^q(T)$ be defined by

$$A_1(x) = A(x) \text{ for all } x \in D_1 = \left\{x \in W^{1,p}(T) : A(x) \in L^q(T)\right\}.$$

Proposition 3. *$A_1 : D_1 \subseteq L^p(T) \to L^q(T)$ is maximal monotone.*

Proof. Let $J : L^p(T) \to L^q(T)$ be defined by $J(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot)$. We claim that it suffices to show that $R(A_1 + J) = L^q(T)$. Indeed, suppose that this surjectivity condition holds, and let $v \in L^p(T)$, $v^* \in L^q(T)$ be such that

$$(A_1(x) - v^*, x - v)_{pq} \geq 0 \tag{3}$$

for all $x \in D_1$. Here, by $(\cdot,\cdot)$, we denote the duality brackets for the pair $(L^p(T), L^q(T))$. Since by hypothesis $R(A_1 + J) = L^q(T)$, we can find $x \in D_1$ such that $A_1(x) + J(x) = v^* + J(v)$. Using this in (3) above, we obtain

$$(J(v) - J(x), x - v)_{pq} \geq 0.$$

But using the inequality mentioned at the end of Section 2, we can easily check that $J(\cdot)$ is strictly monotone. Hence $x = v$, and so $A_1(x) = v^*$. This proves the maximality of $A_1(\cdot)$. Thus we have to show that $R(A_1 + J) = L^q(T)$. Let $\widehat{J} = J|_{W^{1,p}(T)}$ (the restriction of $J(\cdot)$ on $W^{1,p}(T)$). Then $\widehat{J} : W^{1,p}(T) \to W^{1,p}(T)^*$ (recall that $L^q(T)$ is embedded continuously in $W^{1,p}(T)^*$). Evidently $\widehat{J}(\cdot)$ is monotone and continuous, thus maximal monotone. Combining this fact with Proposition 2, we have that $A + \widehat{J} : W^{1,p}(T) \to W^{1,p}(T)^*$ is maximal monotone (see Zeidler [7], Theorem 32.1, p. 888). Moreover, we have

$$\langle A(x) + \widehat{J}(x), x\rangle = \langle A(x), x\rangle + (\widehat{J}(x), x)_{pq} \geq c_1\|x'\|_p^p + \|x\|_p^p = \widehat{c}_1\|x\|_{1,p}^p$$

where $\widehat{c} = \min\{c_1, 1\}$ and $\|\cdot\|_{1,p}$ denotes the norm of the Sobolev space $W^{1,p}(T)$. Therefore, $(A + \widehat{J})(\cdot)$ is coercive. But a maximal monotone, coercive operator is surjective (see Zeidler [7], Corollary 32.35, p. 887). Thus $R(A + \widehat{J}) = W^{1,p}(T)^*$. Therefore, given any $g \in L^q(T)$, we can find $x \in W^{1,p}(T) \subseteq L^p(T)$ such that $A(x) + \widehat{J}(x) = g \Rightarrow A(x) = g - \widehat{J}(x) \in L^q(T) \Rightarrow A(x) = A_1(x)$. Hence, $A_1(x) + J(x) = g$. Because $g \in L^q(T)$ was arbitrary, we conclude that $R(A_1 + J) = L^q(T)$, and, as we already showed, this implies the maximal monotonicity of A_1. □

The next two propositions give us a complete description of the range of $A_1(\cdot)$.

Proposition 4. *If $g \in R(A_1)$, then for some $x \in D_1$, we have $(a(|x'(t)|^2)x'(t))' = g(t)$ a.e. on T and $x'(0) = x'(b) = 0$.*

Proof. Since $g \in R(A_1)$, we have that $g = A_1(x)$ for some $x \in D_1$. Then for every $\phi \in C_0^\infty(T)$, we have

$$(A_1(x), \phi)_{pq} = (g, \phi)_{pq}, \quad \text{hence} \quad \int_0^b a\left(|x'(t)|^2\right) x'(t)\phi'(t)\, dt = \int_0^b g(t)\phi(t)\, dt \quad (4)$$

