Methods and Applications of Analysis
5 (2) 1998, pp. 125–142

ISSN 1073–2772

A SIMPLE PROOF OF AN AOMOTO-TYPE EXTENSION OF GUSTAFSON'S ASKEY-WILSON SELBERG q-INTEGRAL

Kevin W. J. Kadell

ABSTRACT. Selberg has given an important multivariable beta integral which is related to constant term identities associated with root systems, multivariate statistical analysis, the energy levels of complex physical systems, and other topics. Aomoto and Anderson have given additional proofs which have been used by many workers to extend Selberg's integral and give constant term identities associated with root systems. Gustafson evaluated several Selberg-type integrals which extend certain beta integrals including the Askey-Wilson integral. We give a simple proof of an Aomoto-type extension of a constant term formulation of Gustafson's Askey-Wilson Selberg q-integral. We require an elementary algebraic result which we call the q-transportation theory for the root system BC_n and which is the core of a more complex proof modeled on Aomoto's argument.

1. Introduction and summary

Selberg [28] has given an important multivariable beta integral which is related to constant term identities associated with root systems, multivariate statistical analysis, the energy levels of complex physical systems, and other topics. Morris [26] and Macdonald [21] gave a basic conjecture whose proof by many authors has led to an ongoing development of orthogonal polynomials associated with root systems. See Richards [27], Stanley [29], Kadell [20], and Macdonald [22, Ch. VI; 23] who gives many references to the current work of Opdam, Heckman, Cherednik, and others. The Wishart distribution of the eigenvalues of the variance-covariance matrix and the zonal polynomials are related to the case $k = 1/2$ where the beta distribution is replaced by the limiting normal distribution; see Wishart [31] and Wilks [30, Ch. 18]. Dyson [9] introduced the connection with the energy levels of complex physical systems and Selberg-type integrals around the unit circle; see also Mehta and Dyson [25] and Mehta [24]. See Askey [6] for an extended discussion of these and other topics related to Selberg's integral.

Aomoto [5] and Anderson [1, 2] have given additional proofs of Selberg's integral which have been used by many workers to extend Selberg's integral and give constant term identities associated with root systems.

Let n, k, and m be integers with $n \geq 1$, $k \geq 0$, and $0 \leq m \leq n$. Let $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$. We omit m when $m = 0$, and we omit k when $n = 1$.

Aomoto's extension of Selberg's integral is given by the following theorem.

Received July 21, 1997, revised December 1, 1997.

1991 *Mathematics Subject Classification*: 33D70, 33D80, 05A30.

Key words and phrases: Selberg's integral, Aomoto-type extension, Askey-Wilson integral, q-transportation theory, root system BC_n.

Theorem 1 (Aomoto [5]).

$$I_{n,m}^{k}(x,y)=\int_0^1\cdots\int_0^1\prod_{i=1}^{n}t_i^{(x-1)+\chi(i\le m)}(1-t_i)^{(y-1)}\,\Delta_n^{2k}(t_1,\ldots,t_n)\,dt_1\cdots dt_n \tag{1.1}$$

$$=\prod_{i=1}^{n}\frac{\Gamma(x+(n-i)k+\chi(i\le m))\,\Gamma(y+(n-i)k)}{\Gamma(x+y+(2n-i-1)k+\chi(i\le m)}\,\frac{\Gamma(1+ik)}{\Gamma(1+k)}$$

where $\chi(A)$ is 1 or 0 according to whether A is true or false, respectively, and

$$\Delta_n(t_1,\ldots,t_n)=\prod_{1\le i<j\le n}(t_i-t_j) \tag{1.2}$$

denotes the Vandermonde determinant.

When $m=0$, we obtain Selberg's integral in which the integrand is symmetric in $t_1,\ldots,t_n$. Observe that the effect of the parameter m is to introduce the product $t_1\cdots t_m$ into the integrand.

Let q be complex with $0<q<1$ and set $(x;q)_n=\prod_{i=0}^{n-1}(1-xq^i)$, $n\ge 0$. Following Jackson [15], we have the q-gamma function

$$\Gamma_q(x)=(1-q)^{1-x}\,\frac{(q;q)_\infty}{(q^x;q)_\infty}, \tag{1.3}$$

and we define the q-integral by

$$\int_0^a f(t)\,d_qt=a(1-q)\sum_{n=0}^{\infty}q^n\,f(aq^n), \tag{1.4}$$

$$\int_a^b f(t)\,d_qt=\int_0^b f(t)\,d_qt-\int_0^a f(t)\,d_qt. \tag{1.5}$$

Let $[w]\,f$ denote the coefficient of the monomial w in the Laurent expansion of f. We have (see Andrews [4]) the q-beta integral

$$\int_0^1 t^{x-1}\,\frac{(qt;q)_\infty}{(q^yt;q)_\infty}\,d_qt=\frac{\Gamma_q(x)\,\Gamma_q(y)}{\Gamma_q(x+y)} \tag{1.6}$$

and the equivalent constant term formulation of the q-binomial theorem

$$[1]\,(s;q)_a(q/s;q)_b=\frac{(q;q)_{a+b}}{(q;q)_a\,(q;q)_b}. \tag{1.7}$$

We have (see Andrews and Askey [3]) the q-beta integral

$$\int_{-a}^{b}\frac{(qt/a;q)_\infty}{(q^xt/a;q)_\infty}\,\frac{(qt/b;q)_\infty}{(q^yt/b;q)_\infty}\,d_qt=\frac{ab}{(a+b)}\,\frac{\Gamma_q(x)\,\Gamma_q(y)}{\Gamma_q(x+y)}\,\frac{(-b/a;q)_\infty}{(-q^xb/a;q)_\infty}\,\frac{(-a/b;q)_\infty}{(-q^ya/b;q)_\infty} \tag{1.8}$$

where there are no zero factors in the denominator of the integral.

Askey [6] conjectured a number of Selberg q-integrals which featured the function

$${}_q\Delta_n^{2k}(t_1,\ldots,t_n)=\prod_{1\le i<j\le n}t_i^{2k}\left(q^{1-k}\frac{t_j}{t_i};q\right)_{2k}. \tag{1.9}$$

Observe that the $2k$ on the left side of (1.9) is only formally an exponent. Askey observed that (1.9) is a q-analogue of $\Delta_n^{2k}(t_1,\ldots,t_n)$ which "vanishes when $t_i=t_j$ and on k lines on one side of this line and on $k-1$ lines on the other side."

Habsieger [14] and Kadell [16] independently proved Askey's first conjectured Selberg q-integral, which is based on (1.6), and an equivalent constant term identity conjectured by Morris [26], which is based on (1.7). Habsieger used Selberg's proof [28] together with a clever asymptotic argument in place of the symmetry in x and y while Kadell used a telescoping sum in place of the fundamental theorem of calculus to extend Aomoto's argument [5] to the q-case.

Evans [10] used the proof technique developed by Anderson [1, 2] to prove Askey's last conjectured [6] Selberg q-integral, which is based on (1.8); see [19] for a proof using our techniques.

Let a, b, c, and d be nonnegative integers. The Askey-Wilson integral [7] is given by

$$\frac{1}{2\pi i}\int_C \frac{(t^2;q)_\infty\,(t^{-2};q)_\infty}{(at;q)_\infty(a/t;q)_\infty\,(bt;q)_\infty(b/t;q)_\infty\,(ct;q)_\infty(c/t;q)_\infty\,(dt;q)_\infty(d/t;q)_\infty}\,\frac{dt}{t} = \frac{2}{(q;q)_\infty}\,\frac{(abcd;q)_\infty}{(ab;q)_\infty\,(ac;q)_\infty\,(ad;q)_\infty\,(bc;q)_\infty\,(bd;q)_\infty\,(cd;q)_\infty} \tag{1.10}$$

where the contour C is the unit circle traversed in the positive direction, but with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to infinity.