From the definition of the distributional derivative and from (4), we infer that $a(|x'(\cdot)|^2)x'(\cdot) \in W^{1,p}(T)$ and $-(a(|x'(t)|^2)x'(t))' = g(t)$ a.e. on T. Since $W^{1,p}(T)$ is continuously embedded in $C(T)$, we have that $a(|x'(\cdot)|^2)x'(\cdot) \in C(T)$. Exploiting the strict monotonicity of $r \to a(r^2)r$, we obtain that $x'(\cdot) \in C(T)$. For every $y \in W^{1,p}(T)$, from Green's formula (integration by parts), we have

$$\begin{aligned}\int_0^b g(t)y(t)\, dt &= \int_0^b -\left(a\left(|x'(t)|^2\right)x'(t)\right)' y(t)\, dt \\ &= \int_0^b a\left(|x'(t)^2|\right)x'(t)y'(t)\, dt \\ &\quad - a\left(|x'(b)|^2\right)x'(b)y(b) + a\left(|x'(0)|^2\right)x'(0)y(0).\end{aligned}$$

Using the definition of A_1, we obtain

$$a\left(|x'(b)|^2\right)x'(b)y(b) = a\left(|x'(0)|^2\right)x'(0)y(0). \quad (5)$$

Let $y \in W^{1,p}(T)$ be such that $y(0) = y(b) = 1$. We have

$$a\left(|x'(b)|^2\right)x'(b) = a\left(|x'(0)|^2\right)x'(0).$$

Once again via the strict monotonicity of $t \to a(t^2)t$, we obtain that $x'(0) = x'(b)$. Using this in (5) and recalling that $y \in W^{1,p}(T)$ is arbitrary, we conclude that $x'(0) = x'(b) = 0$. □

Remark. A byproduct of this proof is that $D_1 \subseteq C^1(T)$.

Proposition 5. *Let $S = \left\{ g \in L^q(T) : \int_0^b g(t)\, dt = 0 \right\}$. Then $R(A_1) = S$.*

Proof. Let $g \in R(A_1)$. Then we can find $x \in D_1$ such that $A_1(x) = g$. Taking $y \equiv 1 \in W^{1,p}(T)$, we have

$$\int_0^b g(t)\, dt = (g, 1)_{pq} = (A_1, 1)_{pq} = 0,$$

$$\text{therefore } R(A_1) \subseteq S. \quad (6)$$

Next we will show that the opposite inclusion is also true. To this end, let $V = W^{1,p}(T)/_{\mathbf{R}}$ (i.e., we factor out of $W^{1,p}(T)$ the space of constant functions). Let $p : W^{1,p}(T) \to \mathbf{R}$ be the projection map defined by $p(x) = \int_0^b x(t)\, dt$. The quotient norm on V is given by $\|\,[x]\,\|_V = \|x - p(x)\|_p + \|x'\|_p$. Our claim is that there exists $c > 0$ such that $\|x - p(x)\|_p \leq c\|x'\|_p$. Suppose that this is not the case. Then we can find $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(T)$ such that $\|x_n - p(x_n)\|_p = 1$ and $\|x_n'\|_p \overset{n\to\infty}{\longrightarrow} 0$. Let $y_n = x_n - p(x_n)$, $n \geq 1$. Evidently $\{y_n\}_{n\geq 1}$ is bounded in $W^{1,p}(T)$, and because $W^{1,p}(T)$ is embedded compactly in $L^p(T)$, we also have that $y_n \overset{n\to\infty}{\longrightarrow} y$ in $L^p(T)$.

Observe that $\int_0^b y(t)\,dt = 0$. Also from the weak lower semicontinuity of the norm functional, we have

$$\|y'\|_p \leq \underline{\lim}\, \|y_n'\|_p = \underline{\lim}\, \|x_n'\|_p = 0,$$

therefore $y' = 0$, i.e. $y(t) = c \in \mathbf{R}$ for all $t \in T$.

Because $\int_0^b y(t)\,dt = 0$, we deduce that $c = 0$. Hence $\|y\|_p = 0$. But

$$1 = \|x_n - p(x_n)\|_p = \|y_n\|_p \overset{n\to\infty}{\longrightarrow} 0,$$

a contradiction. So $[x] \to \|x'\|_p$ is an equivalent norm on V. Next let $E : V \to V^*$ be the nonlinear operator defined by

$$(\!(E([x]), [y])\!) = \int_0^b a\left(|x'(t)|^2\right) x'(t)y'(t)\,dt \text{ with } [x], [y] \in V$$

where by $(\!(\,\cdot\,,\cdot\,)\!)$, we denote the duality brackets for the dual pair of Banach spaces (V, V^*). Clearly $E(\cdot)$ is well defined. Using the fact that $[x] \to \|x'\|_p$ is an equivalent norm on V and arguing as in the proof of Proposition 2, we can check that $E(\cdot)$ is monotone, demicontinuous, coercive, thus surjective (i.e., $R(E) = V^*$). Let $g \in S$ and consider the map $[x] \to \int_0^b g(t)x(t)\,dt$. Since $\int_0^b g(t)\,dt = 0$, we see that this map is well defined, linear, and continuous. Thus it belongs to V^*. Because $R(E) = V^*$, we infer that there exists $x \in W^{1,p}(T)$ such that

$$\int_0^b g(t)y(t)\,dt = \int_0^b a\left(|x'(t)|^2\right) x'(t)y'(t)\,dt \quad \text{for all } y \in W^{1,p}(T),$$

therefore $\langle g - A(x), y\rangle = 0$ for all $y \in W^{1,p}(T)$, so $A(x) = g$, i.e., $g \in R(A_1)$.

Thus we have proved that

$$S \subseteq R(A_1). \tag{7}$$

From (6) and (7), we conclude that $R(A_1) = S$. □

We introduce the penalty function $\beta : T \times \mathbf{R} \to \mathbf{R}$ defined by

$$\beta(t,x) = \begin{cases} (x - u(t))^{p-1} & \text{if } u(t) \leq x, \\ 0 & \text{if } -u(t) \leq x \leq u(t), \\ -(-u(t) - x)^{p-1} & \text{if } x \leq -u(t). \end{cases}$$

It is clear from this definition that the following is true:

Proposition 6. *$\beta(t,x)$ is measurable in $t \in T$, continuous in $x \in \mathbf{R}$ (i.e., β is a Caratheodory function), and $|\beta(t,x)| \leq a_1(t) + c_1|x|^{p-1}$ a.e. on T, with $a_1 \in L^q(T), c_1 > 0$.*

Then we decompose $f(t,x)$ as $f = f_1 + f_2$ with $f_1 : T \times \mathbf{R} \to \mathbf{R}$ defined by

$$f_1(t,x) = \begin{cases} (\inf_{y \geq x} f(t,y)) \wedge \beta(t,x) & \text{if } x \geq 0, \\ (\sup_{y \leq x} f(t,y)) \vee \beta(t,x) & \text{if } x \leq 0, \end{cases}$$

and $f_2(t,x) = f(t,x) - f_1(t,x)$. Also, we set $f_+(t) = \overline{\lim}_{x\to\infty} f(t,x)$ and $f_-(t) = \underline{\lim}_{x\to-\infty} f(t,x)$.

The next proposition establishes the properties of $f_1(t,x)$ and of the two limit functions f_+ and f_-.

Proposition 7. *For every $x \in \mathbf{R}$, $t \to f_1(t,x)$ is measurable and for almost all $t \in T$, $x \to f_1(t,x)$ is continuous, nondecreasing, and $\lim_{x\to\infty} f_1(t,x) = f_+(t)$, $\lim_{x\to-\infty} f_1(t,x) = f_-(t)$. So both functions f_+ and f_- are measurable.*

Proof. Fix $x \in \mathbf{R}$ and let $\{y_n\}_{n\geq 1}$ be dense in the half line $\{y \in \mathbf{R} : x \leq y\}$. We have

$$\inf_{y\geq x} f(t,y) = \inf_{n\geq 1} f(t,y_n),$$

therefore $t \to \inf_{y\geq x} f(t,y)$ is measurable.

Similarly we have that $t \to \sup_{y\leq x} f(t,y)$ is measurable, and so we conclude that $t \to f_1(t,x)$ is measurable.

Next let $t \in T\backslash N$ where $N \subseteq T$ is the Lebesgue null set outside of which $f(t,\cdot)$ is continuous (see hypothesis $\mathbf{H(f)}$(ii)). Let $H(x) = \{y \in \mathbf{R} : x \leq y\}$. If $x_n \to x$ in $\mathbf{R}$, then $H(x_n) \xrightarrow{K} H(x)$ as $n \to \infty$ where by $\xrightarrow{K}$, we denote the Kuratowski convergence of sets (see, for example, Dal Maso [2], p. 42). From this, we have that $\delta_{H(x_n)} \xrightarrow{e} \delta_{H(x)}$ as $n \to \infty$ where by $\xrightarrow{e}$, we denote the epigraphical (or $\Gamma-$) convergence (see Dal Maso [2], Proposition 4.15, p. 43) and for any $C \subseteq \mathbf{R}$,