Gustafson [12] independently developed a proof technique which is very similar to that of Anderson [1, 2]. He evaluated several Selberg-type integrals which extend certain beta integrals including the Askey-Wilson integral (1.10). He observed [13] that while results such as (1.6)–(1.8) are related to the root system A_{n-1}, certain well poised summation formulas and the Askey-Wilson integral are related to the root system BC_n. Some readers may wish to read the review of the basic algebraic and geometric properties of root systems in Section 2 before continuing.

We give a simple proof of an Aomoto-type extension of a constant term formulation of Gustafson's Askey-Wilson Selberg q-integral. We require an elementary algebraic result which we call the q-transportation theory for the root system BC_n. This result emerges as the core of a more complex proof modeled on Aomoto's argument. The importance of our simple proof is that it reflects the algebraic and geometric properties of simple roots and reflections and the manner in which BC_n is built up from A_{n-1}.

Let α be a root in the root system R. Let $k(\alpha) \geq 0$ be an integer which depends only on the length of the root α. Let e^α denote a formal exponential defined on a root system R. Without loss of generality, we denote the standard orthonormal basis of n-dimensional Euclidean space by $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ and take $e^{\mathrm{e}_i} = t_i$.

The basic object of study in the q-case is the constant term in the Laurent expansion of the function

$$\prod_{\alpha>0} (e^\alpha;q)_{k(\alpha)}\,(q/e^\alpha;q)_{k(\alpha)}. \tag{1.11}$$

Interpreting the empty product as one, we see that (1.11) associates the functions

$${}_q a^k_{n-1}(t_1,\ldots,t_n) = \prod_{1\leq i<j\leq n} \left(\frac{t_i}{t_j};q\right)_k \left(q\frac{t_j}{t_i};q\right)_k \tag{1.12}$$

and

$$ {}_q d_n^k(t_1,\ldots,t_n) = {}_q a_{n-1}^k(t_1,\ldots,t_n) \prod_{1\le i<j\le n} (t_it_j;q)_k\Big(\frac{q}{t_it_j};q\Big)_k $$
$$ = \prod_{1\le i<j\le n} (t_it_j;q)_k\Big(\frac{q}{t_it_j};q\Big)_k\Big(\frac{t_i}{t_j};q\Big)_k\Big(q\frac{t_j}{t_i};q\Big)_k \tag{1.13} $$

with the root systems A_{n-1} and D_n, respectively.

Using the well-known identity

$$ (x;q)_n = (-x)^n\, q^{\binom{n}{2}}\left(\frac{q^{1-n}}{x};q\right)_n \tag{1.14} $$

for reversing a finite q-product, we have

$$ \begin{aligned} s^{2k}\Big(q^{1-k}\frac{t}{s};q\Big)_{2k} &= s^k\left(q^{1-k}\frac{t}{s};q\right)_k s^k\left(q\frac{t}{s};q\right)_k \\ &= s^k\left(-q^{1-k}\frac{t}{s}\right)^k q^{\binom{k}{2}}\left(\frac{s}{t};q\right)_k s^k\left(q\frac{t}{s};q\right)_k \\ &= (-st)^k\, q^{-\binom{k}{2}}\left(\frac{s}{t};q\right)_k\Big(q\frac{t}{s};q\Big)_k. \end{aligned} \tag{1.15} $$

Setting $s=t_i$, $t=t_j$ in (1.15), we have

$$ \begin{aligned} {}_q\Delta_n^{2k}(t_1,\ldots,t_n) &= \prod_{1\le i<j\le n} (-t_it_j)^k\, q^{-\binom{k}{2}}\left(\frac{t_i}{t_j};q\right)_k\Big(q\frac{t_j}{t_i};q\Big)_k \\ &= (-1)^{k\binom{n}{2}}\, q^{-\binom{k}{2}\binom{n}{2}} \prod_{i=1}^{n} t_i^{k(n-1)}\ {}_q a_{n-1}^k(t_1,\ldots,t_n). \end{aligned} \tag{1.16} $$

Thus the functions ${}_q\Delta_n^{2k}(t_1,\ldots,t_n)$ and ${}_q a_{n-1}^k(t_1,\ldots,t_n)$ have the same zeros except for $t_1=0,\ldots,t_n=0$.

We require the constant term formulation

$$ \begin{aligned} &[1]\,(t;q)_a\Big(\frac{q}{t};q\Big)_a(-t;q)_b\Big(-\frac{q}{t};q\Big)_b(\sqrt{q}t;q)_c\Big(\frac{\sqrt{q}}{t};q\Big)_c(-\sqrt{q}t;q)_d\Big(-\frac{\sqrt{q}}{t};q\Big)_d \\ &= \frac{(-q;q)_{a+b}\,(\sqrt{q};q)_{a+c)}\,(-\sqrt{q};q)_{a+d)}\,(-\sqrt{q};q)_{b+c}\,(\sqrt{q};q)_{b+d}\,(-q;q)_{c+d}}{(q;q)_{a+b+c+d}} \end{aligned} \tag{1.17} $$

of the Askey-Wilson integral (1.10).

We refer to a function whose constant term is being extracted as the extractee. Observe that the integrand of (1.10) is invariant under the substitution $t\to 1/t$ while the extractee of (1.17) does not have a corresponding invariance.

Following Gustafson [12], we set

$$ \begin{aligned} {}_q aw_{n,m}^k(a,b,c,d;t_1,\ldots,t_n) &= {}_q d_n^k(t_1,\ldots,t_n)\prod_{i=1}^{n}(t_i;q)_{a+\chi(n-i+1\le m)}\Big(\frac{q}{t_i};q\Big)_a \\ &\times(-t_i;q)_b\Big(-\frac{q}{t_i};q\Big)_b(\sqrt{q}t_i;q)_c\Big(\frac{\sqrt{q}}{t_i};q\Big)_c(-\sqrt{q}t_i;q)_d\Big(-\frac{\sqrt{q}}{t_i};q\Big)_d, \end{aligned} \tag{1.18} $$

$$ \begin{aligned} {}_q \overline{aw}_{n,m}^k(a,b,c,d;t_1,\ldots,t_n) &= {}_q d_n^k(t_1,\ldots,t_n)\prod_{i=1}^{n}(t_i;q)_{a+1}\Big(\frac{q}{t_i};q\Big)_{a+\chi(i\le m)} \\ &\times(-t_i;q)_b\Big(-\frac{q}{t_i};q\Big)_b(\sqrt{q}t_i;q)_c\Big(\frac{\sqrt{q}}{t_i};q\Big)_c(-\sqrt{q}t_i;q)_d\Big(-\frac{\sqrt{q}}{t_i};q\Big)_d, \end{aligned} \tag{1.19} $$

and we use capital letters to denote the constant terms

$$ {}_qAW^k_{n,m}(a,b,c,d) = [1]_q aw^k_{n,m}(a,b,c,d;t_1,\dots,t_n), \tag{1.20}$$

$$ {}_q\overline{AW}^k_{n,m}(a,b,c,d) = [1]_q \overline{aw}^k_{n,m}(a,b,c,d;t_1,\dots,t_n). \tag{1.21}$$

Gustafson's theorem is the $m = 0$ case of the first part of the following theorem, which is our main result.

Theorem 2 (Gustafson [12]).

$$\begin{aligned} {}_qAW^k_{n,m}(a,b,c,d) &= \prod_{i=1}^{n} \frac{(-q;q)_{a+b+(n-i)k}}{(-q;q)_{(n-i)k}} \frac{(\sqrt{q};q)_{a+c+\chi(i\le m)+(n-i)k}}{(\sqrt{q};q)_{(n-i)k}} \frac{(-\sqrt{q};q)_{a+d+\chi(i\le m)+(n-i)k}}{(-\sqrt{q};q)_{(n-i)k}} \\ &\quad\times \frac{(-\sqrt{q};q)_{b+c+(n-i)k}}{(-\sqrt{q};q)_{(n-i)k}} \frac{(\sqrt{q};q)_{b+d+(n-i)k}}{(\sqrt{q};q)_{(n-i)k}} \frac{(-q;q)_{c+d+(n-i)k}}{(-q;q)_{(n-i)k}} \\ &\quad\times \frac{1}{(q;q)_{a+b+c+d+\chi(i\le m)+(2n-i-1)k}} \frac{(q;q)^2_{2(n-i)k}}{(q;q)^2_{(n-i)k}} \frac{(q;q)_{ik}}{(q;q)_k}, \end{aligned} \tag{1.22}$$

$$\begin{aligned} {}_q\overline{AW}^k_{n,m}(a,b,c,d) &= \prod_{i=1}^{n} \frac{(-q;q)_{a+b+\chi(n-i+1\le m)+(n-i)k}}{(-q;q)_{(n-i)k}} \frac{(\sqrt{q};q)_{a+c+1+(n-i)k}}{(\sqrt{q};q)_{(n-i)k}} \frac{(-\sqrt{q};q)_{a+d+1+(n-i)k}}{(-\sqrt{q};q)_{(n-i)k}} \\ &\quad\times \frac{(-\sqrt{q};q)_{b+c+(n-i)k}}{(-\sqrt{q};q)_{(n-i)k}} \frac{(\sqrt{q};q)_{b+d+(n-i)k}}{(\sqrt{q};q)_{(n-i)k}} \frac{(-q;q)_{c+d+(n-i)k}}{(-q;q)_{(n-i)k}} \\ &\quad\times \frac{1}{(q;q)_{a+b+c+d+1+(2n-i-1)k}} \frac{(q;q)^2_{2(n-i)k}}{(q;q)^2_{(n-i)k}} \frac{(q;q)_{ik}}{(q;q)_k}. \end{aligned} \tag{1.23}$$

Comparing (1.18) and (1.19), we see that

$$ {}_q\overline{aw}^k_{n}(a,b,c,d;t_1,\dots,t_n) = {}_qaw^k_{n,n}(a,b,c,d;t_1,\dots,t_n) \tag{1.24}$$

and

$$ {}_qaw^k_{n}(a+1,b,c,d;t_1,\dots,t_n) = {}_q\overline{aw}^k_{n,n}(a,b,c,d;t_1,\dots,t_n). \tag{1.25}$$

Taking the constant terms in (1.24) and (1.25), we have

$$ {}_q\overline{AW}^k_{n}(a,b,c,d) = {}_qAW^k_{n,n}(a,b,c,d) \tag{1.26}$$

and

$$ {}_qAW^k_{n}(a+1,b,c,d) = {}_q\overline{AW}^k_{n,n}(a,b,c,d), \tag{1.27}$$

respectively.

In Section 2, we review the basic algebraic and geometric properties of root systems: positive, simple roots and reflections, the Weyl group, and the crystallographic condition.

In Section 3, we give the q-transportation theories for the root systems A_{n-1} and BC_n, which we explicitly express in terms of the function ${}_qaw^k_n(a,b,c,d;t_1,\dots,t_n)$.

In Section 4, we establish the dependence of the constant term ${}_qAW^k_{n,m}(a,b,c,d)$ on the parameter m.

In Section 5, we establish the dependence of the constant term ${}_q\overline{AW}^k_{n,m}(a,b,c,d)$ on the parameter m.

In Section 6, we combine our results to establish the dependence of the constant term ${}_qAW^k_n(a,b,c,d)$ on the parameter a.

In Section 7, we complete the proof of Theorem 2 by using certain symmetries of ${}_qAW_n(a,b,c,d)$ and the q-Macdonald-Morris conjecture for D_n, proven by Kadell [17] and subsequently by Gustafson [12].

2. Root systems

In this section, we review the basic algebraic and geometric properties of root systems: positive, simple roots and reflections, the Weyl group, and the crystallographic condition.

Let V be a finite dimensional vector space over the reals with zero vector (0), dimension ℓ, an inner product $(\ ,\ )$, and let $\alpha,\ v \in V$. The reflection of v along α, which is given by

$$\sigma_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha, \tag{2.1}$$

is obtained by reflecting v in the hyperplane perpendicular to α. Observe that σ_α is a linear mapping.

A root system is a finite set $R \subseteq V - \{(0)\}$ of nonzero vectors, which are called roots, such that R is a basis for V and $\sigma_\alpha(v) \in R$ whenever $\alpha,\ v \in R$. We say that R has rank ℓ. The group $\langle \sigma_\alpha \mid \alpha \in R\rangle$ generated by the reflections along the roots of R is a finite group W, called the Weyl group, which acts on the root system R.

Since R is finite, we may choose a hyperplane $\mathcal{H}$ with $(0) \in \mathcal{H}$ and $\mathcal{H} \cap R = \emptyset$. Choosing $\rho \notin \mathcal{H}$, we have the system of positive roots

$$R^+ = \{\alpha \in R \mid (\alpha,\rho) > 0\}. \tag{2.2}$$

We generally take

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha, \tag{2.3}$$

and letting $R^- = \{-\alpha \mid \alpha \in R^+\} = \{\alpha \in R \mid (\alpha,\rho) < 0\}$, we have

$$R = R^+ \cup R^-. \tag{2.4}$$

We may write $\alpha > 0$ or $\alpha < 0$ for $\alpha \in R^+$ or $\alpha \in R^-$, respectively.

Observe from (2.1) that

$$\sigma_\alpha(\alpha) = -\alpha. \tag{2.5}$$

We may find a system $\{\alpha_1,\ldots,\alpha_\ell\} \subseteq R^+$ of positive, simple roots such that for $1 \le i \le \ell$, we have $\alpha > 0$ and $\alpha \ne \alpha_i \implies \sigma_i(\alpha) > 0$ where $\sigma_i = \sigma_{\alpha_i}$ denotes the simple reflection along the simple root α_i. Thus σ_i sends α_i to $-\alpha_i$ and, since the Weyl group W acts on R, permutes the other positive roots. Hence we have the basic property

$$\sigma_i(\alpha_i) = -\alpha_i \text{ and } v > 0,\ v \ne \alpha_i \implies \sigma_i(v) > 0, \qquad 1 \le i \le \ell, \tag{2.6}$$

of the positive, simple roots.

A system of positive roots for the root system A_{n-1} is given by

$$A_{n-1}^{+} = \{\mathrm{e}_i - \mathrm{e}_j \mid 1 \le i < j \le n\}. \tag{2.7}$$

Fix i and j with $1 \le i < j \le n$. Then (2.1) gives

$$\sigma_{\mathrm{e}_i - \mathrm{e}_j}(\mathrm{e}_i) = \mathrm{e}_j, \quad \sigma_{\mathrm{e}_i - \mathrm{e}_j}(\mathrm{e}_j) = \mathrm{e}_i, \tag{2.8}$$

and $h \neq i, j$ implies that $\sigma_{\mathrm{e}_i - \mathrm{e}_j}(\mathrm{e}_h) = \mathrm{e}_h$. Thus, $\sigma_{\mathrm{e}_i - \mathrm{e}_j}$ interchanges e_i and e_j. Hence, we see that the Weyl group W of A_{n-1} is isomorphic to the symmetric group S_n.

A system of positive, simple roots for A_{n-1} is given by

$$\alpha_i = \mathrm{e}_i - \mathrm{e}_{i+1}, \qquad 1 \le i \le n-1. \tag{2.9}$$

Observe that the simple reflections of A_{n-1} are the adjacent transpositions.

A system of positive roots for the root system D_n is given by

$$D_n^{+} = A_{n-1}^{+} \cup \{\mathrm{e}_i + \mathrm{e}_j \mid 1 \le i < j \le n\}. \tag{2.10}$$

Fix i and j with $1 \le i < j \le n$. Then (2.1) gives

$$\sigma_{\mathrm{e}_i + \mathrm{e}_j}(\mathrm{e}_i) = -\mathrm{e}_j, \ \sigma_{\mathrm{e}_i + \mathrm{e}_j}(\mathrm{e}_j) = -\mathrm{e}_i, \tag{2.11}$$

and $h \neq i, j$ implies that $\sigma_{\mathrm{e}_i + \mathrm{e}_j}(\mathrm{e}_h) = \mathrm{e}_h$. Thus, $\sigma_{\mathrm{e}_i + \mathrm{e}_j}$ interchanges e_i and e_j and multiplies by minus one. Hence, we see that the Weyl group W of D_n is isomorphic to the semi-direct product of the symmetric group S_n and the group of sign changes in an even number of the first n coordinates.