$$\delta_C = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

Note that for all $x' \in \mathbf{R}$, $\inf_{z\geq x'} = \inf[f(t,z) + \delta_{H(x')}(z) : z \in \mathbf{R}]$. Using Proposition 6.20, p. 62 of Dal Maso [2], we have that $f(t,\cdot) + \delta_{H(x_n)} \xrightarrow{e} f(t,\cdot) + \delta_{H(x)}$ as $n \to \infty$, so finally we can apply Theorem 7.4, p. 69 of Dal Maso [2] and have that $\inf_{y\geq x_n} f(t,y) \stackrel{n\to\infty}{\longrightarrow} \inf_{y\geq x} f(t,y)$. This proves the continuity of $x \to \inf_{y\geq x} f(t,y)$. Similarly we establish the continuity of $x \to \sup y \leq x f(t,y)$. So combining these facts with Proposition 6, we have that $x \to (\inf_{y\geq x} f(t,y)) \wedge \beta(t,x)$ and $x \to (\sup_{y\leq x} f(t,y)) \vee \beta(t,x)$ are both continuous. Therefore, $x \to f_1(t,x)$ is continuous, and it is clear that it is nondecreasing. Moreover, directly from the definition of $f_1(t,x)$, we see that $\lim_{x\to\infty} f_1(t,x) = \overline{\lim}_{x\to\infty} f(t,x) = f_+(t)$ and $\lim_{x\to-\infty} f_1(t,x) = \underline{\lim}_{x\to-\infty} f(t,x) = f_-(t)$. Therefore, $t \to f_+(t)$, $f_-(t)$ are both measurable functions. □

Then $f_2(t,x) = f(t,x) - f_1(t,x)$ is measurable in t and for almost all $t \in T$ continuous in x (a Caratheodory function). Moreover, for almost all $t \in T$ and all $|x| \geq u(t)$, we have $f_2(t,x)x \geq 0$. By virtue of hypothesis $\mathbf{H(f)}$(iii) and Proposition 6, we can find β_1, $\beta_2 \in L^q(T)$ and γ_1, $\gamma_2 > 0$ such that for almost all $t \in T$ and all $x \in \mathbf{R}$, we have

$$|f_1(t,x)| \leq \beta_1(t) + \gamma_1|x|^{p-1} \quad \text{and} \quad |f_2(t,x)| \leq \beta_2(t) + \gamma_2|x|^{p-1}$$

Now let B_1, $B_2 : L^p(T) \to L^q(T)$ be the Nemitsky (superposition) operators corresponding to the functions f_1 and f_2, respectively; i.e., $B_1(x)(\cdot) = f_1(\cdot, x(\cdot))$ and $B_2(x)(\cdot) = f_2(\cdot, x(\cdot))$. From Krasnoselskii's theorem, we know that B_1, B_2 are continuous and bounded (see Zeidler [7], Proposition 26.6, p. 561). Also by virtue of Proposition 7 and Proposition 32.44, p. 905 of Zeidler [7], we have that B_1 is maximal monotone and 3-monotone.

Proposition 8. *If hypotheses $\mathbf{H(a)}$ and $\mathbf{H(f)}$ hold, then*

$$A_1 + B_1 : D_1 \subseteq L^p(T) \to L^q(T)$$

is maximal monotone and boundedly inversely compact.

Proof. The maximal monotonicity of A_1+B_1 follows from Proposition 3, the maximal monotonicity of B_1, the fact that B_1 is defined on all of $L^p(T)$ and Theorem 32.I, p. 888 of Zeidler [7]. Now we will show that $(A_1+B_1)(\cdot)$ is boundedly inversely compact. Let $C^* \subseteq L^q(T)$ and $C \subseteq L^p(T)$ be bounded sets. Let $G = C \cap (A_1+B_1)^{-1}(C^*) \subseteq L^p(T)$. Take $x \in G$. Then, by definition, $A_1(X)+B_1(X) = w \in C^*$. Exploiting the monotonicity of B_1, we have

$$\begin{aligned} c_0\|x'\|_p^p &\le (A_1(x),x)_{pq} \le (A_1(x)+B_1(x)-B_1(0),x)_{pq} \\ &= (w-B_1(0),x)_{pq} \le (\|w\|_q+\|\beta_1\|_q)\|x\|_p \\ &\le (|C^*|+\|\beta_1\|_q)|C| = M \le \infty, \end{aligned}$$

therefore G is bounded in $W^{1,p}(T)$.