A system of positive, simple roots for D_n is given by those (2.9) for A_{n-1} together with

$$\alpha_n = \mathrm{e}_1 + \mathrm{e}_2. \tag{2.12}$$

Observe that the simple reflection σ_n of D_n sends e_1 to $-\mathrm{e}_2$ and e_2 to $-\mathrm{e}_1$.

Systems of positive roots for the root systems B_n, C_n, and BC_n are given by

$$B_n^{+} = D_n^{+} \cup \{\mathrm{e}_1, \dots, \mathrm{e}_n\}, \tag{2.13}$$

$$C_n^{+} = D_n^{+} \cup \{2\mathrm{e}_1, \dots, 2\mathrm{e}_n\}, \tag{2.14}$$

and

$$BC_n^{+} = D_n^{+} \cup \{\mathrm{e}_1, \dots, \mathrm{e}_n\} \cup \{2\mathrm{e}_1, \dots, 2\mathrm{e}_n\}, \tag{2.15}$$

respectively.

Fix i with $1 \le i \le n$. Then by (2.1), we see that σ_{e_i} and $\sigma_{2\mathrm{e}_i}$ are equal as mappings, that they send e_i to $-\mathrm{e}_i$, and that $h \neq i$ implies that $\sigma_{\mathrm{e}_i}(\mathrm{e}_h) = \sigma_{2\mathrm{e}_i}(\mathrm{e}_h) = \mathrm{e}_h$. Hence, we see that the Weyl groups W of B_n, C_n, and BC_n are isomorphic to the semi-direct product of the symmetric group S_n and the group of sign changes in the first n coordinates.

Systems of positive, simple roots for B_n and C_n are given by those (2.9) for A_{n-1} together with

$$\alpha_n = \mathrm{e}_1 \text{ for } B_n \tag{2.16}$$

and

$$\alpha_n = 2\mathrm{e}_1 \text{ for } C_n. \tag{2.17}$$

Observe that in either case the simple reflection σ_n represents the change of sign in the first coordinate.

Let the dual root system be given by

$$R^\vee = \{\alpha^\vee \mid \alpha \in R\} \tag{2.18}$$

where the co-roots are given by

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}. \tag{2.19}$$

Then we have

$$B_n^\vee = C_n, \quad C_n^\vee = B_n. \tag{2.20}$$

Thus, the root systems B_n and C_n are dual to each other. They enjoy a compatibility which allows us to form the self dual root system

$$BC_n = B_n \cup C_n. \tag{2.21}$$

We say that the root system R is reduced if $\alpha,\ \lambda\alpha \in R$ implies that $\lambda = \pm 1$. Observe that A_{n-1}, D_n, B_n, and C_n are reduced root systems while BC_n, which contains $\mathrm{e}_1, \dots, \mathrm{e}_n$ and their doubles $2\mathrm{e}_1, \dots, 2\mathrm{e}_n$, is not a reduced root system.

Observe that the product

$$\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \tag{2.22}$$

is linear only in the first coordinate. We have the crystallographic condition

$$\alpha,\ \beta \in R \implies \langle \beta, \alpha \rangle \text{ is an integer}, \tag{2.23}$$

which allows the nonreduced root system BC_n.

The interested reader may consult Grove and Benson [11] or Carter [8] for further information about root systems and related topics.

3. The q-transportation theories for the root systems A_{n-1} and BC_n

In this section, we give the q-transportation theories for the root systems A_{n-1} and BC_n, which we explicitly express in terms of the function ${}_qaw_n^k(a,b,c,d;t_1,\dots,t_n)$.

The following lemma recalls [17, Lemma 10; 18, Lemma 2] the local q-transportation theory for the root system A_{n-1}.

Lemma 3 ([17, Lemma 10; 18, Lemma 2]). *Let*

$$\Upsilon(t,s) = \Upsilon(s,t) \tag{3.1}$$

be symmetric in s and t and have a Laurent expansion at $s = t = 0$. Then we have

$$[1]\, t \left(1 - \frac{s}{t}\right)\left(1 - Q\frac{t}{s}\right) \Upsilon(s,t) = Q\,[1]\, s \left(1 - \frac{s}{t}\right)\left(1 - Q\frac{t}{s}\right) \Upsilon(s,t). \tag{3.2}$$

We saw in [18] that (3.2) follows easily from the fact that the difference between the two sides of (3.2) is the constant term of an antisymmetric function and hence must be zero.

Observe that for $\pi \in S_n$, we have

$${}_qa_{n-1}^k(t_{\pi(1)},\dots,t_{\pi(n)}) = \prod_{\substack{1\le i<j\le n \\ \pi^{-1}(i)<\pi^{-1}(j)}} \left(\frac{t_i}{t_j};q\right)_k \left(q\frac{t_j}{t_i};q\right)_k \prod_{\substack{1\le i<j\le n \\ \pi^{-1}(i)>\pi^{-1}(j)}} \left(q\frac{t_i}{t_j};q\right)_k \left(\frac{t_j}{t_i};q\right)_k. \tag{3.3}$$

Let $2 \le v \le n$. We see from (3.3) that the function ${}_q a_{n-1}^k(t_1, \dots, t_n)$ is symmetric under the adjacent transposition $t_v \longleftrightarrow t_{v-1}$ except for the factor

$$\left(\frac{t_{v-1}}{t_v}; q\right)_k \left(q\frac{t_v}{t_{v-1}}; q\right)_k = \left(1 - \frac{t_{v-1}}{t_v}\right)\left(1 - q^k\frac{t_v}{t_{v-1}}\right)\left(q\frac{t_{v-1}}{t_v}; q\right)_{k-1}\left(q\frac{t_v}{t_{v-1}}; q\right)_{k-1}. \tag{3.4}$$

Taking $s = t_{v-1}$, $t = t_v$ in Lemma 3 (3.2) and incorporating the function $(qt_{v-1}/t_v;\ q)_{k-1}(qt_v/t_{v-1}; q)_{k-1}$ into $\Upsilon(s,t)$, which we rename $\theta(t_1, \dots, t_n)$, we obtain the q-transportation theory for A_{n-1}

$$[1]\, t_v\, \theta(t_1, \dots, t_n)\ {}_q a_{n-1}^k(t_1, \dots, t_n) = q^k\, [1]\, t_{v-1}\, \theta(t_1, \dots, t_n)\ {}_q a_{n-1}^k(t_1, \dots, t_n) \tag{3.5}$$

where

$$\theta(t_1, \dots, t_n) = \theta(t_1, \dots, t_{v-2}, t_v, t_{v-1}, t_{v+1}, \dots, t_n) \tag{3.6}$$

is symmetric in t_{v-1} and t_v; see [17, Lemma 11; 18, Lemma 3].

The following lemma explicitly expresses the q-transportation theory for A_{n-1} (Lemma 3, (3.2) and (3.5)) in terms of the function ${}_q aw_n^k(a, b, c, d; t_1, \dots, t_n)$.

Lemma 4. *Let $2 \le v \le n$, and let $\theta(t_1, \dots, t_n)$ be symmetric in t_{v-1} and t_v; see* (3.6). *Then we have*

$$\begin{aligned}&[1]\, t_v\, \theta(t_1, \dots, t_n)\ {}_q aw_n^k(a, b, c, d; t_1, \dots, t_n)\\ &\qquad = q^k\, [1]\, t_{v-1}\, \theta(t_1, \dots, t_n)\ {}_q aw_n^k(a, b, c, d; t_1, \dots, t_n),\end{aligned} \tag{3.7}$$

$$\begin{aligned}&[1]\, \frac{1}{t_{v-1}}\, \theta(t_1, \dots, t_n)\ {}_q aw_n^k(a, b, c, d; t_1, \dots, t_n)\\ &\qquad = q^k\, [1]\, \frac{1}{t_v}\, \theta(t_1, \dots, t_n)\ {}_q aw_n^k(a, b, c, d; t_1, \dots, t_n).\end{aligned} \tag{3.8}$$

Proof. Observe that

$$\begin{aligned}{}_q aw_n^k(a, b, c, d; t_1, \dots, t_n) &= {}_q d_n^k(t_1, \dots, t_n) \prod_{i=1}^n (t_i; q)_a \left(\frac{q}{t_i}; q\right)_a (-t_i; q)_b \left(-\frac{q}{t_i}; q\right)_b\\ &\quad \times (\sqrt{q}t_i; q)_c \left(\frac{\sqrt{q}}{t_i}; q\right)_c (-\sqrt{q}t_i; q)_d \left(-\frac{\sqrt{q}}{t_i}; q\right)_d.\end{aligned} \tag{3.9}$$

Observe from (1.13) and (3.9) that ${}_q aw_n^k(a, b, c, d; t_1, \dots, t_n)$ equals ${}_q a_{n-1}^k(t_1, \dots, t_n)$ times a function which is symmetric in $t_1, \dots, t_n$. Incorporating the symmetric function into $\theta(t_1, \dots, t_n)$, we see that (3.5) gives the result (3.7).

Observe that $t_v(1/t_{v-1}t_v) = 1/t_{v-1}$ and $t_{v-1}(1/t_{v-1}t_v) = 1/t_v$. Incorporating the function $1/t_{v-1}t_v$ into $\theta(t_1, \dots, t_n)$, we see that (3.5) gives the result (3.8). □

The proof reflects the fact that when we extend A_{n-1} to D_n, B_n, or C_n, the simple roots of A_{n-1} remain simple roots.

The following lemma recalls the local q-transportation theory [17, Lemma 12] for B_n.

Lemma 5 (17, Lemma 12). *Let $T(t)$ be invariant under the substitution $t \longleftrightarrow 1/t$,*

$$\mathrm{T}(t) = \mathrm{T}\left(\frac{1}{t}\right), \tag{3.10}$$

and have a Laurent expansion at $t=0$. *Then we have*

$$[1]\left(1-\frac{Q}{t}\right)\left(1+\frac{1}{t}\right)(1-t)\,\mathrm{T}(t)=Q\,[1]\,t\left(1-\frac{Q}{t}\right)\left(1+\frac{1}{t}\right)(1-t)\,\mathrm{T}(t). \tag{3.11}$$

Proof. Observe that the left-hand side of (3.11) minus the right-hand side of (3.11) equals

$$[1]\,(1-Qt)\left(1-\frac{Q}{t}\right)\left(1+\frac{1}{t}\right)(1-t)\,\mathrm{T}(t), \tag{3.12}$$

which is zero since the function

$$\left(1+\frac{1}{t}\right)(1-t)=\frac{1}{t}-t \tag{3.13}$$

is antisymmetric under the substitution $t \longleftrightarrow 1/t$. The result (3.11) then follows. □

Setting $Q=-q^b$ in (3.11) gives

$$[1]\left(1+\frac{q^b}{t}\right)\left(1+\frac{1}{t}\right)(1-t)\,\mathrm{T}(t)=-q^b\,[1]\,t\left(1+\frac{q^b}{t}\right)\left(1+\frac{1}{t}\right)(1-t)\,\mathrm{T}(t). \tag{3.14}$$

Observe that

$$\begin{aligned}{}_q aw_1(a,b,c,d;t) &= (t;q)_a\left(\frac{q}{t};q\right)_a(-t;q)_b\left(-\frac{q}{t};q\right)_b\\ &\times(\sqrt{q}t;q)_c\left(\frac{\sqrt{q}}{t};q\right)_c(-\sqrt{q}t;q)_d\left(-\frac{\sqrt{q}}{t};q\right)_d.\end{aligned} \tag{3.15}$$

Hence, we obtain

$$\begin{aligned}\frac{1}{t}\,(1-q^a t)\,{}_q aw_1(a,b,c,d;t) &= (1-t)\,(qt;q)_a\left(\frac{q}{t};q\right)_a\frac{1}{t}\,(1+t)\left(1+\frac{q^b}{t}\right)\\ &\times(-qt;q)_{b-1}\left(-\frac{q}{t};q\right)_{b-1}(\sqrt{q}t;q)_c\left(\frac{\sqrt{q}}{t};q\right)_c(-\sqrt{q}t;q)_d\left(-\frac{\sqrt{q}}{t};q\right)_d.\end{aligned} \tag{3.16}$$

Incorporating the function $(qt;q)_a(q/t;q)_a(-qt;q)_{b-1}(-q/t;q)_{b-1}(\sqrt{q}t;q)_c(\sqrt{q}/t;q)_c \times(-\sqrt{q}t;q)_d(-\sqrt{q}/t;q)_d$ into $\mathrm{T}(t)$, we see that (3.14) gives

$$[1]\,\frac{1}{t}\,(1-q^a t)\,\mathrm{T}(t)\,{}_q aw_1(a,b,c,d;t)=-q^b\,[1]\,(1-q^a t)\,\mathrm{T}(t)\,{}_q aw_1(a,b,c,d;t). \tag{3.17}$$

Observe that

$$\mathcal{D}(s,t)=(st;q)_k\left(\frac{q}{st};q\right)_k\left(\frac{s}{t};q\right)_k\left(q\frac{t}{s};q\right)_k \tag{3.18}$$

satisfies the symmetries

$$\mathcal{D}(s,t)=\mathcal{D}(-s,-t)=\mathcal{D}\left(\frac{\sqrt{q}}{t},\frac{\sqrt{q}}{s}\right)=\mathcal{D}\left(\frac{q}{s},t\right)=\mathcal{D}\left(s,\frac{1}{t}\right). \tag{3.19}$$

Since

$${}_q d_n^k(t_1,\ldots,t_n)=\prod_{1\le i<j\le n}\mathcal{D}(t_i,t_j), \tag{3.20}$$

we have the symmetries

$$ {}_q d_n^k(t_1,\dots,t_n) = {}_q d_n^k(-t_1,\dots,-t_n) \tag{3.21} $$

$$ = {}_q d_n^k\Big(\frac{\sqrt{q}}{t_n},\dots,\frac{\sqrt{q}}{t_1}\Big) \tag{3.22} $$

$$ = {}_q d_n^k\Big(\frac{q}{t_1},t_2,\dots,t_n\Big) \tag{3.23} $$

$$ = {}_q d_n^k(t_1,\dots,t_{n-1},\frac{1}{t_n}). \tag{3.24} $$

The following lemma explicitly expresses Lemma 5, (3.11), in terms of the function ${}_q aw_n^k(a,b,c,d;t_1,\dots,t_n)$.

Lemma 6. *If $\beta(t_1,\dots,t_n)$ is invariant under the substitution $t_n \longleftrightarrow 1/t_n$, i.e.,*

$$ \beta\Big(t_1,\dots,t_{n-1},\frac{1}{t_n}\Big) = \beta(t_1,\dots,t_n), \tag{3.25} $$

then we have

$$ [1]\,\frac{1}{t_n}\,(1-q^a t_n)\,\beta(t_1,\dots,t_n)\,{}_q aw_n^k(a,b,c,d;t_1,\dots,t_n) $$
$$ = -q^b\,[1]\,(1-q^a t_n)\,\beta(t_1,\dots,t_n)\,{}_q aw_n^k(a,b,c,d;t_1,\dots,t_n). \tag{3.26} $$

Proof. Observe from (3.9) and (3.24) that ${}_q aw_n^k(a,b,c,d;t_1,\dots,t_n)$ is equal to ${}_q aw_1(a,b,\ c,d;t_n)$ times a function which is invariant under the substitution $t_n \longleftrightarrow 1/t_n$. Setting $t=t_n$ and incorporating the invariant function into $\beta(t_1,\dots,t_n)$, we see that (3.17) gives the result (3.26). □

Observe by (3.9), (3.21), and (3.22) that we have the symmetries

$$ {}_q aw_1(a,b,c,d;t) = {}_q aw_1(b,a,d,c;-t) \tag{3.27} $$

$$ = {}_q aw_1\Big(c,d,a,b;\frac{\sqrt{q}}{t}\Big). \tag{3.28} $$

Combining (3.21) and (3.22) with (3.27) and (3.28), respectively, we have the symmetries

$$ {}_q aw_n^k(a,b,c,d;t_1,\dots,t_n) = {}_q aw_n^k(b,a,d,c;-t_1,\dots,-t_n) \tag{3.29} $$

$$ = {}_q aw_n^k\Big(c,d,a,b;\frac{\sqrt{q}}{t_n},\dots,\frac{\sqrt{q}}{t_1}\Big). \tag{3.30} $$

The following lemma uses the symmetry (3.30) to translate Lemma 6.