Because $W^{1,p}(T)$ is compactly embedded in $L^p(T)$, we conclude that G is relatively compact in $L^p(T)$. Therefore, $(A_1+B_1)(\cdot)$ is boundedly inversely compact. □

Proposition 9. *If hypotheses* **H(f)** *hold, then for every $k \ge 0$, we can find such that for all $x \in L^p(T)$, we have*

$$k\|B_2(x)\|_q - c(k) \le (B_2(x),x)_{pq}.$$

Proof. We have

$$\begin{aligned} (B_2(x),x)_{pq} &= \int_0^b f_2(t,x(t))x(t)\,dt \\ &= \int_{\{|x|\ge u(t)\}} f_2(t,x(t))x(t)\,dt + \int_{\{|x|<u(t)\}} f_2(t,x(t))x(t)\,dt\ . \end{aligned}$$

By virtue of the fact that $|f_2(t,x)| \le \beta_2(t)+\gamma_2|x|^{p-1}$ a.e. with $\beta_2 \in L^q(T)$, $\gamma_2>0$, we have

$$\left|\int_{\{|x|<u(t)\}} f_2(t,x(t))x(t)\,dt\right| \le \int_{\{|x|<u(t)\}} |f_2(t,x(t))|\ |x(t)|\,dt \le M_1 < \infty\ .$$

Moreover, we have

$$\begin{aligned} |f_2(t,x(t))|^{q/p} &\le \left(\beta_2(t)+\gamma_2|x|^{p/q}\right)^{q/p} \\ &\le \beta_2(t)^{q/p}+\gamma_2^{q/p}|x| \text{ a.e. on } T \end{aligned}$$

(recall that if a, $c>0$ and $0\le r\le 1$, then $(a+c)^r \le a^r+c^r$; here $r=\frac{q}{p}<1$ since $p\ge 2$ and $\frac{1}{p}+\frac{1}{q}=1$). Also, for almost all $t\in T$ and all $|x|\ge u(t)$, we have

$$\begin{aligned} \int_{\{|x|\ge u(t)\}} f_2(t,x(t))x(t)\,dt &= \int_{\{|x|\ge u(t)\}} |f_2(t,x(t))|\ |x(t)|\,dt \\ &\ge \frac{1}{\gamma_3}\int_{\{|x|\ge u(t)\}} |f_2(t,x(t))|\left(|f_2(t,x(t))|^{q/p}-\beta_2(t)^{q/p}\right)dt \quad (\text{with } \gamma_3=\gamma_2^{p/q}) \\ &= \frac{1}{\gamma_3}\int_{\{|x|\ge u(t)\}} \left(|f_2(t,x(t))|^{1+q/p}-\beta_2(t)^{q/p}|f_2(t,x(t))|\right)dt\ . \end{aligned}$$

Using Young's inequality with $\varepsilon>0$, we obtain

$$\beta_1(t)^{q/p}|f_2(t,x(t))| \le \frac{1}{\varepsilon^p p}\beta_1(t)^q + \frac{\varepsilon^q}{q}|f_2(t,x(t))|^q\ .$$

Choose $\varepsilon > 0$ such that $1 - \varepsilon^q/q > 0$. We have

$$\int_{\{|x|\geq u(t)\}} f_2(t,x(t))x(t)\,dt \geq \frac{1}{\gamma_3}\left[\int_{\{|x|\geq u(t)\}} \left(1-\frac{\varepsilon^q}{q}\right)|f_2(t,x(t))|^q - \frac{1}{\varepsilon^p p}\beta_1(t)^q\right] dt$$

$$= c_4(\varepsilon)\int_0^b |f_2(t,x(t))|^q\,dt - c_5(\varepsilon)\|\beta_1\|_q^q - c_6(\varepsilon)\int_{\{|x|<u(t)\}} |f_2(t,x(t))|^q\,dt$$

for some $c_4(\varepsilon)$, $c_5(\varepsilon)$, $c_6(\varepsilon) \geq 0$. On $\{|x| < u(t)\}$, we have $|f_2(t,x(t))| \leq \eta(t)$ a.e. on T with $\eta \in L^q(T)$. So

$$-c_6(\varepsilon)\int_{\{|x|<u(t)\}} |f_2(t,x(t)|^q\,dt \geq -c_6(\varepsilon)\|_q^q\ .$$