Lemma 7. *If $\gamma(t_1,\dots,t_n)$ is invariant under the substitution $t_1 \longleftrightarrow q/t_1$, i.e.,*

$$ \gamma\Big(\frac{q}{t_1},t_2,\dots,t_n\Big) = \gamma(t_1,\dots,t_n), \tag{3.31} $$

then we have

$$ [1]\,t_1\left(1-\frac{q^{c+1/2}}{t_1}\right)\gamma(t_1,\dots,t_n)\,{}_q aw_n^k(a,b,c,d;t_1,\dots,t_n) $$
$$ = -q^{d+1/2}\,[1]\left(1-\frac{q^{c+1/2}}{t_1}\right)\gamma(t_1,\dots,t_n)\,{}_q aw_n^k(a,b,c,d;t_1,\dots,t_n). \tag{3.32} $$

Proof. Observe that if the function $\gamma(t_1,\dots,t_n)$ satisfies the invariance (3.31), then the function

$$\beta(t_1,\dots,t_n)=\gamma\Big(\frac{\sqrt{q}}{t_n},\dots,\frac{\sqrt{q}}{t_1}\Big) \tag{3.33}$$

satisfies the invariance (3.25). Observe that if we make the substitutions

$$(t_1,\dots,t_n)\longleftrightarrow\Big(\frac{\sqrt{q}}{t_n},\dots,\frac{\sqrt{q}}{t_1}\Big) \tag{3.34}$$

and

$$(a,b,c,d)\longleftrightarrow(c,d,a,b) \tag{3.35}$$

in Lemma 6 (3.26), then we obtain the result (3.32) divided by $\sqrt{q}$. □

4. The dependence of ${}_qAW^k_{n,m}(a,b,c,d)$ on the parameter m

In this section, we establish the dependence of the constant term ${}_qAW^k_{n,m}(a,b,c,d)$ on the parameter m.

Let $1\le m\le n$. Taking $\beta(t_1,\dots,t_n)=\prod_{i=n-m+1}^{n-1}(1-q^at_i)$, which is independent of t_n, in Lemma 6 (3.26), we obtain

$$\begin{aligned}{}_qAW^k_{n,m}(a,b,c,d)&=[1]\prod_{i=n-m+1}^{n}(1-q^at_i)\;{}_qaw^k_n(a,b,c,d;t_1,\dots,t_n)\\&=-q^{-b}\,[1]\,\frac{1}{t_n}\prod_{i=n-m+1}^{n}(1-q^at_i)\;{}_qaw^k_n(a,b,c,d;t_1,\dots,t_n).\end{aligned} \tag{4.1}$$

Using the q-transportation theory for A_{n-1}, Lemma 4 (3.8), $m-1$ times with v running from n to $n-m+2$, we obtain

$$\begin{aligned}&{}_qAW^k_{n,m}(a,b,c,d)\\&\quad=-q^{-b-(m-1)k}\,[1]\,\frac{1}{t_{n-m+1}}\prod_{i=n-m+1}^{n}(1-q^at_i)\;{}_qaw^k_n(a,b,c,d;t_1,\dots,t_n).\end{aligned} \tag{4.2}$$

We have

$$\frac{1}{t_{n-m+1}}\,(1-q^at_{n-m+1})=-q^a+\frac{1}{t_{n-m+1}}. \tag{4.3}$$

Substituting (4.3) into (4.2) and using the q-transportation theory, Lemma 4 (3.8), for A_{n-1} $n-m$ times with v running from $n-m+1$ to 2, we obtain

$$\begin{aligned}{}_qAW^k_{n,m}(a,b,c,d)&=q^{a-b-(m-1)k}\,[1]\prod_{i=n-m+2}^{n}(1-q^at_i)\;{}_qaw^k_n(a,b,c,d;t_1,\dots,t_n)\\&\quad-q^{-b-(m-1)k}\,[1]\,\frac{1}{t_{n-m+1}}\prod_{i=n-m+2}^{n}(1-q^at_i)\;{}_qaw^k_n(a,b,c,d;t_1,\dots,t_n)\\&=q^{a-b-(m-1)k}\;{}_qAW^k_{n,m-1}(a,b,c,d)\\&\quad-q^{-b-(n-1)k}\,[1]\,\frac{1}{t_1}\prod_{i=n-m+2}^{n}(1-q^at_i)\;{}_qaw^k_n(a,b,c,d;t_1,\dots,t_n).\end{aligned} \tag{4.4}$$

Observe that

$$\frac{1}{t_1} = -q^{-c-1/2}\left(1 - \frac{q^{c+1/2}}{t_1}\right) + q^{-c-1/2}. \tag{4.5}$$

Substituting (4.5) into (4.4) gives

$$\begin{aligned}
{}_qAW^k_{n,m}(a,b,c,d) &= q^{a-b-(m-1)k}\, {}_qAW^k_{n,m-1}(a,b,c,d) \\
&\quad + q^{-b-c-1/2-(n-1)k}\,[1]\left(1 - \frac{q^{c+1/2}}{t_1}\right) \prod_{i=n-m+2}^{n} (1-q^a t_i)\, {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n) \\
&\quad - q^{-b-c-1/2-(n-1)k}\,[1] \prod_{i=n-m+2}^{n} (1-q^a t_i)\ {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n) \\
&= \left(q^{a-b-(m-1)k} - q^{-b-c-1/2-(n-1)k}\right) {}_qAW^k_{n,m-1}(a,b,c,d) \\
&\quad + q^{-b-c-1/2-(n-1)k}\,[1]\left(1 - \frac{q^{c+1/2}}{t_1}\right) \prod_{i=n-m+2}^{n} (1-q^a t_i)\, {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n).
\end{aligned} \tag{4.6}$$

Taking $\gamma(t_1,\dots,t_n) = -q^{-b-c-d-1-(n-1)k}\prod_{i=n-m+2}^{n}(1-q^a t_i)$, which is independent of t_1 since $m \le n$ in Lemma 7 (3.32), and using $t_1(1-q^{c+1/2}/t_1) = t_1 - q^{c+1/2}$, we obtain

$$\begin{aligned}
{}_qAW^k_{n,m}(a,b,c,d) &= \left(q^{a-b-(m-1)k} - q^{-b-c-1/2-(n-1)k}\right) {}_qAW^k_{n,m-1}(a,b,c,d) \\
&\quad - q^{-b-c-d-1-(n-1)k}\,[1]\,(t_1 - q^{c+1/2}) \prod_{i=n-m+2}^{n} (1-q^a t_i) \\
&\qquad \times\ {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n) \\
&= \left(q^{a-b-(m-1)k} - q^{-b-c-1/2-(n-1)k} + q^{-b-d-1/2-(n-1)k}\right) \\
&\qquad \times\ {}_qAW^k_{n,m-1}(a,b,c,d) \\
&\quad - q^{-b-c-d-1-(n-1)k}\,[1]\,t_1 \prod_{i=n-m+2}^{n} (1-q^a t_i)\ {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n).
\end{aligned} \tag{4.7}$$

We may change the factor t_1 to t_{n-m+1} by using the q-transportation theory, Lemma 4 (3.7), for A_{n-1} $n-m$ times with v running from 2 to $n-m+1$. This gives

$$\begin{aligned}
{}_qAW^k_{n,m}(a,b,c,d) &= \left(q^{a-b-(m-1)k} - q^{-b-c-1/2-(n-1)k} + q^{-b-d-1/2-(n-1)k}\right) \\
&\qquad \times\ {}_qAW^k_{n,m-1}(a,b,c,d) \\
&\qquad - q^{-b-c-d-1-(2n-m-1)k}\,[1]\,t_{n-m+1} \prod_{i=n-m+2}^{n} (1-q^a t_i) \\
&\qquad \times\ {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n).
\end{aligned} \tag{4.8}$$

Observe that

$$t_{n-m+1} = -q^{-a}\left(1 - q^{a} t_{n-m+1}\right) + q^{-a}. \tag{4.9}$$

Substituting (4.9) into (4.8), we obtain

$$\begin{aligned}
&{}_qAW^k_{n,m}(a,b,c,d)\\
&\quad = \left(q^{a-b-(m-1)k} - q^{-b-c-1/2-(n-1)k} + q^{-b-d-1/2-(n-1)k}\right) {}_qAW^k_{n,m-1}(a,b,c,d)\\
&\qquad + q^{-a-b-c-d-1-(2n-m-1)k}\,[1] \prod_{i=n-m+1}^{n} \left(1 - q^{a} t_i\right) {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n)\\
&\qquad - q^{-a-b-c-d-1-(2n-m-1)k}\,[1] \prod_{i=n-m+2}^{n} \left(1 - q^{a} t_i\right) {}_qaw^k_n(a,b,c,d;t_1,\dots,t_n)\\
&\quad = \Big(q^{a-b-(m-1)k} - q^{-b-c-1/2-(n-1)k}\\
&\qquad + q^{-b-d-1/2-(n-1)k} - q^{-a-b-c-d-1-(2n-m-1)k}\Big) {}_qAW^k_{n,m-1}(a,b,c,d)\\
&\qquad + q^{-a-b-c-d-1-(2n-m-1)k}\, {}_qAW^k_{n,m}(a,b,c,d;t_1,\dots,t_n).
\end{aligned} \tag{4.10}$$

Multiplying (4.10) by $-q^{a+b+c+d+1+(2n-m-1)k}$ and rearranging gives

$$\begin{aligned}
&\left(1 - q^{a+b+c+d+1+(2n-m-1)k}\right) {}_qAW^k_{n,m}(a,b,c,d)\\
&\quad = \left(1 - q^{a+c+1/2+(n-m)k} + q^{a+d+1/2+(n-m)k} - q^{2a+c+d+1+2(n-m)k}\right)\\
&\qquad \times {}_qAW^k_{n,m-1}(a,b,c,d)\\
&\quad = \left(1 - q^{a+c+1/2+(n-m)k}\right)\left(1 + q^{a+d+1/2+(n-m)k}\right) {}_qAW^k_{n,m-1}(a,b,c,d).
\end{aligned} \tag{4.11}$$

We then obtain

$$\begin{aligned}
{}_qAW^k_{n,m}(a,b,c,d) = {}& \frac{\left(1 - q^{a+c+1/2+(n-m)k}\right)\left(1 + q^{a+d+1/2+(n-m)k}\right)}{\left(1 - q^{a+b+c+d+1+(2n-m-1)k}\right)}\\
&\times {}_qAW^k_{n,m-1}(a,b,c,d),
\end{aligned} \tag{4.12}$$

which gives the dependence of the constant term ${}_qAW^k_{n,m}(a,b,c,d)$ on the parameter m.

5. The dependence of ${}_q\overline{AW}^k_{n,m}(a,b,c,d)$ on the parameter m

In this section, we establish the dependence of the constant term ${}_q\overline{AW}^k_{n,m}(a,b,c,d)$ on the parameter m.

Let $1 \le m \le n$. We have

$$
\begin{aligned}
{}_q\overline{AW}^k_{n,m}(a,b,c,d) &= [1] \prod_{i=1}^{m} \left(1 - \frac{q^{a+1}}{t_i}\right) {}_q\overline{aw}^k_n(a,b,c,d;t_1,\dots,t_n) \\
&= [1] \prod_{i=1}^{m-1} \left(1 - \frac{q^{a+1}}{t_i}\right) {}_q\overline{aw}^k_n(a,b,c,d;t_1,\dots,t_n) \\
&\quad - q^{a+1}\,[1]\,\frac{1}{t_m} \prod_{i=1}^{m-1} \left(1 - \frac{q^{a+1}}{t_i}\right) {}_q\overline{aw}^k_n(a,b,c,d;t_1,\dots,t_n) \\
&= {}_q\overline{AW}^k_{n,m-1}(a,b,c,d) - q^{a+1}\,[1]\,\frac{1}{t_m}\, {}_q\overline{aw}^k_{n,m-1}(a,b,c,d;t_1,\dots,t_n).
\end{aligned}
\tag{5.1}
$$

Using the q-transportation theory, Lemma 4 (3.8), for A_{n-1} $n-m$ times with v running from $m+1$ to n, we obtain

$$
\begin{aligned}
{}_q\overline{AW}^k_{n,m}(a,b,c,d) &= {}_q\overline{AW}^k_{n,m-1}(a,b,c,d) \\
&\quad - q^{a+1+(n-m)k}\,[1]\,\frac{1}{t_n}\, {}_q\overline{aw}^k_{n,m-1}(a,b,c,d;t_1,\dots,t_n).
\end{aligned}
\tag{5.2}
$$

Taking $\beta(t_1,\dots,t_n) = \prod_{i=1}^{m-1}(1-q^{a+1}/t_i)\prod_{i=1}^{n-1}(1-q^a t_i)$, which is independent of t_n since $m \le n$, in Lemma 6 (3.26), we obtain

$$
\begin{aligned}
{}_q\overline{AW}^k_{n,m}(a,b,c,d) &= {}_q\overline{AW}^k_{n,m-1}(a,b,c,d) \\
&\quad + q^{a+b+1+(n-m)k}\,[1]\, {}_q\overline{aw}^k_{n,m-1}(a,b,c,d;t_1,\dots,t_n) \\
&= \left(1 + q^{a+b+1+(n-m)k}\right) {}_q\overline{AW}^k_{n,m-1}(a,b,c,d),
\end{aligned}
\tag{5.3}
$$

which gives the dependence of the constant term ${}_q\overline{AW}^k_{n,m}(a,b,c,d)$ on the parameter m.

6. The dependence of ${}_qAW^k_n(a,b,c,d)$ on the parameter a

In this section, we combine our results to establish the dependence of the constant term ${}_qAW^k_n(a,b,c,d)$ on the parameter a.

Repeated use of the recurrence relation (4.12) gives

$$
{}_qAW^k_{n,n}(a,b,c,d) = \prod_{i=1}^{n} \frac{\left(1 - q^{a+c+1/2+(n-i)k}\right)\left(1 + q^{a+d+1/2+(n-i)k}\right)}{\left(1 - q^{a+b+c+d+1+(2n-i-1)k}\right)}\, {}_qAW^k_n(a,b,c,d).
\tag{6.1}
$$

Repeated use of the recurrence relation (5.3) gives

$$
{}_q\overline{AW}^k_{n,n}(a,b,c,d) = \prod_{i=1}^{n} \left(1 + q^{a+b+1+(n-i)k}\right) {}_q\overline{AW}^k_n(a,b,c,d).
\tag{6.2}
$$

Combining our results (1.26), (1.27), (6.1), and (6.2), we obtain

$$
{}_qAW_n^k(a+1,b,c,d) = {}_qAW_n^k(a,b,c,d) \times \prod_{i=1}^{n}(1+q^{a+b+1+(n-i)k}) \frac{(1-q^{a+c+1/2+(n-i)k})\,(1+q^{a+d+1/2+(n-i)k})}{(1-q^{a+b+c+d+1+(2n-i-1)k})}, \qquad (6.3)
$$

which gives the dependence of the constant term ${}_qAW_n^k(a,b,c,d)$ on the parameter a.