Therefore we have

$$(B_2(x),x)_{pq} \geq -c_4(\varepsilon)\|B_2(x)\|_q^q - c_7(\varepsilon)$$

for some $c_7(\varepsilon) > 0$ (recall that we have fixed $\varepsilon > 0$ so that $1 - \varepsilon^q/q > 0$). A new application of Young's inequality tells us that given $k' > 0$, we can find $c_8(k')$ such that

$$\|B_2(x)\|_q \leq \frac{c_4(\varepsilon)}{k'}\|B_2(x)\|_q^q - c_8(k'),$$

hence

$$k'\|B_2(x)\|_q - c_8(k') \leq c_4(\varepsilon)\|B_2(x)\|_q^q\ .$$

Therefore, finally, we have that for every $k \geq 0$, there exists $c(k) = c_8(kc_4(\varepsilon))/c_4(\varepsilon)$ such that

$$(B(x),x)_{pq} \geq k\|B_2(x)\|_q - c(k)\ .$$

□

4. Existence theorem

In this section, we prove an existence theorem for problem (1).

Definition. By a *solution of* (1), we mean a function $x \in C^1(T)$ such that $a(|x'(\cdot)|^2)x'(\cdot) \in W^{1,q}(T)$, $-(a(|x'(t)|^2)x'(t))' + f(t,x(t)) = v(t)$ a.e. on T, and $x'(0) = x'(b) = 0$.

Using the auxiliary results of Section 3 and Theorem 1 in Section 2, we can prove the following existence theorem.

Theorem 10. *If hypotheses* **H(a)**, **H(f)** *hold,* $v \in L^q(T)$ *and*

$$\int_0^b f_-(t)\,dt < \int_0^b v(t)\,dt < \int_0^b f_+(t)\,dt,$$

then problem (1) *has a solution.*

Proof. Since for almost all $t \in T$, $f_1(t,\cdot)$ is nondecreasing, $\lim_{x\to\infty} f_1(t,x) = f_+(t)$ and $\lim_{x\to-\infty} f_1(t,x) = f_-(t)$ (see Proposition 7) from the monotone convergence theorem, given $\delta > 0$, we can find $n \geq 1$ large enough such that

$$\int_0^b f_1(t,-n)\,dt < \int_0^b (v(t)+h(t))\,dt < \int_0^b f_1(t,n)\,dt$$

for all $h \in L^q(T)$ with $\|h\|_q < \delta$. Fix such an h. From the continuity of $x \to \int_0^b f_1(t,x)\,dt$ and the intermediate value theorem, we deduce that there exists $x_0 \in [-n,n]$ such that

$$\int (v(t)+h(t))\,dt = \int_0^b f_1(t,x_0)\,dt\,.$$

We write

$$\begin{aligned} v(t)+h(t) &= v(t)+h(t) - \frac{1}{b}\int_0^b (v(t)+h(t))\,dt + \frac{1}{b}\int_0^b (v(t)+h(t))\,dt \\ &= v(t)+h(t) - \frac{1}{b}\int_0^b (v(t)+h(t))\,dt + \frac{1}{b}\int_0^b f_1(t,x_0)\,dt - f_1(t,x_0) + f_1(t,x_0). \end{aligned}$$

Let $w(t) = v(t)+h(t) - \frac{1}{b}\int_0^b (v(t)+h(t))\,dt + \frac{1}{b}\int_0^b f_1(t,x_0)\,dt - f_1(t,x_0)$. Evidently, $w \in L^q(T)$ and $\int_0^b w(t)\,dt = 0$. Thus $w \in R(A_1)$ (Proposition 5). So $v+h = w + f_1(\cdot\,,x_0) \in R(A_1)+R(B_1)$. Since $h \in L^q(T)$, $\|h\|_q < \delta$ was arbitrary, we see that $v \in \operatorname{int}(R(A_1)+R(B_1))$. Apply Theorem 1 to conclude that $v \in \operatorname{int} R(A_1+B_1+B_2)$. Hence there exists $x \in D_1 \subseteq C^1(T)$ such that $A_1(x)+B_1(x)+B_2(x) = v$. Finally using Proposition 4, we conclude that

$$\begin{aligned} -\left(a(|x'(t)|^2)x'(t)\right)' + f(t,x(t)) = v(t) \quad \text{a.e. on } T \\ x'(0) = x'(b) = 0. \end{aligned} \tag{8}$$

Thus, $x(\cdot)$ is the desired solution of (1). □

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