7. A proof of Theorem 2

In this section, we complete the proof of Theorem 2 by using certain symmetries of ${}_qAW_n(a,b,c,d)$ and the q-Macdonald-Morris conjecture for D_n, proven by Kadell [17] and subsequently by Gustafson [12].

Taking the constant terms in (3.29) and (3.30), we see that

$$
{}_qAW_n^k(a,b,c,d) = {}_qAW_n^k(b,a,d,c) \qquad (7.1)
$$

$$
= {}_qAW_n^k(c,d,a,b). \qquad (7.2)
$$

Combining (7.1) and (7.2), we have

$$
{}_qAW_n^k(a,b,c,d) = {}_qAW_n^k(d,c,b,a). \qquad (7.3)
$$

Using the symmetries (7.1)–(7.3) of ${}_qAW_n^k(a,b,c,d)$, we see that (6.3) gives

$$
{}_qAW_n^k(a,b+1,c,d) = {}_qAW_n^k(a,b,c,d) \times \prod_{i=1}^{n}(1+q^{a+b+1+(n-i)k}) \frac{(1+q^{b+c+1/2+(n-i)k})\,(1-q^{b+d+1/2+(n-i)k})}{(1-q^{a+b+c+d+1+(2n-i-1)k})},
$$

$$
{}_qAW_n^k(a,b,c+1,d) = {}_qAW_n^k(a,b,c,d) \times \prod_{i=1}^{n}(1+q^{c+d+1+(n-i)k}) \frac{(1-q^{a+c+1/2+(n-i)k})\,(1+q^{b+c+1/2+(n-i)k})}{(1-q^{a+b+c+d+1+(2n-i-1)k})}, \qquad (7.4)
$$

and

$$
{}_qAW_n^k(a,b,c,d+1) = {}_qAW_n^k(a,b,c,d) \times \prod_{i=1}^{n}(1+q^{c+d+1+(n-i)k}) \frac{(1+q^{a+d+1/2+(n-i)k})\,(1-q^{b+d+1/2+(n-i)k})}{(1-q^{a+b+c+d+1+(2n-i-1)k})}, \qquad (7.5)
$$

which give the dependence of the constant term ${}_qAW_n^k(a,b,c,d)$ on the parameters b, c, and d, respectively.

The case $a=b=c=d=m=0$ of Theorem 2 (1.22) is given by

$$
{}_qAW_n^k(0,0,0,0) = [1] \prod_{1\le i<j\le n} (t_it_j;q)_k \left(\frac{q}{t_it_j};q\right)_k \left(\frac{t_i}{t_j};q\right)_k \left(q\frac{t_j}{t_i};q\right)_k
$$

$$
= \prod_{i=1}^{n} \frac{1}{(q;q)_{(2n-i-1)k}} \frac{(q;q)_{2(n-i)k}^2}{(q;q)_{(n-i)k}^2} \frac{(q;q)_{ik}}{(q;q)_k}, \qquad (7.6)
$$

which is the q-Macdonald-Morris conjecture for D_n, proven by Kadell [17] and subsequently by Gustafson [12].

Observe that the functions on the right-hand sides of (1.22) and (1.23) satisfy the recurrence relations (1.26), (1.27), (4.12), (5.3), (6.3), and (7.4)–(7.6). Using (6.3), and (7.4)–(7.6), we may establish the case $m = 0$ of (1.22) by induction on $a+b+c+d$. Using (4.12), (1.26), and (5.3), we may establish (1.22) and (1.23) by induction on m. This completes the proof of Theorem 2.

We note that the verification of (7.7) masks the fact that our proof requires the analytic continuation of the constant term ${}_qAW_n^k(a,0,0,0)$ at $a = -1-(n-1)k$. The details of how to do this are given in [17, Section 10].

Acknowledgments. Research supported by National Science Foundation grant DMS 9002043.

References

1. G. Anderson, *The evaluation of Selberg sums*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 469–472.
2. ———, *A short proof of Selberg's generalized beta formula*, Forum Math. **3** (1991), 415–417.
3. G. E. Andrews and R. Askey, *Another q-extension of the beta function*, Proc. of the A.M.S. **81** (1981), 97–102.
4. G. E. Andrews, *The Theory of Partitions.* In: *Encyclopedia of Mathematics and Its Applications*, Vol.2, (Ed. G.-C. Rota), Addison-Wesley, Reading, Mass., 1976.
5. K. Aomoto, *Jacobi polynomials associated with Selberg's integral*, SIAM J. Math. Anal. **18** (1987), 545–549.
6. R. Askey, *Some basic hypergeometric extensions of integrals of Selberg and Andrews*, SIAM. J. Math. Anal. **11** (1980), 938–951.
7. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), No. 319.
8. R. W. Carter, *Simple Groups of Lie Type*, John-Wiley, London, New York, 1972.
9. F. J. Dyson, *Statistical theory of the energy levels of complex systems I*, J. Math. Phys. **3** (1962), 140–156.
10. R. Evans, *Multidimensional beta and gamma integrals*, Contemp. Math. **166** (1994), 341–357.
11. L. C. Grove and C. T. Benson, *Finite Reflection Groups*, 2nd. ed., Springer-Verlag, New York, 1985.
12. R. A. Gustafson, *A generalization of Selberg's beta integral*, Bull. Amer. Math. Soc. **22** (1990), 97–105.
13. ———, private communications.
14. L. Habsieger, *Une q-intégrale de Selberget-Askey*, SIAM J. Math. Anal. **19** (1988), 1475–1489.
15. F. H. Jackson, *Certain q-identities*, Quart. J. Math. **12** (1941), 167–172.
16. K. W. J. Kadell, *A proof of Askey's conjectured q-analogue of Selberg's integral and a conjecture of Morris*, SIAM J. Math. Anal. **19** (1988), 969–986.
17. ———, *A proof of the q-Macdonald-Morris conjecture for BC_n*, Mem. Amer. Math. Soc. **108** (1994), No. 516.
18. ———, *A simple proof of an Aomoto type extension of the q-Morris theorem*, Contemp. Math. **166** (1994), 167–181.
19. ———, *A simple proof of an Aomoto-type extension of Askey's last conjectured Selberg q-integral*, preprint.
20. ———, *The Selberg-Jack symmetric functions*, Adv. Math. **130** (1997), 33–101.
21. I. G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13** (1982), 988–1007.
22. ———, *Symmetric Functions and Hall Polynomials*, 2nd. ed., Oxford University Press, Oxford, 1995.
23. ———, *Symmetric Functions and Orthogonal Polynomials*, University Lecture Series, Vol.12, American Mathematical Society, Providence, 1998.
24. M. L. Metha, *Random Matrices and the Statistical Theory of Energy Levels*, Academic Press, New York, 1967.
25. M. L. Mehta and F. J. Dyson, *Statistical theory of the energy levels of complex systems: V*, J. Math. Phys. **4** (1963), 713–719.
26. W. G. Morris, II, *Constant term identities for finite and affine root systems: conjectures and theorems*, Ph.D. dissertation, University of Wisconsin-Madison, January 1982.

27. D. St. P. Richards, *Analogs and extensions of Selberg's integral.* In: Proceedings IMA Workshop on q-Series and Partitions, March 1988.
28. A. Selberg, *Bemerkninger om et multipelt integral*, Norsk Mat. Tidsskr. **26** (1944), 71–78.
29. R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), 76–115.
30. S. S. Wilks, *Mathematical Statistics*, Wiley, New York, 1962.
31. J. Wishart, *The generalized product moment distribution in samples from a normal multivariate population*, Biometrika **20A** (1928), 32–43.

Department of Mathematics, Arizona State University, Tempe, AZ 85278-1804, USA